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Synchronizing chaotic systems using backstepping design

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Abstract

Backstepping design is a recursive procedure that combines the choice of a Lyapunov function with the design of a controller. In this paper it is proposed for synchronizing chaotic systems. There are several advantages in this method for synchronizing chaotic systems: (a) it presents a systematic procedure for selecting a proper controller in chaos synchronization; (b) it can be applied to a variety of chaotic systems whether they contain external excitation or not; (c) it needs only one controller to realize synchronization between chaotic systems; (d) there is no derivatives in controller, so it is easy to be complemented. Examples of Lorenz system, Chua's circuit and Duffing system are presented.

1. Introduction

Synchronizing chaotic systems and circuits has received great interest in recent years since the seminal paper by Ott–Grebogi–Yorke [1]. Generally the two chaotic systems in synchronization are called drive system and response system respectively. The idea of synchronization is to use the output of the drive system to control the response system, so that the output of the response system follows the output of the drive system asymptotically. Some attempts to solve the problem have been made recently [2–7]. However, although they derive some methods for solving the problem, some of them can be only applied to chaotic systems with two dimensions like Duffing system [7] and some of them need several controllers to realize synchronization [4]. Some focus on the condition with unknown parameters and disturbances [4,6]. This paper focuses on increasing the effectiveness of the method to a wider variety of chaotic systems by using only one controller. So it has not studied the condition with unknown parameters and disturbances. The method here to use is backstepping design [8]. It consists in a recursive procedure that links the choice of a Lyapunov function with the design of a controller. It has been successfully used to stabilize and track chaotic systems by Saverio Mascolo and Giuseppe Grassi [9]. The paper is organized as follows: In Section 2 the class of chaotic systems considered in this work and the problem formulation are presented. In Section 3 the tool is utilized for several systems such as Lorenz system, Chua's circuit and Duffing system. Numerical simulations are carried out to confirm the validity of the proposed theoretical approach. In Section 4 conclusion is presented.

2. Problem formulation

In general, typical dynamics of chaotic systems such as Lorenz system, Rössler system, Chua's circuit and Duffing system all belong to the system as following:

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$$
\begin{aligned}
\dot{x}_1 &= f_1(x_1, x_2) \\
\dot{x}_2 &= f_2(x_1, x_2, x_3) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + f_{n+1}(t)
\end{aligned} \tag{1}
$$

where f_1 is a linear function and f_i $(i = 2, 3, \ldots, n+1)$ are nonlinear functions and when it comes to Lorenz system, Rössler system and Chua's circuit, $f_{n+l}(t) = 0$.

Assume that drive system is expressed as Eq. (1). Then response system which is coupled with system (1) by u is as following:

$$
\begin{aligned}
\dot{y}_1 &= f_1(y_1, y_2) \\
\dot{y}_2 &= f_2(y_1, y_2, y_3) \\
&\vdots \\
\dot{y}_n &= f_n(y_1, y_2, \ldots, y_n) + f'_{n+1}(t) + u
\end{aligned} \tag{2}
$$

where $f'_{n+1}(t)$ has similar characteristics as $f_{n+1}(t)$.

By properly choosing u, synchronization between drive system and response system can be achieved.

Let us define the state errors between the response system and the drive system as

$$e_1 = y_1 - x_1 \quad e_2 = y_2 - x_2 \cdots \quad e_n = y_n - x_n \tag{3}$$

namely,

$$y_1 = e_1 + x_1 \quad y_2 = e_2 + x_2 \cdots \quad y_n = e_n + x_n \tag{4}$$

Subtract (1) from (2). Notice Eqs. (3) and (4), finally error system can be derived as

$$
\begin{aligned}
\dot{e}_1 &= g_1(e_1, e_2) \\
\dot{e}_2 &= g_2(x_1, x_2, x_3, e_1, e_2, e_3) \\
&\vdots \\
\dot{e}_n &= g_n(x_1, x_2, \ldots, x_n, e_1, e_2, \ldots, e_n) + f'_{n+1}(t) - f_{n+1}(t) + u
\end{aligned} \tag{5}
$$

where g_1 is a linear function and g_i $(i = 2, 3, \ldots, n)$ are nonlinear functions with input $(x_1, x_2, \ldots, x_n)$ from system (1). Apparently g_i $(i = 2, 3, \ldots, n)$ depends not only on state variables but also on time t. The problem to realize the synchronization between two chaotic systems now transforms to another problem on how to choose a control law u to make e_i $(i = 1, 2, \ldots, n)$ generally converge to zero with time increasing. Here backstepping design is used to achieve the objective.

3. Adaptive synchronization via a backstepping design

In this section, Lorenz system, Chua's circuit and Duffing system are presented for synchronizing by backstepping design.

3.1. Lorenz system

In Lorenz system, external excitation does not exit and drive Lorenz system and response Lorenz system can be described respectively as (6) and (7)

$$
\begin{aligned}
\dot{x}_1 &= \sigma(y_1 - x_1) \\
\dot{y}_1 &= \rho x_1 - y_1 - x_1 z_1 \\
\dot{z}_1 &= -\beta z_1 + x_1 y_1
\end{aligned} \tag{6}
$$

$$\begin{aligned}\dot{x}_2 &= \sigma(y_2 - x_2)\\ \dot{y}_2 &= \rho x_2 - y_2 - x_2 z_2\\ \dot{z}_2 &= -\beta z_2 + x_2 y_2 + u\end{aligned} \tag{7}$$

where $\sigma, \rho, \beta > 0$. Let

$$e_x = x_2 - x_1 \quad e_y = y_2 - y_1 \quad e_z = z_2 - z_1 \tag{8}$$

namely,

$$x_2 = e_x + x_1 \quad y_2 = e_y + y_1 \quad z_2 = e_z + z_1 \tag{9}$$

Subtract Eq. (6) from Eq. (7) and consider Eqs. (8) and (9)

$$\begin{aligned}\dot{e}_x &= \sigma(e_y - e_x)\\ \dot{e}_y &= \rho e_x - e_y - e_x e_z - e_x z_1 - x_1 e_z\\ \dot{e}_z &= -\beta e_z + e_x e_y + e_x y_1 + x_1 e_y + u\end{aligned} \tag{10}$$

variables x_1, y_1, z_1 in error system (10) can be considered as input signals from system (6). If system (10) did not have u, it would have an equilibrium (0,0,0). If we choose a u which would not change the equilibrium (0,0,0), problem of synchronization between drive and response system can be transformed into a problem on how to realize the asymptotical stabilization of system (10). Now the objective is to find a control law u for stabilizing the error variables of system (10) at the origin.

First we consider the stability of system (11)

$$\dot{e}_x = \sigma(e_y - e_x) \tag{11}$$

where e_y is regarded as a controller. Choose Lyapunov function V_1 as following:

$$V_1(e_x) = \frac{1}{2} e_x^2 \tag{12}$$

The derivative of V_1 is as following:

$$\dot{V}_1 = -\sigma e_x^2 + \sigma e_x e_y \tag{13}$$

Assume controller $e_y = \alpha_1(e_x)$, Eq. (13) can be rewritten as

$$\dot{V}_1 = -\sigma e_x^2 + \sigma e_x \alpha_1(e_x) \tag{14}$$

if $\alpha_1(e_x) = 0$ (controller must be as simple as possible), then

$$\dot{V}_1 = -\sigma e_x^2 < 0 \tag{15}$$

makes system (11) asymptotically stable. Function $\alpha_1(e_x)$ is an estimative function when e_y is considered as a controller. The error between e_y and $\alpha_1(e_x)$ is

$$w_2 = e_y - \alpha_1(e_x) \tag{16}$$

Study (e_x, w_2) system (17)

$$\begin{aligned}\dot{e}_x &= \sigma(w_2 - e_x)\\ \dot{w}_2 &= \rho e_x - w_2 - e_x e_z - e_x z_1 - x_1 e_z\end{aligned} \tag{17}$$

Consider e_z as a controller in system (17). Assume when it is equal to $\alpha_2(e_x, w_2)$, it makes system (17) asymptotically stable. Choose Lyapunov function $V_2(e_x, w_2) = V_1(e_x) + (1/2)w_2^2$. The derivative of V_2 is

$$\dot{V}_2 = -\sigma e_x^2 - w_2^2 + w_2(\sigma e_x + \rho e_x - z_1 e_x - e_x e_z - x_1 e_z) \tag{18}$$

If $\alpha_2(e_x, w_2) = (\sigma + \rho - z_1)e_x/(e_x + x_1)$ (according to paper [10] we have already known that x_1 and z_1 have boundaries respectively), $\dot{V}_2$ is

$$\dot{V}_2 = -\sigma e_x^2 - w_2^2 < 0 \tag{19}$$

negative definite. Define the error variable w_3 as

$$w_3 = e_z - \alpha_2(e_x, w_2) \tag{20}$$

Study full dimension (e_x, w_2, w_3) system

$$\begin{aligned}
\dot{e}_x &= \sigma(w_2 - e_x) \\
\dot{w}_2 &= \rho e_x - w_2 - e_x(w_3 + \alpha_2) - e_x z_1 - x_1(w_3 + \alpha_2) \\
\dot{w}_3 &= -\beta(w_3 + \alpha_2) + e_x w_2 + e_x y_1 + x_1 w_2 + u - \frac{\mathrm{d}\alpha_2}{\mathrm{d}t}
\end{aligned} \tag{21}$$

where

$$\frac{\mathrm{d}\alpha_2}{\mathrm{d}t} = \frac{\sigma(\sigma + \rho - z_1)(w_2 x_1 - e_x y_1) + (\beta z_1 - x_1 y_1)(e_x^2 + e_x x_1)}{(e_x + x_1)^2}$$

Choose Lyapunov function $V_3(e_x, w_2, w_3) = V_2(e_x, w_2) + (1/2)w_3^2$. The derivative of V_3 is

$$\dot{V}_3 = -\sigma e_x^2 - w_2^2 + w_2[\sigma e_x + \rho e_x - z_1 e_x - (x_1 + e_x)\alpha_2] + w_3\left[-\beta(w_3 + \alpha_2) + e_x y_1 + u - \frac{\mathrm{d}\alpha_2}{\mathrm{d}t}\right] \tag{22}$$

Considering $\sigma e_x + \rho e_x - z_1 e_x - (x_1 + e_x)\alpha_2 = 0$, Eq. (22) can be rewritten as

$$\dot{V}_3 = -\sigma e_x^2 - w_2^2 + w_3\left[-\beta(w_3 + \alpha_2) + e_x y_1 + u - \frac{\mathrm{d}\alpha_2}{\mathrm{d}t}\right] \tag{23}$$

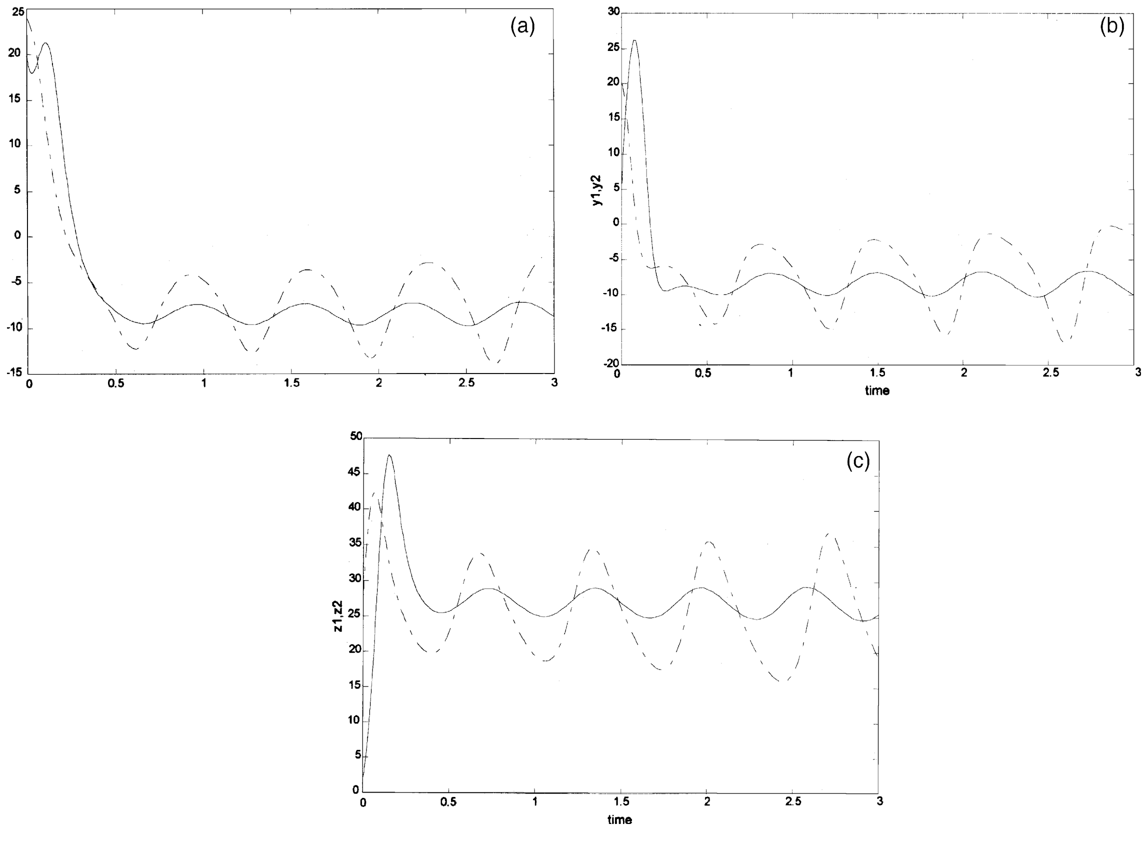

Fig. 1. The control of Lorenz system is switched off: (a) comparison of time waveform of x; (b) comparison of time waveform of y; (c) comparison of time waveform of z.

Let $u = \beta\alpha_2 - e_x y_1 + (\mathrm{d}\alpha_2/\mathrm{d}t)$, and $\dot{V}_3$ can be described as

$$\dot{V}_3 = -\sigma e_x^2 - w_2^2 - \beta w_3^2 < 0$$

negative definite.

Substitute $(\beta\alpha_2 - e_x y_1 + \mathrm{d}\alpha_2/\mathrm{d}t)$ for u in Eq. (10), equilibrium is still (0,0,0) and has not been changed. So following above procedure we can conclude that equilibrium (0,0,0) of system (10) is asymptotically stable. As for an arbitrary initial error between systems (6) and (7), after a finite period of time, the initial error will converge to zero and synchronization between two Lorenz systems will be achieved.

By taking $\sigma = 10$, $\beta = 8/3$, $\rho = 28$ and giving initial condition ($x_1(0) = 20$, $y_1(0) = 5$, $z_1(0) = 2$, $x_2(0) = 24$, $y_2(0) = 20$, $z_2(0) = 28$) and the numerical resolves are reported in Fig. 1. Fig. 1 shows the time waveforms of x, y and z without u and Fig. 2 shows the waveforms of x, y and z with u.

3.2. Chua's circuit

In order to further test the effectiveness of the method Chua's circuit, which was the first physical dynamical system capable of generating chaotic phenomena in the laboratory, is proposed for synchronizing. The circuit considered here contains a cubic nonlinearity and the drive system (24) and response system (25) are described by the following set of differential equations:

$$\begin{aligned} \dot{x}_1 &= \alpha(y_1 - x_1^3 - cx_1) \\ \dot{y}_1 &= x_1 - y_1 + z_1 \\ \dot{z}_1 &= -\beta y_1 \end{aligned} \tag{24}$$

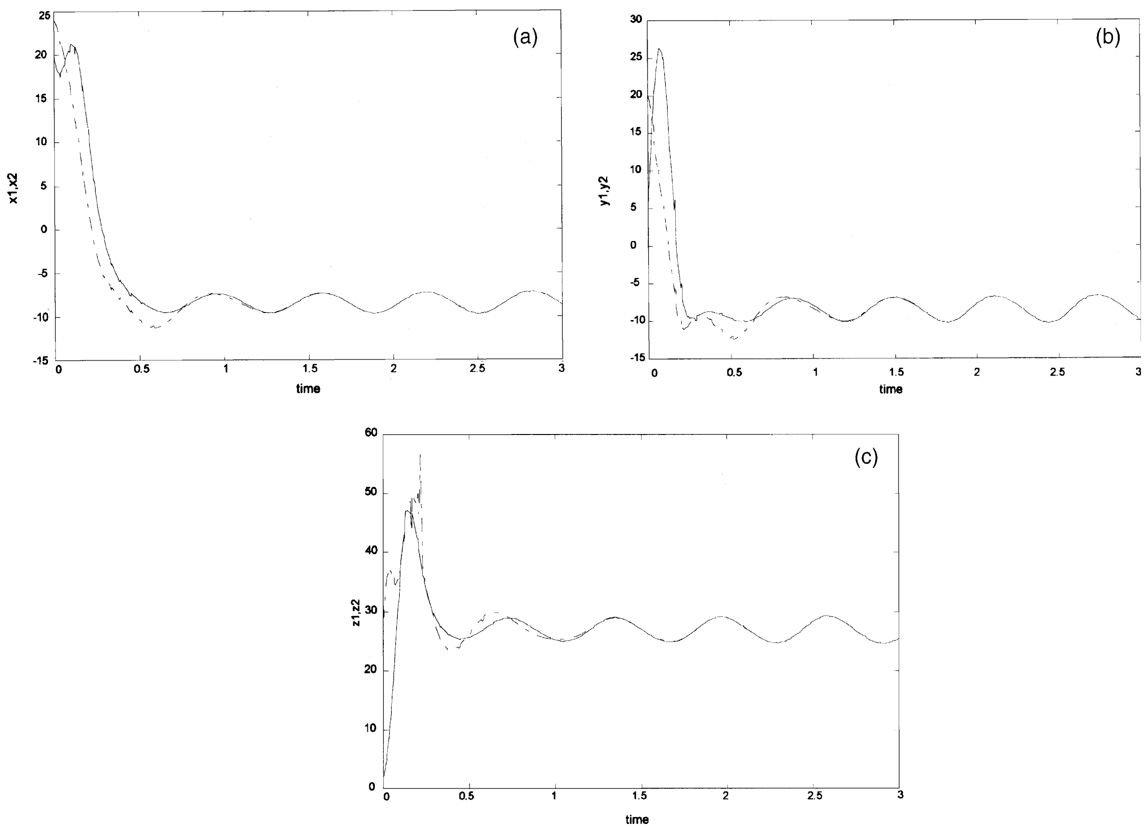

Fig. 2. The control of Lorenz system is switched on: (a) comparison of time waveform of x; (b) comparison of time waveform of y; (c) comparison of time waveform of z.

$$\begin{aligned} \dot{x}_2 &= \alpha(y_2 - x_2^3 - cx_2) + u \\ \dot{y}_2 &= x_2 - y_2 + z_2 \\ \dot{z}_2 &= -\beta y_2 \end{aligned} \tag{25}$$

where α, c and β are the circuit parameters.

Subtract (24) from (25) and rearrange the order of the equations, the error system can be written as

$$\begin{aligned} \dot{e}_z &= -\beta e_y \\ \dot{e}_y &= e_x - e_y + e_z \\ \dot{e}_x &= \alpha e_y - \alpha e_x(e_x^2 + 3x_1 e_x + 3x_1^2) - \alpha c e_x + u \end{aligned} \tag{26}$$

where $e_x = x_2 - x_1$; $e_y = y_2 - y_1$; $e_z = z_2 - z_1$. The objective is to find a control law u so that system (26) is stabilized at the origin. Starting from the first equation of system (26), an estimative stabilizing function $\alpha_1(e_z)$ has to be designed for the virtual control e_y in order to make the derivative of $V_1(e_z) = (1/2)e_z^2$, namely $\dot{V}_1 = -\beta e_z \alpha_1(e_z)$, negative definite when $\alpha_1(e_z) = e_z$. Define the error variable w_2 as

$$w_2 = e_y - \alpha_1(e_z) \tag{27}$$

Study (e_z, w_2) system (28)

$$\begin{aligned} \dot{e}_z &= -\beta(w_2 + e_z) \\ \dot{w}_2 &= e_x - w_2 + \beta(w_2 + e_z) \end{aligned} \tag{28}$$

Consider e_x as a controller in system (28). Assume when it is equal to $\alpha_2(e_z, w_2)$, it makes system (28) asymptotically stable. Choose Lyapunov function $V_2(e_z, w_2) = V_1(e_z) + (1/2)w_2^2$. The derivative of V_2 is

$$\dot{V}_2 = -\beta e_z^2 - w_2^2 + w_2(e_x + \beta w_2) \tag{29}$$

If $\alpha_2(e_z, w_2) = -\beta w_2$, $\dot{V}_2$ is

$$\dot{V}_2 = -\beta e_z^2 - w_2^2 < 0 \tag{30}$$

negative definite. Define the error variable w_3 as

$$w_3 = e_x - \alpha_2(e_z, w_2) \tag{31}$$

Study full dimension (e_z, w_2, w_3) system

$$\begin{aligned} \dot{e}_z &= -\beta(w_2 + e_z) \\ \dot{w}_2 &= w_3 - w_2 + \beta e_z \\ \dot{w}_3 &= \alpha(w_2 + e_z) + \beta(w_3 - w_2 + \beta e_z) - \alpha e_x(e_x^2 + 3e_x x_1 + 3x_1^2) - \alpha c(w_3 - \beta w_2) + u \end{aligned} \tag{32}$$

Choose Lyapunov function $V_3(e_z, w_2, w_3) = V_2(e_z, w_2) + (1/2)w_3^2$. The derivative of V_3 is

$$\dot{V}_3 = -\beta e_z^2 - w_2^2 + w_3\lfloor w_2 + \alpha(w_2 + e_z) + \beta(w_3 - w_2 + \beta e_z) - \alpha e_x(e_x^2 + 3e_x x_1 + 3x_1^2) - \alpha c(w_3 - \beta w_2) + u\rfloor. \tag{33}$$

Let $u = -w_2 - w_3 - \alpha(w_2 + e_z) - \beta(w_3 - w_2 + \beta e_z) + \alpha e_x(e_x^2 + 3x_1 e_x + 3x_1^2) + \alpha c(w_3 - \beta w_2)$, and $\dot{V}_3$ can be described as

$$\dot{V}_3 = -\beta e_z^2 - w_2^2 - w_3^2 < 0$$

negative definite.

Following above procedure we have chosen a control law u. As for an initial error between systems (24) and (25), after a finite period of time, the initial error will converge to zero and synchronization between two Chua's circuits will be achieved (Fig 3).

By taking $\alpha = 10$, $\beta = 16$, $c = -0.143$ and giving initial condition ($x_1(0) = 1$, $y_1(0) = 2$, $z_1(0) = 1$, $x_2(0) = 10$, $y_2(0) = 5$, $z_2(0) = 5$), the numerical resolves such as time waveforms of x, y and z are reported in Fig. 1. The control law u is switched on at $t = 5$.

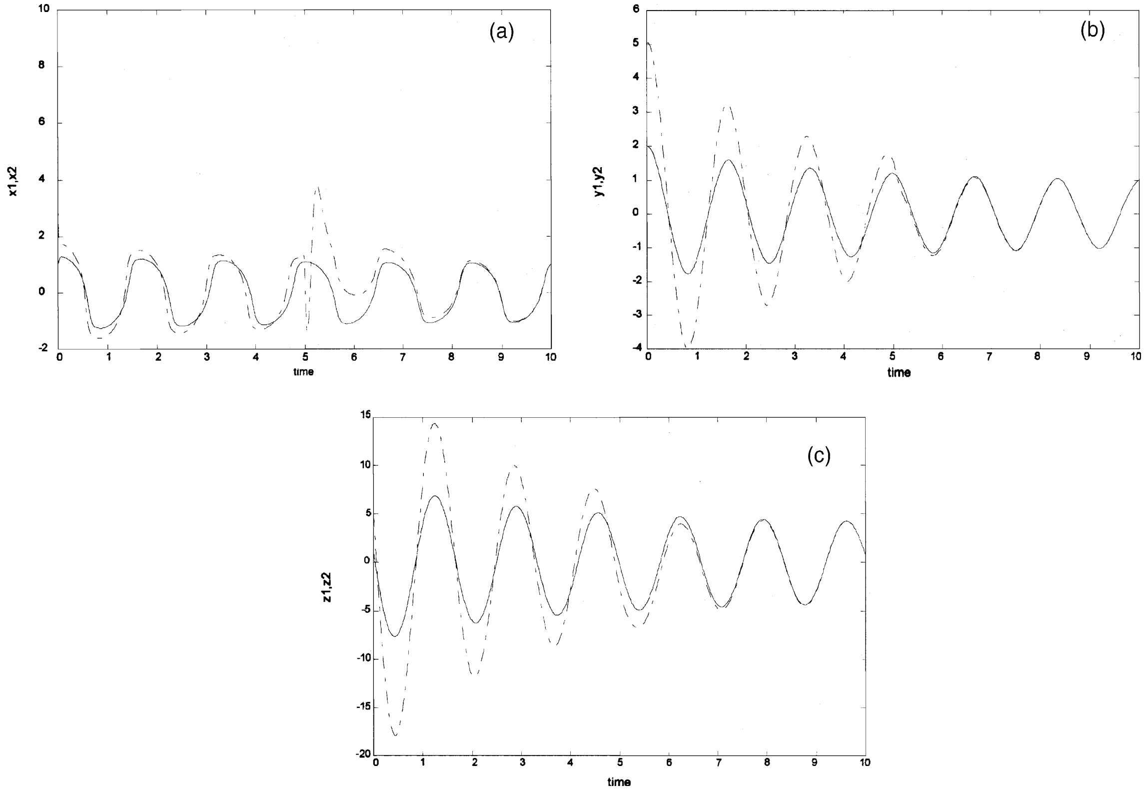

Fig. 3. Synchronization for Chua's circuit using the backstepping design. The control is switched on at $t = 5$: (a) comparison of time waveform of x; (b) comparison of time waveform of y; (c) comparison of time waveform of z.

3.3. Duffing system

Lorenz system and Chua's circuit discussed above can generate chaotic phenomena under no external excitation condition while Duffing system can generate chaotic phenomena only under external excitation. So here we classify Duffing system to another chaotic system and make Duffing system an example to illustrate how to use this method to synchronize chaotic systems with external excitation. The following set of differential equations formulates two Duffing systems. The first is drive system and the second response system

$$\begin{aligned} \dot{x}_1 &= y_1 \\ \dot{y}_1 &= ax_1 + by_1 - x_1^3 + c\cos(0.4t) \end{aligned} \tag{34}$$

$$\begin{aligned} \dot{x}_2 &= y_2 \\ \dot{y}_2 &= ax_2 + by_2 - x_2^3 + c\cos t + u \end{aligned} \tag{35}$$

where a, b and c are known parameters.

Subtract (34) from (35), the error system can be written as

$$\begin{aligned} \dot{e}_x &= e_y \\ \dot{e}_y &= ae_x + be_y - e_x(e_x^2 + 3x_1e_x + 3x_1^2) + c[\cos t - \cos(0.4t)] + u \end{aligned} \tag{36}$$

where $e_x = x_2 - x_1$; $e_y = y_2 - y_1$; $e_z = z_2 - z_1$. The objective is to find a control law u so that system (36) is stabilized at the origin. Starting from the first equation of system (36), an estimative stabilizing function $\alpha_1(e_x)$ has to be designed for the virtual control e_y in order to make the derivative of $V_1(e_x) = (1/2)e_x^2$, namely $\dot{V}_1 = e_x\alpha_1(e_x)$, negative definite when $\alpha_1(e_z) = -e_x$. Define the error variable w_2 as

$$w_2 = e_y - \alpha_1(e_x) \tag{37}$$

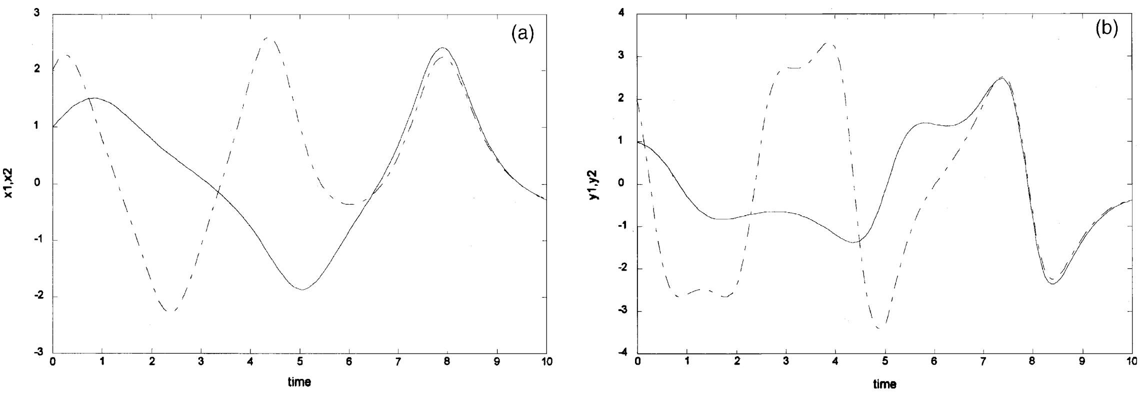

Fig. 4. Synchronization for Duffing system using the backstepping design. The control is switched on at $t = 5$: (a) comparison of time waveform of x; (b) comparison of time waveform of y.

Study (e_x, w_2) system (38):

$$\begin{aligned} \dot{e}_x &= w_2 - e_x \\ \dot{w}_2 &= (b+1)w_2 + (a-b-1)e_x - e_x(e_x^2 + 3x_1e_x + 3x_1^2) + c[\cos t - \cos(0.4t)] + u \end{aligned} \tag{38}$$

Choose Lyapunov function $V_2(e_x, w_2) = V_1(e_x) + (1/2)w_2^2$. The derivative of V_2 is

$$\dot{V}_2 = -e_x^2 + w_2\lfloor (b+1)w_2 + (a-b)e_x - e_x(e_x^2 + 3e_xx_1 + 3x_1^2) + c[\cos t - \cos(0.4t)] + u \rfloor \tag{39}$$

Let $u = -(b+2)w_2 - (a-b)e_x + e_x(e_x^2 + 3e_xx_1 + 3x_1^2) - c[\cos t - \cos(0.4t)]$, and $\dot{V}_2$ can be described as

$$\dot{V}_2 = -e_x^2 - w_2^2 < 0$$

negative definite.

Following above procedure we have chosen a control law u. As for an initial error between systems (34) and (35), after a finite period of time, the initial error will converge to zero and synchronization between two Duffing systems will be achieved.

By taking $\alpha = 1.8$, $\beta = -0.1$, $c = -1.1$ and giving initial condition ($x_1(0) = 1$, $y_1(0) = 1$, $x_2(0) = 2$, $y_2(0) = 2$), the numerical resolves such as time waveforms of x and y are reported in Fig. 4. The control law u is switched on at $t = 5$.

4. Conclusion

In this paper, backstepping design has been used to synchronize chaotic systems. The advantages of this method can be summarized as follows: (a) it is a systematic procedure for synchronizing chaotic systems; (b) it can be applied to a variety of chaotic systems no matter whether it contains external excitation or not; (c) it needs only one controller to realize synchronization no matter how much dimensions the chaotic system contains; (d) there is no derivatives in controller, so it is easy to be complemented. The technique has been successfully applied to the Lorenz system, Chua's circuit and Duffing system. Numerical simulations have verified the effectiveness of the method.

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