Synchronization of chaotic structurally nonequivalent systems

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(Received 29 November 1999)

Synchronization features are explored for a pair of chaotic high-dimensional bidirectionally coupled structurally nonequivalent systems. We find two regimes of synchronization in dependence on the coupling strength: creation of a lower dimensional chaotic state, and for larger coupling a transition toward a stable periodic motion. We characterize this new state, showing that it is associated with an abrupt transition in the Lyapunov spectrum. The robustness of this state against noise is discussed, and the use of this dynamical property as a possible approach for the control of chaos is outlined.

PACS number(s): 05.45.Xt, 05.45.Jn

In the last decade, synchronization of chaos has become a field of great interest. So far, four types of synchronization have been studied for chaotic concentrated systems, namely complete synchronization (CS) [1], phase (PS) [2], lag (LS) synchronization [3], and generalized synchronization (GS) [4]. CS implies a perfect linking of the chaotic trajectories, so as they remain in step with each other in the course of the time. This mechanism occurs when two identical chaotic systems evolving from different initial conditions are coupled through a signal, provided that the sub-Lyapunov exponents of the subsystem to be synchronized are all negative [1].

In fact, when one considers two nonidentical systems, one can reach a dynamical regime (PS), wherein a perfect locking of the phases of the two signals is realized already for small coupling, while the two amplitudes remain uncorrelated [2].

A third type of synchronization is LS, consisting of the fact that the two signals become identical in phases and amplitudes, but shifted in time of a lag time [3].

Finally, GS implies the hooking of the amplitude of one system to a given function of the amplitude of the other system [4].

Reference [3] describes the situation of two symmetrically coupled nonidentical chaotic oscillators, wherein consecutive transitions between PS, LS, and CS are observed when increasing the coupling strength. These transitions can be identified in terms of changes in the Lyapunov spectrum.

Recently, experimental verifications of these theoretical findings have been offered, e.g., in the cardiorespiratory system [5], in the human brain [6], in the cells of paddlefish [7], and in communication with chaotic lasers [8].

Synchronization features in high dimensional systems have been so far mostly limited to the case of structurally equivalent systems, i.e., systems where the nonidenticity resulted in a rather small parameter mismatch.

In nature we cannot expect to have coupled low-dimensional systems which are structurally equivalent. Therefore, our intention in this paper is to study the coupling of structurally nonequivalent systems, i.e., systems generating chaotic attractors with high and different fractal dimensions. As an example, let us consider the symmetric coupling of two chaotic systems, the first giving rise to a solution x_1 with fractal dimension D_1 and the second to a solution x_2 with fractal dimension D_2 (with a sufficient high difference between D_1 and D_2). Synchronization is associated with the building of some kind of dynamical relations between the two signals. We show that synchronized states can be realized either in a chaotic manifold, which can be very low dimensional if compared with the dimensionality of the two uncoupled systems, or even in a periodic manifold. The transition between these two synchronization manifolds as a function of the coupling parameter is associated with a large change in the dimensionality of the system.

In the following, we will specialize our analysis to two delayed dynamical systems, since they constitute prototypic examples of high dimensional chaotic systems. Such systems, indeed, even provide a link with space extended systems by means of a two variable representation of the time [9], and the formation and propagation of space-time structures, as defects and/or spatiotemporal intermittency has been here identified and controlled [10].

Let us then make reference to symmetrically coupled Mackey-Glass equations:

$$\dot{x}_{1,2} = -0.1x_{1,2}(t) + 0.2\frac{x_{1,2}(t-T_{1,2})}{1+x_{1,2}(t-T_{1,2})^{10}} + \varepsilon[x_{2,1}(t) - x_{1,2}(t)], \tag{1}$$

where the dot denotes the temporal derivative, $x_{1,2}$ are real variables, $T_{1,2}$ are distinct delay times, and $0 \leqslant \varepsilon < 1$ is the coupling strength. Synchronization features have been explored for identical delayed dynamical systems ($T_1 = T_2$) in Ref. [11], even in a high dimensional chaotic case. It is well known that the fractal dimensions of system (1) are proportional to the delay time [12]. If so, selecting $T_1 \neq T_2$ and choosing $T_{1,2}$ sufficiently large, implies that the two systems generate high dimensional chaotic signals with quite different fractal dimensions, thus confined within structurally different chaotic attractors.

The purpose of the present paper is studying the effect of $\varepsilon \neq 0$ in Eq. (1). Even though the scenario that we will describe is a general feature of system (1), regardless of the particular choice of the delay times, for the sake of exempli-

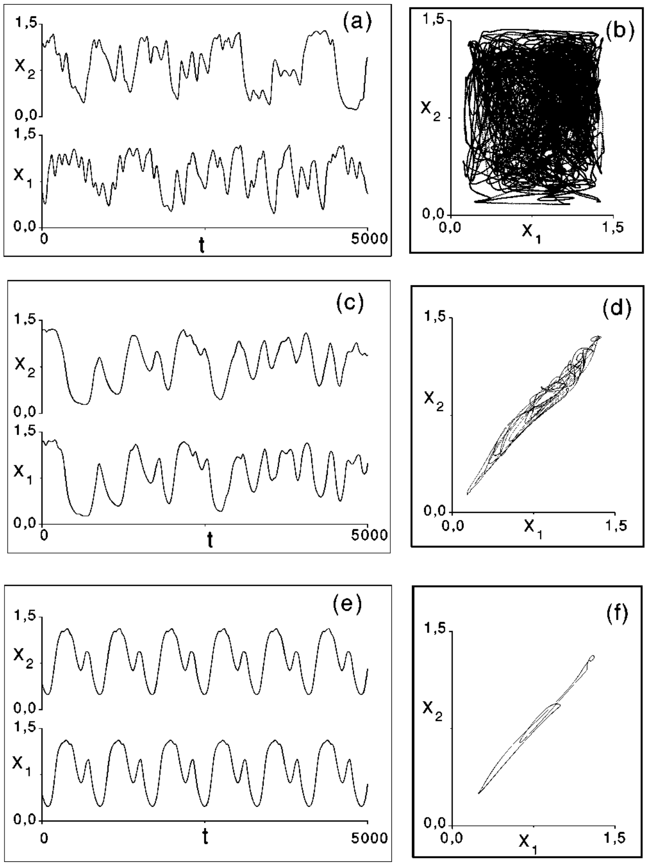

FIG. 1. (a),(c),(e) Time evolution of signals x_1 and x_2 for $\varepsilon=0$ [(a), uncoupled case], $\varepsilon=0.3$ [(c), synchronized chaotic state], and $\varepsilon=0.65$ [(e), synchronized periodic state]. In all cases, time is in arbitrary units. (b),(d),(f) Projections of the attractor of the system (1) on the plane (x_1,x_2) for $\varepsilon=0$ (b), $\varepsilon=0.3$ (d), and $\varepsilon=0.65$ (e).

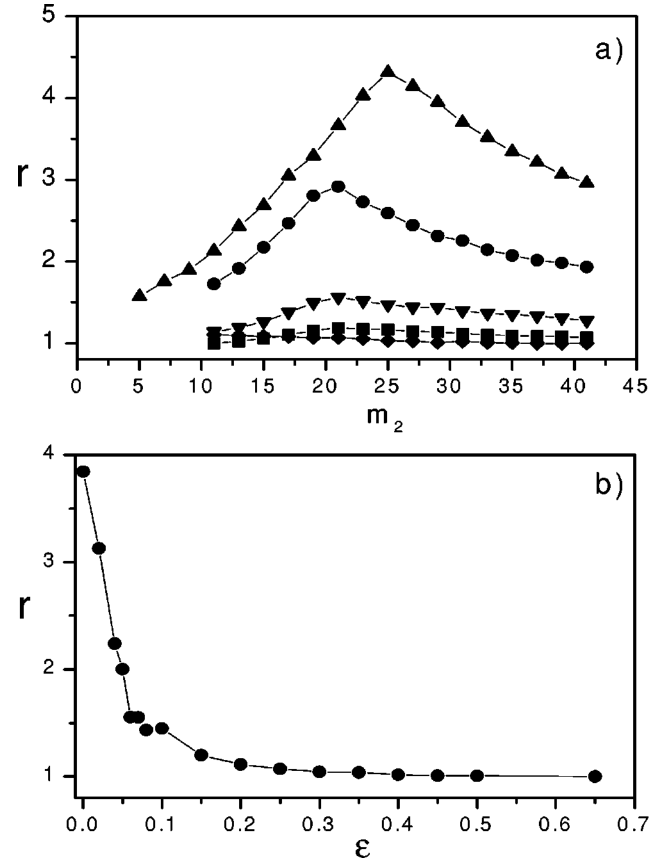

FIG. 2. (a) MFNN parameter [4,13] (dimensionless quantity, see text for definition) as a function of m_2 (dimensionless quantity) for $\varepsilon=0$ (upper triangles), $\varepsilon=0.05$ (circles), $\varepsilon=0.1$ (lower triangles), $\varepsilon=0.2$ (squares), and $\varepsilon=0.5$ (diamonds). The calculations have been done with 500 000 data points from the solution of system (1), and taking $n=5000$ randomly selected state vectors for the averaging process of *r*. $m_1=25$ for $\varepsilon\leqslant 0.1$. $m_1=15$ for $\varepsilon>0.1$. (b) MFNN parameter as a function of ε for a fixed $m_2=35$, and the other parameters as above.

fication, in the following we will select $T_1=100$ and $T_2=90$. This implies that, at $\varepsilon=0$, $x_1(t)$ [$x_2(t)$] develops into a chaotic attractor of fractal dimension $D_1\simeq 12.2$ [$D_2\simeq 10.1$]. Figure 1(a) shows the two signals $x_1(t)$ and $x_2(t)$ at $\varepsilon=0$. The two signals are clearly uncorrelated [Fig. 1(b)].

By gradually increasing ε, the system begins to build up correlations between x_1 and x_2. This result is consistent with what was already observed in Ref. [2] in the case of a symmetric coupling between a chaotic and a hyperchaotic Rössler system. At variance with what was previously reported, there are no intermediate PS states, and phases of the two signals adjust as the result of a gradual transition toward a synchronized state [Fig. 1(c), $\varepsilon=0.3$], which is yet chaotic.

Finally, a transition is observed toward a periodic state, which is reached for large ε values [Fig. 1(e), $\varepsilon=0.65$]. At this stage, the coupled system of Eq. (1) realizes a simple periodic attractor. Therefore, a large structural change in system (1) is associated with the increasing of ε, since the fractal dimensions have passed from $D=22.3$ at $\varepsilon=0$ to $D=1$ at $\varepsilon=0.65$.

To study the appearance of synchronization in system (1) quantitatively, we make use of the *mutual false nearest neighbors* (MFNN) parameter [4,13]. We consider three embedding spaces, namely S_1, S_2, and S_3. S_1 is the embedding space of $x_1(t)$ at the fixed embedding dimension m_1, S_2 is the embedding space of $x_2(t)$ at variable embedding dimension m_2, and S_3 is the embedding space of $x_2(t)$ at the fixed embedding dimension m_1. We then pick up randomly *n* state vectors $\mathbf{x}_1^n$ in S_1 and consider the corresponding images $\mathbf{x}_2^n$ and $\mathbf{x}_3^n$ in S_2 and S_3. We call $\mathbf{x}_{1,NN1}^n$ ($\mathbf{x}_{3,NN3}^n$) the nearest neighbor to $\mathbf{x}_1^n$ ($\mathbf{x}_3^n$) in S_1 (S_3). In the same way, we consider the nearest neighbor $\mathbf{x}_{2,NN2}^n$ to $\mathbf{x}_2^n$ in S_2, and we call $\mathbf{x}_{1,NN2}^n$ ($\mathbf{x}_{3,NN2}^n$) the image of $\mathbf{x}_{2,NN2}^n$ in S_1 (S_3). The MFNN parameter is then defined as [4,13]

$$r=\left\langle \frac{|\mathbf{x}_1^n-\mathbf{x}_{1,NN2}^n|^2}{|\mathbf{x}_1^n-\mathbf{x}_{1,NN1}^n|^2}\,\frac{|\mathbf{x}_3^n-\mathbf{x}_{3,NN3}^n|^2}{|\mathbf{x}_3^n-\mathbf{x}_{3,NN2}^n|^2}\right\rangle_n , \tag{2}$$

where $\langle\cdots\rangle_n$ denotes averaging over *n*. It has been shown that $r\equiv 1$ for systems showing GS, whereas $r\neq 1$ when the systems are not synchronized (for more details, see Ref. [4,13]).

Figure 2(a) shows the dependence of *r* upon m_2, by fixing $m_1=25$ for $\varepsilon\leqslant 0.1$, and $m_1=15$ for $\varepsilon>0.1$. Figure 2(b) reports *r* as a function of ε for $m_2=35$, and $m_1=25$ ($\varepsilon\leqslant 0.1$), $m_1=15$ ($\varepsilon>0.1$). The primer of a synchronized state appears evident at $\varepsilon\simeq 0.15$.

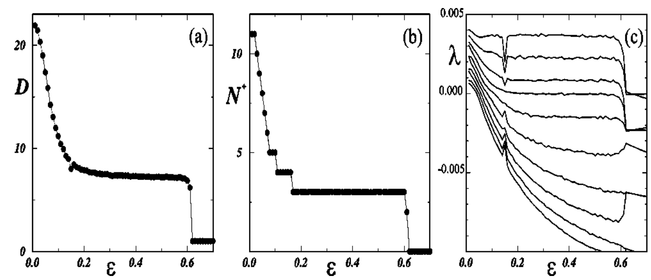

FIG. 3. (a) Kaplan-Yorke dimension of Eq. (1) as a function of the coupling strength ε. (b) Number of positive exponents in the Lyapunov spectrum vs ε. (c) Largest ten exponents in the Lyapunov spectrum vs ε. In all cases the calculations have been performed over a time $\overline{t}=1\,000\,000$, corresponding to 10 000 delay units of the system with larger delay. The transition toward a periodic synchronized state around $\varepsilon \simeq 0.6$ is marked by a sudden change in the Kaplan-Yorke dimension (a) and by the fact that many positive Lyapunov exponents goes to negative value at once (c).

The occurrence of the periodic synchronized state is associated with a transition in the Lyapunov spectrum, wherein many positive Lyapunov exponents passes to negative values at once. Figure 3 reports the measurement of the Kaplan-Yorke or Lyapunov dimension of Eq. (1) [Fig. 3(a)] as well as the number of positive Lyapunov exponents [Fig. 3(b)], as functions of ε. For small ε, one observes a slow continuous decreasing process of the Lyapunov dimension driving sequentially positive Lyapunov exponents from positive to negative values, consistent with what was already described for structurally equivalent systems. Indeed, a slow continuous variation in the Lyapunov dimension is signature of a slow continuous variation in the Lyapunov spectrum.

At larger couplings, two different dynamical regimes can be isolated. This first corresponds to the appearance of GS ($0.15<\varepsilon$). At the beginning of GS, a plateau in the Lyapunov dimension around $D \simeq 7.2$–7.5 sets in for $0.15<\varepsilon<0.6$. This situation indicates that GS is initially realized over a high dimensional chaotic state. Correspondingly, the number of positive Lyapunov exponents does not change.

Finally, a second regime is encountered ($0.6<\varepsilon$). Here, we find a transition in the Lyapunov dimension, leading to the stabilization of a final periodic state. Around such a transition point we observe a sudden change in the Lyapunov spectrum, wherein all residual positive Lyapunov exponents suddenly jump to negative values at once [Fig. 3(c)]. This phenomenon constitutes a remarkable difference with the synchronization features studied so far in the literature.

These findings are further confirmed by looking at the ten largest Lyapunov exponents of Eq. (1) as functions of the coupling parameter ε [Fig. 3(c)]. All calculations have been performed over a time $\overline{t}=1\,000\,000$, corresponding to 10 000 delay units of the system with larger delay.

Focusing on the transition from a hyperchaotic state to a periodic orbit near $\varepsilon \approx 0.6$, we find an intermittent behavior. The system switches in time between two qualitative different types of dynamics: a motion close to the periodic orbit; and a vastly irregular motion far away from the periodic orbit. A similar desynchronization scenario has been characterized as on-off intermittency in the case of identical DDS [11].

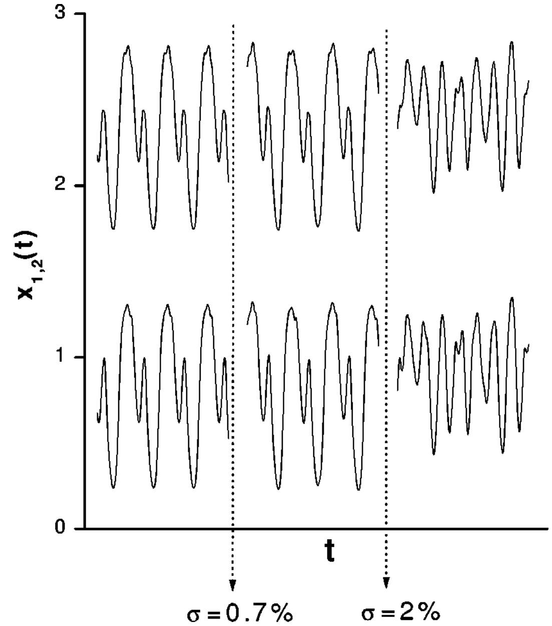

FIG. 4. Dynamical effect of noise in Eq. (1). $\varepsilon=0.65$. Horizontal axis reports time in arbitrary units, vertical axis reports the two signals $x_{1,2}$. The signal x_2 is vertically shifted of 1.5. The noise amplitude σ is measured in percent of the rms of the signal oscillations. Three different situations are shown: $\sigma=0\%$ (first part of the plot), $\sigma=0.7\%$ (middle part of the plot), and $\sigma=2\%$ (last part of the plot). Therefore, the scale for the horizontal axis is not set. While the synchronized periodic state is robust against relatively small noise amplitudes, larger noise amplitudes recover the chaotic synchronized state.

A heuristic argument for this dynamical transition can be offered. From Fig. 1(d) one can easily realize that the coupling is strongly reduced in the synchronized state, since $x_1(t) \simeq x_2(t)$. Therefore, the two systems should adjust themselves in some dynamical solution compatible with both attractors, that is they must adjust onto a trajectory contained within the intersection of the two independent chaotic attractors. Now, it is a well known property of chaotic systems that the trajectory moves within the attractor such as to shadow an infinite number of unstable periodic orbits (UPO's) [14]. The global Lyapunov exponents (Λ's) are the average over the trajectory of the local Lyapunov exponents (λ's). Ergodicity of chaos allows one to calculate such average over all λ's corresponding to the different UPO's. Now, different UPO's may have different λ's. Therefore, even though the spectrum is composed by many positive Λ's, it can occur that a given UPO possesses a single positive Lyapunov exponent $\overline{\lambda}$. From the other side, the coupling strength gradually reduces all λ's. Therefore, when ε is such that $\overline{\lambda}$ becomes negative, the corresponding periodic orbit becomes linearly stable. In this case, the attractor contains a linearly stable orbit embedded within an infinity of other UPO's. Then, apart for the transient time needed by the natural evolution for shadowing that particular orbit (the so called target time), the system is asymptotically trapped on this orbit.

This qualitative picture is confirmed by the measurement of the robustness of the periodic synchronized state against noise, performed by adding a noisy source $\xi_{1,2}(t)$ to the right-hand side of Eq. (1). The noisy perturbation has zero average, and it is δ correlated in time. The results are reported in Fig. 4. After setting $\varepsilon = 0.65$ such as the original system (1) realizes the synchronized periodic state, we set different noise amplitudes. A relevant result is that there exists a critical noise amplitude above which the modified system recovers the same chaotic synchronized state naturally occurring at smaller ε (see the last part of the plot in Fig. 4). This feature is consistent with what was discussed above. Indeed, when the noisy perturbation is sufficiently large, so as to lead the trajectory outside the phase-space region where linear properties hold, the trajectory gets recaptured in the infinity of other UPO's, and therefore the motion becomes again chaotic, but still preserving the synchronization features.

Finally, let us briefly discuss the relevance of the described phenomenon. The stabilization process of a previously unstable periodic motion is a consequence here of a sufficiently large coupling strength between the two systems, but *it is not generated by external perturbations*, as in the case of the usual chaos control theory [15]. Therefore, the presented mechanism can be used as an alternative approach for the *internal* stabilization of unstable periodic orbits, in all cases in which external interventions would be not desirable, nor possible.

The authors acknowledge Martin Bünner for help in calculating the Lyapunov exponents, and Santiago de San Roman, Ying Chen Lai, Celso Grebogi, Antonio Politi, and Holger Kantz for fruitful discussions. This work was partly supported by the Ministerio de Educacion y Ciencia, Spain (Grant No. PB95-0578), Universidad de Navarra, Spain (PIUNA), and Integrated Action Italy-Spain HI97-30. S.B. acknowledges financial support from EU Contract No. ERBFM-BICT983466. D.L.V. acknowledges financial support from an AECI grant. J.K. acknowledges financial support from the SFB (Contract No. SFB555).

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