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Synchronization of two coupled fractional-order chaotic oscillators

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Abstract

The dynamics of fractional-order systems have attracted increasing attentions in recent years. In this paper, the synchronization of two coupled nonlinear fractional order chaotic oscillators is numerically demonstrated using the master–slave synchronization scheme. It is shown that fractional-order chaotic oscillators can be synchronized with appropriate coupling strength.

1. Introduction

Fractional calculus is a 300-year-old topic, but its applications to physics and engineering are just a recent focus of interest. Many systems are known to display fractional-order dynamics, such as viscoelastic systems [1], electrode–electrolyte polarization [2], and electromagnetic waves [3]. More recently, there is a new trend to investigate the control and dynamics of fractional order dynamical systems. Such as Chua's system [4], Wien bridge oscillator [5], Rössler equation [6] and "jerk" model [7] etc. Ahmad [7] present a conjecture that third-order chaotic nonlinear systems can still produce chaotic behavior with a total system order of $2 + \varepsilon$, $1 > \varepsilon > 0$, using the appropriate control parameters. In Ref. [4], it is shown that the fractional-order Chua's circuit of order as low as 2.7 can produce a chaotic attractor. In Ref. [7], chaotic behaviors of the fractional-order "jerk" model are studied, in which chaotic attractor is generated with the system orders as low as 2.1. In Ref. [6], chaos and hyperchaos in the fractional-order equations were studied, in which chaos can exist in the fractional-order equation with order as low as 2.4, and hyperchaos exists in the fractional order Rössler hyperchaos equation with order as low as 3.8. And chaotic control [8,9] of fractional-order systems are investigated.

On the other hand, synchronization of chaotic systems has attracted much attention since the seminal paper by Pecora and Carroll [10]. In this paper, we study the synchronization technique and apply it to the synchronization of two coupled nonlinear fractional-order chaotic oscillators. Simulations are shown that two coupled fractional-order chaotic oscillators can be brought to an exact synchronization with appropriate coupling strength. We can know that the synchronization rate of the fractional-order chaotic oscillators is slower than its integer-order counterpart, however, as the

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increasing of system order, the curves of synchronization error can be evidently smooth, which indicates that the master–slave synchronization of two coupled fractional order ($q > 3$) oscillators can be smooth and stable.

2. Fractional derivative and its approximation

The idea of fractional integrals and derivatives has been known since the development of the regular calculus, with the first reference probably being associated with Leibniz in 1695. Two commonly used definitions for the general fractional differintegral are the Grunwald–Letnikov (GL) definition and the Riemann–Liouville (RL) definition [11]. The RL definition is given here

$$\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q} = \frac{1}{\Gamma(n-q)} \frac{\mathrm{d}^n}{\mathrm{d}t^n} \int_0^t \frac{f(\tau)}{(t-\tau)^{q-n+1}} \mathrm{d}\tau \tag{1}$$

where $n-1 \leqslant q < n$ and $\Gamma(\bullet)$ is the Euler's *gamma* function. Upon considering all the initial conditions to be zero, the Laplace transform of the Riemann–Liouville fractional derivative is

$$L\left\{\frac{\mathrm{d}^q f(t)}{\mathrm{d}t^q}\right\} = s^q L\{f(t)\} \tag{2}$$

Thus, the fractional integral operator of order "q" can be represented by the transfer function $F(s) = 1/s^q$ in the frequency domain.

The standard definitions of fractional diffintegral do not allow direct implementation of the operator in time-domain simulations. An effective method to deal with this problem is to approximate fractional operators by using the standard integer order operators. The approximation approach taken here is that of Ref. [12]. Basically the idea is to approximate the system behavior in the frequency domain, using a specified error in decibels and a bandwidth to generate a continuous sequence of pole-zero pairs for the system with a single fractional power pole. This approximation is created by choosing an initial breakpoint, the line with slop of $-20q$ dB/decade is approximated by a number of zig-zig lines connected together with alternate slops of 0 dB/decade and -20 dB/decade. Thus an approximation of any desired accuracy over any frequency band can be achieved. In Table 1 of Ref. [7], approximations for $1/s^q$ with $q = 0.1 \sim 0.9$ in steps of 0.1 were given with errors of approximately 2 dB. We will mainly use these approximations in the following simulations.

3. Synchronization of two coupled fractional order chaotic oscillators

The model studied in this paper is an electronic chaotic oscillator of canonical structure, and one control parameter. It has been reported in [7], and conjectured as the simplest possible for a chaotic oscillator. This oscillator is known to give a double-scroll-like chaotic attractor in the range $1.0 > a > 0.49$.

Consider the master–slave synchronization scheme of two fractional-order chaotic oscillators with the master oscillator M and the slave oscillator S.

M:

$$\begin{aligned}
\frac{\mathrm{d}^q x_1}{\mathrm{d}t^q} &= y_1 \\
\frac{\mathrm{d}^q y_1}{\mathrm{d}t^q} &= z_1 \\
\frac{\mathrm{d}^q z_1}{\mathrm{d}t^q} &= -a[x_1 + z_1 - f(x_1)]
\end{aligned} \tag{3}$$

S:

$$\begin{aligned}
\frac{\mathrm{d}^q x_2}{\mathrm{d}t^q} &= y_2 + c(x_1 - x_2) \\
\frac{\mathrm{d}^q y_2}{\mathrm{d}t^q} &= z_2 \\
\frac{\mathrm{d}^q z_2}{\mathrm{d}t^q} &= -a[x_2 + y_2 + z_2 - f(x_2)]
\end{aligned} \tag{4}$$

where

$$f(x) = \mathrm{sgn}(x) = \left\{ \begin{array}{ll} 1, & x \geqslant 0 \\ -1, & x < 0 \end{array} \right\} \tag{5}$$

$q > 0$ is the fractional order, $c > 0$ is the coupling strength. For the sake of simplicity, only the first variable x is used for coupling the two fractional-order chaotic oscillators. Define the error signal as $e = x - y$, the aim of the synchronization scheme is to design the coupling strength such that $\|e(t)\| \to 0$ as $t \to \infty$. This scheme is similar to the master–slave synchronization of classical integer-order chaotic systems.

When $q = 0.9$ and $a = 0.21$, the fractional-order oscillator is chaotic. The phase plot of x and z is shown in Fig. 1.

Next, we numerically study the synchronization of fractional-order chaotic oscillators. To obtain a critical value of c to make the two oscillators synchronized, we continuously increase the coupling strength c. When $c < 0.5$, no synchronous phenomenon is observed. When $c = 0.5$, the curve of the synchronization error $J(t) = \log(\|e(t)\|)$ is shown in Fig. 2(a), which indicates that the master–slave synchronization is achieved. In Fig. 2(b), we show the curve of the synchronization error when $c = 2$, in which the synchronization effect is better than that of $c = 0.5$.

In Fig. 3(a) and (b), we show the synchronization of two nonlinear chaotic oscillators with different initial conditions, the control is applied at the time $t = 80$.

For the purpose of comparison, we also plot the curves of synchronization error of the integer order ($a = 0.8$) and fractional order ($q = 3.2, a = 1$) chaotic oscillators in Figs. 4 and 5. Comparing Fig. 2 with Figs. 4 and 5, we can know that the synchronization rate of the fractional-order chaotic oscillators is slower than its integer-order counterpart. However, as the increasing of system order, the curves of synchronization error can be evidently smooth, which indicates that the master–slave synchronization of two coupled fractional order ($q > 3$) oscillators can be smooth and stable.

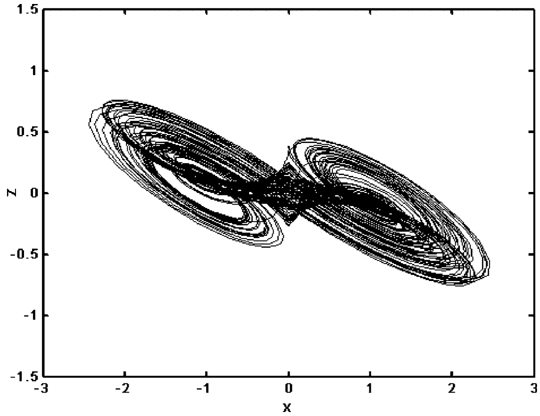

Fig. 1. Phase plot of the fractional-order oscillator with $q = 0.9$.

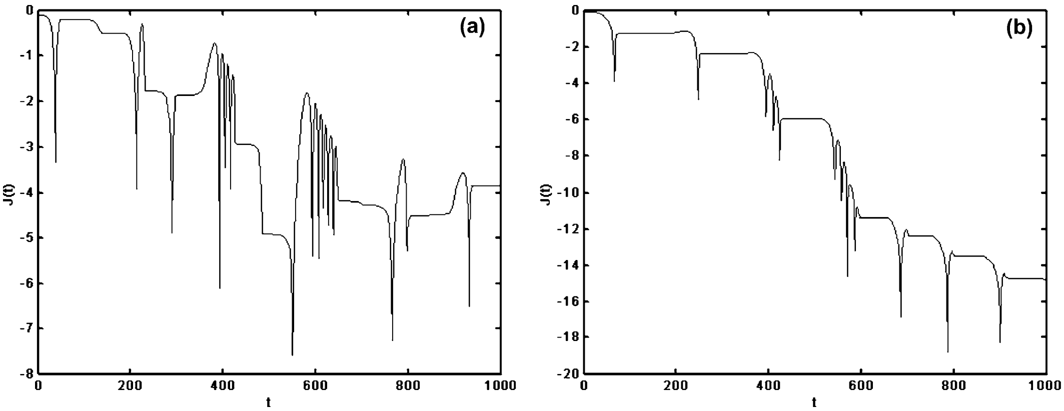

Fig. 2. Synchronization error of the fractional-order chaotic oscillators with $q = 0.9$: (a) $c = 0.5$, (b) $c = 2$.

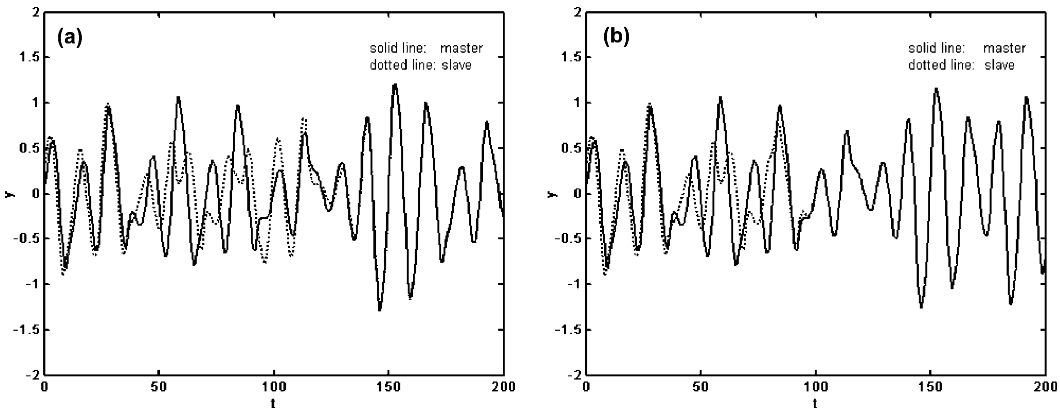

Fig. 3. Synchronization of the fractional-order chaotic oscillators with $q = 0.9$: (a) $c = 0.5$, (b) $c = 2$.

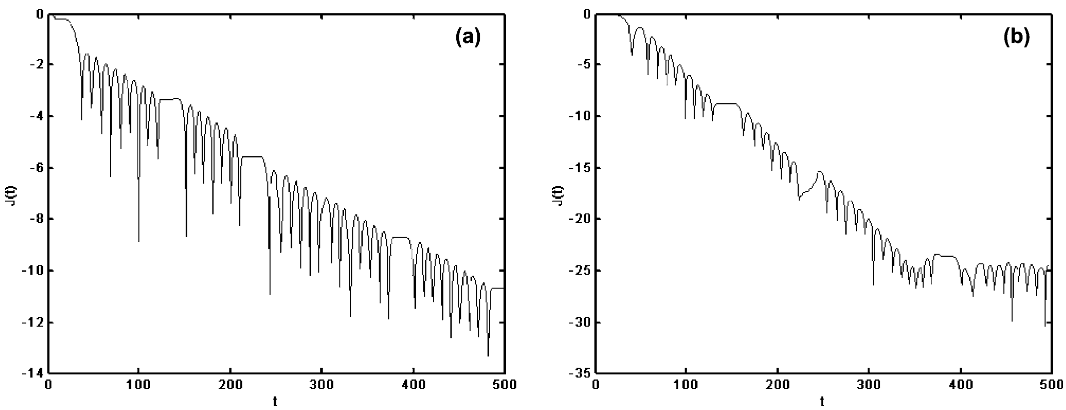

Fig. 4. Synchronization error of the integer-order chaotic oscillators: (a) $c = 0.5$, (b) $c = 2$.

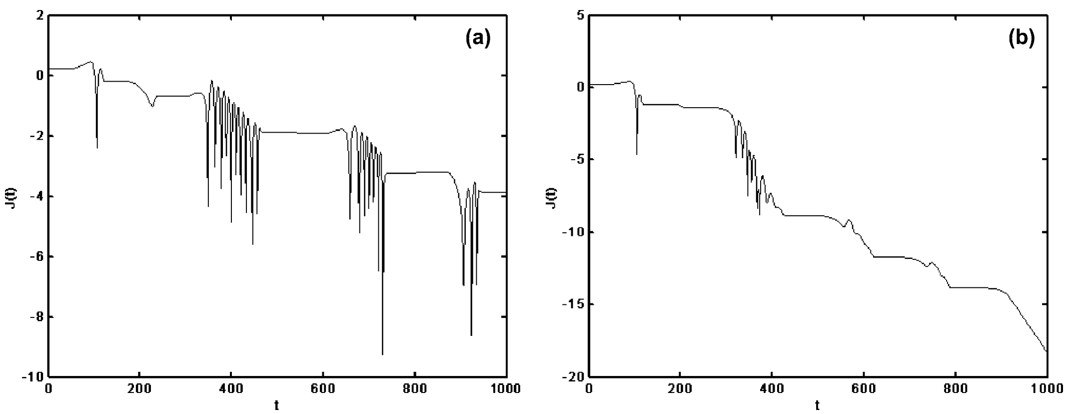

Fig. 5. Synchronization error of the fractional-order chaotic oscillators with $q = 3.2$: (a) $c = 0.5$, (b) $c = 2$.

4. Conclusions

In this paper, we have studied the master–slave synchronization of coupled fractional-order chaotic oscillators. We find that two fractional-order chaotic oscillators can be brought to an exact synchronization with appropriate coupling strength. As the increasing of system order, the process of synchronization of two coupled fractional order ($q > 3$) oscillators can be smooth and stable. Chaotic synchronization in fractional-order systems is intricate. Future works regarding this topic include the investigation of some other types of synchronization of fractional-order chaotic systems.

Acknowledgments

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