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Synchronization of complex dynamical networks with time delays☆

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Abstract

In the present paper, two kinds of dynamical complex networks are considered. The first is that elements of every node have different time delays but all nodes in such networks have the same time-delay vector. The second is that different nodes have different time-delay vectors, and the elements of each node also have different time delays. Corresponding synchronization theorems are established. Numerical examples show the efficiency of the derived theorems.

Keywords: Synchronization; Complex dynamical networks; Time delay

1. Introduction

It has been shown that complex networks exist in various fields of real world, such as in the Internet, the World Wide Web, telephone call graphs, food webs, neural networks, electrical power grids, cellular and metabolic networks, scientific citation web, living organisms, etc. The nature of complex networks is their complexity,

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including topological structure, dynamical evolution, node diversity and meta-complication, etc. ([1] and many references cited therein).

Among all kinds of complex networks, the random networks, small-world networks and scale-free networks are most noticeable. The random networks were first introduced by Erdös and Rényi [2]. Their random model has dominated the mathematical research of complex networks for nearly half a century, mainly due to the absence of super-computational power and detailed topological information about various large-scale real-world networks. The small-world networks were introduced by Watts and Strogatz [3,4] in the process of investigating the transition from regular networks to random ones. Such networks behave a high degree of clustering as in the regular networks and a small average distance among nodes as the random distribution of the network peaks at an average value and decays exponentially (e.g., Refs. [5,6] and references therein). The latter ones were introduced by Barabási and Albert [7], which exhibit power-law distribution. Generally speaking, power law is regarded to be equivalent to the scale free property, see Ref. [1]. However, as far as we know, there is no rigorous mathematical proof for this equivalence. There are many interesting works on such networks [8,9].

Time delays commonly exist in the world, some of them are trivial so can be ignorant whilst some of them can not be ignored, such as in long-distance communication and traffic congestions, etc. Recently, Masoller and her cooperators have considered the coupled map lattice with time delay [10–12]. More recently, Li and Chen have discussed continuous complex dynamical networks with (same) time delays in the whole networks [13], where the stability theorem of synchronization is established by constructing a Lyapunov–Krasovskii functional [14] which is often difficult to be found.

In the present paper, we study the complex dynamical networks which have different time delays. We show that the stability theorems of synchronization are easily derived just by using the definition of matrix measure [15,16]. The layout of the paper is organized as follows. In Section 2, two kinds of complex dynamical networks with time delays are considered and some new theorems of synchronization for networks with time delays are established. Numerical examples are given in Section 3, and some remarks are included in Section 4.

2. Synchronization stability for networks with time delays

In Ref. [13], Li and Chen studied the following networks:

$$\dot{x}_i = f(x_i) + \varepsilon \sum_{j=1}^{N} c_{ij} A x_j(t-\tau), \quad i = 1, 2, \ldots, N, \tag{1}$$

where $f : \mathfrak{R}^n \to \mathfrak{R}^n$ is a continuously differentiable function, $x_i = (x_{i1}, x_{i2}, \ldots, x_{in})^{\mathrm{T}} \in \mathfrak{R}^n$ are the state variables of node i, $\varepsilon > 0$ represents the coupling strength, $A = (a_{ij})_{n\times n} \in \mathfrak{R}^{n\times n}$ indicates inner-coupling between the elements of the node itself, while $C = (c_{ij})_{N\times N}$ denotes the outer-coupling between the nodes of the whole network (it is often assumed that there is at most one connection between node i and another node j, and that there are no isolated clusters, i.e., C is an irreducible matrix). The entries c_{ij} are defined as follows: if there is a connection between node i and node $j\,(j\neq i)$, then we set $c_{ij}=1$; otherwise $c_{ij}=0\,(j\neq i)$, and the diagonal elements of C are defined by $c_{ii}=-\sum_{j=1,j\neq i}^{N}c_{ij}$, $i=1,2,\ldots,N$, τ is the time delay. Obviously all time delays in the network are the same. And the outer coupling configuration matrix C is a real symmetric and irreducible one. The following lemma can be found in Refs. [8,17]:

Lemma 1. *Suppose that* $C=(c_{ij})_{N\times N}$ *is a real symmetric and irreducible matrix, where* $c_{ij}\geqslant 0$ $(i\neq j)$, $c_{ii}=-\sum_{j=1,j\neq i}^{N}c_{ij}$, *then*

(1) 0 *is an eigenvalue of* C *with multiplicity* 1, *associated with the eigenvector* $(1,1,\ldots,1)^{\mathrm{T}}$;
(2) *all the other eigenvalues of* C *are less than* 0;
(3) *there exists a unitary matrix*, $\Phi=(\phi_1,\phi_2,\ldots,\phi_N)$ *such that*

$$C^{\mathrm{T}}\phi_k=\lambda_k\phi_k,\quad k=1,2,\ldots,N\,,$$

where $0=\lambda_1>\lambda_2\geqslant\lambda_3\geqslant\cdots\geqslant\lambda_N$ *are the eigenvalues of* C.

At present, the following more general complex networks are analyzed,

$$\begin{aligned}\dot{x}_i&=f(x_i)+\varepsilon\sum_{j=1}^{N}c_{ij}A(x_{j1}(t-\tau_1),x_{j2}(t-\tau_2),\ldots,x_{jn}(t-\tau_n))^{\mathrm{T}}\triangleq f(x_i)\\&\quad+\varepsilon\sum_{j=1}^{N}c_{ij}A\cdot\overline{x_j(t-\tau)}\end{aligned}\tag{2}$$

in which f, ε, x_i $(i=1,2,\ldots,N)$, C and A have the same meanings as those in (1). The only difference is that in Eq. (2) every node has the same retardation time vector $(\tau_1,\tau_2,\ldots,\tau_n)$.

Hereafter, the network (2) with delays is said to achieve (asymptotical) synchronization if

$$x_i(t)=s(t)\text{ as }t\to+\infty,\quad i=1,2,\ldots,N\,,\tag{3}$$

in which $s(t)\in\Re^n$ satisfies

$$\dot{s}(t)=f(s(t))\,,$$

where $s(t)$ can be either an equilibrium point, or a (quasi-)periodic orbit, or an orbit of a chaotic attractor.

Before the establishment of the synchronization theorem, the definition of the "matrix measure" [15,16] is introduced below.

Definition 1. Suppose that $\|\cdot\|_i$ is an induced matrix norm in $\mathscr{C}^{n\times n}$, then the corresponding matrix measure of a given matrix A is a function $\mathscr{M}_i:\mathscr{C}^{n\times n}\to\Re$, defined by

$$\mathscr{M}_i(A) = \lim_{\delta \to 0^+} \frac{\|I + \delta A\|_i - 1}{\delta} ,$$

in which I is the identity matrix.

In general, the matrix measure is difficult to calculate. But for some special norms, it is easy to find the associate matrix measures. For example, $\mathscr{M}_2(A)$ is half of the maximum eigenvalue of $\bar{A}' + A$. Here $\|A\|_2$ is the spectral norm of A, and $\bar{A}'$ is the transpose and complex conjugate of A.

Theorem 1. *Consider the delayed dynamical network* (2), *let* $0 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_N$ *be the eigenvalues of the coupling configuration matrix C. If the following* $(N-1)$ *systems of n-dimensional linear time-varying delayed differential equations are asymptotically stable about their zero solutions*:

$$\dot{\eta}(t) = Df(s(t))\eta(t) + \varepsilon\lambda_i A \cdot \overline{\eta(t-\tau)}, \quad i = 2, \ldots, N , \tag{4}$$

where $Df(s(t)) \in \Re^{n\times n}$ *is the Jacobian of* $f(x(t))$ *at* $s(t)$, $\eta(t) \in \Re^n$, $\overline{\eta(t-\tau)} = (\eta_1(t-\tau_1), \ldots, \eta_n(t-\tau_n))^{\mathrm{T}} \in \Re^n$, *then the synchronized states of* (3) *are asymptotically stable.*

Proof. To investigate the stability of the synchronized states (3), let

$$x_i(t) = s(t) + e_i(t) . \tag{5}$$

Substituting in (2) gives

$$\dot{e}_i(t) = f(s(t) + e_i(t)) - f(s(t)) + \varepsilon \sum_{j=1}^{N} c_{ij} A \cdot \overline{e_j(t-\tau)}, \quad 1 \leqslant i \leqslant N \tag{6}$$

in which $\overline{e_j(t-\tau)} = (e_{j1}(t-\tau_1), \ldots, e_{jn}(t-\tau_n))^{\mathrm{T}} \in \Re^n$. Its linearized system reads as

$$\dot{e}_i(t) = Df(s(t))e_i(t) + \varepsilon A \cdot (\overline{e_1(t-\tau)}, \overline{e_2(t-\tau)}, \ldots, \overline{e_N(t-\tau)})(c_{i1}, \ldots, c_{iN})^{\mathrm{T}} . \tag{7}$$

Let $e(t) = (e_1(t), e_2(t), \ldots, e_N(t)) \in \Re^{n\times N}$, one gets

$$\dot{e}(t) = Df(s(t))e(t) + \varepsilon A \cdot \overline{e(t-\tau)}\, C^{\mathrm{T}} , \tag{8}$$

where $\overline{e(t-\tau)} = (\overline{e_1(t-\tau)}, \overline{e_2(t-\tau)}, \ldots, \overline{e_N(t-\tau)}) \in \Re^{n\times N}$.

By Lemma 1, there exists a nonsingular matrix Φ, such that $C^{\mathrm{T}}\Phi = \Phi\Gamma, \Gamma = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$. If one sets $e(t)\Phi = v(t) = (v_1(t), v_2(t), \ldots, v_N(t)) \in \Re^{n\times N}$, then (8) can be transformed into the following matrix equation

$$\dot{v}(t) = Df(s(t))v(t) + \varepsilon A\overline{v(t-\tau)}\Gamma , \tag{9}$$

that is,

$$\dot{v}_i(t) = Df(s(t))v_i(t) + \varepsilon\lambda_i A \cdot \overline{v_i(t-\tau)}, \quad i = 1, \ldots, N , \tag{10}$$

in which $\overline{v_i(t-\tau)} = (v_{i1}(t-\tau_1), \ldots, v_{in}(t-\tau_n))^{\mathrm{T}} \in \Re^n$.

Note that $\lambda_1 = 0$ corresponds to the synchronization of the system (3). If the following $(N-1)$ pieces of the n-dimensional linear time-varying delayed differential equations

$$\dot{v}_i(t) = Df(s(t))v_i(t) + \varepsilon\lambda_i A \cdot \overline{v_i(t-\tau)}, \quad i = 2, \ldots, N$$

are asymptotically stable, then $e(t)$ will tend to zero asymptotically, which shows that the synchronized states (3) are asymptotically stable. This completes the proof. □

Theorem 2. *Assume that all eigenvalues of the matrix* C *in* (2) *are listed in order,* $0 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_N$. *If there exist an* i *and a* j *such that* $\begin{pmatrix} -\lambda_j(A + A^{\mathrm{T}}) & \lambda_i A \\ \lambda_i A^{\mathrm{T}} & 0 \end{pmatrix} \leqslant 0$, $\mathscr{M}_2[Df(s(t)) + \lambda_j A] \leqslant a < 0$ *for all* $t \geqslant t_0$, *where* $i = 2, 3, \ldots, n$, $j = 1, 2, \ldots, n$, *then the synchronous states* (3) *of the dynamical network* (2) *are exponentially stable for any fixed delay* $\tau_k > 0\,(k = 1, 2, \ldots, n)$.

Proof. Consider the linear time-varying system

$$\dot{x}(t) = Df(s(t))x(t) + \varepsilon\lambda_i A \cdot \overline{x(t-\tau)}, \quad i = 2, 3, \ldots,$$

where $\overline{x(t-\tau)} = (x_1(t-\tau_1), x_2(t-\tau_2), \ldots, x_n(t-\tau_n))^{\mathrm{T}} \in \Re^n$. Since

$$\begin{aligned}
&\mathrm{d}\|x(t)\|^2/\mathrm{d}t \\
&\quad = [(x(t)^{\mathrm{T}} Df(s(t))x(t) + \varepsilon\lambda_i x(t)^{\mathrm{T}} A \cdot \overline{x(t-\tau)} + x(t)^{\mathrm{T}}(Df(s(t)))^{\mathrm{T}} x(t) \\
&\qquad + \varepsilon\lambda_i \overline{x(t-\tau)}^{\mathrm{T}} A^{\mathrm{T}} x(t)] \\
&\quad = x(t)^{\mathrm{T}}\{[Df(s(t)) + \lambda_j A]^{\mathrm{T}} + [Df(s(t)) + \lambda_j A]\}x(t) \\
&\qquad + \varepsilon \begin{pmatrix} x(t) \\ \overline{x(t-\tau)} \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} -\lambda_j(A + A^{\mathrm{T}}) & \lambda_i A \\ \lambda_i A^{\mathrm{T}} & 0 \end{pmatrix} \begin{pmatrix} x(t) \\ \overline{x(t-\tau)} \end{pmatrix} \\
&\quad \leqslant x(t)^{\mathrm{T}}\{[Df(s(t)) + \lambda_j A]^{\mathrm{T}} + [Df(s(t)) + \lambda_j A]\}x(t) \\
&\quad \leqslant 2x(t)^{\mathrm{T}} \mathscr{M}_2[Df(s(t)) + \lambda_j A]x(t) \leqslant 2a\|x(t)\|^2,
\end{aligned}$$

one has $\|x(t)\| \leqslant \|x(t_0)\|\mathrm{e}^{2at}$. That is, all the linear time-varying systems (4) are exponentially stable around their zero solutions. From Theorem 1, the synchronous states of dynamical network (2) are exponentially stable. The proof is thus complete. □

In network (2), the elements of any node have different time delays, but all nodes in the whole network have the same time-delay vector, i.e., the time-delay vector is $(\tau_1, \tau_2, \ldots, \tau_n)$. In the following, a somewhat more general network can be similarly studied. Consider

$$\begin{aligned}
\dot{x}_i &= f(x_i) + \varepsilon \sum_{j=1}^{N} c_{ij} A(x_{j1}(t-\tau_{j1}), x_{j2}(t-\tau_{j2}), \ldots, x_{jn}(t-\tau_{jn}))^{\mathrm{T}} \triangleq f(x_i) \\
&\quad + \varepsilon \sum_{j=1}^{N} c_{ij} A \cdot \overline{x_j(t-\tau_j)},
\end{aligned} \tag{11}$$

where f, ε, x_i $(i = 1, 2, \ldots, N)$, C and A have the same meanings as those in (2), the unique difference is that in (11) a different node j has the different time-delay vector, $(\tau_{j1}, \tau_{j2}, \ldots, \tau_{jn})$. The synchronous states of (11) are also defined as (3). Similar to Theorems 1 and 2, one gets

Theorem 3. *Consider the delayed dynamical network* (11). *Let* $0 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_N$ *be the eigenvalues of the coupling configuration matrix C. If the following* $(N-1)$ *systems of n-dimensional linear time-varying delayed differential equations are asymptotically stable about their zero solutions*

$$\dot{\eta}(t) = Df(s(t))\eta(t) + \varepsilon\lambda_i A \cdot \overline{\eta(t-\tau_i)}, \quad i = 2, \ldots, N, \tag{12}$$

where $Df(s(t)) \in \Re^{n\times n}$ *is the Jacobian of* $f(x(t))$ *at* $s(t)$, $\eta(t) \in \Re^n$, $\overline{\eta(t-\tau_i)} = (\eta_1(t-\tau_{i1}), \ldots, \eta_n(t-\tau_{in}))^{\mathrm{T}} \in \Re^n$, *then the synchronized states of the equilibria to* (11) *are asymptotically stable.*

Theorem 4. *Assume that all eigenvalues of the matrix C in* (11) *are listed in order*, $0 = \lambda_1 > \lambda_2 \geqslant \cdots \geqslant \lambda_N$. *If there exist an i and a j such that*

$$\begin{pmatrix} -\lambda_j(A + A^{\mathrm{T}}) & \lambda_i A \\ \lambda_i A^{\mathrm{T}} & 0 \end{pmatrix} \leqslant 0, \mathscr{M}_2[Df(s(t)) + \lambda_j A] \leqslant a < 0$$

for all $t \geqslant t_0$, *where* $i = 2, 3, \ldots, N$, $j = 1, 2, \ldots, N$, *then the synchronous states of* (3) *the equilibria to the dynamical network* (11) *are exponentially stable for any fixed delay* $\tau_{kl} > 0\,(k, l = 1, 2, \ldots, n)$.

The proofs of Theorems 3 and 4 are almost the same as those of Theorems 1 and 2, so are omitted here. Some notes on network models (1) and (2) are presented in Ref. [18]. And some other related results are studied in Refs. [19,20].

3. Several illustrative examples

The following simulations illustrate the theoretical results derived in Section 2. For simplicity, we consider a four-node network, in which each node is a simple 3-dimensional stable linear system described in Ref. [13] $(\dot{x}_{i1}, \dot{x}_{i2}, \dot{x}_{i3})^{\mathrm{T}} = (-x_{i1}, -2x_{i2}, -3x_{i3})^{\mathrm{T}}$, and its Jacobian matrix is $Df(s(t)) = \mathrm{diag}(-1, -2, -3) \triangleq J$. Assume that the coupled configuration matrix $(c_{ij})_{N\times N}$ is

$$C = \begin{pmatrix} -2 & 1 & 1 & 0 \\ 1 & -3 & 1 & 1 \\ 1 & 1 & -3 & 1 \\ 0 & 1 & 1 & -2 \end{pmatrix},$$

the coupling strength $\varepsilon = 0.2$ and the inner-coupling matrix $A = \mathrm{diag}(1, 1, 1)$.

At first, the time-delay vector $(1, 2, 3)$ is considered. So the corresponding network reads below

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \mathrm{diag}(J, J, J, J) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \varepsilon \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \tag{13}$$

in which $x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}} \in R^3$, $i = 1, 2, 3, 4$, and

$$y_1 = \begin{pmatrix} -2x_{11}(t-1) + x_{21}(t-1) + x_{31}(t-1) \\ -2x_{12}(t-2) + x_{22}(t-2) + x_{32}(t-2) \\ -2x_{13}(t-3) + x_{23}(t-3) + x_{33}(t-3) \end{pmatrix},$$

$$y_2 = \begin{pmatrix} x_{11}(t-1) - 3x_{21}(t-1) + x_{31}(t-1) + x_{41}(t-1) \\ x_{12}(t-2) - 3x_{22}(t-2) + x_{32}(t-2) + x_{42}(t-2) \\ x_{13}(t-3) - 3x_{23}(t-3) + x_{33}(t-3) + x_{43}(t-3) \end{pmatrix},$$

$$y_3 = \begin{pmatrix} x_{11}(t-1) + x_{21}(t-1) - 3x_{31}(t-1) + x_{41}(t-1) \\ x_{12}(t-2) + x_{22}(t-2) - 3x_{32}(t-2) + x_{42}(t-2) \\ x_{13}(t-3) + x_{23}(t-3) - 3x_{33}(t-3) + x_{43}(t-3) \end{pmatrix},$$

$$y_4 = \begin{pmatrix} x_{21}(t-1) + x_{31}(t-1) - 2x_{41}(t-1) \\ x_{22}(t-2) + x_{32}(t-2) - 2x_{42}(t-2) \\ x_{23}(t-3) + x_{33}(t-3) - 2x_{43}(t-3) \end{pmatrix}.$$

In the following, we show its synchronous states $R^3 \ni x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}} = s(t) = \mathbf{0}$ is asymptotically stable by numerical simulations. Set $e_{i1} = x_{i+1,1} - x_{i1}$, $e_{i2} = x_{i+1,2} - x_{i2}$, $e_{i3} = x_{i+1,3} - x_{i3}$. The curves of the synchronous states for (13) are plotted in Fig. 1.

Next, we only use Theorem 1 to verify it. Consider the following system in R^3,

$$\begin{pmatrix} \dot{\eta}_1 \\ \dot{\eta}_2 \\ \dot{\eta}_3 \end{pmatrix} = \begin{pmatrix} -\eta_1 \\ -2\eta_2 \\ -3\eta_3 \end{pmatrix} + \lambda_i \begin{pmatrix} \eta_1(t-1) \\ \eta_2(t-2) \\ \eta_3(t-3) \end{pmatrix}, \quad i = 2, 3, 4, \tag{14}$$

where $\lambda_2 = -2, \lambda_3 = \lambda_4 = -4$ are three nonzero eigenvalues of C (the first one is zero).

The characteristic matrix [21] of (14) is $\mathrm{diag}(\lambda - \lambda_i \mathrm{e}^{-\lambda} + 1, \lambda - \lambda_i \mathrm{e}^{-2\lambda} + 2, \lambda - \lambda_i \mathrm{e}^{-3\lambda} + 3)$, all its eigenvalues are negative, so the trivial solution to (14) is asymptotically stable for $\lambda_2, \lambda_3, \lambda_4$. So by Theorem 1, the synchronous states (3) for (13) are asymptotically stable, which coincides with the above numerical results.

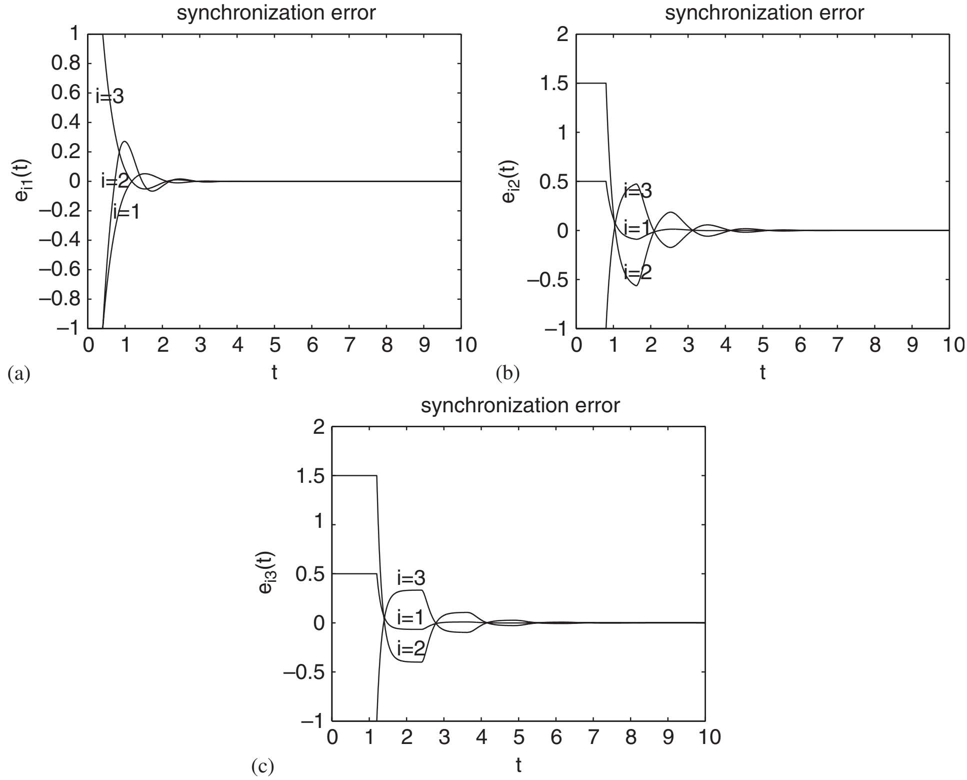

Fig. 1. Synchronization errors for the delayed networks (13): (a) $e_{i1}(t)$ vs t, (b) $e_{i2}(t)$ vs t, (c) $e_{i3}(t)$ vs t.

In the following, a more general case is considered. The time-delay matrix is read as,

$$\begin{pmatrix} 1.1 & 1.2 & 1.3 \\ 2.1 & 2.2 & 2.3 \\ 3.1 & 3.2 & 3.3 \\ 4.1 & 4.2 & 4.3 \end{pmatrix} \begin{array}{lll} \leftarrow & \text{1st} & \text{node}\,, \\ \leftarrow & \text{2nd} & \text{node}\,, \\ \leftarrow & \text{3rd} & \text{node}\,, \\ \leftarrow & \text{4th} & \text{node}\,. \end{array}$$

The network is explicitly expressed as follows,

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \mathrm{diag}(J, J, J, J) \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \varepsilon \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}, \tag{15}$$

in which $x_i = (x_{i1}, x_{i2}, x_{i3})^{\mathrm{T}} \in R^3,\ i = 1, 2, 3, 4,\ \varepsilon = 0.2$, and

$$y_1 = \begin{pmatrix} -2x_{11}(t-1.1) + x_{21}(t-2.1) + x_{31}(t-3.1) \\ -2x_{12}(t-1.2) + x_{22}(t-2.2) + x_{32}(t-3.2) \\ -2x_{13}(t-1.3) + x_{23}(t-2.3) + x_{33}(t-3.3) \end{pmatrix},$$

$$y_2 = \begin{pmatrix} x_{11}(t-1.1) - 3x_{21}(t-2.1) + x_{31}(t-3.1) + x_{41}(t-4.1) \\ x_{12}(t-1.2) - 3x_{22}(t-2.2) + x_{32}(t-3.2) + x_{42}(t-4.2) \\ x_{13}(t-1.3) - 3x_{23}(t-2.3) + x_{33}(t-3.3) + x_{43}(t-4.3) \end{pmatrix},$$

$$y_3 = \begin{pmatrix} x_{11}(t-1.1) + x_{21}(t-2.1) - 3x_{31}(t-3.1) + x_{41}(t-4.1) \\ x_{12}(t-1.2) + x_{22}(t-2.2) - 3x_{32}(t-3.2) + x_{42}(t-4.2) \\ x_{13}(t-1.3) + x_{23}(t-2.3) - 3x_{33}(t-3.3) + x_{43}(t-4.3) \end{pmatrix},$$

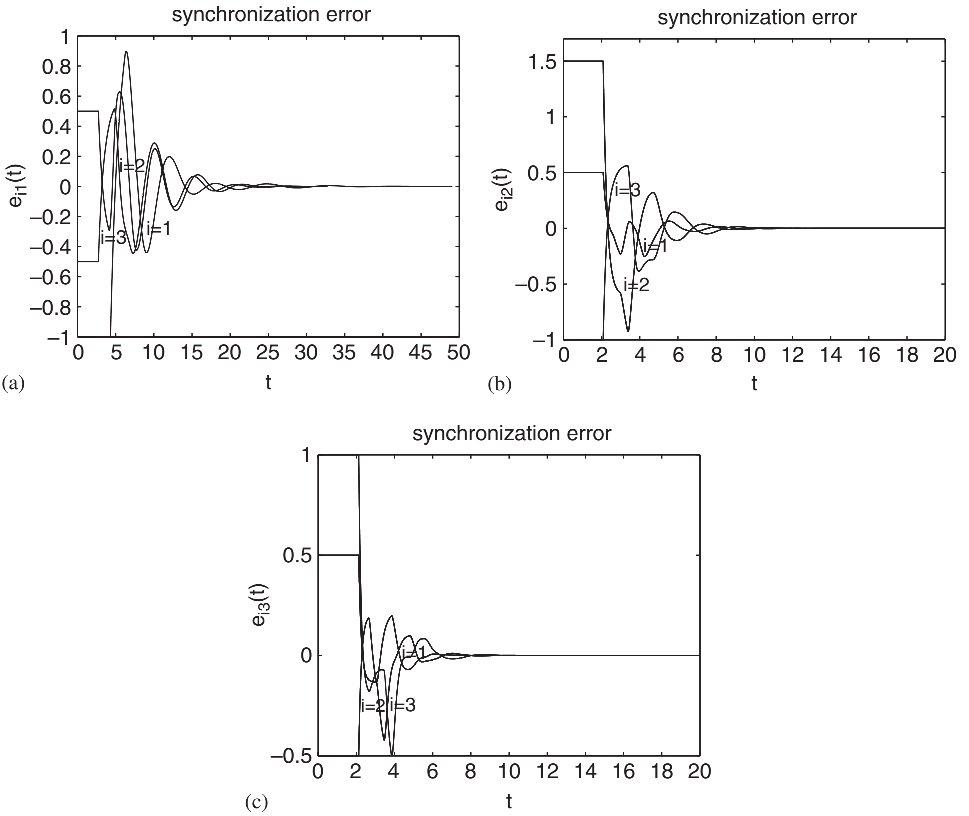

Fig. 2. Synchronization errors of networks (15): (a) $e_{i1}(t)$ vs t, (b) $e_{i2}(t)$ vs t, (c) $e_{i3}(t)$ vs t.

$$y_4 = \begin{pmatrix} x_{21}(t-2.1) + x_{31}(t-3.1) - 2x_{41}(t-4.1) \\ x_{22}(t-2.2) + x_{32}(t-3.2) - 2x_{42}(t-4.2) \\ x_{23}(t-2.3) + x_{33}(t-3.3) - 2x_{43}(t-4.3) \end{pmatrix}.$$

Set again $e_{i1} = x_{i+1,1} - x_{i1}$, $e_{i2} = x_{i+1,2} - x_{i2}$, $e_{i3} = x_{i+1,3} - x_{i3}$.

From the simulations of this network (Fig. 2), one can see that the synchronous states (3) for (15) are asymptotically stable. Next, we use Theorem 4 to verify it.

$$\begin{pmatrix} -\lambda_1(A + A^{\mathrm{T}}) & \lambda_2 A \\ \lambda_2 A^{\mathrm{T}} & 0 \end{pmatrix}_{6\times 6} \leqslant 0$$

holds, since all its eigenvalues are less than or equal to zero. And $\mathscr{M}_2[Df(s(t)) + \lambda_1 A] \leqslant -1 < 0$. So by Theorem 4, the synchronous states (3) for (15) are shown to be asymptotically stable, which is in line with the above numerical calculations.

With the increase of the node number N, it is difficult even impossible to directly judge the synchronous states (3) for networks (2) and (11), but by the aid of the theorems derived here, it becomes easier.

4. Conclusion

In the present paper, general dynamical networks with different time delays are considered. Several theorems on the synchronization properties are established. Two illustrative examples are presented, which show the efficiency of the derived results. Generally speaking, the number N of the coupled scale of a complex dynamical network is often very large (e.g., the metabolic system), so the considered network is fairly high dimensional with nN dimensions if the dimension of the identical node is n. To directly investigate the synchronous states of such a high dimensional system with/without time delay is quite difficult even impossible. From our theorems, we only consider the stability of the attractor of an n dimensional system, which makes the consideration of the synchronous state much easier. Further research will be contributed to directed networks (where C is not symmetric) and time-varying outer coupling networks (where C depends upon the time t).

References

[1] R. Albert, A.L. Barabási, Rev. Mod. Phys. 74 (2002) 47.
[2] P. Erdös, A. Rényi, Publ. Math. Inst. Hung. Acad. Sci. 5 (1960) 17.
[3] D.J. Watts, S.H. Strogatz, Nature 393 (1998) 440.
[4] D.J. Watts, Small-Worlds: The Dynamics of Networks between Order and Randomness, Princeton University Press, Princeton, NJ, 1999.
[5] X.F. Wang, G. Chen, Int. J. Bifurc. Chaos 12 (2002) 187.
[6] J.H. Lü, X. Yu, G. Chen, D. Cheng, IEEE Trans. CAS-I 51 (2004) 787.
[7] A.L. Barabási, R. Albert, Science 286 (1999) 509.
[8] X.F. Wang, G. Chen, IEEE Trans. CAS-I 49 (2002) 54.

[9] X. Li, X.F. Wang, G. Chen, IEEE Trans. CAS-I 51 (2004) 2074.
[10] C. Masoller, A.C. Martí, D.H. Zanette, Phys. A 325 (2003) 186.
[11] A.C. Martí, C. Masoller, Phys. Rev. E 67 (2003) 056219.
[12] A.C. Martí, C. Masoller, Phys. A 342 (2004) 344.
[13] C.G. Li, G. Chen, Phys. A 343 (2004) 263.
[14] J.K. Hale, S.M. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.
[15] F. Vidyasagar, Nonlinear Systems Analysis, Prentice-Hall, Englewood Cliffs, NJ, 1978.
[16] C.P. Li, X. Xia, Chaos 14 (2004) 557.
[17] C.W. Wu, L.O. Chua, IEEE Trans. CAS-I 42 (1995) 430.
[18] C.P. Li, W.G. Sun, The 24th Chinese Control Conference, Guangzhou, July 15–18, 2005, p. 202.
[19] C.P. Li, G. Chen, T.S. Zhou, Some remarks on synchronization of complex networks with nonlinear inner-coupling functions, The Second Chinese Academic Forum on Complex Dynamical Networks, Beijing, October 16–19, 2005, in press.
[20] C.P. Li, W.G. Sun, D. Xu, Synchronization of complex dynamical networks with nonlinear inner-coupling functions and time delays, Prog. Theor. Phys. 114 (4) (2005), in press.
[21] W.H. Deng, Y.J. Wu, C.P. Li, Stability analysis of differential equations with time-dependent delays, Int. J. Bifurc. Chaos 16 (2) (2006), in press.