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Abstract

We study the synchronization problem in discrete-time via an extended Kalman filter (EKF). That is, synchronization is obtained of transmitter and receiver dynamics in case the receiver is given via an extended Kalman filter that is driven by a noisy drive signal from the transmitter. Extensive computer simulations show that the filter is indeed suitable for synchronization of (noisy) chaotic transmitter dynamics. An application to secure communication is also given.

Keywords: Synchronization, Extended Kalman filtering, Secure communication, Parameter estimation, Discrete-time systems.

1991 Mathematics Subject Classification: 65C99, 70K50, 93C55, 93E11, 94A99.

1 Introduction

Synchronization is a fascinating phenomenon and has been observed in many diverse systems. Synchronization in chaotic systems may bring many interesting possibilities in practical applications. For example, it is believed that synchronization plays a crucial role in information processing in living organisms and could lead to important applications in speech and image processing (Ogorzalek [17]). Another area where synchronization may play an important role, is (secure) communication. Due to the fact that chaotic signals are noise-like and unpredictable in nature, such signals can possibly be used as potential carriers for secure communication ([20], [12], [18],

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[6], [10]). Moreover, the synchronization property of chaotic circuits has revealed potential applications to secure communications (see, e.g., [5], Kocarev *et al.*, [12]; Parlitz *et al.*, [18]; Cuomo *et al.*, [6]; Halle *et al.*, [10]).

However, the problem of obtaining two chaotic systems oscillating in a synchronized way is a nontrivial question. This is because any small difference in initial conditions would be exponentially amplified and thus the motions of the systems rapidly become uncorrelated (Ogorzalek [17]). Despite this fact, Pecora and Carroll discovered (see [5], [19]), that in particular cases it is very well possible that synchronization between a chaotic transmitter and driven receiver is possible. Much further work about synchronization can be found in the special issue [24].

In this paper we will concentrate on synchronization for systems in discrete time. Mostly, synchronization has been studied for systems in continuous time but many ideas go through in discrete time. On the other hand, the theory is less far developed for discrete time systems -even despite the fact that in many cases the actual implementation is done in discrete time.

Consider an autonomous, discrete-time, dynamical system

$$x(k+1) = f(x(k)), \qquad (1)$$

where $x(k)$ is a n-dimensional vector. The Pecora-Carroll scheme for synchronization can be described as follows. Divide the system into two subsystems via $x(k) = (x_1(k), x_2(k))^T$, where $x_1(k)$ is n_1-dimensional and $x_2(k)$ is n_2-dimensional, with $n_1 + n_2 = n$ (often $n_1 = 1$, see [5], [6], [19]). We refer to $x_1(k)$ as the *drive signal.* So, the transmitter system (1) can be written as

$$x_1(k+1) = f_1(x_1(k), x_2(k)), \qquad (2)$$
$$x_2(k+1) = f_2(x_1(k), x_2(k)) \qquad (3)$$

where $f(x) = (f_1(x_1, x_2), f_2(x_1, x_2))^T$. The driven replica subsystem is described by

$$x_2^R(k+1) = f_2\left(x_1(k), x_2^R(k)\right), \qquad (4)$$

the so-called receiver dynamics. In (4), $x_2^R(k)$ is the response variable and (4) is known as the response subsystem.

The receiver system (4) *synchronizes* with the transmitter system (2) and (3), if

$$\lim_{k\to\infty} \| x_2(k) - x_2^R(k) \| = 0, \qquad (5)$$

no matter which initial values $x_1(0)$, $x_2(0)$ and $x_2^R(0)$ have.

In the context of communication the drive signal is transmitted from the transmitter system to the receiver system. The full state $x(k)$ of the transmitter is unknown at the receiver. By driving with the known signal $x_1(k)$ to a replica response subsystem (4), we can then obtain $x_2(k)$, if the copy synchronizes with the full system (2)-(3) (according to (5)).

In general, no matter if f_i is chaotic or complex, no complete general answer exists to the problem whether a replica system (4) would achieve synchronization according to (5). For that reason several attempts for achieving synchronization of signals like $x_2(k)$ and $x_2^R(k)$ have been proposed. The above idea of synchronization by decomposition into subsystems (by using a replica (4)) has first been given in Pecora and Carroll [19]; the ocurrence of such synchronization is conditioned on whether the conditional Lyapunov exponents for (4) are negative. In such case, the system (1) is said to possess a self-synchronizing property. Note however, the negativity of the conditional Lyapunov exponents is not a guarantee for the successful synchronization, cf. Badola *et al.* [2]. However, recent research has shown, that one is not necessarily constrained to the use of a replica (4) when choosing a system to achieve synchronization (at least when one is partly free to choose the receiver dynamics-driven by the transmitter's drive signal), i.e., it is also possible if, instead of (4), we utilize a different response system

$$x_2^R(k+1) = f_2^R\left(x_1(k), x_2^R(k)\right). \tag{6}$$

This may leads to considerably more flexibility in applications like secure communication. This added flexibility may facilitate potential improvement in synchronization, e.g., Ding and Ott [8] have obtained synchronization using (6) in case a replica (4) does not synchronize. Another adventage of using (6) is to improve the convergence of the synchronization. In particular, we like to recall the (reduced) observer viewpoint advocated by Nijmeijer and Mareels in [15] which basically admits -under suitable assumptions- the construction of dynamics (6) such that (5) holds, whatever initial conditions (2), (3) and (6) have. A possible extension of [15] in discrete time has been given by Huijberts *et al.* [11].

On the other hand, for communication purposes (specifically, additive signal masking and recovery), it is required that the synchronization is not affected by some noise in the synchronizing drive signal. In other words, the use of synchronized chaotic systems for secure communications relies on the robustness of the synchronization to perturbations in the drive signal and in the system dynamics, [6], see also [20]. For certain synchronized chaotic systems, the ability to synchronize is robust. In the Lorenz system, for example, one can observe such property (Cuomo *et al.* [6]) for an exact replica (4).

On the basis of these considerations, we propose a system of the form

$$\hat{x}(k+1) = f^R(\bar{x}_1(k), \hat{x}(k)), \tag{7}$$

where $\bar{x}_1(k)$ is the drive signal corrupted by noise from the transmitter dynamics i.e., $\bar{x}_1(k) = x_1(k) + v(k)$, with $v(k)$ the noise signal, and $\hat{x}(k)$ is the state estimate for the original system (1), provided by an extended Kalman filter (EKF). In the next section, we give the particular equations for the EKF and further background on it. Among some advantages of using (7) as receiver/estimator for (1) are:

- The EKF possesses some natural robustness property to additive Gaussian noise in the drive signal (see Cuomo *et al.* [6]).
- The EKF is easily implemented.
- Flexibility in applications since a decomposition into two subsystems (2)-(3) is not necessary.

The idea of using an EKF to estimate the states of a chaotic system is given perhaps for the first time by Fowler [9]. Cuomo *et al.* [6] shown that the EKF estimates of a Lorenz system approach its true states. Recently in [21], Sobiski and Thorp describe a method of synchronizing two chaotic systems by implementing an EKF for continuous-time systems.

Motivated by these papers and by its implication in secure communications, we study synchronization between the discrete-time systems (1) and (7) when the EKF is used as receiver driven by a noisy drive signal. Besides synchronization per se, we also study the utility of using the EKF for reconstruction of a binary message. The idea is that some parameter in the transmitter can be used as the (binary signal) message carrier. In the EKF we include in this case as extra state a parameter-estimator, see also [21] where a similar idea was used in continuous time. At the same time, we mention some differences with [21]:

- The context we use here is discrete-time. This on the one hand makes the presentation even simpler than in continuous time. On the other hand, to the best of our (knowledge) using the EKF for synchronization in discrete time is new.
- By means of extensive simulations we make an attempt to give an evaluation of the performance of the EKF by changing filter-parameters (covariance noise, initial error, etc.).
- For the given examples a rigorous result that ensures the *local* (optimal) convergence of the EKF is given.

The organization of this paper is as follows: in Section 2 we present our approach to achieve synchronization of discrete-time systems via an extended Kalman filter which is driven by a noisy drive signal from the transmitter. By using computer simulations, the approach used in this study is illustrated with two examples in Subsection 3.1, while in Subsection 3.2 an application of these results to secure communication is given. Finally, Section 4 contains some concluding remarks.

2 An extended Kalman filter as receiver

Briefly, given a stochastic (linear) model description, the Kalman filtering problem is to produce an estimate $\hat{x}(k)$ of the state model $x(k)$, using measurements till time

k so as to minimize the mean-square error between estimate and state. From this viewpoint, the Kalman filter is the *optimal* linear filter since it produces an estimate minimizing a mean-square error. For application to nonlinear models, the so-called extended Kalman filter is often used in practice, but, in general, no guarantee in this case of producing a good (optimal) state estimate can be given. In this case the nonlinear system is linearized by employing the best estimates of the state as the reference values used at each stage for the linearization, i.e., the EKF consists of using the classical Kalman filter equations for the first-order approximation of the nonlinear system about the last estimate. As a direct consequence of taking this approximation, the EKF is no longer linear or optimal. For further details we refer the reader to Anderson and Moore [1] and references therein.

We consider transmitter dynamics of the form

$$x(k+1) = f(x(k)) + w(k), \quad x(0) = x_0, \tag{8}$$

with transmitted signal

$$y(k) = h(x(k)) + v(k). \tag{9}$$

Typically, $x(k)$ is an n-dimensional vector and $y(k)$ is a scalar signal (although much of what follows can be extended to a vector signal $y(k)$).

In (8) $w(k)$ represents the noise in the dynamics of the transmitter which is assumed to be a zero mean noise process with $E\left[w(k)\, w^T(l)\right] = Q\delta_{kl} > 0$, with δ_{kl} the Kronecker delta function. Also $v(k)$ is a zero mean noise process with $E\left[v(k)\, v(l)\right] = R\delta_{kl} > 0$; $v(k)$ and $w(k)$ are assumed to be independent.

Remark 1 *Although it may not be necessary to introduce the dynamics noise $w(k)$ in (8), we find it convenient and more flexible to do so. In fact from certain perspective it can be argued that an error free dynamics would be an over-idealization, which may prohibit a successful synchronization through the EKF, see Sorenson [23]. In the theoretic developments to follow one might think about the covariance matix Q as being reasonably small, mimicking at least a very accurate modeling. For convenience we have assumed in the examples of Section 3 that $w(k) = 0$ for all k.*

The receiver dynamics we propose is a filter that will produce an estimate for the state $x(k)$ given the measurements $y(k)$ according to (9). The EKF that we use here as the receiver dynamics for (8) and (9) is given as follows, cf. [1].

1) Measurement update equations:

$$\hat{x}(k) = \hat{x}(k/k-1) + K_{\hat{x}}(k)\left[y(k) - h(\hat{x}(k/k-1))\right], \tag{10}$$

the vector $\hat{x}(k)$ is referred to as the *filtered* estimate of the state $x(k)$. The covariance of the error in $\hat{x}(k)$ is given by

$$P_{\hat{x}}(k) = \left[I - K_{\hat{x}}(k) H_{\hat{x}}(k)\right] P_{\hat{x}}(k/k-1). \tag{11}$$

2) Time update equations:

The (one-step ahead) *predictor* of $x(k+1)$ is given by

$$\hat{x}(k+1/k) = f(\hat{x}(k)), \tag{12}$$

the covariance matrix of the prediction error is

$$P_{\hat{x}}(k+1/k) = F_{\hat{x}}(k) P_{\hat{x}}(k) F_{\hat{x}}^T(k) + Q, \tag{13}$$

where

$$K_{\hat{x}}(k) = P_{\hat{x}}(k/k-1) H_{\hat{x}}^T(k) \left[H_{\hat{x}}(k) P_{\hat{x}}(k/k-1) H_{\hat{x}}^T(k) + R \right]^{-1} \tag{14}$$

is known as the Kalman gain matrix, and

$$F_{\hat{x}}(k) = \left. \frac{\partial f(x(k))}{\partial x(k)} \right|_{x(k)=\hat{x}(k)}, \tag{15}$$

$$H_{\hat{x}}(k) = \left. \frac{\partial h(x(k))}{\partial x(k)} \right|_{x(k)=\hat{x}(k/k-1)}. \tag{16}$$

Remark 2 *In practice it might be impossible to determine $x(0)$ exactly. In this case, $x(0)$ is assumed to be a Gaussian random variable of known mean value $E\{x(0)\} = \bar{x}(0)$ and known covariance matrix $E\left\{[x(0) - \bar{x}(0)][x(0) - \bar{x}(0)]^T\right\} = P(0)$, and it is independent of $w(k)$ and $v(k)$.*

The filter is initialized by setting $\hat{x}(0) = \bar{x}_0$ and $P(0) = P_0 = P_0^T > 0$. Thus, $x(0)$ is given and we choose arbitrarily $\bar{x}_0$, $P_0 = P_0^T > 0$.

The definition of synchronization given in the introduction, i.e., equation (5), can be extended to include approximate or noisy synchronization to accommodate inaccurate parameters and non-ideal signal transmission. In this case, the receiver (10)-(13) can not synchronize with the transmitter (8) in the way that condition (5) is fulfilled so we need to replace it by a weaker condition

$$\lim_{k\to\infty} \| x(k) - \hat{x}(k) \| \leq \rho, \quad \forall k \geq \tau, \tag{17}$$

where ρ should be related to R and is a constant of the synchronization error. If for a given ρ there exists a time instant τ (to be called the synchronization time) such that condition (17) is fulfilled, then the transmitter (8) and the EKF receiver (10)-(13) are *approximately synchronized.*

Remark 3 *Also one might consider as an adequate condition for approximate synchronization in the noisy context*

$$\lim_{k\to\infty} \| E(x(k) - \hat{x}(k)) \| \leq \rho, \quad \forall k \geq \tau.$$

In particular this may be a more relevant requirement if $w(k)$ is not necessarily bounded. Since we will later on assume that both $v(k)$ and $w(k)$ are bounded it suffices to take (17) as the definition for approximate synchronization.

We define for $e_i(k) = x_i(k) - \hat{x}_i(k)$, $i = 1, 2, \ldots, n$

$$\tau_i(\rho) = \min_{\tau} \{|e_i(k)| < \rho, \quad k = \tau, \tau + 1, \ldots\}, \tag{18}$$

and the synchronization time by

$$\tau = \max(\tau_i), \quad i = 1, 2, \ldots, n. \tag{19}$$

Remark 4 *From (17) note that there exists a compromise between the quantities ρ and τ, since if ρ increases then τ decreases, and vice versa.*

The EKF is often used to design observers (to deal with state estimation) for forced or non forced nonlinear systems. In spite of the fact that only local convergence is ensured, this method is widely used in practice and often gives convincing results (for a summary of the theory and applications see, e.g. Boutayeb *et al.* [4], Baras *et al.* [3], La Scala *et al.* [13], Ljung [14], Song and Grizzle [22] and references therein). The convergence aspects of the EKF when it is used as a deterministic observer for discrete-time system, are analyzed through a Lyapunov approach in Boutayeb *et al.* [4], and Song and Grizzle [22]. We follow the approach proposed in La Scala *et al.* [13], for the establishing the convergence of the EKF when applied to a stochastic, discrete-time nonlinear system with a linear output map. To this end, define the error in the filtered state as

$$e(k)\, x(k) - \hat{x}(k). \tag{20}$$

From (10) we have

$$e(k) = [I - K_{\hat{x}}(k) H_{\hat{x}}(k)]\, e(k/k-1) - K_{\hat{x}}(k)\, v(k),$$

where $e(k/k-1)\, x(k) - \hat{x}(k/k-1)$ is the error in the predicted state estimate, thus,

$$e(k+1/k) = \frac{\partial f}{\partial x}(x(k)) \cdot e(k) - \mathcal{O}_f(x(k), -e(k)) + w(k)$$

$\mathcal{O}_f$ is the remainder term from the Taylor series expansion of f, i.e.,

$$\mathcal{O}_f(a, b) = f(a+b) - f(a) - \frac{\partial f}{\partial a}(a) \cdot b.$$

So, for the (corrupted) drive signal (9), we have the error dynamics equation

$$\begin{aligned} e(k) &= [I - K_{\hat{x}}(k) H_{\hat{x}}(k)]\, F_x(k-1)\, e(k-1) \\ &\quad - [I - K_{\hat{x}}(k) H_{\hat{x}}(k)]\, \mathcal{O}_f(x(k-1), -e(k-1)) \\ &\quad + [I - K_{\hat{x}}(k) H_{\hat{x}}(k)]\, w(k-1) - K_{\hat{x}}(k)\, v(k). \end{aligned} \tag{21}$$

From the last equation, we see that the dynamics for the filtering error of the EKF driven by a noisy drive signal from the transmitter, is composed by the sum of the error dynamics for the deterministic case (neglecting linearization errors), and nonlinear perturbation terms driven by the noise processes and remainder term from the Taylor series expansion of f.

Consider the time-varying linear system

$$\begin{aligned} \xi(k+1) &= F_z(k)\,\xi(k) + w(k), \\ Y(k) &= H\xi(k) + v(k), \end{aligned} \tag{22}$$

with $F_z(k) = (\partial f/\partial x)(\psi(k) - \phi(k))$. Define the observability Gramian of $\left[F_z,\ R^{-\frac{1}{2}}H\right]$ along a trajectory $\{z(k)\}$ of (22) as

$$\mathcal{O}(k,M) = \sum_{i=k-M}^{k} \Phi^T(i,k)\,H^T(i)\,R^{-1}H(i)\,\Phi(i,k)$$

for some $M \geq 0$ and for all $k \geq M$, where $\Phi(k_2,k_1) = F_z(k_2-1)$ $F_z(k_2-2)\cdots F_z(k_1)$. Similarly, the controllability Gramian of $[F_z, Q]$ along a trajectory $\{z(k)\}$ of (22) as

$$\mathcal{C}(k,M) = \sum_{i=k-M}^{k-1} \Phi(k,i+1)\,Q\Phi^T(k,i+1).$$

A system is said to be controllable (observable) along a trajectory $\{z(k)\}$ if there exists M such that for all $R_x > 0$ there exist $0 < \varepsilon_r < R_x$, $a_i(R_x,\varepsilon_r,M)$ and $b_i(R_x,\varepsilon_r,M)$, $i=1,2$, such that for some arbitrary sequence $\{\psi(k)\}$, $\|\psi(k)\| \leq R_x$, and for all $\{\phi(k)\}$ such that $\|\phi(k)\| \leq \varepsilon_r$

$$a_1 I \geq \mathcal{C}(k,M) \geq a_2 I, \qquad 0 < a_2 \leq a_1 < \infty, \tag{23}$$

$$b_1 I \leq \mathcal{O}(k,M) \leq b_2 I, \qquad 0 < b_1 \leq b_2 < \infty \tag{24}$$

where these Gramians are evaluated along the trajectory $z(k) = \psi(k) - \phi(k)$ of (22).

The following assumptions are needed, cf. [13]:

1) The transmitted signal (9) is a linear in x, i.e.,

$$y(k) = Hx(k) + v(k). \tag{25}$$

2) $f \in C^3(\mathbb{R}^n,\mathbb{R}^n)$, $(\partial f/\partial x)(x)$ is invertible for all $x \in \mathbb{R}^n$, and

3) for all k (see Remark 3)

$$\|x(k)\| \leq R_x, \tag{26}$$

$$\|w(k)\| \leq \|w\| < \infty \qquad \text{and} \qquad \|v(k)\| \leq \|v\| < \infty. \tag{27}$$

From Assumption 2), we can find $\rho_i > 0$, $i = 1, 2, 3$ such that

$$\left\| \frac{\partial f}{\partial x}(x) \right\| \leq \rho_1, \quad \left\| \frac{\partial^2 f}{\partial x^2}(x) \right\| \leq \rho_2, \quad \left\| \frac{\partial^3 f}{\partial x^3}(x) \right\| \leq \rho_3$$

for all $\|x\| \leq R_x + \varepsilon_r$. Furthermore, by the continuity of f, there exists a $\rho_4 > 0$ such that

$$\left\| \frac{\partial f}{\partial x}(x_1) - \frac{\partial f}{\partial x}(x_2) \right\| \leq \rho_4 \|x_1 - x_2\|$$

for all $\|x_1\|$, $\|x_2\| \leq R_x$, and

$$\left\| \frac{\partial f}{\partial x}(x_1) - \frac{\partial f}{\partial x}(x_2) - \frac{\partial^2 f}{\partial x^2}(x_2) \cdot (x_1 - x_2) \right\| \leq \frac{1}{2}\rho_4 \|x_1 - x_2\|^2$$

(see, La Scala *et al.* [13]). Let

$$p = a_1 + \frac{1}{b_1}, \quad q = \frac{1}{a_2} + b_2, \quad s = \frac{1}{a_2} + b_2 + \rho_1^2 \delta_1^{-1}.$$

Since $P_z(k)$ and $P_z(k + 1/k)$ are defined by means of the linear system (22), we find, using [7] the bounds (depending on ε_r, R_x and M)

$$\begin{array}{rcccl} q^{-1}I & \leq & P_z(k) & \leq & pI, \\ q^{-1}I & \leq & P_z(k + 1/k) & \leq & sI. \end{array}$$

Let

$$\varepsilon_z = \min \left\{ \varepsilon_r \, , \, \frac{\sqrt{2}}{\sqrt{\rho_4}} \left(-\rho_1 + \sqrt{\rho_1^2 + \frac{1}{q}\left(\frac{1}{sp^2} - \gamma \right)} \right)^{\frac{1}{2}} \right\}, \tag{28}$$

where $0 < \gamma < 1/sp^2$, provided that $-1/sp^2 + \rho_1 \rho_4 q \|z(k)\|^2 + \frac{1}{4}\rho_4^2 q \|z(k)\|^4 \leq -\gamma$. Define $\alpha, \beta > 0$ via

$$\beta \alpha^k = (pq)^{\frac{1}{2}} \left(1 - \frac{\gamma}{q} \right)^{\frac{k}{2}} \tag{29}$$

and consider

$$\bar{z}(k+1) = A(k)\, \bar{z}(k) + \bar{f}_2(k, \bar{z}(k)) \tag{30}$$

where

$$A(k) = [I - K_{\bar{z}}(k+1)\, H]\, F_x(k),$$

$$\bar{f}_2(k, \bar{z}(k)) = -\frac{\partial K_{\bar{z}}(k+1)}{\partial z(k)} H F_x(k)\, z(k)\, \bar{z}(k)$$

and

$$\tilde{z}(k) = x(k) - z(k).$$

Note that (30) is the linearized, undriven component of the EKF error dynamics, neglecting linearization errors. In La Scala *et al.* [13] an explicit expression for an upper bound on $\bar{f}_2$ is given as

$$\left\|\bar{f}_2\left(k, \bar{z}\left(k\right)\right)\right\| \leq \zeta_{\bar{z}} = \delta_k \rho_1 \rho_5 \varepsilon_z \tag{31}$$

where $\delta_k = \delta_p \rho_5 \delta_2^{-1}\left(1 + s\rho_5^2 \delta_2^{-1}\right)$, $\|H\| \leq \rho_5$ and $\delta_p = 2\rho_1\rho_2 p$.

Let

$$\begin{aligned} \eta \quad & \max\left\{pq\left(\|w\| + \frac{1}{2}\rho_4\varepsilon_r^2\right) + p\rho_5\delta_2^{-1}\|v\|,\right. \\ & \left.\delta_k\left(\rho_5\|w\| + \|v\| + \frac{1}{2}\rho_4\rho_5\varepsilon_r^2\right) + 2pq\rho_1\right\}, \end{aligned} \tag{32}$$

and

$$\zeta = 2\rho_1\rho_5\delta_k + \rho_1\rho_5\delta_{k2}\varepsilon_r \tag{33}$$

with $\delta_{k2}\left(\rho_1, \rho_2, \rho_3, \rho_5, \delta_2, p\right) > 0$ such that

$$\left\|\frac{\partial^2 K_{\bar{z}}\left(k+1\right)}{\partial z^2}\right\| \leq \delta_{k2}.$$

Assume there exist η, $\zeta > 0$ as defined in (32) and (33) such that

$$\left\|\left[I - K_{\bar{z}}\left(k+1\right)H\right]\left[w\left(k\right) - O_f\left(x\left(k\right), e\left(k\right)\right)\right] - K_{\bar{z}}\left(k\right)v\left(k\right)\right\| \leq \eta\varepsilon_r,$$

and

$$\left\|\frac{\partial^2}{\partial z^2}\left[I - K_{\bar{z}}\left(k+1\right)H\right]F_x\left(k\right)\cdot z\left(k\right)\right\| \leq \zeta$$

for all $\|z\left(k\right)\| \leq \varepsilon_r$ and for all $k \geq 0$.

Theorem 5 [EKF Stability] (La Scala *et al.* [13]) *Consider the error dynamics of the EKF given in (21) when the EKF is applied to a signal model (8)-(25) which satisfies the standing assumptions 1)-3). Select M and $0 < \varepsilon_r < R_x$ such that the controllability and observability conditions (23) and (24) are satisfied. Then if*

$$\beta\left(\zeta + \eta\right) + \alpha < 1, \tag{34}$$

$$\beta\zeta_{\bar{z}} + \alpha < 1, \tag{35}$$

$$\|e\left(0\right)\| < \varepsilon_z\left(pq\right)^{-\frac{1}{2}}, \tag{36}$$

where ε_z, α and β, $\zeta_{\bar{z}}$, η , and ζ are given in (28), (29), (31), and (32), and (33), respectively, we have that the error dynamics satisfies

$$\|e\left(k\right)\| \leq \beta\left(\alpha^k\left(\alpha + \zeta\beta\right)^k\right)\|e_0\| \tag{37}$$

for all $0 \le k < \tau$, *and*

$$\|e(k)\| \le \frac{\beta\eta\left(R_x - \beta\varepsilon_z\right)}{1-(\alpha+\zeta\beta)} \le \varepsilon_r \tag{38}$$

for all $k \ge \tau$.

It is interesting to observe that the above theorem implies that under suitable technical conditions on the system dynamics f we obtain the 'practical stability' condition (37)-(38). It is clear that in the given noisy context this is the best we can hope far; convergence to zero is obviously impossible. The constant ε_r will play in the next section the approximate synchronization constant, see (17). Similarly the synchronization time τ in (19) is related to the integer M.

3 Examples

3.1 Synchronization

Example 1 (see Badola *et al.* [2])

Consider two coupled logistic maps as the transmitter dynamics

$$\begin{aligned} x_1(k+1) &= (1-\epsilon)\,\mu x_1(k)\,(1-x_1(k)) + \epsilon x_2(k), \\ x_2(k+1) &= (1-\epsilon)\,\mu x_2(k)\,(1-x_2(k)) + \epsilon x_1(k). \end{aligned} \tag{39}$$

Treating $y(k) = x_2(k)$ as the (ideal) drive signal ($n_1 = n_2 = 1$), Badola *et al.*, in [2] investigated the synchronization of $x_1(k)$ and the receiver signal $x_1^R(k)$ of which the dynamics were taken as

$$x_1^R(k+1) = (1-\epsilon)\,\mu x_1^R(k)\left(1-x_1^R(k)\right) + \epsilon x_2(k). \tag{40}$$

In Badola *et al.* [2] it turned out that only for particular initial conditions synchronization between (39) and (40) occurs. We therefore reconsider (39) in the frame of Section 2. To do this, we use an EKF presented in the previous section as receiver dynamics for the noisy transmitter

$$\begin{aligned} x_1(k+1) &= (1-\epsilon)\,\mu x_1(k)\,(1-x_1(k)) + \epsilon x_2(k) + w_1(k), \\ x_2(k+1) &= (1-\epsilon)\,\mu x_2(k)\,(1-x_2(k)) + \epsilon x_1(k) + w_2(k) \end{aligned} \tag{41}$$

and the (corrupted) drive signal

$$y(k) = x_2(k) + v(k). \tag{42}$$

The EKF will yield the state estimates $\hat{x}_1(k)$ and $\hat{x}_2(k)$ for the signals $x_1(k)$ and $x_2(k)$. The structure of the EKF is given by

$$\begin{aligned} \hat{x}_1(k+1) &= (1-\epsilon)\,\mu\hat{x}_1(k)\,(1-\hat{x}_1(k)) + \epsilon\hat{x}_2(k) \\ &\quad + k_1(k)\,[y(k) - \hat{x}_2(k)], \\ \hat{x}_2(k+1) &= (1-\epsilon)\,\mu\hat{x}_2(k)\,(1-\hat{x}_2(k)) + \epsilon\hat{x}_1(k) \\ &\quad + k_2(k)\,[y(k) - \hat{x}_2(k)] \end{aligned} \tag{43}$$

where the gain vector $(k_1(k), k_2(k))^T$ is given via equations (10)-(14).

We investigate the evolution of the estimation process created by the EKF with the assumption that the initial matrix P_0 is of the form $P_0 = \text{diag}\{p_{0i}\}$, $i = 1, 2$. Also, the variance of the noise R was fixed at 0.00005 and $p_{01} = p_{02} = 100$. To simplify the presentation we realized the dynamics noise as being identical zero or, which is equivalent, the covariance Q of $(w_1, w_2)^T$ was supposed to be extremely small of the form $Q = \text{diag}\{q_i\}$, $i = 1, 2$, with $q_i = 10^{-8}$.

For the parameter values of $\epsilon = 0.09$ and $\mu = 3.7$ and initial conditions in $[0,1] \times [0,1]$, we have that $\|x(k)\| \leq R_x = \sqrt{1.9}$ for all $k \geq 0$. Although $\partial f(x)/\partial x$ is not invertible everywhere, it turns out that when initializing (41) at $x(0) = (0.4, 0.7)$ the Jacobian remained nonsingular along the trajectory. The controllability and observability conditions hold for all $k \geq 1$. Finally, we take $\varepsilon_r = 1$, $\rho_1 = 4.76$, $\rho_2 = \rho_3 = \rho_4 = 3.36$ and $\rho_5 = 1$ to satisfy the Theorem 5.

Initial conditions $x(0) = (0.4, 0.7)$ have been used for the subsequent simulations. For the above parameter values of μ and ϵ, the transmitter (41) is apparently chaotic. Following [2], $x_1(k)$ and $x_1^R(k)$ do not synchronize for these parameter values and initial conditions $x_1^R(0) = \hat{x}_1(0) = 0.65$ while we obtain synchronization (according to (17)) using the EKF as receiver. Figure 1 shows the synchronization error between transmitter and receiver dynamics. We see that, after some transient behavior, the approximate synchronization is clearly visible; according to (19) it is obtained when $\tau = 3$ when $\rho = 0.04$ is considered (see Figure 2).

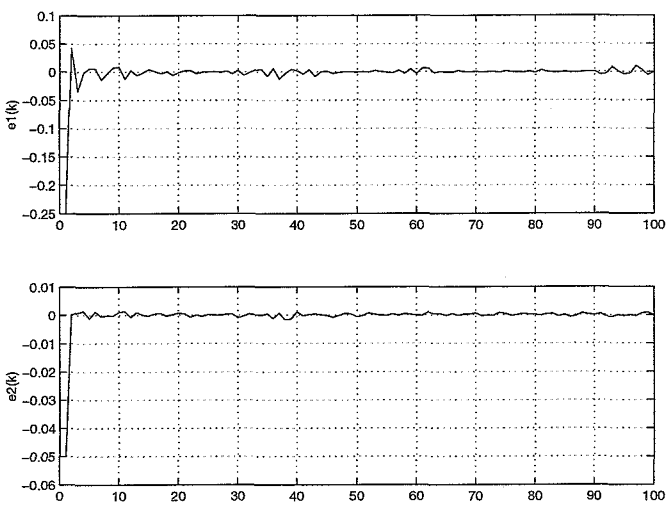

Figure 1: Synchronization errors $e_i(k) = x_i(k) - \hat{x}_i(k)$ $(i = 1, 2)$ for transmitter (41) and EKF receiver (43) for Example 1: $e(0) = (-0.25, -0.05)$, $R = 0.00005$, $\mu = 3.7$, and $\epsilon = 0.09$.

To evaluate the performance of the EKF from the point of view of sensitivity

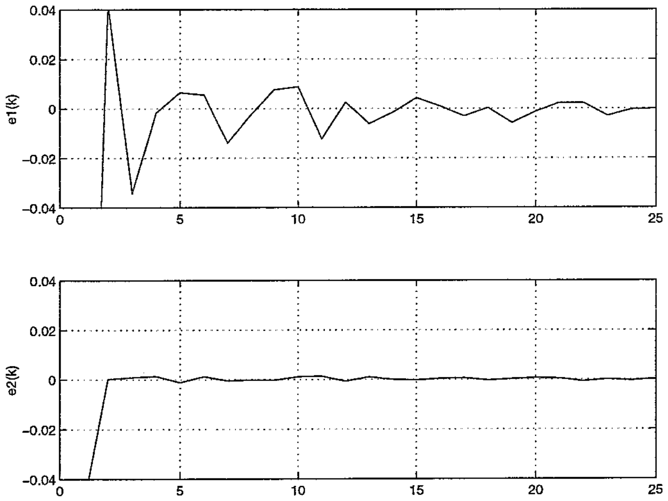

Figure 2: Approximate synchronization when $\rho = 0.04$ is considered for Example 1: $R = 0.00005$, $\mu = 3.7$, and $\epsilon = 0.09$.

to initial errors, twenty different Monte Carlo runs were taken in order to obtain root-mean-square error statistics. The results are summarized in Table 1; where sd_i $(i = 1, 2)$ is the sum of square errors given by

$$sd_i = \sum_{i=1}^{N} \left(x_i(k) - \hat{x}_i(k)\right)^2, \quad k = 0, 1, \ldots, N \tag{44}$$

where $x_i(k)$ and $\hat{x}_i(k)$ are the true and estimated states, respectively, and N the number of time steps. Thus, the mean square error (MSE) is obtained by $\frac{sd_i}{N}$. With the purpose to know the same statistics, when the transient has died out we define the truncated mean-square error as

$$TMSE = \frac{1}{N + 1 - \tau} \sum_{i=\tau}^{N} \left(x(i) - \hat{x}(i)\right)^2. \tag{45}$$

The results in Table 1 show the good performance of the EKF for the system (41). In particular, it should be observed that even with larger initial errors, the truncated mean-square errors remain within similar ranges.

Example 2 (see Huijberts *et al.* [11]) Consider the third order transmitter

$$\begin{aligned} x_1(k+1) &= (1-\epsilon)\,\mu x_1(k)\,(1 - x_1(k)) + \epsilon x_2(k)\,, \\ x_2(k+1) &= (1-\epsilon)\,\mu x_2(k)\,(1 - x_2(k)) + \epsilon x_3(k)\,, \\ x_3(k+1) &= (1-\epsilon)\,\mu x_3(k)\,(1 - x_3(k)) + \epsilon x_1(k) \end{aligned} \tag{46}$$

$x(0)$	$\hat{x}(0)$	$x(0)-\hat{x}(0)$	sd_1	sd_2	τ	tsd_1	tsd_2
(0.4,0.7)	(0.4,0.7)	(0,0)	0.0019	0.0002	0	0.0019	0.0002
-	(0.39,0.69)	(0.01,0.01)	0.0021	0.0003	0	0.0021	0.0003
-	(0.37,0.67)	(0.03,0.03)	0.0022	0.0003	0	0.0022	0.0003
-	(0.35,0.85)	(0.05,-0.15)	0.0052	0.0225	2	0.0027	2.4625e-05
-	(0.35,0.90)	(0.05,-0.20)	0.0057	0.0400	2	0.0030	2.7650e-05
-	(0.35,0.95)	(0.05,-0.25)	0.0062	0.0625	4	0.0032	2.8954e-05
-	(0.35,1)	(0.05,-0.30)	0.0082	0.0900	6	0.0035	3.3452e-05
-	(0.30,0.85)	(0.10,-0.15)	0.0214	0.0225	3	0.0028	2.7802e-05
-	(0.25,0.85)	(0.15,-0.15)	0.0745	0.0226	8	0.0036	4.3795e-05
-	(0.20,0.85)	(0.20,-0.15)	0.1697	0.0226	8	0.0046	5.0678e-05
-	(0.65,0.75)	(-0.25,-0.05)	0.0666	0.0025	3	0.0023	2.6513e-05
-	(0.70,0.75)	(-0.30,-0.05)	0.1026	0.0025	3	0.0024	4.1756e-05
-	(0.75,0.75)	(-0.35,-0.05)	0.1870	0.0025	8	0.0056	7.0948e-05

Table 1: Dependence of the synchronization time on the initial condition and truncated mean-square error according to (45) for Example 1: $p_{0i} = 100$, $i = 1,2$, $R = 0.00005$, $\mu = 3.7$, $\epsilon = 0.09$, $\rho = 0.04$, $N = 100$.

as an extension of the system (39). Similarly that the last example we consider the noisy transmitter

$$\begin{aligned} x_1(k+1) &= (1-\epsilon)\mu x_1(k)(1-x_1(k)) + \epsilon x_2(k) + w_1(k), \\ x_2(k+1) &= (1-\epsilon)\mu x_2(k)(1-x_2(k)) + \epsilon x_3(k) + w_2(k), \\ x_3(k+1) &= (1-\epsilon)\mu x_3(k)(1-x_3(k)) + \epsilon x_1(k) + w_3(k), \end{aligned} \tag{47}$$

in this case the (corrupted) drive signal is

$$y(k) = x_2(k) + v(k). \tag{48}$$

The equations for the EKF (receiver system for (46)) are

$$\begin{aligned} \hat{x}_1(k+1) &= (1-\epsilon)\mu\hat{x}_1(k)(1-\hat{x}_1(k)) + \epsilon\hat{x}_2(k) \\ &\quad + k_1(k)[y(k)-\hat{x}_2(k)], \\ \hat{x}_2(k+1) &= (1-\epsilon)\mu\hat{x}_2(k)(1-\hat{x}_2(k)) + \epsilon\hat{x}_3(k) \\ &\quad + k_2(k)[y(k)-\hat{x}_2(k)], \\ \hat{x}_3(k+1) &= (1-\epsilon)\mu\hat{x}_3(k)(1-\hat{x}_3(k)) + \epsilon\hat{x}_1(k) \\ &\quad + k_3(k)[y(k)-\hat{x}_2(k)] \end{aligned} \tag{49}$$

with gain vector $(k_1(k), k_2(k), k_3(k))^T$ given via equations (10)-(14).

In the following simulations we take $x(0) = (0.2, 0.4, 0.6)$, $R = 0.0001$, $P_0 = \text{diag}\{p_{0i}\}$, $p_{0i} = 100$, $i = 1,2,3$, $\mu = 3.7$, and $\epsilon = 0.35$[1] have been used. Again, to

[1] In this case, we take this value since estimating the signals $x_1(k)$ and $x_3(k)$ is more difficult because $x_1(k)$ is only indirectly influenced via $x_3(k)$.

simplify the presentation we realized the dynamics noise as being identical zero or, which is equivalent, the covariance Q of $(w_1, w_2, w_3)^T$ was supposed to be extremely small of the form $Q = \text{diag}\{q_i\}$, $i = 1, 2, 3$, $q_i = 10^{-8}$.

For the above parameter values of ϵ and μ and initial condition in $[0, 1.2] \times [0, 1.2] \times [0, 1.2]$, we have that $\|x(k)\| \leq R_x = \sqrt{3.15}$ for all $k \geq 0$. Again, we have that $f(x)/\partial x$ is not invertible everywhere, it turns out that when initializing (47) at $(0.2, 0.4, 0.6)$ the Jacobian remained nonsingular along the trajectory. The controllability and observability conditions hold for all $k \geq 1$. Finally, we take $\varepsilon_r = 1.2$, $\rho_1 = 5.94$, $\rho_2 = \rho_3 = \rho_4 = 4.2$ and $\rho_5 = 1$ in order to satisfy the Theorem 5.

Figure 3 shows the synchronization error evolution between (47) and (49) for $\hat{x}(0) = (1.3, 6.4, 0)$. Notice that we have assumed here a rather large initial error as to see the clear effect that the EKF needs more time for approximate synchronization. Again, we can see, after some transient behavior, that approximate synchronization is achieved; according to (19) it is obtained at $\tau = 8$ when $\rho = 0.04$ is considered (see Figure 4).

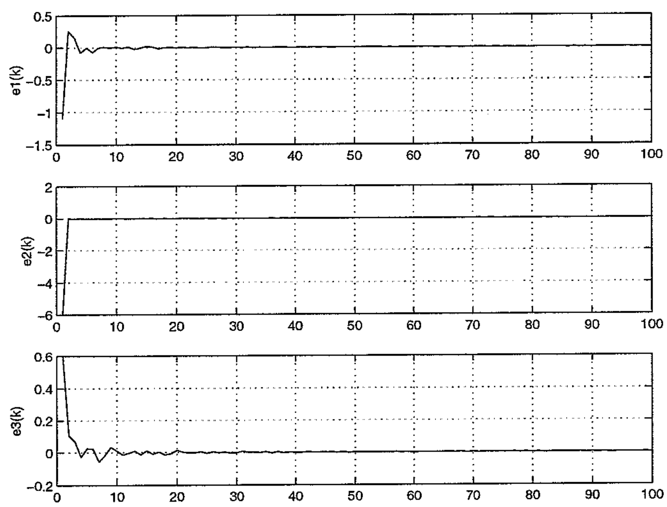

Figure 3: Synchronization errors $e_i(k) = x_i(k) - \hat{x}_i(k)$ $(i = 1, 2, 3)$ for transmitter (47) and EKF receiver (49) for Example 2: $e(0) = (-1.1, -6, 0.6)$, $R = 0.0001$, $\mu = 3.7$, and $\epsilon = 0.35$.

Twenty different Monte Carlo runs were taken in order to obtain root-mean-square error statistics. The results are summarized in Table 2 (sd_i and τ) and Table 3 (tsd_i).

3.2 Secure communication

Finally, in this subsection, we want to present an illustration of the potential use of synchronized systems through the EKF in secure communications.

$x(0)$	$\hat{x}(0)$	$x(0)-\hat{x}(0)$	sd_1	sd_2	sd_3	τ
(0.2,0.2,0.6)	(0.2,0.2,0.6)	(0,0,0)	0.0025	0.0010	0.0037	0
-	(0.21,0.21,0.61)	(-0.01,-0.01,-0.01)	0.0028	0.0012	0.0040	0
-	(0.22,0.22,0.62)	(-0.02,-0.02,-0.02)	0.0030	0.0014	0.0044	0
-	(0.23,0.23,0.63)	(-0.03,-0.03,-0.03)	0.0035	0.0020	0.0049	0
-	(0.24,0.24,0.64)	(-0.04,-0.04,-0.04)	0.0051	0.0026	0.0057	3
-	(0.25,0.25,0.65)	(-0.05,-0.05,-0.05)	0.0078	0.0035	0.0089	3
-	(0.26,0.26,0.66)	(-0.06,-0.06,-0.06)	0.0112	0.0044	0.0107	5
-	(0.28,0.28,0.68)	(-0.08,-0.08,-0.08)	0.0191	0.0072	0.0126	5
-	(0.3,0.3,0.7)	(-0.1,-0.1,-0.1)	0.0277	0.0108	0.0158	5
-	(0.4,0.4,0.8)	(-0.2,-0.2,-0.2)	0.1130	0.0439	0.1377	10
-	(-0.05,-0.05,0.35)	(0.25,0.25,0.25)	0.6244	0.0825	0.3338	17
(0.2,0.4,0.6)	(1.1,2.2,0.1)	(-0.9,-1.8,0.5)	0.7098	3.2463	0.3684	6
-	(1.1,3.9,0)	(-0.9,-3.5,0.6)	0.8112	12.2512	0.3984	6
-	(1.3,6.4,0)	(-1.1,-6,0.6)	1.3486	36.0032	0.4523	8
-	(1.3,7.4,-0.1)	(-1.1,-7,0.7)	1.4776	49.0023	0.7087	11

Table 2: Dependence of the synchronization time on the initial condition for Example 2: $p_{0i} = 100$, $i = 1,2,3$, $R = 0.0001$, $\mu = 3.7$, $\epsilon = 0.35$, $\rho = 0.04$, $N = 100$.

$x(0)$	$\hat{x}(0)$	$x(0)-\hat{x}(0)$	tsd_1	tsd_2	tsd_3
(0.2,0.2,0.6)	(0.2,0.2,0.6)	(0,0,0)	0.0025	0.0010	0.0037
-	(0.21,0.21,0.61)	(-0.01,-0.01,-0.01)	0.0028	0.0012	0.0040
-	(0.22,0.22,0.62)	(-0.02,-0.02,-0.02)	0.0030	0.0014	0.0044
-	(0.23,0.23,0.63)	(-.03,-.03,-.03)	0.0035	0.0020	0.0049
-	(0.24,0.24,0.64)	(-0.04,-0.04,-0.04)	0.0010	0.0004	0.0017
-	(0.25,0.25,0.65)	(-0.05,-0.05,-0.05)	0.0013	0.0005	0.0020
-	(0.26,0.26,0.66)	(-0.06,-0.06,-0.06)	0.0017	0.0007	0.0022
-	(0.28,0.28,0.68)	(-0.08,-0.08,-0.08)	0.0019	0.0009	0.0024
-	(0.3,0.3,0.7)	(-0.1,-0.1,-0.1)	0.0030	0.0011	0.0025
-	(0.4,0.4,0.8)	(-0.2,-0.2,-0.2)	0.0017	0.0016	0.0028
-	(-0.05,-0.05,0.35)	(0.25,0.25,0.25)	0.0100	0.0059	0.0062
(0.2,0.4,0.6)	(1.1,2.2,0.1)	(-0.9,-1.8,0.5)	0.0005	0.0005	0.0010
-	(1.1,3.9,0)	(-0.9,-3.5,0.6)	0.0005	0.0007	0.0011
-	(1.3,6.4,0)	(-1.1,-6,0.6)	0.0018	0.0014	0.0020
-	(1.3,7.4,-0.1)	(-1.1,-7,0.7)	0.0024	0.0015	0.0026

Table 3: Truncated mean-square error according to (45) for Example 2: $p_{0i} = 100$, $i = 1,2,3$, $R = 0.0001$, $\mu = 3.7$, $\epsilon = 0.35$, $\rho = 0.04$, $N = 100$.

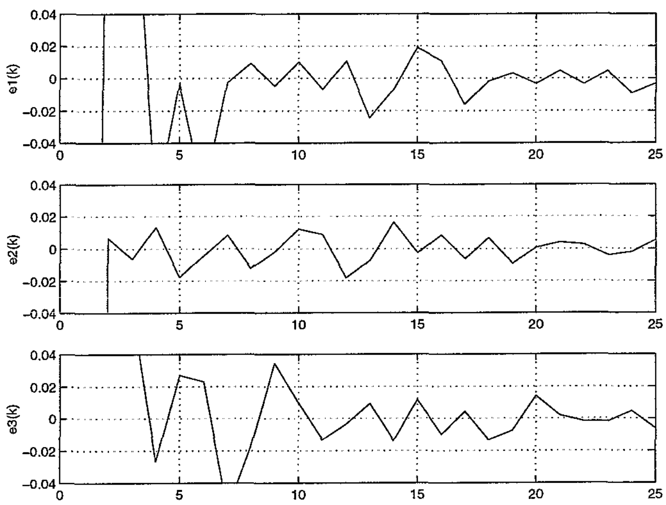

Figure 4: Approximate synchronization when $\rho = 0.04$ is considered for Example 2: $R = 0.0001$, $\mu = 3.7$, and $\epsilon = 0.35$.

Parameter switching is the simplest form of chaotic parameter modulation. In this method the message $s(k)$ is supposed to be binary, and is used to modulate one or more parameters of the (switching) transmitter, i.e., $s(k)$ controls a switch whose action changes the parameter values of the transmitter. Thus, according to the value of $s(k)$ at any given instant k, the transmitter has either the parameter set value p or the parameter set value $\bar{p}$.

At the receiver, $s(k)$ is decoded by using the synchronization error to decide whether the received signal corresponds to one parameter value, or the other (it can be interpreted as a zero or one).

The usefulness of this simple idea has been demonstrated by Parlitz *et al.* [18] and Cuomo *et al.* [6] for a replica.

In our case, to transmit $s(k)$ via parameter modulation scheme, the EKF is modified to estimate the value of this parameter. Thus, a combined state and parameter estimation is made by the extension of the state vector with the unknown parameter. Let μ be the parameter to be modulated in the transmitter dynamics (41) and (47); in both examples, the parameter value ϵ was fixed. If no other apriori information is available, an additional state $\mu(k)$ is used to extend the original state vector by treating μ as a function of time according to $\mu(k+1) = \mu(k)$. So, the noisy transmitter dynamics (41) and (47) become for Example 1:

$$\begin{aligned} x_1(k+1) &= (1-\epsilon)\,\mu(k)\,x_1(k)\,(1-x_1(k)) + \epsilon x_2(k) + w_1(k), \\ x_2(k+1) &= (1-\epsilon)\,\mu(k)\,x_2(k)\,(1-x_2(k)) + \epsilon x_1(k) + w_2(k), \\ \mu(k+1) &= \mu(k) + w_3(k), \end{aligned} \tag{50}$$

and for Example 2:

$$\begin{aligned} x_1(k+1) &= (1-\epsilon)\mu(k)x_1(k)(1-x_1(k)) + \epsilon x_2(k) + w_1(k), \\ x_2(k+1) &= (1-\epsilon)\mu(k)x_2(k)(1-x_2(k)) + \epsilon x_3(k) + w_2(k), \\ x_3(k+1) &= (1-\epsilon)\mu(k)x_3(k)(1-x_3(k)) + \epsilon x_1(k) + w_3(k), \\ \mu(k+1) &= \mu(k) + w_4(k), \end{aligned} \tag{51}$$

respectively, with the (corrupted) transmitted signal

$$y(k) = x_2(k) + v(k). \tag{52}$$

We note that both examples satisfy the conditions of Theorem 5; for communication purposes we take values of R_x smaller than $\sqrt{1.9}$ and $\sqrt{3.15}$ for Examples 1 and 2, respectively. We use a 'modulation rule' to modulate $s(k)$ in the parameter μ of the transmitter (50) and (51). Then the EKF used as receiver maintains synchronization by estimating the changes in the modulated parameter μ (while the parameter ϵ is fixed at the same value as in the transmitter). So, $s(k)$ can be recovered by the estimation given through the EKF. The modulation rule is given by

$$\mu(k) = \mu + a \cdot s(k), \qquad \hat{\mu}(k) = \mu + a \cdot \tilde{s}(k) \tag{53}$$

where a is a constant and $\tilde{s}(k)$ is the recovered message. The message is defined as follows

$$s(k) = \begin{cases} 0, & 0 \leq k < 200, \\ 1, & 200 \leq k < 400, \\ 0, & 400 \leq k < 600, \\ 1, & 600 \leq k < 800, \\ 0, & 800 \leq k < 1000. \end{cases}$$

An illustration for the binary communication of Example 1, via modulation and estimation of parameter μ with $a = 0.08$, i.e., when μ is switched between $\mu(0) = 3.7$ and $\mu(1) = 3.78$ is shown in Figure 5.

Figure 6 shows binary communication for Example 2, via modulation and estimation of parameter μ with $a = 0.1$, i.e., when μ is switched between $\mu(0) = 3.7$ and $\mu(1) = 3.8$.

4 Concluding remarks

We have discussed the use (in discrete-time) of an extended Kalman filter (EKF) as receiver system for chaotic synchronization purposes. Synchronization is obtained between transmitter and receiver dynamics when the EKF is driven by a noisy drive signal from the transmitter.

The computer simulation results presented, show that our chaotic synchronization approach is robust to additive Gaussian noise. Besides synchronization per se, we have

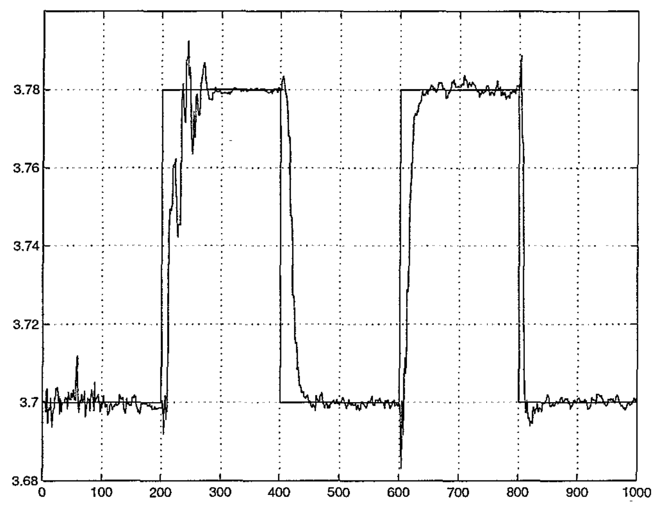

Figure 5: Estimated and true value μ for Example 1: $x(0) = \hat{x}(0) = (0.4, 0.7, 3.7)$, R=0.00005, and $a = 0.08$.

presented the utility of using the EKF for reconstruction of a binary message. In this case, by modulating a parameter in the transmitter and estimating this parameter via a modified EKF. Thus, we expect that it can be possibly applied to experimental systems, especially, for secure communication systems based on signal masking and parameter modulation.

Complementary simulations showed that synchronization and binary communication are also possible in case $x(0) \neq \hat{x}(0)$. Although, the synchronization time depends on the initial conditions and is different to the case $x(0) = \hat{x}(0)$. Obviously, the synchronization time is smaller for $x(0) \approx \hat{x}(0)$ and for smaller noise variance.

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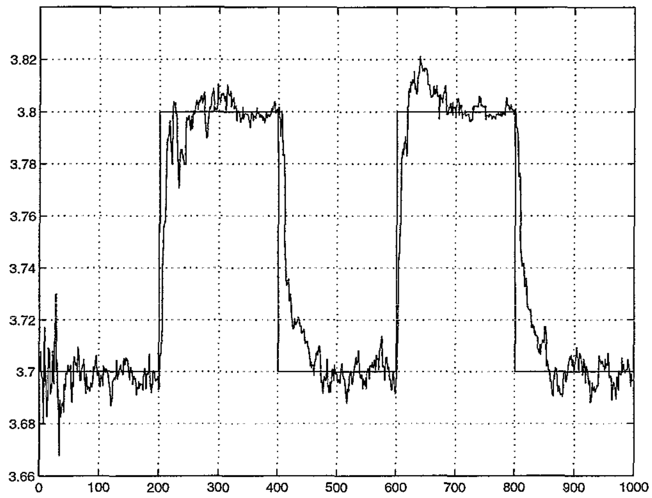

Figure 6: Estimated and true value μ for Example 2: $x(0) = \hat{x}(0) = (0.2, 0.4, 0.6, 3.7)$, R=0.0001, and $a = 0.1$.

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