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Synchronization of bidirectional N-coupled fractional-order chaotic systems with ring connection based on antisymmetric structure

Cuimei Jiang[1], Akbar Zada[2], M. Tamer Şenel[3] and Tongxing Li[4,5*]

*Correspondence:
litongx2007@163.com
[4]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, P.R. China
[5]School of Information Science and Engineering, Linyi University, Linyi, P.R. China
Full list of author information is available at the end of the article

Abstract
This paper discusses the synchronization problem of N-coupled fractional-order chaotic systems with ring connection via bidirectional coupling. On the basis of the direct design method, we design the appropriate controllers to transform the fractional-order error dynamical system into a nonlinear system with antisymmetric structure. By choosing appropriate fractional-order Lyapunov functions and employing the fractional-order Lyapunov-based stability theory, several sufficient conditions are obtained to ensure the asymptotical stabilization of the fractional-order error system at the origin. The proposed method is universal, simple, and theoretically rigorous. Finally, some numerical examples are presented to illustrate the validity of theoretical results.

MSC: 34C28; 34D06

Keywords: Chaos synchronization; Fractional-order chaotic system; Fractional-order Lyapunov function; Ring connection; Antisymmetric structure

1 Introduction

In recent years, more and more attention has been diverted towards the study of fractional-order chaotic systems due to their potential applications in the fields of secure communication, encryption, signal and control processing [1–5]. Various fractional-order dynamical systems, such as the fractional-order Lorenz system [6], the fractional-order Lü system [7], the fractional-order financial system [8], the fractional-order permanent magnet synchronous motor (PMSM) system [9], the fractional-order Rabinovich system [10], the fractional-order microscopic chemical system [11], the fractional-order hyperchaotic PWC system [12], have chaotic or hyperchaotic behavior. Meanwhile, several different types of synchronization for fractional-order chaotic systems have been observed and developed in the previous works. For instance, complete synchronization [13], phase synchronization [14], impulsive synchronization [15], and lag projective synchronization [16], just to enumerate a few examples.

Coupled chaotic systems, as a special class of nonlinear systems, have been extensively studied in theoretical physics and other fields of natural sciences and engineering. Multiple chaotic systems can be coupled in a ring which makes them correlative. Synchronization of coupled chaotic systems with ring connection extends the traditional mode of synchronization from one-to-one into one-to-many. Then this synchronization is realized at lower cost and has better flexibility and practicability. In the meantime, this method provides the possibility to realize simultaneous multiparty communications. Therefore, many researchers have paid attention to investigate synchronization of coupled chaotic systems with ring connection. For instance, in [17], it was shown that chaos synchronization of three coupled oscillators with ring connection was accomplished. Yu and Zhang [18] realized global synchronization of three coupled chaotic systems with ring connection. The adaptive synchronization method for coupled systems was proposed for multi-Lorenz systems family in [19]. Based on special antisymmetric structure, Chen et al. [20] studied synchronization of N-coupled integer-order chaotic systems via unidirectional coupling. With respect to other recent representative works on this topic, we refer the reader to [21–25] and the references cited therein.

Most of the above-mentioned works mainly focused on the coupled integer-order chaotic systems, not involving coupled fractional-order chaotic systems. Compared with the integer-order chaotic systems, the fractional-order chaotic systems can display much richer dynamical behaviors and therefore they are considered as the powerful tools in secure communication for their capability of improving the security of chaotic communication systems. As a result, it is meaningful and challenging to explore various synchronization schemes of coupled fractional-order chaotic systems with ring connection. As is well-known, N fractional-order chaotic systems with ring connection can be coupled in the unidirectional and bidirectional ways, which are two typical coupling cases. Zhou and Li [26] presented the theory for synchronization problems in an ω-symmetrically coupled fractional differential system. Synchronization of N-coupled fractional-order chaotic systems with unidirectional coupling and bidirectional coupling was studied in [27, 28]. Ouannas et al. [29] investigated Q-S synchronization in coupled chaotic incommensurate fractional-order systems. As shown in [30], synchronization of unidirectional N-coupled fractional-order chaotic systems can be derived. However, there are two problems to be addressed: one is shall we design the general controllers to achieve synchronization of the N-coupled fractional-order chaotic systems with ring connection via bidirectional coupling; the other is how to transform the fractional-order error system into a special antisymmetric structure. As far as we know, there is little work with consideration of these two problems.

On the basis of the above discussion, this paper is concerned with synchronization of bidirectional N-coupled fractional-order chaotic systems with ring connection. By virtue of the direct method, the general controllers are designed to transform the fractional-order error system into a nonlinear system with special antisymmetric structure and achieve synchronization via bidirectional coupling, which extends the existing results proposed in [27]. Noticeably, fractional-order Lyapunov functions are constructed to analyze the stability of fractional-order error dynamical systems, which is different from the synchronization schemes proposed in the previous literature.

The remainder of this paper is organized as follows. In the next section, some preliminaries are presented. Section 3 designs the general controllers to achieve synchronization of N-coupled fractional-order chaotic system via bidirectional coupling. Section 4 provides numerical examples to exhibit the feasibility and effectiveness of the proposed control technique. Finally, conclusions are drawn in Sect. 5.

2 Preliminaries

Fractional calculus is a generalization of integration and differentiation to arbitrary non-integer orders. Several existing definitions of fractional derivatives are given in [31], where the Caputo definition is used in engineering applications extensively. We firstly introduce the following Caputo definition.

Definition 1 (see [31]) For a function f, the Caputo fractional derivative of fractional-order α is defined as follows:

$$D^{\alpha}f(t)=\frac{1}{\Gamma(n-\alpha)}\int_{t_0}^{t}\frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha-n+1}}\,d\tau,\quad t>t_0,$$

where $m-1<\alpha<m$, $m=[\alpha]+1$, $[\alpha]$ denotes the integer part of α, Γ stands for gamma function, and D^{α} is generally called α-order Caputo differential operator.

The main advantage of Caputo definition is that Caputo derivative of a constant is equal to zero. Throughout, the fractional-order chaotic systems will be described by utilizing Caputo definition with lower limit of integral $t_0=0$ and the order $0<\alpha<1$.

From Definition 1, it is clear that the Caputo fractional derivative satisfies the linearity property

$$D^{\alpha}\big(\lambda f(t)+\mu g(t)\big)=\lambda D^{\alpha}f(t)+\mu D^{\alpha}g(t),$$

where λ and μ are real constants.

The property for Caputo fractional derivative of a general quadratic function is stated as follows.

Lemma 1 (see [32]) *Assume that $\alpha\in(0,1]$, $x(t)=(x_1(t),x_2(t),\dots,x_n(t))\in\mathbb{R}^n$ and $x_i(t)$ $(i=1,2,\dots,n)$ are continuous and derivable functions. Then*

$$D^{\alpha}\big(x^T(t)Px(t)\big)\le\big(D^{\alpha}x(t)\big)^TPx(t)+x^T(t)PD^{\alpha}x(t)$$

for any time instant $t>0$, where $P\in\mathbb{R}^{n\times n}$ is a positive definite matrix.

There are some stability results for various classes of fractional-order systems [33, 34]. Next, we recall the following useful results of the Lyapunov-based stability theory.

Definition 2 (see [33]) A continuous function $\gamma:[0,t)\to[0,\infty)$ is said to be the class-K function if it is strictly increasing and $\gamma(0)=0$.

Lemma 2 (see [33]) *Suppose that $x(t)=0$ is the equilibrium point of the fractional-order system $D^{\alpha}x(t)=f(x,t)$, $x\in\mathbb{R}^n$, where $0<\alpha<1$. If there exist a Lyapunov function $V(t,x(t))$ and three class-K functions γ_i $(i=1,2,3)$ satisfying*

$$\gamma_1\big(\|x\|\big)\le V\big(t,x(t)\big)\le\gamma_2\big(\|x\|\big)\quad\text{and}\quad D^{\alpha}V\big(t,x(t)\big)\le-\gamma_3\big(\|x\|\big),$$

then $x(t)=0$ is asymptotically stable.

3 Synchronization scheme

Consider a general form of N fractional-order chaotic systems which can be expressed as

$$\begin{cases} D^\alpha X_1 = A_1X_1 + G_1(X_1), \\ D^\alpha X_2 = A_2X_2 + G_2(X_2), \\ \vdots \\ D^\alpha X_N = A_NX_N + G_N(X_N), \end{cases} \tag{1}$$

where $X_1, X_2, \ldots, X_N \in \mathbb{R}^n$ ($N > 2$) are the state vectors of the chaotic systems; $A_1, A_2, \ldots, A_N \in \mathbb{R}^{n\times n}$ are constant matrices ($A_i \neq A_j$, $i \neq j$); $G_i(X_i)$ ($i = 1, 2, \ldots, N$) are the continuous nonlinear functions ($G_i \neq G_j$, $i \neq j$). Then, the structure for ring connection between N systems via bidirectional coupling is displayed in Fig. 1. Therefore, a bidirectional coupling scheme among N fractional-order chaotic systems (1) with ring connection can be expressed as follows:

$$\begin{cases} D^\alpha X_1 = A_1X_1 + G_1(X_1) + Q_1(X_2 + X_N - 2X_1), \\ D^\alpha X_2 = A_2X_2 + G_2(X_2) + Q_2(X_1 + X_3 - 2X_2), \\ D^\alpha X_3 = A_3X_3 + G_3(X_3) + Q_3(X_2 + X_4 - 2X_3), \\ \vdots \\ D^\alpha X_N = A_NX_N + G_N(X_N) + Q_N(X_{N-1} + X_1 - 2X_N), \end{cases} \tag{2}$$

where $Q_i = \operatorname{diag}(q_{i1}, q_{i2}, \ldots, q_{in})$ are n-dimensional diagonal matrices and $q_{ij} \geq 0$ are the ideal gains which represent the coupled parameters.

Suppose that the first system is chosen as the aim system, and add the controller to the remaining systems. Thus, model (2) takes the form

$$\begin{cases} D^\alpha X_1 = A_1X_1 + G_1(X_1) + Q_1(X_2 + X_N - 2X_1), \\ D^\alpha X_2 = A_2X_2 + G_2(X_2) + Q_2(X_1 + X_3 - 2X_2) + U_1, \\ D^\alpha X_3 = A_3X_3 + G_3(X_3) + Q_3(X_2 + X_4 - 2X_3) + U_2, \\ \vdots \\ D^\alpha X_N = A_NX_N + G_N(X_N) + Q_N(X_{N-1} + X_1 - 2X_N) + U_{N-1}. \end{cases} \tag{3}$$

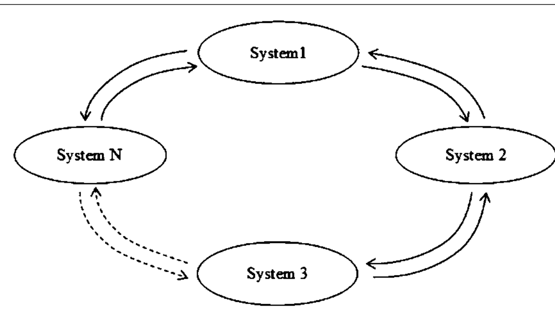


Figure 1 The coupled mode of N systems via bidirectional coupling

Define the synchronization error vectors as $e_i = X_{i+1} - X_i \in \mathbb{R}^n$ ($i = 1, 2, \ldots, N-1$). Then we have the error system

$$e = \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \end{pmatrix}.$$

Taking into account system (3), we can obtain the error dynamical system as follows:

$$\begin{aligned} D^\alpha e &= \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \end{pmatrix} \\ &\quad + \begin{pmatrix} (A_2 - A_1)X_1 + G_2(X_2) - G_1(X_1) + U_1 \\ (A_3 - A_2)X_2 + G_3(X_3) - G_2(X_2) + U_2 - U_1 \\ \vdots \\ (A_N - A_{N-1})X_{N-1} + G_N(X_N) - G_{N-1}(X_{N-1}) + U_{N-1} - U_{N-2} \end{pmatrix}, \end{aligned} \tag{4}$$

where

$$\Sigma_1 = \begin{pmatrix} A_2 - 2Q_1 - Q_2 & Q_2 - Q_1 & -Q_1 & \ldots & -Q_1 & -Q_1 \\ Q_2 & A_3 - Q_3 - Q_2 & Q_3 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & Q_{N-2} & A_{N-1} - Q_{N-1} - Q_{N-2} \end{pmatrix},$$

$$\Sigma_2 = [-Q_1 \quad 0 \quad 0 \quad \ldots \quad 0 \quad Q_{N-1}]^T,$$

$$\Sigma_3 = [-Q_N \quad -Q_N \quad -Q_N \quad \ldots \quad -Q_N \quad Q_{N-1} - Q_N],$$

$$\Sigma_4 = A_N - 2Q_N - Q_{N-1}.$$

The objective of this paper is to find the suitable and effective controllers U_i ($i = 1, 2, \ldots, N-1$) such that the error dynamical system (4) is not only transformed into a nonlinear system with antisymmetric structure but also asymptotically stable at the origin. That is, we can achieve synchronization of bidirectional N-coupled fractional-order chaotic systems with ring connection based on antisymmetric structure.

In [35–37], the authors proposed a direct design method to achieve chaos synchronization. Two advantages of this method are to present an easy procedure for choosing proper controllers in chaos synchronization and construct simple controllers easily to implement. Therefore, in this paper, this method is used to realize our objective.

First, the controllers U_i are chosen as

$$\begin{cases} U_1 = V_1 - (A_2 - A_1)X_1 - G_2(X_2) + G_1(X_1), \\ U_2 = V_2 - (A_3 - A_2)X_2 - G_3(X_3) + G_2(X_2) + U_1, \\ \vdots \\ U_{N-1} = V_{N-1} - (A_N - A_{N-1})X_{N-1} - G_N(X_N) + G_{N-1}(X_{N-1}) + U_{N-2}, \end{cases} \tag{5}$$

where

$$\begin{pmatrix} V_1 \\ V_2 \\ \vdots \\ V_{N-1} \end{pmatrix} = \Theta \begin{pmatrix} e_1 \\ e_2 \\ \vdots \\ e_{N-1} \end{pmatrix} \tag{6}$$

and Θ is a coefficient matrix. Then, the error dynamical system (4) can be represented as

$$D^\alpha e(t) = \Phi(t)e(t), \tag{7}$$

where

$$\Phi(t) = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix} + \Theta.$$

Clearly, there are many possible choices for Θ to guarantee that the error dynamical system (7) is asymptotically stable at the origin. Without loss of generality, Θ is defined by a state dependent coefficient matrix. As a result, the sufficient stability conditions of system (7) are obtained by transforming it into a stable system with antisymmetric structure. The main result is shown in the following.

Theorem 1 *Synchronization of bidirectional N-coupled fractional-order system* (3) *can be achieved by utilizing the control laws in equation* (5), *if the state dependent coefficient matrix* $\Phi(t) = \Phi_1(t) + \Phi_2$ *satisfies the hypotheses*

$$\Phi_1^T(t) = -\Phi_1(t) \quad \text{and} \quad \Phi_2 = \operatorname{diag}(-\varphi_1, -\varphi_2, \dots, -\varphi_{N-1}), \tag{8}$$

where $\varphi_i = \operatorname{diag}(\varphi_{i1}, \varphi_{i2}, \dots, \varphi_{in})$ $(i = 1, 2, \dots, N-1)$ *and* $\varphi_{ij} > 0$ $(j = 1, 2, \dots, n)$.

Proof Consider a positive definite Lyapunov candidate function as

$$V(e,t) = \frac{1}{2}e^T(t)e(t).$$

Calculating the Caputo derivative of $V(e,t)$, we derive from Lemma 1 and system (7) that

$$\begin{aligned} D^\alpha V(e,t) &\le \frac{1}{2}\big(\big(D^\alpha e(t)\big)^T e(t) + e^T(t)D^\alpha e(t)\big) \\ &= \frac{1}{2}e^T(t)\big(\Phi^T(t) + \Phi(t)\big)e(t). \end{aligned}$$

Substituting (8) into the latter inequality, one can conclude that

$$\begin{aligned} D^\alpha V(e,t) &\le \frac{1}{2}e^T(t)\big(\Phi_2^T + \Phi_2\big)e(t) = e^T(t)\Phi_2 e(t) \\ &= \big(e_1^T(t), e_2^T(t), \dots, e_{N-1}^T(t)\big) \begin{pmatrix} -\varphi_1 & 0 & 0 & \dots & 0 & 0 \\ 0 & -\varphi_2 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & -\varphi_{N-1} \end{pmatrix} \begin{pmatrix} e_1(t) \\ e_2(t) \\ \vdots \\ e_{N-1}(t) \end{pmatrix} \end{aligned}$$

$$\begin{aligned}
&= -e_1^T(t)\varphi_1 e_1(t) - e_2^T(t)\varphi_2 e_2(t) - \cdots - e_{N-1}^T(t)\varphi_{N-1}e_{N-1}(t) \\
&\leq -\varphi_{\min}\bigl(e_1^T(t)e_1(t) + e_2^T(t)e_2(t) + \cdots + e_{N-1}^T(t)e_{N-1}(t)\bigr) \\
&= -\varphi_{\min}\bigl\|e(t)\bigr\|^2,
\end{aligned}$$

where $\varphi_{\min} = \min_{1\leq i\leq N, 1\leq j\leq n}\{\varphi_{ij}\}$. An application of Lemma 2 yields that the closed-loop system (7) is globally asymptotically stable. Therefore, synchronization of bidirectional N-coupled fractional-order chaotic system (3) is accomplished. This completes the proof. □

As a consequence of Theorem 1, we can obtain the following corollary.

Corollary 1 *If the structures of general fractional-order chaotic systems are identical, i.e., $A_i = A_j = A$ and $G_i(\cdot) = G_j(\cdot) = G(\cdot)$, then the controllers U_i can be designed as*

$$\begin{cases}
U_1 = V_1 - G(X_2) + G(X_1), \\
U_2 = V_2 - G(X_3) + G(X_2) + U_1, \\
\vdots \\
U_{N-1} = V_{N-1} - G(X_N) + G(X_{N-1}) + U_{N-2},
\end{cases}$$

where V_i $(i = 1, 2, \ldots, N-1)$ are defined as in (6) and Θ is a coefficient matrix. Similarly to the aforementioned discussion, we can achieve synchronization of bidirectional N-coupled identical fractional-order chaotic systems with ring connection based on antisymmetric structure.

Remark 1 The fractional-order error dynamical system (4) is transformed into system (7) by virtue of the control laws U_i. Furthermore, system (7) can be transformed into a stable system by choosing the appropriate coefficient matrix Θ which ensures that $\Phi(t)$ is an antisymmetric structure. Therefore, selecting of the coefficient matrices plays an important role in achieving synchronization of bidirectional N-coupled fractional-order chaotic systems with ring connection.

4 Numerical simulations

To illustrate the effectiveness of the proposed synchronization scheme, two groups of examples are considered and their numerical simulations are performed. Two cases including 3-coupled identical and nonidentical fractional-order chaotic systems are investigated, respectively.

4.1 Synchronization of 3-coupled identical fractional-order chaotic systems

In this case, we consider synchronization of 3-coupled fractional-order Lü systems via bidirectional coupling. Lü system connects Lorenz system and Chen system and represents the transition from one to the other. In 2006, Lu studied the chaotic dynamics of fractional-order Lü system and found that the lowest order for this system to have chaos is 0.3; see [7] for more details. The bidirectional 3-coupled fractional-order Lü systems with designed controllers are introduced in the form as

$$\begin{cases}
D^\alpha x_{11} = a(x_{12} - x_{11}) + q_{11}(x_{21} + x_{31} - 2x_{11}), \\
D^\alpha x_{12} = bx_{12} - x_{11}x_{13} + q_{12}(x_{22} + x_{32} - 2x_{12}), \\
D^\alpha x_{13} = x_{11}x_{12} - cx_{13} + q_{13}(x_{23} + x_{33} - 2x_{13}),
\end{cases}$$

$$\begin{cases} D^\alpha x_{21} = a(x_{22} - x_{21}) + q_{21}(x_{11} + x_{31} - 2x_{21}) + u_{11}, \\ D^\alpha x_{22} = bx_{22} - x_{21}x_{23} + q_{22}(x_{12} + x_{32} - 2x_{22}) + u_{12}, \\ D^\alpha x_{23} = x_{21}x_{22} - cx_{23} + q_{23}(x_{13} + x_{33} - 2x_{23}) + u_{13}, \end{cases}$$

and

$$\begin{cases} D^\alpha x_{31} = a(x_{32} - x_{31}) + q_{31}(x_{21} + x_{11} - 2x_{31}) + u_{21}, \\ D^\alpha x_{32} = bx_{32} - x_{31}x_{33} + q_{32}(x_{22} + x_{12} - 2x_{32}) + u_{22}, \\ D^\alpha x_{33} = x_{21}x_{22} - cx_{23} + q_{23}(x_{23} + x_{13} - 2x_{33}) + u_{23}, \end{cases}$$

where

$$A_1 = A_2 = A_3 = A = \begin{pmatrix} -a & a & 0 \\ 0 & b & 0 \\ 0 & 0 & -c \end{pmatrix}, \qquad G_i(X_i) = \begin{pmatrix} 0 \\ -x_{i1}x_{i3} \\ x_{i1}x_{i2} \end{pmatrix} \quad (i = 1, 2, 3),$$

$Q_1 = \operatorname{diag}(q_{11}, q_{12}, q_{13})$, $Q_2 = \operatorname{diag}(q_{21}, q_{22}, q_{23})$, $Q_3 = \operatorname{diag}(q_{31}, q_{32}, q_{33})$ are the coupled matrices, $U_1 = (u_{11}, u_{12}, u_{13})^T$ and $U_2 = (u_{21}, u_{22}, u_{23})^T$ are the control inputs.

Next, we consider the synchronization error vectors $e_i = X_{i+1} - X_i$ ($i = 1, 2$) and obtain the error dynamical system as follows:

$$\begin{aligned} D^\alpha e &= \begin{pmatrix} D^\alpha e_1 \\ D^\alpha e_2 \end{pmatrix} \\ &= \begin{pmatrix} A - 2Q_1 - Q_2 & Q_2 - Q_1 \\ Q_2 - Q_3 & A - 2Q_3 - Q_2 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} u_{11} \\ -x_{21}x_{23} + x_{11}x_{13} + u_{12} \\ x_{21}x_{22} - x_{11}x_{12} + u_{13} \\ u_{21} - u_{11} \\ -x_{31}x_{33} + x_{21}x_{23} + u_{22} - u_{12} \\ x_{31}x_{32} - x_{21}x_{22} + u_{23} - u_{13} \end{pmatrix}, \end{aligned}$$

where

$$A - 2Q_1 - Q_2 = \begin{pmatrix} -a - 2q_{11} - q_{21} & a & 0 \\ 0 & b - 2q_{12} - q_{22} & 0 \\ 0 & 0 & -c - 2q_{13} - q_{23} \end{pmatrix},$$

$$A - 2Q_3 - Q_2 = \begin{pmatrix} -a - 2q_{31} - q_{21} & a & 0 \\ 0 & b - 2q_{32} - q_{22} & 0 \\ 0 & 0 & -c - 2q_{33} - q_{23} \end{pmatrix},$$

$$e_1 = \begin{pmatrix} e_{11} \\ e_{12} \\ e_{13} \end{pmatrix}, \quad \text{and} \quad e_2 = \begin{pmatrix} e_{21} \\ e_{22} \\ e_{23} \end{pmatrix}.$$

Choosing the coefficient matrix

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$$

as follows:

$$\Theta_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ -a & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\Theta_2 = \begin{pmatrix} q_{11} - 2q_{21} + q_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{12} - 2q_{22} + q_{32} & 0 & -a & 0 & 0 \\ 0 & 0 & q_{13} - 2q_{23} + q_{33} & 0 & 0 & 0 \end{pmatrix},$$

we can design the controllers U_1 and U_2 as

$$U_1 = \begin{pmatrix} u_{11} \\ u_{12} \\ u_{13} \end{pmatrix} = \begin{pmatrix} 0 \\ -ae_{11} + x_{21}x_{23} - x_{11}x_{13} \\ -x_{21}x_{22} + x_{11}x_{12} \end{pmatrix},$$

$$U_2 = \begin{pmatrix} u_{21} \\ u_{22} \\ u_{23} \end{pmatrix} = \begin{pmatrix} (q_{11} - 2q_{21} + q_{31})e_{11} \\ (q_{12} - 2q_{22} + q_{32})e_{12} - a(e_{11} + e_{21}) + x_{31}x_{33} - x_{11}x_{13} \\ (q_{13} - 2q_{23} + q_{33})e_{13} - x_{31}x_{32} + x_{11}x_{12} \end{pmatrix}.$$

Then, we obtain the error system as $D^\alpha e(t) = (\Phi_1(t) + \Phi_2)e(t)$, where $\Phi_2 = \operatorname{diag}(-a - 2q_{11} - q_{21}, b - 2q_{12} - q_{22}, -c - 2q_{13} - q_{23}, -a - 2q_{31} - q_{21}, b - 2q_{32} - q_{22}, -c - 2q_{33} - q_{23})$ and

$$\Phi_1(t) = \begin{pmatrix} 0 & a & 0 & q_{21} - q_{11} & 0 & 0 \\ -a & 0 & 0 & 0 & q_{22} - q_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{23} - q_{13} \\ q_{11} - q_{21} & 0 & 0 & 0 & a & 0 \\ 0 & q_{12} - q_{22} & 0 & -a & 0 & 0 \\ 0 & 0 & q_{13} - q_{23} & 0 & 0 & 0 \end{pmatrix}.$$

Assume that conditions

$$\begin{aligned} &-a - 2q_{11} - q_{21} < 0, \qquad b - 2q_{12} - q_{22} < 0, \qquad -c - 2q_{13} - q_{23} < 0, \\ &-a - 2q_{31} - q_{21} < 0, \qquad b - 2q_{32} - q_{22} < 0, \qquad -c - 2q_{33} - q_{23} < 0 \end{aligned}$$

are satisfied. Thus, according to Corollary 1, the error system is asymptotically stable with the controllers U_1 and U_2. That is, synchronization of 3-coupled fractional-order Lü systems via bidirectional coupling is accomplished.

The Adams–Bashforth–Moulton predictor-corrector scheme [38] is used to obtain the simulation results illustrated with the initial condition $X_1(0) = (1,2,3)^T$, $X_2(0) = (30,9,-10)^T$, $X_3(0) = (4,30,2)^T$, and $\alpha = 0.96$. The fractional-order Lü system with $(a,b,c) = (35,28,3)$ can generate chaotic attractor; see Fig. 2. Further, selecting $q_{11} = q_{13} = q_{21} = q_{23} = q_{33} = 2$, $q_{12} = q_{22} = 23$, $q_{31} = 5$, and $q_{32} = 26$, we obtain simulation results as displayed in Figs. 3 and 4. The state variables of the fractional-order Lü system with designed controllers are demonstrated in Fig. 3. Figure 4 illustrates that the errors of synchronization converge asymptotically to zero in a quite short period, i.e., synchronization of bidirectional 3-coupled fractional-order Lü systems can be realized.

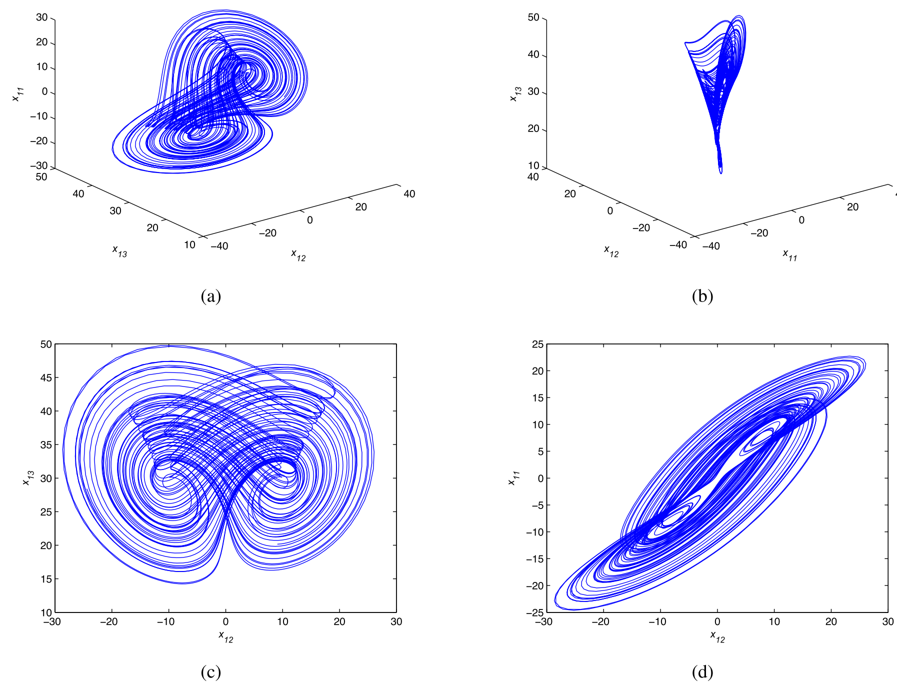


Figure 2 Chaotic attractors of the fractional-order Lü system

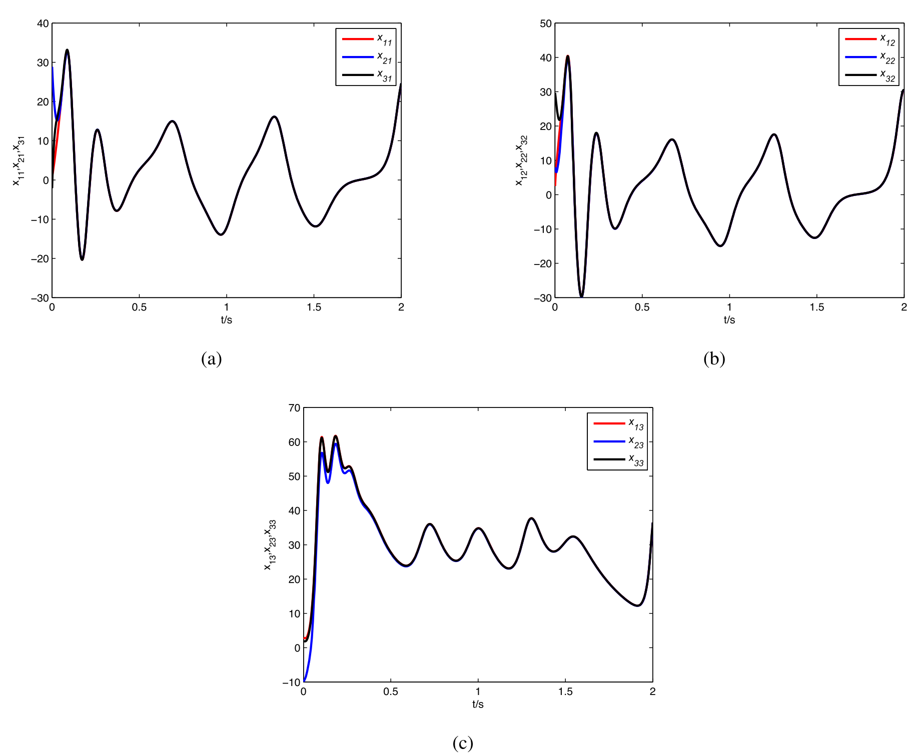


Figure 3 The state trajectories x_{i1}, x_{i2}, and x_{i3} ($i = 1, 2, 3$) of the fractional-order Lü system

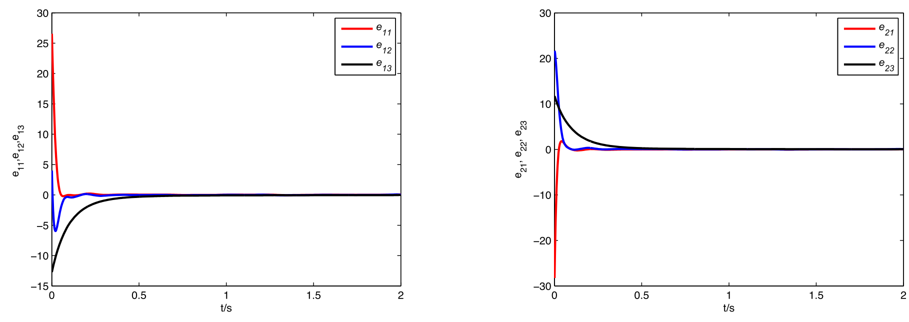


Figure 4 The error dynamics of synchronization among three fractional-order Lü systems

4.2 Synchronization of 3-coupled nonidentical fractional-order chaotic systems

In the following, we consider synchronization of 3-coupled nonidentical fractional-order chaotic systems, which are the fractional-order Lorenz system [6], the fractional-order financial system [8], and the fractional-order PMSM system [9]. They are expressed as

$$\begin{cases} D^\alpha x_{11} = a_1(x_{12} - x_{11}) + q_{11}(x_{21} + x_{31} - 2x_{11}), \\ D^\alpha x_{12} = a_2 x_{11} - x_{12} - x_{11}x_{13} + q_{12}(x_{22} + x_{32} - 2x_{12}), \\ D^\alpha x_{13} = x_{11}x_{12} - a_3 x_{13} + q_{13}(x_{23} + x_{33} - 2x_{13}), \end{cases} \tag{9}$$

$$\begin{cases} D^\alpha x_{21} = x_{23} + (x_{22} - b_1)x_{21} + q_{21}(x_{11} + x_{31} - 2x_{21}) + u_{11}, \\ D^\alpha x_{22} = 1 - b_2 x_{22} - x_{21}^2 + q_{22}(x_{12} + x_{32} - 2x_{22}) + u_{12}, \\ D^\alpha x_{23} = -x_{21} - b_3 x_{23} + q_{23}(x_{13} + x_{33} - 2x_{23}) + u_{13}, \end{cases} \tag{10}$$

and

$$\begin{cases} D^\alpha x_{31} = -x_{31} + x_{32}x_{33} + q_{31}(x_{21} + x_{11} - 2x_{31}) + u_{21}, \\ D^\alpha x_{32} = -x_{32} - x_{31}x_{33} + c_1 x_{33} + q_{32}(x_{22} + x_{12} - 2x_{32}) + u_{22}, \\ D^\alpha x_{33} = c_2(x_{32} - x_{33}) + q_{33}(x_{23} + x_{13} - 2x_{33}) + u_{23}, \end{cases} \tag{11}$$

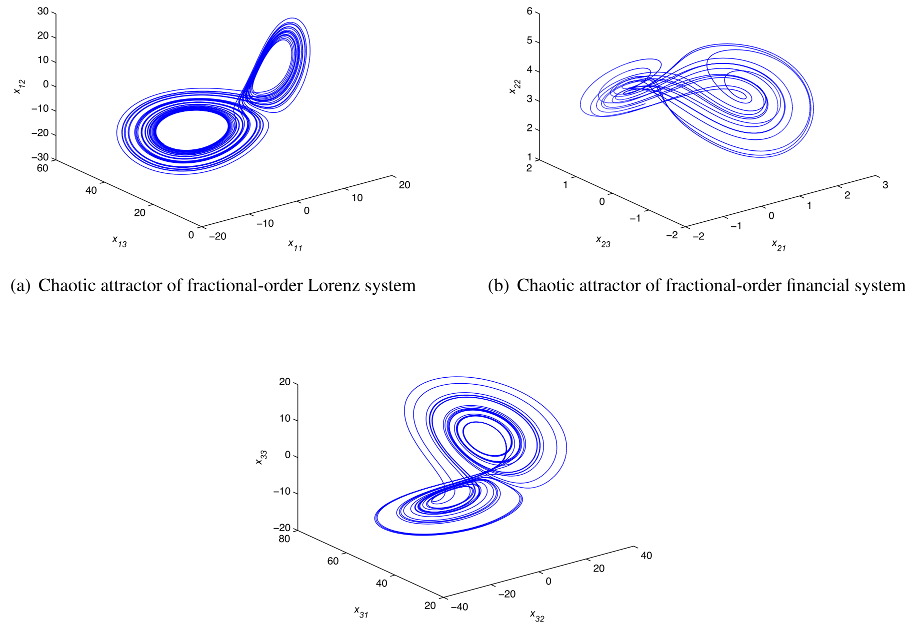
where

$$A_1 = \begin{pmatrix} -a_1 & a_1 & 0 \\ a_2 & -1 & 0 \\ 0 & 0 & -a_3 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} -b_1 & 0 & 1 \\ 0 & -b_2 & 0 \\ -1 & 0 & -b_3 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & c_1 \\ 0 & c_2 & -c_2 \end{pmatrix},$$

$$G_1(X_1) = \begin{pmatrix} 0 \\ -x_{11}x_{13} \\ x_{11}x_{12} \end{pmatrix}, \qquad G_2(X_2) = \begin{pmatrix} x_{21}x_{22} \\ 1 - x_{21}^2 \\ 0 \end{pmatrix}, \qquad G_3(X_3) = \begin{pmatrix} x_{32}x_{33} \\ -x_{31}x_{33} \\ 0 \end{pmatrix},$$

$Q_1 = \mathrm{diag}(q_{11}, q_{12}, q_{13})$, $Q_2 = \mathrm{diag}(q_{21}, q_{22}, q_{23})$, $Q_3 = \mathrm{diag}(q_{31}, q_{32}, q_{33})$ are the coupled matrices, $U_1 = (u_{11}, u_{12}, u_{13})^T$ and $U_2 = (u_{21}, u_{22}, u_{23})^T$ are the control inputs. Let $\alpha = 0.99$,



(a) Chaotic attractor of fractional-order Lorenz system

(b) Chaotic attractor of fractional-order financial system

(c) Chaotic attractor of fractional-order PMSM system

Figure 5 Chaotic attractors of systems (9)–(11)

$(a_1, a_2, a_3) = (10, 28, 8/3)$, $(b_1, b_2, b_3) = (3, 0.1, 1)$, and $(c_1, c_2) = (50, 4)$. In the absence of the controllers, fractional-order systems (9)–(11) without the coupling terms behave chaotically as shown in Fig. 5; see [6, 8, 9] for more details.

The synchronization error can be presented as $e_i = X_{i+1} - X_i$ $(i = 1, 2)$. Then, in view of systems (9)–(11), the error dynamical system reduces to

$$D^\alpha e = \begin{pmatrix} \Sigma_1 & \Sigma_2 \\ \Sigma_3 & \Sigma_4 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + \begin{pmatrix} (a_1 - b_1)x_{11} - a_1 x_{12} + x_{13} + x_{21}x_{22} + u_{11} \\ -a_2 x_{11} + (1 - b_2)x_{12} + 1 - x_{21}^2 + x_{11}x_{13} + u_{12} \\ -x_{11} + (a_3 - b_3)x_{13} - x_{11}x_{12} + u_{13} \\ (b_1 - 1)x_{21} - x_{23} + x_{32}x_{33} - x_{21}x_{22} + u_{21} - u_{11} \\ (b_2 - 1)x_{22} + c_1 x_{23} - x_{31}x_{33} - 1 + x_{21}^2 + u_{22} - u_{12} \\ x_{21} + c_2 x_{22} + (b_3 - c_2)x_{23} + u_{23} - u_{13} \end{pmatrix},$$

where

$$\Sigma_1 = A_2 - 2Q_1 - Q_2 = \begin{pmatrix} -b_1 - 2q_{11} - q_{21} & 0 & 1 \\ 0 & -b_2 - 2q_{12} - q_{22} & 0 \\ -1 & 0 & -b_3 - 2q_{13} - q_{23} \end{pmatrix},$$

$$\Sigma_2 = Q_2 - Q_1 = \begin{pmatrix} q_{21} - q_{11} & 0 & 0 \\ 0 & q_{22} - q_{12} & 0 \\ 0 & 0 & q_{23} - q_{13} \end{pmatrix},$$

$$\Sigma_3 = Q_2 - Q_3 = \begin{pmatrix} q_{21} - q_{31} & 0 & 0 \\ 0 & q_{22} - q_{32} & 0 \\ 0 & 0 & q_{23} - q_{33} \end{pmatrix},$$

$$\Sigma_4 = A_3 - 2Q_3 - Q_2 = \begin{pmatrix} -1-2q_{31}-q_{21} & 0 & 0 \\ 0 & -1-2q_{32}-q_{22} & c_1 \\ 0 & c_2 & -c_2-2q_{33}-q_{23} \end{pmatrix}.$$

Choosing the coefficient matrix

$$\Theta = \begin{pmatrix} \Theta_1 \\ \Theta_2 \end{pmatrix}$$

as follows:

$$\Theta_1 = \begin{pmatrix} 0 & 0 & 0 & q_{31}-q_{21} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{32}-q_{22} & 0 \\ 0 & 0 & 0 & 0 & 0 & q_{33}-q_{23} \end{pmatrix},$$

$$\Theta_2 = \begin{pmatrix} q_{11}-q_{21} & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{12}-q_{22} & 0 & 0 & 0 & -c_1-c_2 \\ 0 & 0 & q_{13}-q_{23} & 0 & 0 & 0 \end{pmatrix},$$

we can design the controllers in the form

$$\begin{cases} u_{11} = (q_{31}-q_{21})e_{21} - (a_1-b_1)x_{11} + a_1x_{12} - x_{13} - x_{21}x_{22}, \\ u_{12} = (q_{32}-q_{22})e_{22} + a_2x_{11} - (1-b_2)x_{12} - 1 + x_{21}^2 - x_{11}x_{13}, \\ u_{13} = (q_{33}-q_{23})e_{23} + x_{11} - (a_3-b_3)x_{13} + x_{11}x_{12}, \end{cases}$$

and

$$\begin{cases} u_{21} = (q_{11}-q_{21})e_{11} + (q_{31}-q_{21})e_{21} - (b_1-1)x_{21} \\ \qquad + x_{23} - x_{32}x_{33} - (a_1-b_1)x_{11} + a_1x_{12} - x_{13}, \\ u_{22} = (q_{12}-q_{22})e_{12} + (q_{32}-q_{22})e_{22} - (c_1+c_2)e_{23} - (b_2-1)x_{22} \\ \qquad - c_1x_{23} + a_2x_{11} - (1-b_2)x_{12} + x_{31}x_{33} - x_{11}x_{13}, \\ u_{23} = (q_{13}-q_{23})e_{13} + (q_{33}-q_{23})e_{23} - x_{21} \\ \qquad - c_2x_{22} - (b_3-c_2)x_{23} + x_{11} - (a_3-b_3)x_{13} + x_{11}x_{12}. \end{cases}$$

Then, we have the error system as $D^\alpha e(t) = (\Phi_1(t) + \Phi_2)e(t)$, where $\Phi_2 = \operatorname{diag}(-b_1-2q_{11}-q_{21}, -b_2-2q_{12}-q_{22}, -b_3-2q_{13}-q_{23}, -1-2q_{31}-q_{21}, -1-2q_{32}-q_{22}, -c_2-2q_{33}-q_{23})$ and

$$\Phi_1(t) = \begin{pmatrix} 0 & 0 & 1 & q_{31}-q_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 & q_{32}-q_{12} & 0 \\ -1 & 0 & 0 & 0 & 0 & q_{33}-q_{13} \\ q_{11}-q_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & q_{12}-q_{32} & 0 & 0 & 0 & -c_2 \\ 0 & 0 & q_{13}-q_{33} & 0 & c_2 & 0 \end{pmatrix}.$$

If conditions

$$\begin{aligned} &-b_1-2q_{11}-q_{21} < 0, \qquad -b_2-2q_{12}-q_{22} < 0, \qquad -b_3-2q_{13}-q_{23} < 0, \\ &-1-2q_{31}-q_{21} < 0, \qquad -1-2q_{32}-q_{22} < 0, \qquad -c_2-2q_{33}-q_{23} < 0 \end{aligned}$$

are satisfied, then the error system is asymptotically stable under the designed controllers by means of Theorem 1. Hence, synchronization of systems (9)–(11) is realized.

The simulation results are illustrated with the initial condition $X_1(0) = (1,4,5)^T$, $X_2(0) = (2,3,2)^T$, $X_3(0) = (6,2,4)^T$, and $\alpha = 0.99$. Furthermore, selecting $q_{11} = q_{12} = q_{13} = 2$, $q_{21} = q_{22} = q_{23} = 5$, $q_{11} = q_{12} = q_{13} = q_{31} = q_{32} = q_{33} = 8$, we obtain simulation results as displayed in Figs. 6 and 7. Figure 6 shows the state variables of the 3-coupled fractional-order systems (9)–(11). From Fig. 7, it is clear that the errors of synchronization converge asymptotically to zero in a quite short period. As expected, synchronization of bidirectional 3-coupled nonidentical fractional-order chaotic systems can be realized.

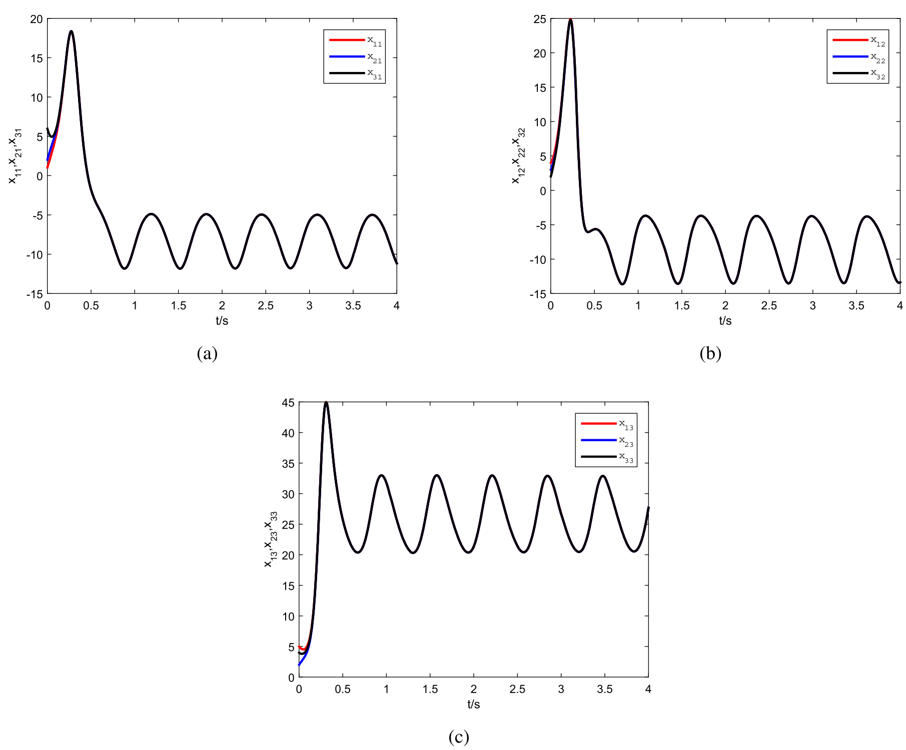


Figure 6 The state trajectories x_{i1}, x_{i2}, and x_{i3} ($i = 1,2,3$) of 3-coupled fractional-order chaotic systems (9)–(11)

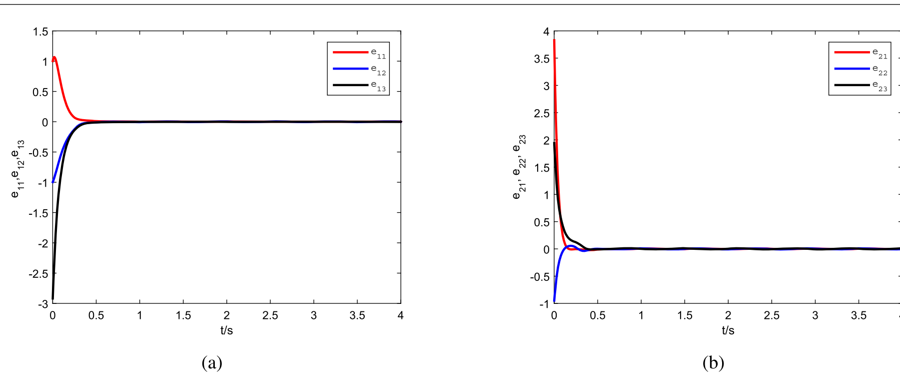


Figure 7 The error dynamics of synchronization among 3-coupled fractional-order chaotic systems (9)–(11)

5 Conclusions

In this paper, we introduce, analyze, and validate synchronization of N-coupled fractional-order chaotic systems with ring connection by utilizing the bidirectional coupling. By virtue of the direct design method, the proper controllers are designed to transform the fractional-order error dynamical system into a nonlinear system with antisymmetric structure. Thus, on the basis of quadratic Lyapunov functions and the Lyapunov stability theory of fractional-order systems, we obtain several sufficient conditions which ensure the occurrence of synchronization among N-coupled fractional-order chaotic systems. Additionally, the synchronization scheme is applicable to all fractional-order chaotic systems, including those that can exhibit hyperchaotic behavior. Furthermore, the proposed results pave the way for new directions in the study of various kinds of chaos synchronization among N-coupled fractional-order systems with ring connection. For example, considering the unknown parameters, external disturbances, and the effect of noise on such synchronization schemes would be interesting and valuable directions for future work.

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Availability of data and materials
Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All four authors contributed equally to this work. They all read and approved the final version of the manuscript.

Author details
[1]School of Mathematics and Statistics, Qilu University of Technology (Shandong Academy of Sciences), Jinan, P.R. China. [2]Department of Mathematics, University of Peshawar, Peshawar, Pakistan. [3]Department of Mathematics, Faculty of Sciences, Erciyes University, Kayseri, Turkey. [4]LinDa Institute of Shandong Provincial Key Laboratory of Network Based Intelligent Computing, Linyi University, Linyi, P.R. China. [5]School of Information Science and Engineering, Linyi University, Linyi, P.R. China.

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