Synchronization in small-world networks

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In this paper we consider complete synchronization in small-world networks of identical Rössler oscillators. By applying a simple but effective dynamical optimization coupling scheme, we realize complete synchronization in networks with undelayed or delayed couplings, as well as ensuring that all oscillators have uniform intensities during the transition to synchronization. Further, we obtain the coupling matrix with much better synchronizability in a certain range of the probability p for adding long-range connections. Direct numerical simulations fully verify the efficiency of our mechanism. © *2008 American Institute of Physics*. [DOI: 10.1063/1.2939136]

In the past decade, synchronization in complex networks, especially the question of synchronizability, attracts a lot of interest. The works on synchronizability in networks with a given topology can be divided into two classes according to the coupling matrix in networks. One is the static mechanism, where the coupling matrix remains fixed during the transition to synchronization. From the degree and load based weighted networks, the synchronizability becomes optimal when the intensities of all oscillators become uniform. The other one is the dynamical mechanism, where the coupling matrix evolves in time by introducing adaptive strengths between connected oscillators. The adaption process can enhance synchronization by modifying the coupling matrix in networks, but the synchronizability is still far from being optimal. This is because the resulting networks have nonuniform intensities even for networks with homogeneous degrees. In this paper we consider complete synchronization in small-world networks of identical Rössler oscillators by applying a simple but effective dynamical optimization coupling scheme. We realize complete synchronization in networks with undelayed or delayed couplings, as well as ensure that all oscillators have uniform intensities during the transition to synchronization. Moreover, we obtain the coupling matrix with much better synchronizability in a certain range of the probability *p* for adding long-range connections.

I. INTRODUCTION

In the past decade, the dynamics of complex networks has been extensively investigated, with special emphasis on the interplay between the complexity in the overall topology and the local dynamical properties of the coupled oscillators.[1–28] As a typical kind of dynamics on complex networks, synchronization, especially the ability of networks to become synchronized (synchronizability), has attracted a lot of interest in multidisciplinary fields.[8–28] The works on synchronizability in networks with a given topology can be divided into two classes according to the coupling matrix. One is the static mechanism, where the coupling matrix remains fixed during the transition to synchronization. The character is that the coupling matrix unidirectionally affects synchronization.[9,10,13–20] It has been recently shown that for randomly enough unweighted and weighted networks,[20] the synchronizability is controlled by $S_{\max}/S_{\min}$, where $S_{\max}$ and $S_{\min}$ are the maximum and minimum of the intensities S_i, defined by the sum of the couplings for oscillator i. For unweighted scale-free networks (SFNs) generated by the Barabási–Albert model,[2] $S_{\max}/S_{\min}=k_{\max}/k_{\min}\sim N^{1/2}$, where N is the network size, $k_{\max}$ and $k_{\min}$ are the maximal and minimal degrees, respectively. From the degree[15,16] and load[13,14] based weighted networks, the synchronizability becomes optimal when the intensities of all oscillators become uniform.

The other is the dynamical mechanism, where the coupling matrix evolves in time by introducing adaptive strengths between connected oscillators. The adaptation process can enhance synchronization by modifying the coupling matrix. However, during the transition to synchronization, the dynamical mechanism[21,22] cannot ensure uniform intensities even for small-world networks (SWNs), which is not consistent with the necessary condition for the optimal synchronizability in the static mechanism.[13–16,19] Zhou and Kurths proposed a dynamical mechanism using local information among each oscillator and its neighbors.[21] In the corresponding networks the connections between different oscillators are strengthened. The adaptive process drives the network into the direction of a more homogeneous topology, ongoing with an enhanced ability for synchronization. Thereby it is possible to synchronize networks that exceed by several orders of magnitude the size of the largest com-

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parable random graph that is still synchronizable.[29] For simplicity, we call this mechanism the Zhou–Kurths method. It shows that the Zhou–Kurths method is very effective to realize synchronization in SFNs, and can enhance the synchronizability in SFNs substantially. After the adaptation of the couplings, the weights of incoming links V_i scale with the degree k of the corresponding oscillator $\mathbf{x}_i$ as $V(k) \sim k^{-\theta}$, and the synchronizability is characterized by $S_{\max}/S_{\min} \sim N^{\beta/2}$ with $\beta = 1-\theta$ and $\theta = 0.54 \pm 0.01$ for SFNs of Rössler oscillators, and the average intensity $S(k)$ over oscillators with degree k increases as $S(k) \sim k^{\beta}$ (Ref. 21).

In this paper we consider complete synchronization in SWNs, especially the Newman–Watts (NW) model,[28] by introducing a simple but effective dynamical mechanism. Our aims are to (i) realize complete synchronization in SWNs with undelayed or delayed couplings, whose oscillators all have uniform intensities during the transition to synchronization, and (ii) to assign the coupling matrix with enhanced synchronizability in certain cases.

By applying the *dynamical optimization* (DO) mechanism,[24] we will achieve the above aims. The DO mechanism adjusts the coupling strengths based on the "winner-take-all" strategy. It realizes complete synchronization in SFNs with undelayed couplings, as well as enhances the synchronizability greatly.[24] In this paper, we extend the DO mechanism to NW networks with undelayed or delayed couplings. We show that the DO mechanism is more effective in realizing synchronization in NW networks than the Zhou–Kurths method. Since the DO mechanism can ensure the uniform intensities of all oscillators, it can also effectively realize synchronization in NW networks with delayed couplings. But the Zhou–Kurths method cannot realize synchronization in networks with delayed couplings. Moreover, in a certain range of the probability p for adding long-range connections, we design a coupling matrix for NW networks, which has much better synchronizability than unweighted networks, degree based weighted networks and the Zhou–Kurths method.

This paper is organized as follows: In the next section, by applying the DO mechanism, we can realize complete synchronization in NW networks of identical Rössler oscillators, as well as ensure the uniform intensities of all oscillators during the transition to synchronization. In Sec. III, we enhance the synchronizability in NW networks by designing the coupling matrix. We draw up our conclusions in the last section.

II. SYNCHRONIZATION IN SMALL-WORLD NETWORKS

Our general model for networks consisting of N coupled identical Rössler oscillators with a time-varying coupling matrix is given by

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sum_{j=1}^{N} G_{ij}\mathbf{H}(\mathbf{x}_j,\mathbf{x}_i), \tag{1}$$

where $\mathbf{x}_i$ is the state, $\mathbf{F}(\mathbf{x}_i)$ is the dynamics of the individual oscillator $\mathbf{x}_i$, $\mathbf{H}(\mathbf{x}_j,\mathbf{x}_i)$ is the inner coupling function, $G = (G_{ij})$ is the outer coupling matrix. $G_{ij} = A_{ij}W_{ij}$, where $A = (A_{ij})$ is the binary adjacency matrix, W_{ij} is the coupling strength of the incoming link $(\mathbf{x}_i,\mathbf{x}_j)$ pointing from oscillator $\mathbf{x}_j$ to oscillator $\mathbf{x}_i$ if they are connected, $G_{ii} = -\Sigma_{j\in K_i} A_{ij}W_{ij}$, K_i is the neighbor set of oscillator $\mathbf{x}_i$.

In this paper we consider complete synchronization in network (1) in two cases. (i) One case is the network (1) with undelayed couplings, where the function $\mathbf{H}(\mathbf{x}_j,\mathbf{x}_i) = \mathbf{H}_0(\mathbf{x}_j) - \mathbf{H}_0(\mathbf{x}_i)$, and $\mathbf{H}_0$ is the output function for each oscillator. (ii) The other case is the network (1) with delayed couplings, in which the function $\mathbf{H}(\mathbf{x}_j,\mathbf{x}_i) = \mathbf{H}_0[\mathbf{x}_j(t-\tau)] - \mathbf{H}_0[\mathbf{x}_i(t)]$ with a time delay $\tau > 0$.

Our aim is to realize complete synchronization in network (1), as well as ensure that all oscillators in network (1) have uniform intensities during the transition to synchronization. Recently, we have already obtained some results on this problem. For different variants of the Kuramoto model, we have proposed a *dynamical gradient network* (DGN) approach to realize phase synchronization.[23] It shows that all the oscillators have uniform intensities during the transition to synchronization. However, the DGN approach is very special in two aspects. One is that it should assign a scale potential to each oscillator within any time interval, which depends on the extent of the local synchronization among itself and its neighbor oscillators. The other is that the incoming link to be adjusted by the DGN approach is often not mostly effective. Inspired by the idea of the DGN approach,[23] we have further introduced a DO mechanism for SFNs.[24] It reflects the "winner-take-all" strategy, where the incoming link to be adjusted is always chosen as a pair of oscillators with the weakest synchronization. This means that the DO mechanism is more effective than the DGN approach. We also show that the DO mechanism has much better synchronizability in SFNs than the Zhou–Kurths method.[24]

In this paper, we apply the DO mechanism to SWNs with undelayed or delayed couplings. Here we first introduce the idea of the DO mechanism. The DO mechanism is to increase the strength of only one incoming link of each oscillator by a small amount ε, at every time step $t_n = t_0 + nT$ for $n \geqslant 1$, where t_0 is the transient time, and $T > 0$ is the length of time intervals. For oscillator $\mathbf{x}_i$ and its neighbor oscillator $\mathbf{x}_j$, the total synchronization difference,

$$E_n(i,j) = \int_{t_{n-1}}^{t_n} \phi(\mathbf{x}_i,\mathbf{x}_j)dt \tag{2}$$

within the interval $[t_{n-1},t_n)$ is evaluated, where ϕ is a synchronization error function. For complete synchronization in SWNs, the function ϕ is a non-negative error function if oscillators i,j are not synchronized, and satisfies $\phi(\mathbf{x}_i,\mathbf{x}_i) = 0$ if oscillators i,j are synchronized. To check complete synchronization in network (1), the function ϕ is chosen as $\phi(\mathbf{x}_i,\mathbf{x}_j) = |x_i - x_j| + |y_i - y_j| + |z_i - z_j|$.

The total synchronization difference $E_n(i,j)$ reflects the competition ability of the incoming link $(\mathbf{x}_i,\mathbf{x}_j)$ within the time interval $[t_{n-1},t_n)$. For oscillator $\mathbf{x}_i$, the incoming link with the weakest synchronization, i.e., $(\mathbf{x}_i,\mathbf{x}_{j^n_{\max}})$, is the winner within the interval $[t_{n-1},t_n)$, where the index $j^n_{\max}$ is decided by the following dynamical optimization problem:

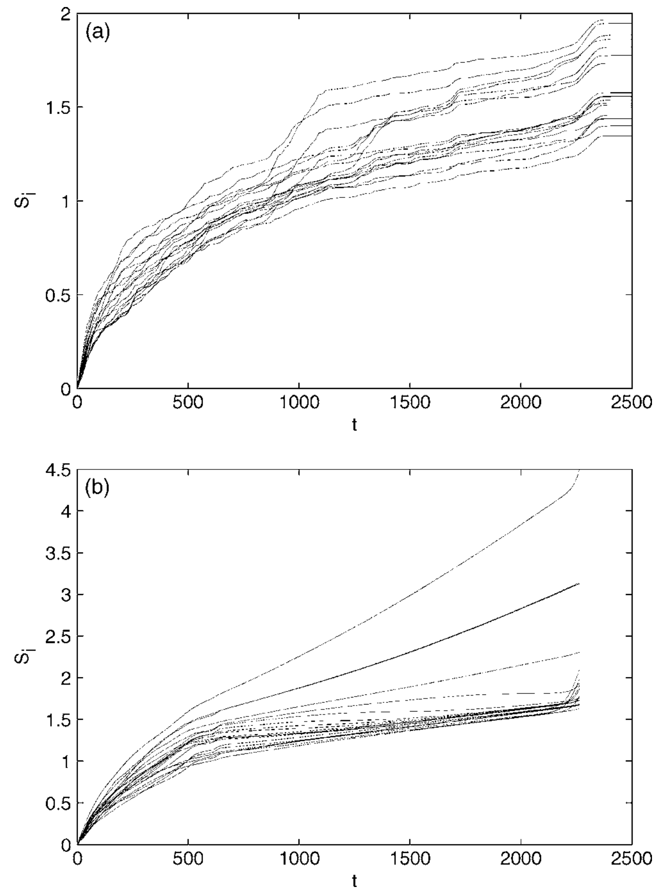

FIG. 1. The intensities S_i as a function of time t for arbitrarily 20 oscillators in networks with undelayed couplings (a), or delayed couplings (b), by the Zhou–Kurths method. The parameters are $N=500$, $K=4$, $p=0.003$, $\gamma=0.002$, $\tau=0.01$ s.

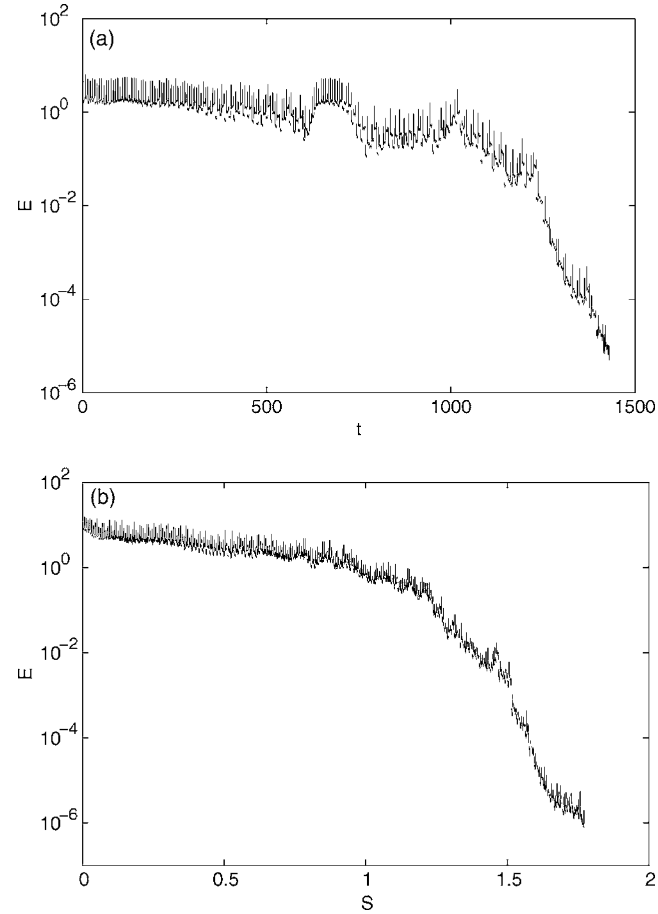

FIG. 2. The average synchronization error E in networks with undelayed couplings as a function of (a) time t, and (b) the intensity S, by the DO mechanism. The parameters are $N=500$, $K=4$, $p=0.003$, $T=1$ s, $\varepsilon=0.001$.

$$j^n_{\max} = \arg\max_{j\in K_i} E_n(i,j). \tag{3}$$

The solutions of the optimization (3) within different intervals are also different, which depends on the dynamics of oscillators. The connection strength is then adjusted dynamically by

$$W^{n+1}_{ij^n_{\max}} = W^n_{ij^n_{\max}} + \varepsilon, \quad W^{n+1}_{ij} = W^n_{ij}, \quad j \neq j^n_{\max}, \tag{4}$$

where $\varepsilon > 0$ is a small value, and W^n_{ij} is the coupling strength in the interval $[t_{n-1}, t_n)$.

From Eqs. (3) and (4), we strengthen the incoming link with the weakest synchronization, namely, the link with the maximal competition ability. For complete synchronization in SWNs with undelayed couplings, the additional term $\varepsilon(\mathbf{x}_{j^n_{\max}} - \mathbf{x}_i)$ can be regarded as the negative feedback term for the unidirectional synchronization from oscillator $\mathbf{x}_{j^n_{\max}}$ to oscillator $\mathbf{x}_i$. This could make the synchronization difference between oscillator $\mathbf{x}_i$ and its neighbor $\mathbf{x}_{j^n_{\max}}$ be smaller, which implies the synchronization in SWNs be realized.

Note that the intensities of all oscillators in network (1) are uniform, since at each step the intensity of each oscillator increases by the same amount ε. This is consistent with the necessary condition for optimal synchronizabiliy in the static mechanism.[13–16,19]

In order to show the effectiveness of the DO mechanism, our analysis and simulations are based on SWNs generated by the NW model.[28] The initial network is a K nearest-neighbor coupled network consisting of N oscillators arranged in a ring, with each oscillator $\mathbf{x}_i$ being adjacent to its neighbor oscillators $\mathbf{x}_{i\pm 1}, \cdots, \mathbf{x}_{i\pm K/2}$, and with K being even. Then one adds with probability p a connection between a pair of oscillators. In the following, network (1) is a network of Rössler oscillators, $\mathbf{x}_i=(x_i, y_i, z_i)$, $\mathbf{F}(\mathbf{x}_i)=[-0.97y_i - z_i, 0.97x_i + 0.15y_i, z_i(x_i - 8.5) + 0.4]$, the function $\mathbf{H}_0(\mathbf{x}_i)=(x_i, 0, 0)$. In order to show complete synchronization, we define the average synchronization error as $E=\frac{1}{N}\sum_{i=1}^{N}\|\mathbf{x}_i - \bar{\mathbf{x}}\|$, where $\bar{\mathbf{x}}=(\bar{x}, \bar{y}, \bar{z})$ is the mean-field of all $\mathbf{x}_i$. In our simulations, the initial coupling strengths for all incoming links are zero, the transient time is $t_0=100$ s, the length of time intervals is $T=1$ s, and the small value is $\varepsilon=0.001$. Further, initial conditions for all oscillators are randomly chosen from the chaotic attractor. The solution of network (1) is solved by using the Euler method with the time step $h=0.01$ s, and our ending condition for the DO mechanism is $E < 10^{-5}$.

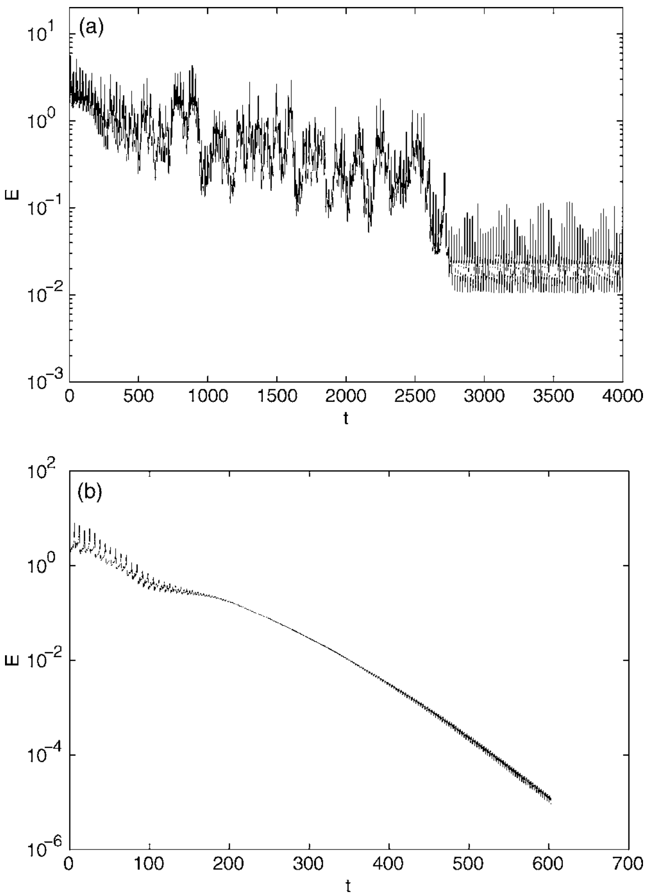

FIG. 3. The average synchronization error E in networks with delayed couplings as a function of time t. (a) The Zhou–Kurths method (τ=0.01 s). (b) The DO mechanism (τ=2 s). The parameters N=500, p=0.003, γ=0.002, T=1 s, ε=0.001.

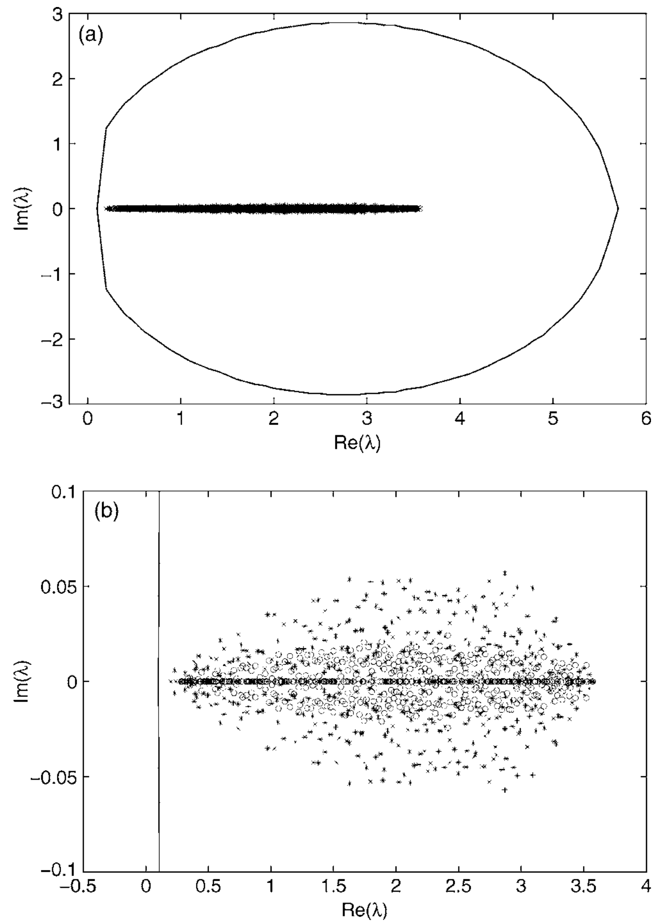

FIG. 4. [(a) and (b)] Distribution of eigenvalues of the Laplacian matrix of σG_{norm} in network (1) with undelayed couplings (○) and delayed couplings (*). Solid line: The stability region $\mathcal{R}$. The parameters are N=500, K=4, p=0.003, T=1 s, ε=0.001, σ=2, τ=1 s.

In this paper we consider complete synchronization in NW networks in two cases. (i) One case is the network (1) with undelayed couplings, where the function $\mathbf{H}(\mathbf{x}_j,\mathbf{x}_i)=\mathbf{H}_0(\mathbf{x}_j)-\mathbf{H}_0(\mathbf{x}_i)$. From recent works,[21,22] the dynamical mechanism can realize complete synchronization both in SFNs with undelayed couplings and in SWNs with undelayed couplings. However, even for NW networks with homogeneous degrees, the dynamical mechanisms cannot ensure uniform intensities if all oscillators have different initial conditions. We plot the intensities S_i, defined by the sum of the coupling strengths of neighbor oscillators of oscillator i (i.e., $S_i=\Sigma_{j\in K_i}G_{ij}$), for 20 arbitrarily chosen oscillators in NW networks according to the Zhou–Kurths method [Fig. 1(a)]. When the adaptation parameter is chosen as γ=0.002 in the Zhou–Kurths method, we find that the Zhou–Kurths method cannot ensure uniform intensities during or after the adaptation. Based on the DO mechanism, complete synchronization in NW networks are realized effectively [Fig. 2(a)], and the intensities are always uniform during the transition to synchronization. From Fig. 2(b), the intensity $S=S_i$ is also a good indicator for synchronization in networks. As S increases to a critical value, a network becomes synchronous.

(ii) The other case is the network (1) with delayed couplings, in which the function $\mathbf{H}(\mathbf{x}_j,\mathbf{x}_i)=\mathbf{H}_0[\mathbf{x}_j(t-\tau)]-\mathbf{H}_0[\mathbf{x}_i(t)]$ with a time delay $\tau>0$. Even for a small time delay τ (such as τ=0.01 s), the Zhou–Kurths method cannot realize synchronization in NW networks [Fig. 3(a)]. The synchronization error between two connected oscillators is about $10^{-2}\times 500=5$ for networks with N=500. Due to the DO mechanism, complete synchronization can be realized effectively when the time delay τ=2 s [Fig. 3(b)]. The synchronization error is about $10^{-5}\times 500=0.005$. Hence the DO mechanism is more effective than the Zhou–Kurths method. The main reason is that the DO mechanism ensures that the intensities are always uniform during the transition to synchronization. But the Zhou–Kurths method cannot ensure uniform intensities even for the small time delay [Fig. 1(b)]. Though the difference of intensities between oscillators is small initially, it becomes large as time increases. The uniformity of intensities is the necessary condition for the existence of a synchronous manifold in NW networks with delayed couplings. After the adaptation, the synchronous manifold is given by $\mathbf{x}_i(t)=\mathbf{x}_0(t)$, $i=1,\ldots,N$, where $\mathbf{x}_0(t)$ is the solution of the isolated dynamics $\dot{\mathbf{x}}_0(t)=\mathbf{F}[\mathbf{x}_0(t)]+S_0\{\mathbf{H}_0[\mathbf{x}_0(t-\tau)]-\mathbf{H}_0[\mathbf{x}_0(t)]\}$, $S_0=\varepsilon n_0$ is the ultimate intensity, and n_0 is the ending adjustment step.

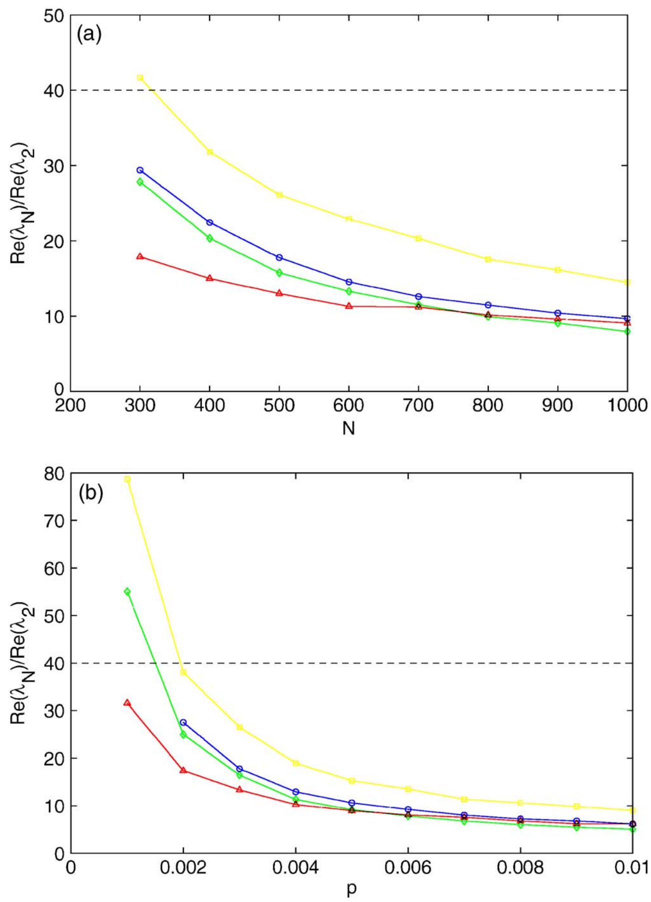

FIG. 5. (Color online) The ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ as a function of network size N for a fixed probability $p=0.003$ (a), and the probability p for a fixed size $N=500$ (b). Yellow line (□): type I networks; green line (◇): type II networks; blue line (○): type III networks; red line (△): type IV networks; black dashed line: the maximal ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ in the region $\mathcal{R}$. The parameters are $K=4$, $\gamma=0.002$, $T=1$ s, $\varepsilon=0.001$. All the estimates are averaged over 20 realizations of networks.

III. SYNCHRONIZABILITY IN SMALL-WORLD NETWORKS

In this section we discuss the synchronizability of NW networks. We first briefly review the stability of networks,

$$\dot{\mathbf{x}}_i = \mathbf{F}(\mathbf{x}_i) + \sigma \sum_{j=1}^{N} G_{ij}^0 [\mathbf{H}_0(\mathbf{x}_j) - \mathbf{H}_0(\mathbf{x}_i)], \qquad (5)$$

where $\mathbf{H}_0$ is the output function, and σ is the overall coupling strength in networks. Without loss of generality, we assume that the coupling matrix $G^0=(G_{ij}^0)$ is asymmetric. The coupling matrix $G^0=(A_{ij}W_{ij}^0)$ is similarly defined as the matrix G in network (1). The variational equation for the synchronous state $\{\mathbf{x}_i=\mathbf{s}, \forall i\}$ is given by $\dot{\xi}_i=[D\mathbf{F}(\mathbf{s}) - \sigma\lambda_l D\mathbf{H}_0(\mathbf{s})]\xi_i$, where $\dot{\mathbf{s}}=\mathbf{F}(\mathbf{s})$ is the dynamics of the isolated oscillator, D is the Jacobian operator, and λ_l is a complex eigenvalue of the Laplacian matrix L $(=-G^0)$, satisfying $\mathrm{Re}(\lambda_1)\leqslant \mathrm{Re}(\lambda_2)\leqslant \cdots \leqslant \mathrm{Re}(\lambda_N)$. The largest Lyapunov exponent (LLE), $\Lambda(\alpha,\beta)$, of the master stability equation $\dot{\eta}=[D\mathbf{F}_0(\mathbf{s})-(\alpha+i\beta)D\mathbf{H}_0(\mathbf{s})]\eta$ is a function of the parameters α and β, which is known as the master stability function (MSF).[25,26] Let $\mathcal{R}$ be the region in the complex plane where the MSF provides a negative LLE [Figs. 4(a) and 4(b): The region $\mathcal{R}$ bounded by the solid line]. The condition for complete synchronization in network (5) is that the set $\{\sigma\lambda_l, \lambda_l \neq 0\}$ is entirely contained in $\mathcal{R}$.[25,26] Here we only consider the case where the region $\mathcal{R}$ is bounded. For the networks of Rössler oscillators in this paper, the stability region $\mathcal{R}$ is shown by the solid line in Fig. 4(a). In order to judge whether the set $\{\sigma\lambda_l, \lambda_l\neq 0\}$ is in the stability region in the case of the complex eigenvalues of the Laplacian matrix L, one should minimize the ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ for a fixed value of $\max|\mathrm{Im}(\lambda_l)|$, and one should minimize $\max|\mathrm{Im}(\lambda_l)|$ for a fixed value of the ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$. Summing up the above minimization, a good condition is that $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ and $\max|\mathrm{Im}(\lambda_l)|$ are simultaneously minimized.[14,17]

In this section we analyze the synchronizability in NW networks by applying the DO mechanism. From our recent work[24] the DO mechanism can ensure uniform intensities of all oscillators in networks, regardless of the initial conditions of the oscillators in networks with undelayed or delayed couplings. Here we design the coupling matrix G^0 in network (5) through the adaptation of the coupling strengths in network (1). After the adaptation by the DO mechanism, the coupling matrix G^0 in network (5) is assigned by the following matrix:

$$G^0 = G_{\mathrm{norm}} = G_{\mathrm{end}}/S_0, \qquad (6)$$

where G_{end} is the coupling matrix of network (1) after the adaptation. For the coupling matrix $G^0=G_{\mathrm{norm}}$, all eigenvalues are fully contained within the unit circle centered at 1.[14,17] So $0\leqslant \mathrm{Re}(\lambda_l)\leqslant 2$, $|\mathrm{Im}(\lambda_l)|\leqslant 1$, and the largest $\mathrm{Re}(\lambda_N)$ will never diverge. During the transition to synchronization in network (1), $S_{\max}/S_{\min}$ always equals 1 in the DO mechanism.

It should be pointed out that $\max|\mathrm{Im}(\lambda_i)|$ is sufficiently small due to the DO mechanism (the maximal value is less than 0.1). Even for a large coupling strength $\sigma=2$, all the nonzero eigenvalues of the Laplacian matrix of σG_{norm} are located in a narrow region around the real axes in the stabil-

TABLE I. The ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ as a function of network size N for a fixed probability $p=0.003$.

	N							
Type	300	400	500	600	700	800	900	1000
I	41.66	31.83	26.10	22.91	20.34	17.55	16.17	14.51
II	27.84	20.36	15.77	13.29	11.53	9.96	9.08	7.97
III	29.38	22.44	17.61	14.56	12.60	11.49	10.42	9.68
IV	17.88	15.00	12.99	11.31	11.22	10.17	9.63	9.08

TABLE II. The ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ as a function of the probability p for a fixed size $N=500$.

	p								
Type	0.002	0.003	0.004	0.005	0.006	0.007	0.008	0.009	0.01
I	38.04	26.53	18.98	15.25	13.52	11.38	10.63	9.88	9.01
II	24.98	16.45	11.32	9.16	7.85	6.78	6.03	5.50	5.07
III	27.52	17.77	12.92	10.61	9.22	8.02	7.23	6.80	6.18
IV	17.37	13.33	10.26	8.96	8.06	7.55	6.79	6.22	6.19

ity region $\mathcal{R}$ [Fig. 4(a)]. Hence the ratio $\mathrm{Re}(\lambda_N)/\mathrm{Re}(\lambda_2)$ indicates the synchronizability in networks. In order to show the enhanced synchronizability in NW networks, we compare the synchronizability in the unweighted network (5) (type I network: $W_{ij}^0=1$), the degree based weighted network (5) (type II network: $W_{ij}^0=1/k_i$), network (5) with adaptive couplings by the Zhou–Kurths method (type III network), and network (5) with the coupling matrix being designed by network (1) with undelayed couplings (type IV networks).

We find that for a fixed small probability p (such as $p=0.003$) for adding long-range connections, the synchronizability in type III networks is better than that in type I networks, but it is worse than that in type II networks, no matter how large the size N of the networks is [Fig. 5(a), Table I]. However, we find that type IV networks have a better synchronizability than both type II networks and type III networks when the size is not too large. Of course, the smaller the probability p, the larger the size of type IV networks with better synchronizability than both type II networks and type III networks. For the fixed size $N=500$, we observe similar results in a certain range of the probability p [Fig. 5(b), Table II]. From Fig. 5 and Tables I and II, we see that the synchronizability in type IV networks is better than those in type II networks and type III networks in some cases. It is reasonable that type IV networks have better synchronizability than type III networks. This is because the DO mechanism ensures uniform intensities of all oscillators in type IV networks. Now we further analyze the reason why type IV networks have better synchronizability than type II networks in a certain range of the probability p.

In order to do so, we slightly modify NW networks. The initial network is a K-nearest-neighbor coupled network consisting of N oscillators arranged in a ring, with each oscillator $\mathbf{x}_i$ being adjacent to its K neighbor oscillators $\mathbf{x}_{i\pm1},\ldots,\mathbf{x}_{i\pm K/2}$, and with K being even. Then one adds with probability p a long-range connection between a pair of os-

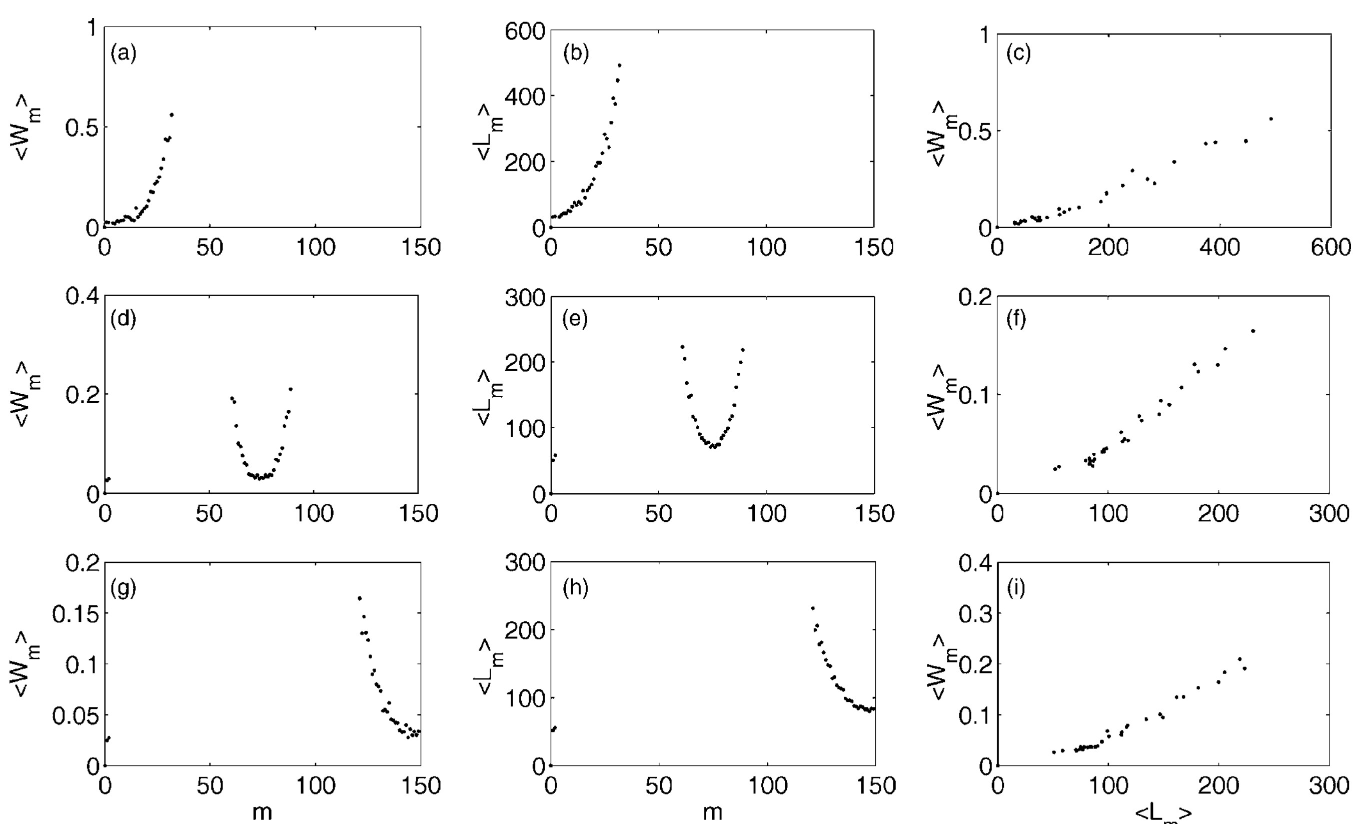

FIG. 6. The dependence of $\langle W_m\rangle$ on m in (a), (d), and (g) of $\langle L_m\rangle$ on m in (b), (e), and (h), and the relationship between $\langle W_m\rangle$ and $\langle L_m\rangle$ in (c), (f), and (i), respectively. $n_1=0$, $n_2=30$ [(a), (b), and (c)]; $n_1=60$, $n_2=90$ [(d), (e), and (f)]; $n_1=120$, $n_2=150$ [(g), (h), and (i)]. The parameters in type V networks are $N=300$, $K=4$, $p=0.2$, $T=1$ s, $\varepsilon=0.001$.

cillators with indices satisfying $n_1 \leqslant \min\{|i-j|, N-|i-j|\} \leqslant n_2$, where $0 \leqslant n_1$, $n_2 \leqslant N/2$ are two positive integers. This kind of network is called type V networks. Based on type V networks, we adjust the coupling strengths by the DO mechanism. After the adaptation, we define the average coupling strength $\langle W_m \rangle$ over the k_W links having the same $m = \min\{|i-j|, N-|i-j|\}$,

$$\langle W_m \rangle = \frac{1}{k_W} \sum G_{ij}. \tag{7}$$

Further, for the unweighted type V networks, the average load $\langle L_m \rangle$ over the k_L links having the same m is given by

$$\langle L_m \rangle = \frac{1}{k_L} \sum L_{ij}, \tag{8}$$

where the load L_{ij} of the link connecting oscillators $\mathbf{x}_i$ and $\mathbf{x}_j$ quantifies the traffic of the shortest paths passing that link. Here the size of type V networks is $N=300$ and the probability $p=0.2$. For different n_1 and n_2, we plot the relationship between $\langle W_m \rangle$ and m [Figs. 6(a), 6(d), and 6(g)], and the relationship between $\langle L_m \rangle$ and m [Figs. 6(b), 6(e), and 6(h)], respectively. From these subfigures, we conclude that $\langle W_m \rangle$ has a similar dependence on m as $\langle L_m \rangle$, which is further verified by the relationship $\langle W_m \rangle \sim \langle L_m \rangle$ [Figs. 6(c), 6(f), and 6(i)]. This implies that the adaptation due to the DO mechanism may lead to a similar synchronizability as the load based weighted networks. This may in part explain why type IV networks have a better synchronizability than type II networks in a certain range of the probability p for adding long-range connections.

IV. CONCLUSION

This paper considers complete synchronization in small-world networks of identical Rössler oscillators. Differing from the exiting dynamical mechanism, we apply a simple but effective DO mechanism to networks with undelayed or delayed couplings. We realize complete synchronization in networks, as well as ensure that all oscillators have uniform intensities during the transition to synchronization. The uniformity of the intensities is consistent with the necessary condition for the optimal synchronizability in the static mechanism. Further, we design a coupling matrix with much better synchronizability in a certain range of the probability p for adding long-range connections.

The DO mechanism can also be applied to the phase synchronization in SWNs with nonidentical oscillators. For example, we consider the phase synchronization in the Kuramoto model.[22,23,30,31] In this case, $\mathbf{x}_i = \theta_i$, $\mathbf{F}(\mathbf{x}_i) = w_i$, w_i are frequencies uniformly distributed in the interval $[-\triangle, \triangle]$ with $\triangle > 0$, $\mathbf{H}(\mathbf{x}_j, \mathbf{x}_i) = \sin(\theta_j - \theta_i)$ for the undelayed couplings and $\mathbf{H}(\mathbf{x}_j, \mathbf{x}_i) = \sin[\theta_j(t-\tau) - \theta_i]$ for the delayed couplings, the error function $\phi(\mathbf{x}_i, \mathbf{x}_j) = 1 - r_n(i,j)$ with $r_n(i,j) e^{\zeta \Psi_n(i,j)} = (e^{\zeta \theta_j} + e^{\zeta \theta_i})/2$ and $\zeta^2 = -1$, where $0 \leqslant r_n(i,j) \leqslant 1$ measures the extent of the synchronization of oscillators i, j, and $\Psi_n(i,j)$ stands for an average phase. Hence $\phi(\mathbf{x}_i, \mathbf{x}_j)$ are non-negative oscillators $\mathbf{x}_i, \mathbf{x}_j$ are not synchronized, and $\phi(\mathbf{x}_i, \mathbf{x}_j) = 0$ if oscillators $\mathbf{x}_i, \mathbf{x}_j$ are synchronized. Of course, our mechanism can be applicable to the synchronization in networks with nonidentical chaotic oscillators (such as Rössler oscillator) provided that the term of "phase" in networks is well-defined.[8]

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