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SYNCHRONIZATION IN THE WINFREE MODEL OF COUPLED NONLINEAR OSCILLATORS

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Abstract

Synchronization is studied in a population of phase oscillators with mean-field coupling—a special case of the more general Winfree model. Each oscillator is coupled to the mean-field with a strength dependent on its phase. The uncoupled frequencies of the oscillators are assumed to be randomly distributed according to a specified population density. The response of this system is considered as a function of two parameters. The first describes the strength of the coupling between each oscillator while the second characterizes the distribution of uncoupled frequencies in the population. As these parameters are varied, the synchronous solution can disappear as oscillators near the edge of the population (large deviation of their natural frequency from the population mean) no longer remain locked to the mean frequency. Using a Poincare-Lindstedt analysis, the bifurcation describing the loss of synchrony is characterized for a general oscillator population. The bifurcation is then calculated for several different frequency distributions, including even polynomial distributions and populations that correspond to a discrete system of N oscillators. By formulating the discrete system in a continuum framework, the analysis can be carried through and the bifurcation curve can be calculated for arbitrarily large discrete populations. Results of the discrete system will be presented from $N = 2$ to $N = 10^6$ oscillators. As the number of discrete oscillators increases, approaching the continuum uniform distribution, the coefficients of the bifurcation curve increase and become singular as N grows without bound.

Key words

Synchronization, Winfree model, Coupled oscillators, Mean-field coupling

1 Introduction

Synchronization describes the collective locking in a population of oscillators to a single frequency. The present work considers this phenomenon in a population of phase oscillators with mean-field coupling—a special case of the more general Winfree model (Winfree, 1967). In this system, each oscillator is coupled to the mean-field with a strength dependent on its phase. This coupling, described through a phase-response curve, can either advance or delay the phase of the oscillator, so that the coupling cannot be described as strictly excitatory or inhibitory. Moreover, the uncoupled frequencies of the oscillators are assumed to be randomly distributed across the population according to a specified population density, leading to an integro-differential equation. The response of this system is described as two parameters are varied. The first describes the strength of the coupling between each oscillator while the second characterizes the distribution of uncoupled frequencies in the population.

This work is based on analysis by Ariaratnam and Strogatz (2001), in which a uniform distribution of natural frequencies was considered. The authors classified several qualitatively different states for the population, including incoherence, synchronization, and oscillator death. As the above parameters are varied, the synchronous solution can disappear as oscillators near the edge of the population (large deviation of their natural frequency from the population mean) no longer remain locked to the mean frequency. However, in Ariaratnam and Strogatz the bifurcation curve describing the loss of synchrony was only determined numerically. For the uniform frequency distribution the bifurcation curve contains a singularity in the coefficients of the perturbation expansion. This analysis is generalized to consider any even distribution for the uncoupled frequencies of the population.

Using a Poincare-Lindstedt analysis, the bifurcation describing the loss of synchrony is characterized in terms of the phase distribution of the oscillators and the coupling to the mean-field. The bifurcation is then calculated for several different frequency distributions, including even polynomial distributions and populations that correspond to a discrete system of N oscillators. One novel feature of this analysis is that by formulating the discrete system in a continuum framework, arbitrarily large discrete populations can be considered. Results of the discrete system will be presented from $N = 2$ to $N = 10^6$ oscillators. As the number of discrete oscillators increases, approaching the continuum uniform distribution, the coefficients of the bifurcation curve increase without bound.

2 Synchronization in the Winfree Model

Consider a population of oscillators whose evolution is governed by the integro-differential equation

$$\frac{\partial\Theta}{\partial t}(t,\nu) = 1 + \Gamma\,\nu - \mathrm{P}\ \sin\Theta(t,\nu)\cdot \cdot\int_{-1}^{1}\Big(1+\cos\Theta(t,\mu)\Big)\ h(\mu)\ d\mu. \tag{1}$$

The term $h(\mu)$ represents the distribution of natural frequencies over the population. Moreover $h(\mu)$ is assumed to be an even function and

$$\int_{-1}^{1} h(\mu)\ d\mu = 1. \tag{2}$$

This system was considered by Ariaratnam and Strogatz (2001) for a population of oscillators with a uniform distribution of natural frequencies. The parameters Γ and P are assumed to be small, corresponding to a system of nearly identical oscillators with weak coupling, so that $\Gamma \to \varepsilon\Gamma$, and $\mathrm{P} \to \varepsilon\mathrm{P}$. For $\varepsilon = 0$ the evolution equation is simply

$$\frac{\partial\Theta}{\partial t}(t,\nu) = 1, \tag{3}$$

which possesses the general solution

$$\Theta(t,\nu) = t + \phi_0(\nu). \tag{4}$$

Therefore, Eq. (1) is subject to the following transformation

$$\Theta(t,\nu) = \Omega\, t + \Big(\Phi(\nu) + \hat{\Theta}(t,\nu)\Big). \tag{5}$$

In this, Ω represents the mean frequency of the population and $\Phi(\nu) + \hat{\Theta}(t,\nu)$ describes the evolution of each oscillator with respect to the mean, so that

$$\int_0^{2\pi} \hat{\Theta}(\tau,\nu)\ d\tau = 0. \tag{6}$$

With this, the evolution equation becomes

$$\begin{aligned}\frac{\partial\hat{\Theta}}{\partial t}(t,\nu) &= (1-\Omega) + \Gamma\,\nu \\ &\quad -\mathrm{P}\ \sin\Big(\Omega\, t + \Phi(\nu) + \hat{\Theta}(t,\nu)\Big)\cdot \\ &\quad \cdot\int_{-1}^{1}\Big(1+\cos\Big(\Omega\, t + \Phi(\mu) + \hat{\Theta}(t,\mu)\Big)\Big)\ h(\mu)\ d\mu.\end{aligned} \tag{7}$$

2.1 Lindstedt's Method

In what follows Lindstedt's method is used to characterize periodic solutions for $\varepsilon \neq 0$ and to identify the bifurcation curve on which these periodic solutions arise. To begin, the time scale $\tau = \Omega\, t$ is introduced together with the following expansions

$$\begin{gathered}\Gamma = \sum_{i=1}^{n}\varepsilon^i\,\gamma_i, \quad \mathrm{P} = \sum_{i=1}^{n}\varepsilon^i\,\rho_i, \\ \Omega = 1 + \sum_{i=1}^{n}\varepsilon^i\,\omega_i, \\ \hat{\Theta}(\tau,\nu) = \sum_{i=1}^{n}\varepsilon^i\,\theta_i(\tau,\nu), \quad \Phi(\nu) = \sum_{i=0}^{n}\varepsilon^i\,\phi_i(\nu),\end{gathered} \tag{8}$$

and these expansions are returned to Eq. (7). At $\mathcal{O}(\varepsilon)$ the resulting equation becomes

$$\frac{\partial\theta_1}{\partial\tau}(\tau,\nu) = -\omega_1 + \gamma_1\,\nu - \rho_1\sin(\tau+\phi_0(\nu))\cdot \cdot\int_{-1}^{1}\Big(1+\cos(\tau+\phi_0(\mu))\Big)\ h(\mu)\ d\mu. \tag{9}$$

The equations at $\mathcal{O}(\varepsilon^2)$ and $\mathcal{O}(\varepsilon^3)$, although straightforward, are lengthy and omitted for brevity. Finally, the following quantities are identified

$$\beta_k\cos\xi_k = \frac{1}{k!}\int_{-1}^{1}\left.\frac{d^k\cos(\Phi(\mu))}{d\varepsilon^k}\right|_{\varepsilon=0} h(\mu)\ d\mu, \tag{10a}$$

$$\beta_k\sin\xi_k = \frac{1}{k!}\int_{-1}^{1}\left.\frac{d^k\sin(\Phi(\mu))}{d\varepsilon^k}\right|_{\varepsilon=0} h(\mu)\ d\mu. \tag{10b}$$

Note that $\mathrm{B} = \sum \varepsilon^i \beta_i$ can be identified with the Kuramoto order parameter (Kuramoto, 1984; Strogatz, 2000). These may be combined to yield

$$\beta_k = \frac{1}{k!}\int_{-1}^{1}\left[\cos\xi_k\left(\left.\frac{d^k\cos(\Phi(\mu))}{d\varepsilon^k}\right|_{\varepsilon=0}\right) + \sin\xi_k\left(\left.\frac{d^k\sin(\Phi(\mu))}{d\varepsilon^k}\right|_{\varepsilon=0}\right)\right] h(\mu)\, d\mu, \quad (11a)$$

$$0 = \frac{1}{k!}\int_{-1}^{1}\left[\sin\xi_k\left(\left.\frac{d^k\cos(\Phi(\mu))}{d\varepsilon^k}\right|_{\varepsilon=0}\right) - \cos\xi_k\left(\left.\frac{d^k\sin(\Phi(\mu))}{d\varepsilon^k}\right|_{\varepsilon=0}\right)\right] h(\mu)\, d\mu. \quad (11b)$$

$\mathcal{O}(\varepsilon)$. At $\mathcal{O}(\varepsilon)$, the equation of motion is

$$\frac{\partial\theta_1}{\partial\tau}(\tau,\nu) = -\omega_1 + \gamma_1\,\nu - \rho_1\sin(\tau+\phi_0(\nu))\left[1+\beta_0\cos(\tau+\xi_0)\right]. \quad (12)$$

In this equation terms that are not periodic in τ lead to secular terms in the solution for θ_1. To remove these undesirable terms, the secular condition is identified as

$$0 = -\omega_1 + \gamma_1\,\nu - \frac{\rho_1\beta_0}{2}\,\sin(\phi_0(\nu)-\xi_0). \quad (13)$$

Also, from Eqs. (11)

$$\beta_0 = \int_{-1}^{1}\cos(\phi_0(\mu)-\xi_0)\,h(\mu)\,d\mu, \quad (14a)$$

$$0 = \int_{-1}^{1}\sin(\phi_0(\mu)-\xi_0)\,h(\mu)\,d\mu. \quad (14b)$$

Solving Eq. (13) for $\sin(\phi_0(\nu)-\xi_0)$ and substituting into the second of these relations, one obtains

$$\omega_1 = 0, \quad (15)$$

and the secular condition reduces to

$$\frac{2\gamma_1}{\rho_1\beta_0}\,\nu = \sin(\psi_0(\nu)), \quad (16)$$

where $\psi_0(\nu) = \phi_0(\nu) - \xi_0$. Returning to the equation of motion at $\mathcal{O}(\varepsilon)$

$$\frac{\partial\theta_1}{\partial\tau}(\tau,\nu) = -\rho_1\sin((\tau+\xi_0)+\psi_0(\nu)) - \frac{\rho_1\beta_0}{2}\sin(2\,(\tau+\xi_0)+\psi_0(\nu)), \quad (17)$$

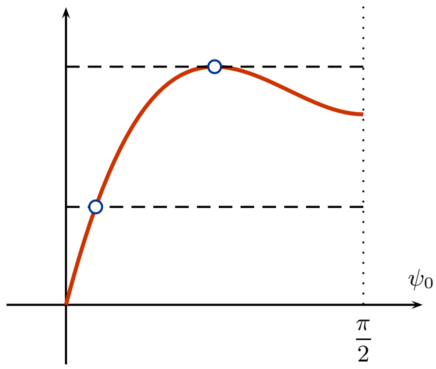

Figure 1. Geometrical solution to Eq. (20). The solid curve represents the right hand side of Eq. (20) while the dashed line is $2\,\gamma_1/\rho_1$ (shown at two different values). Intersections determine the value of $\psi_0^\star(1)$. Note that a saddle-node bifurcation occurs for sufficiently large values of $2\,\gamma_1/\rho_1$.

so that

$$\theta_1(\tau,\nu) = \rho_1\cos((\tau+\xi_0)+\psi_0(\nu)) + \frac{\rho_1\beta_0}{4}\cos(2\,(\tau+\xi_0)+\psi_0(\nu)). \quad (18)$$

To evaluate this solution the value of β_0 must still be found in a self-consistent manner from Eq. (11) so that

$$\beta_0 = \int_{-1}^{1}\cos(\psi_0(\mu))\,h(\mu)\,d\mu. \quad (19)$$

Changing variables from μ to ψ_0

$$\frac{2\,\gamma_1}{\rho_1} = \int_{-\psi_0(1)}^{\psi_0(1)}\cos^2(\psi_0)\,h\left(\frac{\sin(\psi_0)}{\sin(\psi_0(1))}\right)\,d\psi_0, \quad (20)$$

which can be solved for $\psi_0(1)$, as illustrated in Figure 1. Then from Eq. (16), β_0 is determined as

$$\beta_0 = \frac{2\,\gamma_1}{\rho_1\,\sin(\psi_0(1))}. \quad (21)$$

As illustrated in Figure 1, the equilibrium distribution can disappear in a saddle-node bifurcation for sufficiently large values of $2\,\gamma_1/\rho_1$. In addition, from Eq. (16), $\sin(\psi_0(\nu))$ must increase monotonically with increasing ν so that $\psi_0(1)$ must lie in the interval $[0,\pi/2]$, or equivalently $0 \le \beta_0 \le (2\,\gamma_1)/\rho_1$. Therefore, the equilibrium distribution can also be lost as its extent $\psi_0(1)$ reaches the boundary at $\pi/2$.

Saddle-node Bifurcations. One can show that a saddle-node bifurcation occurs if, in addition to Eq. (20), the following is satisfied

$$\beta_0 = \int_{-1}^{1} \frac{\sin^2(\psi_0(\mu))}{\cos(\psi_0(\mu))}\, h(\mu)\, d\mu. \tag{22}$$

These may be combined so that the location of the saddle-node bifurcation is determined from the solution of the equation

$$\begin{aligned} 0 &= \int_{-1}^{1} \frac{\cos(2\,\psi_0(\mu))}{\cos(\psi_0(\mu))}\, h(\mu)\, d\mu, \\ &= \int_{-\psi_0^\star(1)}^{\psi_0^\star(1)} \cos(2\psi_0)\, h\left(\frac{\sin(\psi_0)}{\sin(\psi_0^\star(1))}\right)\, d\psi_0, \end{aligned} \tag{23}$$

where $\psi_0^\star(1)$ corresponds to the extent of the equilibrium distribution at the bifurcation.

$\mathcal{O}(\varepsilon^2)$. At $\mathcal{O}(\varepsilon^2)$, the secularity condition can be expressed as

$$\begin{aligned} 0 = &-\omega_2 + \gamma_2\,\nu \\ &-\frac{\rho_1^2}{2}\left\{\left(1+\frac{\beta_0^2}{8}\right) + \frac{\beta_0}{8}\cos(\psi_0(\nu))\right\} \\ &-\frac{1}{2}\Big\{\rho_2\beta_0\sin(\psi_0(\nu)) \\ &\qquad + \rho_1\,\beta_1\sin(\psi_0(\nu)+(\xi_0-\xi_1)) \\ &\qquad + \rho_1\,\beta_0\,\phi_1(\nu)\;\cos(\psi_0(\nu))\Big\}. \end{aligned} \tag{24}$$

Integrating this condition over the population and solving for ω_2

$$\omega_2 = -\frac{\rho_1^2}{2}\left(1+\frac{\beta_0^2}{4}\right). \tag{25}$$

Using Eq. (16), the secularity condition reduces to

$$\begin{aligned} 0 = &\frac{\rho_1^2}{2}\left[\frac{\beta_0}{8}\Big(\beta_0 - \cos(\psi_0(\nu))\Big)\right] \\ &-\frac{\rho_1\beta_0}{2}\Big[\phi_1(\nu)\;\cos(\psi_0(\nu)) + \frac{\beta_1}{\beta_0}\sin(\psi_0(\nu)-\delta_1) \\ &\qquad + \left(\frac{\rho_2}{\rho_1}-\frac{\gamma_2}{\gamma_1}\right)\sin(\psi_0(\nu))\Big], \end{aligned} \tag{26}$$

where $\delta_1 = \xi_1 - \xi_0$. The self-consistency condition can be expressed as

$$\beta_1\cos(\delta_1) = -\int_{-1}^{1} \phi_1(\mu)\sin(\psi_0(\mu))\, h(\nu)\, d\nu, \tag{27}$$

and at the bifurcation

$$\beta_0\left(\frac{\rho_2}{\rho_1}-\frac{\gamma_2}{\gamma_1}\right) = 0. \tag{28}$$

Subsequently solving the $\mathcal{O}(\varepsilon^2)$ secularity condition

$$\begin{aligned} \phi_1(\nu)\;\cos(\psi_0(\nu)) = &\frac{\rho_1}{8}\Big(\beta_0 - \cos(\psi_0(\nu))\Big) \\ &-\frac{\beta_1}{\beta_0}\;\sin(\psi_0(\nu)-\delta_1). \end{aligned} \tag{29}$$

$\mathcal{O}(\varepsilon^3)$. As was done at $\mathcal{O}(\varepsilon^2)$, integrating the $\mathcal{O}(\varepsilon^3)$ secularity condition (not shown) over the population and solving for ω_3 yields

$$\omega_3 = -\left\{\frac{\rho_1^2\beta_0\beta_1}{4}\cos(\delta_1) + \rho_1\rho_2\left(1+\frac{\beta_0^2}{4}\right)\right\}. \tag{30}$$

At the bifurcation, the self-consistency condition can be reduced to

$$\begin{aligned} 0 = &\Bigg\{\beta_0\left(\frac{\rho_3}{\rho_1}-\frac{\gamma_3}{\gamma_1}\right) \\ &-\frac{\rho_1^2\beta_0}{64}\left[17 + \left(\int_{-1}^{1}\cos(2\,\psi_0(\mu))\,h(\mu)\,d\mu\right)^2\right] \\ &+\frac{\rho_1^2\beta_0}{64}\left[1 - \frac{\beta_0}{2}\int_{-1}^{1}\frac{1}{\cos^3(\psi_0(\nu))}\,h(\nu)\,d\nu\right]\Bigg\} \\ &-\frac{\beta_1^2}{\beta_0^2}\left\{\frac{\cos^2(\delta_1)}{2}\int_{-1}^{1}\frac{1}{\cos^3(\psi_0(\nu))}\,h(\nu)\,d\nu\right\}. \end{aligned} \tag{31}$$

This equation has either zero, one, or two solutions depending on the value of $\beta_1\cos(\delta_1)$. In particular, for $\delta_1 = \pi/2$, the cubic coefficients of P and Γ on the bifurcation curve satisfy

$$\begin{aligned} \left(\frac{\rho_3}{\rho_1}-\frac{\gamma_3}{\gamma_1}\right) = \\ \frac{\rho_1^2}{64}\Bigg[16 + \left(\int_{-1}^{1}\cos(2\psi_0(\nu))\,h(\nu)\,d\nu\right)^2 \\ + \left(\frac{\beta_0}{2}\int_{-1}^{1}\frac{1}{\cos^3(\psi_0(\nu))}\,h(\nu)\,d\nu\right)\Bigg]. \end{aligned} \tag{32}$$

2.2 Uniform Distribution

For a uniform population of oscillators ($h(\nu) = 1/2$), the equilibrium distribution at the bifurcation from the

$\mathcal{O}(\varepsilon)$ analysis can be written as

$$\beta_0 = \frac{2\,\gamma_1}{\rho_1} = \frac{\pi}{4}, \quad \longrightarrow \quad \sin(\psi_0(\nu)) = \nu. \quad (33)$$

Therefore the equilibrium distribution covers the interval $\psi_0 \in [-\pi/2, \pi/2]$. When Eq. (32) is evaluated to determine the cubic coefficients of the bifurcation curve, the integral

$$\int_{-1}^{1} \frac{1}{\cos^3(\psi_0(\nu))}\, h(\nu)\, d\nu, \quad (34)$$

is singular for the uniform distribution, so that the cubic coefficients of P and Γ are undefined. However, in Ariaratnam and Strogatz (2001) numerical results are presented for 800 oscillators (approximating the uniform distribution), and the location of the bifurcation curve is identified in terms of the parameters. Surprisingly, no singularity in the bifurcation curve is observed for the numerical results.

2.3 Discrete Population

Now consider a discrete population of N oscillators, for which the distribution function $h(\nu)$ can be written as a sum of delta functions

$$h(\nu) = \frac{1}{N}\sum_{i=1}^{N} \delta\left(\nu - \frac{N+1-2i}{N-1}\right). \quad (35)$$

In the limit $N \to \infty$, this approaches a uniform distribution with $h(\nu) = 1/2$. With this specific population, the equilibrium distribution at the bifurcation is determined by the solution to (from Eq. (23))

$$\begin{aligned} 0 = \frac{2}{N}\sum_{i=1}^{N}\left[1 - \left(\frac{2\,\gamma_1}{\rho_1\,\beta_0}\frac{N+1-2i}{N-1}\right)^2\right]^{1/2} \quad (36)\\ -\frac{1}{N}\sum_{i=1}^{N}\left[1 - \left(\frac{2\,\gamma_1}{\rho_1\,\beta_0}\frac{N+1-2i}{N-1}\right)^2\right]^{-1/2}. \end{aligned}$$

The solution to this equation can then be used to determine the expansion for γ at the bifurcation.

In Figure 2 the cubic coefficient in the expansion for γ, from Eq. (32), is shown as the number of oscillators in the discrete system increases. As $N \to \infty$ this coefficient increases and numerically appears to increase unbounded, although very slowly—at $N = 10^6$ Eq. (32) is evaluated to yield $\gamma_3 \sim -1.6$. Eq. (36) was solved with a bracketing method in MATLAB, using double precision variables to ensure an accurate solution for large N.

For this system the mean distribution can be written to $\mathcal{O}(\varepsilon)$ as

$$\begin{aligned} \sin(\Psi(\nu)) = \frac{2\gamma_1}{\rho_1\beta_0}\nu \quad (37)\\ +\varepsilon\left\{\frac{\rho_1}{8}\Big(\beta_0 - \cos(\psi_0(\nu))\Big)\right.\\ \left. -\frac{\beta_1}{\beta_0}\sin(\psi_0(\nu) - \delta_1)\right\} + \mathcal{O}(\varepsilon^2). \end{aligned}$$

The value of $|\sin(\Phi(\nu))|$ must be less than one for all values of ν. In particular, for $\nu = 1$, this provides an estimate of the allowable range of ε over which this expansion is valid. That is at the bifurcation value of $\beta_1 = 0$

$$\varepsilon < 8\frac{\rho_1\beta_0 - 2\gamma_1}{\rho_1^2\beta_0\Big(\beta_0 - \cos(\psi_0(\nu))\Big)}. \quad (38)$$

As $N \to \infty$, the quantity $\rho_1\beta_0 - 2\gamma_1$ goes to zero, so that the allowable range of ε over which this expansion is valid vanishes. Viewing Figure 2, although the cubic coefficient of the bifurcation curve blows up as $N \to \infty$, the range of validity of this expansion vanishes as the continuum limit is approached.

2.4 Polynomial Distribution

For a general even continuous population, the frequency distribution can be expressed as

$$h(\nu) = \sum_{k=0}^{\infty} h_k\ |\nu|^k. \quad (39)$$

With this distribution, the saddle-node bifurcation condition becomes

$$\begin{aligned} 0 &= \int_{-1}^{1}\frac{\cos(2\,\psi_0(\mu))}{\cos(\psi_0(\mu))}\sum_{k=0}^{\infty} h_k\ |\mu|^k\ d\mu, \quad (40)\\ &= \sum_{k=0}^{\infty}\frac{2\,h_k}{\sin^k(\psi_0(1))}\int_0^{\psi_0^\star(1)}\cos(2\psi_0)\ \sin^k(\psi_0)\, d\psi_0. \end{aligned}$$

As before, the solution $\psi_0^\star(1)$ must lie in the interval $[0, \pi/2]$. For solutions of this equation for which $\psi_0^\star(1)$ is greater than $\pi/2$ the equilibrium distribution, given by Eq.(16) is not defined, so that these solutions are invalid.

As an example, consider a piecewise linear distribution of the form

$$h(\nu) = h_0 + (1 - 2\,h_0)\ |\nu|. \quad (41)$$

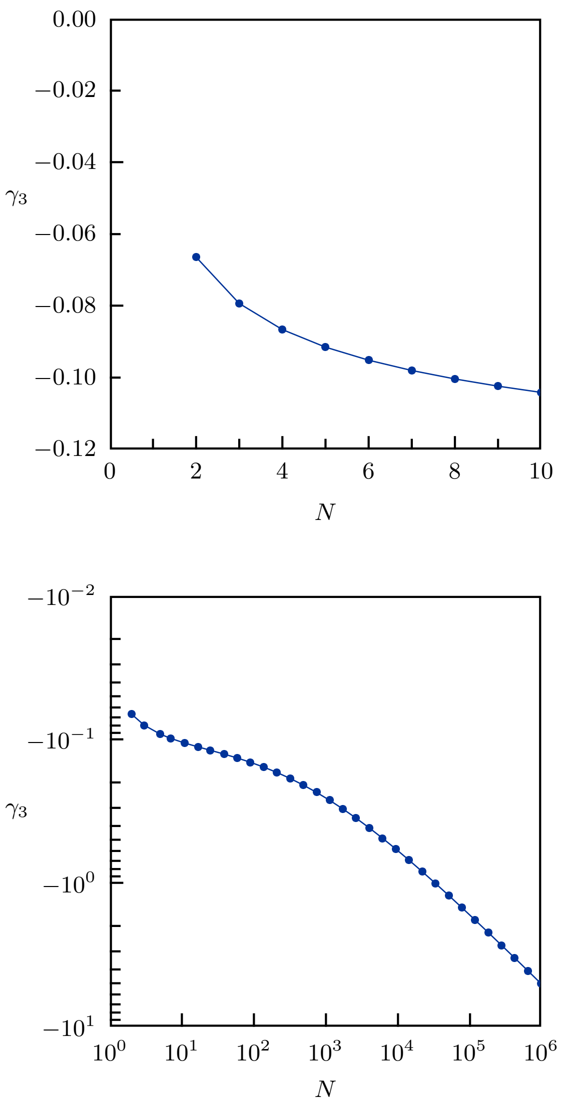

Figure 2. Cubic coeffi cient γ_3 ($\rho_3 = 0$) at the bifurcation as N varies.

As illustrated in Figure 3, for $h_0 = 1$ this can be described as a tent-shaped distribution, while for $h_0 = 1/2$ it reduces to the uniform distribution considered above. With this, the condition for the saddle-node bifurcation reduces to

$$0 = \left(\frac{h_0 - 2}{3}\right) \cos^3(\psi_0^\star(1)) + (1 - h_0)\cos(\psi_0^\star(1)) - \left(\frac{1 - 2\,h_0}{3}\right). \tag{42}$$

One solution to this equation is always given by $\cos(\psi_0^\star(1)) = 1$, while the remaining solutons satisfy

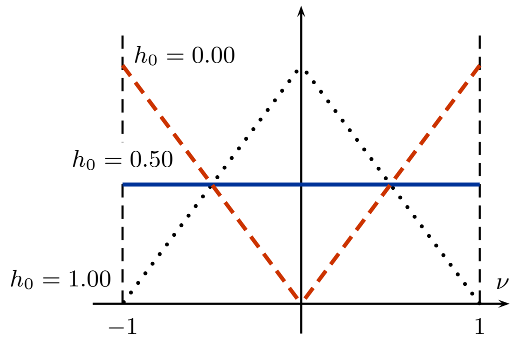

Figure 3. Piecewise linear population densities ($h_1 = 1 - 2\,h_0$).

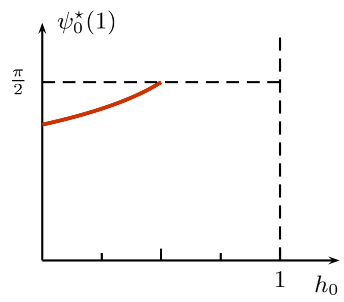

Figure 4. Piecewise linear population densities ($h_1 = 1 - 2\,h_0$).

the equation

$$0 = \cos^2(\psi_0^\star(1)) + \cos(\psi_0^\star(1)) + \left(\frac{2\,h_0 - 1}{2 - h_0}\right), \tag{43}$$

so that

$$\cos(\psi_0^\star(1)) = -\frac{1}{2} \pm \sqrt{\frac{6 - 9\,h_0}{4\,(2 - h_0)}}. \tag{44}$$

As seen in Figure 4 as $h_0 \to 1/2$, the extent of the equilibrium distribution at the bifurcation approaches $\pi/2$. Recall that $\psi_0(1)$ must lie in the interval $[0, \pi/2]$, so that $0 \le \cos(\psi_0(1)) \le 1$. Therefore admissible solutions to this quadratic equation only exist for $0 \le h_0 \le 1/2$. For $1/2 < h_0 \le 1$ no saddle-node bifurcation occurs in the system. Instead, synchrony is lost as the $\mathcal{O}(\varepsilon)$ equilibrium distribution covers the interval $-\pi/2 \le \psi_0(\nu) \le \pi/2$.

3 Conclusions

Synchronization has been considered in a population of phase oscillators with mean-field coupling, where the uncoupled frequencies of the oscillators are distributed across a specified population density. It was found that synchrony can be lost in either a classical saddle-node bifurcation or as the equilibrium distribution covers the interval $[-\pi/2, \pi, 2]$. For the saddle-node bifurcation a Poincaré-Lindstedt method was applied and the coefficients of the resulting bifurcation curve were identified. Finally, a number of specific distributions were evaluated, including the uniform distribution, polynomial distributions, and a population density corresponding to a system of discrete oscillators. By formulating the discrete problem in a continuum framework, the Poincaré-Lindstedt analysis for the saddle node bifurcation was carried out for as many as $N = 10^6$ oscillators.

References

Ariaratnam, Joel T. and Steven H. Strogatz (2001). Phase diagram for the Winfree model of coupled nonlinear oscillators. *Physical Review Letters* **86**(19), 4278–4281.

Kuramoto, Y. (1984). *Chemical Oscillations, Waves, and Turbulence*. Springer-Verlag. Berlin.

Strogatz, S. H. (2000). From Kuramoto to Crawford: Exploring the onset of synchronization in populations of coupled oscillators. *Physica D* **143**, 1–20.

Winfree, A. T. (1967). Biological rhythms and the behavior of populations of coupled oscillators. *Journal of Theoretical Biology* **16**, 15–42.