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Synergies and Price Trends in Sequential Auctions

Flavio Marques Menezes, Paulo Klinger Monteiro

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Synergies and Price Trends in Sequential Auctions \({ }^{\text {x }}\)
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\author{
Flavio M. M enezes \\ Australian National University and \\ EPGE/FGV \\ Email: Flavio.M enezes@anu.edu.au
}

\author{
Paulo K. M onteiro \\ EPGE/FGV \\ Email: pklm@fgv.br
}

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\begin{abstract}
In this paper we consider sequential auctions where an individual's value for a bundle of objects is either greater than the sum of the values for the objects separately (positive synergy) or less than the sum (negative synergy). We show that the existence of positive synergies implies declining expected prices. W hen synergies are negative, expected prices are increasing.

There are several corollaries. First, the seller is indixerent between selling the objects simultaneously as a bundle or sequentially when synergies are positive. Second, when synergies are negative, the expected revenue generated by the simultaneous auction can be larger or smaller than the expected revenue generated by the sequential auction. In addition, in the presence of positive synergies, an option to buy the additional object at the price of the ..rst object is never exercised in the symmetric equilibrium and the seller's revenue is unchanged. Under negative synergies, in contrast, if there is an equilibrium where the option is never exercised, then equilibrium prices may either increase or decrease and, therefore, the net exect on the seller's revenue of the introduction of an option is ambiguous.

F inally, we examine two special cases with asymmetric players. In the ..rst case, players have distinct synergies. In this example, even if one player has positive synergies and the other has negative synergies, it is still possible for expected prices to decline. In the second case, one player wants two objects and the remaining players want one object each. For this example, we show that expected prices may not necessarily decrease as predicted by Branco (1997). The reason is that players with singleunit demand will generally bid less than their true valuations in the ..rst
\end{abstract}

\footnotetext{
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}
period. Therefore, there are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the ..rst auction by the players with single-unit demand. J EL Classi..cation: D44.
K eywords: synergies; sequential auctions, increasing and decreasing expected prices.

\section*{1 Introduction}

Weber (1983) considers a sequential auction of identical objects and shows that expected prices follow a martingale i.e., bidders expect prices will remain constant on average throughout the sequence of auctions within a sale. In Weber's model, bidders only purchase one of a ..xed number of objects. That is, the marginal value for a bidder of a second object is zero.

The essence of Weber's result is that there are two opposite and exactly owsetting exects on price as the auction proceeds; a reduction in competition with fewer buyers puts downward pressure on price, while increased competition with fewer objects put upward pressure on price.

Thereis, however, empirical evidencethat prices are not constant throughout sequential auction sales. A shenfelter (1989) reports that identical cases of wine fetch dixerent prices at sequential auctions in three auction houses from 1985 to 1987. Although the most common pattern was for prices to remain constant, prices were at least twice as likely to decline as to increase. A shenfelter refers to this phenomenon as the "price decline anomaly."

McA fee and Vincent (1993) adopted a similar approach to A shenfelter and examined data from Christie's wine auctions at Chicago in 1987. In addition to pairwise comparisons, they examined triples of identical wine sold in the same auction sale. Their results are very similar to those of A shenfelter.

Similar empirical ..ndings were identi..ed in a number of other markets; cable television licenses (Gandal (1995); condominiums (A shenfelter and Genesove (1992), and Vanderporten (1992-a,b); dairy cattle (Engelbrecht-Wiggans and K ahn (1992)); stamps (Taylor (1991) and Thiel and Petry (1990)) and wool (J ones, Menezes and Vella (1998)). Gandal provides evidence that prices increased in the sale of cable-TV licences in Israel. The price increases are attributed by Gandal to the interdependencies among licenses which may increase competition in the later rounds of the sale. J ones, Menezes and Vella indicate that prices may increase or decrease in sequential auctions of wool (adjusting prices to estimate wool of homogeneous quality).

Most theoretical explanations for price variation in sequential auctions have concentrated on explaining the price decline anomaly. In a two-object model, Black and de M eza (1992) explain the price decline anomaly by the existence of an option that gives the winner of the ..rst auction the rights to purchase the second object at the same price. In particular, for the case where the value of a second object for a player is equal to a fraction of the value of a ..rst object, they show the existence of an equilibrium in which expected prices increase in
the absence of an option to buy and may decrease when the option is present. We will characterize price trends in a more general setting and determine the exect of the option on the seller's revenue. McA fee and Vincent explain the anomaly by considering the exects of risk aversion on bidding strategies. For identical objects they show how bids in the ..rst round are equal to the expected prices in the second round plus a risk premium associated with the risky future price. They assume buyers have nondecreasing risk aversion and can only buy one object.

Von der Fehr (1994) uses participation costs to obtain dixerent net valuations for identical objects. W hen bidders face a cost of participating in each auction of two identical objects sold sequentially, price is lower in the second auction than it is in the ..rst. This follows because the number of buyers who stay for the second auction falls by more than the successful bidder in the ..rst auction. Once again, buyers only buy one object.

Engelbrecht-W iggans (1994) and Bernhardt and Scoones (1994) show how expected prices decline when the objects are statistically identical (i.e., where bidders' valuations for the objects are independent draws from a ..xed distribution) and the distribution of values is bounded. Finally, M enezes and M onteiro (1997) replicate these results for the case when buyers are allowed to buy more than one object but participation is endogenous as bidders face participation costs.

In this paper we examine sequential auctions of identical objects where individuals demand more than one object. A \(n\) individual's value for a bundle of objects is either greater than the sum of the values attributed to the separate objects (positive synergy) or less than the sum (negative synergy) - Black and de M eza consider a special case of negative synergies. Thus, in this paper we explore the type of interdependencies described, for example, by Gandal (1997) in reference to the cable-TV auctions in Israel.

R osenthal and K rishna (1996) al so consider the exects of synergies on bidding behavior. However, they concentrate on simultaneous auctions and consider only a very special type of positive synergy; where a bidder's value for two objects is simply equal to twice his value for an individual object plus a positive constant. (For example, for a player with a value close to zero, the marginal synergy is in..nite). B ranco (1997) provides an example of sequential auctions with positive synergy of the same type of R osenthal and K rishna. In his example, equilibrium behavior implies in a decline in expected prices.

In contrast, we consider synergies of a general form, allowing for positive and negative synergies. We show that the existence of positive synergies implies declining expected prices. When two objects are worth more as a bundle than as separate objects, whoever buys the ..rst object has the opportunity to realize the synergy. Therefore, the price in the ..rst period includes a premium to rełect such opportunity. For the case of negative synergies, expected prices increase.

There are several corollaries. First, the seller is indixerent between selling the objects simultaneously as a bundle or sequentially when synergies are positive. Second, when synergies are negative, the simultaneous auction may yield a higher or smaller expected revenue than the sequential auction. Third, when
the synergy is positive an option to buy the additional object at the price of the ..rst object is never exercised in the symmetric equilibrium. In contrast, if there is an equilibrium where the option is never exercised, then equilibrium prices may either increase or decrease and, therefore, the net exect on the seller's revenue of the introduction of an option is ambiguous. This conforms with the results of Black and de \(M\) eza for the case of constant negative average synergies.

Finally, we present two special cases with asymmetric players. In the ..rst example, players have distinct synergies. In this example, even if one player has positive synergies and the other has negative synergies, it is still possible for expected prices to decline. In the second example, one player wants two objects and the remaining players want one object each. For this example, we show that expected prices may not necessarily decrease as predicted by Branco (1997). The reason is that players with single-unit demand will generally bid less than their true valuations in the ..rst period. Therefore, there are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the ..rst auction by the players with single-unit demand.

\section*{2 Price trends}

We consider the sale of two identical objects sequentially through second-price sealed-bid auctions. Buyer i's utility from one object is given by \(v_{i} ; i=1 ;::: ; n\). The \(v_{i}\) 's are drawn independently from a ..xed distribution \(F(\Phi\) with \(F(0)=0\) and density \(\mathrm{f}>0\). Buyer i's utility from owning the two objects is given by the continuous function \(\#\left(v_{i}\right)\). We suppose that \(\left.\# 0\right), 0\) and that \(\#(x) i x\) is strictly increasing. De..ne \(Y^{1}=\max f v_{j} ; j, 2 g\) and \(Y^{2}\) as the second highest of \(f v_{j} ; j, 2 g\) :

If \(\sharp x)>2 x\); we say that there are positive synergies. If \(\sharp x)<2 x\); we say that there are negative synergies. Otherwise, there are no synergies. The next theorem characterizes bidding strategies in the symmetric equilibrium. As a corollary, we can predict whether prices are likely to increase, decrease or remain the same as a function of the existing synergies. We need the lemma

Lemmac 1 Suppose \(\underset{a}{a} \pm(x)\) i \(x\) ids strictly increasing. Then the function \(b(x)=\) \(E \quad \max \pm(x) ; x ; Y^{2} j Y^{1}=x\) is strictly increasing.

Proof. The proof is in the appendix.
Remark 1 In the lemma above we use the unique continuous version of the conditional expectation. Namely if \(n>2\) we have that
\[
b(x)=Z_{0}^{Z_{x}} \operatorname{maxf} \pm(x) ; x ; y g \frac{(n ; 2) F(y)^{n_{i} 3} f(y)}{F(x)^{n_{i} 2}} d y:
\]

Remark 2 If the synergy is positive then \(b(x)= \pm(x) ; x\) : If the synergy is negative, i.e. if \(\pm(x)\); \(x<x\) then the possibility that \(\pm(x)\); \(x<Y^{2}<Y^{1}=x\) is taken into account in the calculus of \(b(\phi\) :

Theorem 2 Assume that \(\pm(0), 0\) and that \(\pm(x)\) i \(x\) is strictly increasing. Then in the symmetric equilibrium, a player with value \(x\) for one object bids in the ..rst auction
\[
b(x)=E^{f} \max { }^{\mathbb{C}} \pm(x) ; x ; Y^{2^{a}} j Y^{1}=x^{\mathfrak{a}}:
\]

His bid in the second auction equals \(x\) in case he does not win the ..rst object and equals \(\pm(x)\) i \(x\) if he wins the ..rst object.

Proof. Suppose bidders \(\mathrm{i}=2 ;::: ; \mathrm{n}\) bid \(\mathrm{b}(\mathrm{x})\) in the ..rst auction and \(\pm(\mathrm{x}) \mathrm{i} \mathrm{x}\) in case of winning the ..rst auction and \(x\) in case of not winning the ..rst auction. De..ne \(G\left(y_{1} ; y_{2}\right)=\left(v_{i} \max f\left(y_{1}\right) ; y_{1} ; y_{2} g\right)^{+}\): Let us ..nd the best response of bidder 1. If he wins the ..rst object he will bid \(\pm v) ; v\) in the second auction. If he does not get the ..rst object he will bid his signal \(v\) in the second auction. We need only to ..nd his bid in the ..rst auction. The expected utility of bidder one when his signal is \(v\) and he bids \(x\) is

Since \(b(\Phi)\) is a continuous function its range is an interval. Therefore we may suppose without loss of generality that \(x=b(!) ;!, 0\) : Thus de..ning \(h(!)=\) \(H\) (b(!)) we have from lemma (1)that
\[
h(!)=E \hat{A}_{!, Y^{1}} v_{i} b^{i^{1} Y^{1}{ }^{\phi}}+{ }^{i} \pm(v) i v_{i} Y^{1}{ }^{\phi_{+}}{ }^{0}+\hat{A}_{!<Y_{1}} G^{i} Y^{1} ; Y^{2^{4}}:
\]

If \(f_{Y^{1}}\) is the density of \(Y^{1}\) we can write

Therefore the derivative of \(h\) is
\[
\begin{aligned}
& h^{0}(!)={ }^{3} v_{i} b(!)+\left( \pm(v) i v_{i}!\right)^{+} \underset{i}{E}{ }^{\ddagger} G^{i}!; Y^{2}{ }^{\Phi}{ }_{j Y^{1}}=!{ }^{\alpha^{\prime}} f_{Y^{1}}(!)= \\
& { }^{i} \operatorname{maxf} \pm(v) i!; v g i b(!) i \quad E^{£} G^{i}!; Y^{2^{\Phi}}{ }_{j} Y^{1}=!^{\infty} f_{Y^{1}}(!) \text { : }
\end{aligned}
\]

Now since
\[
\begin{aligned}
& E^{£} \max { }^{\mathbb{C}}!; \pm(!) i!; Y^{2^{\underline{a}}} j Y^{1}=!^{\alpha}
\end{aligned}
\]
we can rewrite
\[
h^{0}(!)=E^{£} \max f \pm(v) i!; g_{i} \max { }^{@}!; \pm(!) i!; Y^{2^{\underline{a}}} j Y^{1}=!{ }^{\alpha_{Y^{1}}}(!):
\]

Suppose that! >v: Then
\[
\max f \pm(v) i!; v g<\max f \pm(!) i!;!g \quad \max { }^{©}!; \pm(!) i!; Y^{2^{\underline{a}}}:
\]
 \(\max f \pm(!) \mathrm{i}!;!\mathrm{g}, \max !; \neq(!) \mathrm{i}!; \mathrm{Y}^{2}\) since \(\mathrm{Y}^{2} \quad\) ! given that \(\mathrm{Y}^{1}=\) !: Therefore \(h^{0}(!)>0\) : Thus the maximum of \(h\) is achieved at \(!=v\) : QED
De..ning \(X^{1}\) as the largest of the signals \(f v_{j} ; j, 1 g\) and \(X^{2}\) as the second largest of the signals \(f v_{j} ; j, 1 g, X^{3}\) as the third largest of the signals and \(\left.f X^{2} ; X^{3} ; \sharp X^{1}\right) ; X^{1} g^{2}\) as the second highest among \(\left.f X^{2} ; X^{3} ; \sharp X^{1}\right) ; X^{1} g\) the equilibrium prices in each auction are given by, respectively:
\[
\begin{aligned}
& \mathrm{P}^{1}=\mathrm{b}^{\mathrm{i}} \mathrm{X}^{2^{\dagger}} \\
& \left.\mathrm{P}^{2}=\mathrm{f} \mathrm{X}^{2} ; \mathrm{X}^{3} ; \Psi \mathrm{X}^{1}\right) \mathrm{i} \mathrm{X}^{1} g^{2}:
\end{aligned}
\]

If synergies are positive then \(P^{2}=X^{2} \quad \pm^{i} X^{2}{ }^{\Phi}{ }_{i} X^{2}=P^{1}\) : That is the price in the second auction is not greater than the price in the ..rst auction and is in general smaller. Thus, it follows that equilibrium prices - and therefore expected prices - decrease if the synergy is positive. The price remains the same in the absence of synergies. If \(\mathrm{n}=2\) and the synergy is negative then \(P^{1} \quad P^{2}\) : In general, if the synergy is negative the equilibrium price may either increase or decrease, but expected prices increase as we show below. \({ }^{1}\)

The reason for prices to increase is rather intuitive. In each period, bidder's bid their true marginal valuations. In the ..rst period, a player bids the dixerence between his value for the bundle \(( \pm \mathrm{v})\) ) and his value from owning the ..rst object only (v). In the second period, he bids either his value for one object or again his marginal value, depending whether or not he won the ..rst object. Thus, in the presence of positive synergies, we have \(\pm(\mathrm{v}), 2 \mathrm{v}\) (that is \(\pm(\mathrm{v}) \mathrm{i} v, v\) ), which results in decreasing prices; the price in the ..rst period includes a premium for the opportunity to realize the synergies. For example, with three players, a \(5 \%\) constant average synergy (that is, \(\pm(\mathrm{v})=2: 05 \mathrm{v}\) ) implies a \(4.8 \%\) decrease in the expected price in the second period.
\(W\) ith \({ }_{4}\) positive synergy, the expected revenue of the sequential auction is \(E\left[ \pm X^{2}\right]\), which coincides with the expected revenue when the two objects are sold simultaneously as a bundle. In this case, the revenue equivalence theorem holds because the individual with the highest signal receives both objects in either type of auction.

W hen synergies are negative, however, the revenue equivalence theorem fails to hold because the individual with the highest signal may not win the second object - he always wins the ..rst auction. Let us consider an example with negative synergy.

\footnotetext{
\({ }^{1}\) J eitschko and W olfstetter (1998) consider a model of sequential auctions with two bidders. Their valuations for the second object depend on bidders' histories; they are determined by independent draws from a distribution with the same support of the ..rst object valuations, but the actual distribution depend on whether a particular individual won or not the ..rst object. They obtain a similar result that when there are positive synergies (economies of scale in their terminology), expected prices decline. They also show that in the presence of diseconomies of scale, expected prices increase (in English auctions). In section 5 below we present an example with asymmetric players where these price trends may be reversed.
}

Example 1 We suppose \(\sharp x)=x\). It is not strictly in our aypotheses gut it is the "limit" of \(\sharp x)=(1+)\),\(x when, ! 0\). Then \(b(x)=E \quad Y^{2}{ }^{2} Y^{1}=x\). Thus \(E\left[P^{1}\right]=E\left[b\left(X^{2}\right)\right]=E\left[X^{3}\right]\) and \(P^{2}=X^{3}\) : The revenue from selling in bundles is \(E\left[X^{2}\right]\). It is clear that depending on the distribution, \(E\left[X^{2}\right]\) may be either greater or smaller than \(2 \mathrm{E}\left[\mathrm{X}^{3}\right]\).

In a lemma below we show that if the synergy is always negative, equilibrium prices may either increase or decrease depending upon the realization of \(X^{1} ; X^{2} ; X^{3}\). However expected prices always increase.

Theorem 3 If \(\sharp x\) ) ; \(x \quad x\) for every \(x\) then \(E\left[P^{1}\right] \quad E\left[P^{2}\right]\).
Proof. First note that


The last inequality being true since \(\pm^{i} X^{2}{ }^{\Phi} ; X^{2} \quad \min { }^{@} X^{2} ; \pm X^{i}{ }^{\Phi}{ }_{i} X^{1}{ }^{\underline{a}}\) : This ends the proof. QED

\section*{3 The value of an option to buy}

Black and de M eza (1992) consider the case where the value of a second object for a player is equal to a fraction of the value of a ..rst object. They characterize an equilibrium in which expected prices increase in the absence of an option to buy and may decrease when the option is present. M oreover, they show that the option may increase the seller's expected revenue. Here, however, in the presence of positive synergies, the option to buy is never exercised and the seller's expected revenue is not axected by the introduction of an option. The reason is that when synergies are positive, the individual with the highest signal wins both objects in equilibrium so that the option has no value.

\subsection*{3.1 The option to buy if the synergy is positive.}

The model is the same as in the previous section and so is the notation. The distinction is that now the winner of the ..rst auction has the right to buy the second object at the same price paid for the ..rst object. The timing is as follows. E ach bidder submits a bid in the ..rst auction. The winner of the ..rst auction is given an option to buy the second object at the price paid in the ..rst auction. If this option is exercised, there is no second auction. Otherwise, bidders submit bids for the second object and the winner is determined. The next theorem characterizes equilibrium behavior in the presence of positive synergies.

Theorem 4 Suppose the synergy is always positive. Then the equilibrium strategy de..ned in theorem \(1, b(x)=\sharp x) ; x\) is also an equilibrium strategy when
there is an option to buy both objects in the ..rst auction. In equilibrium the option is never exercised.

Proof. We assume that players 2 ; :::; n bid in the ..rst period according to the function \(b(x)=\sharp(x) ; x\). We suppose also that they never exercise their option to buy. Their behavior in the second auction is the same as in theorem 1. We consider the game from Player 1's perspective. Let us ..nd what is Player 1's best response. Let \(v\) denote 1 's value and \(x\) his bid. We can write his expected pro..ts, \(\mathrm{H}(\mathrm{x})\), as follows: \(\mathrm{H}(\mathrm{x})=\)
\[
\begin{aligned}
& E\left[\hat{A}_{x, b\left(Y^{1}\right)} \max { }^{\circledR} \pm(v) i 2 b^{i} Y^{1}{ }^{\Phi} ; \max { }^{\circledR} v ; \pm(v) i Y^{1^{a}} ; b\left(Y^{1}\right)^{\underline{a}}+\right. \\
& \hat{A}_{x<b\left(Y^{1}\right)} i_{i} \max ^{i} \pm^{i} Y^{1}{ }^{\Phi}{ }_{i} Y^{1} ; Y^{2^{\prime}}{ }^{\$ \Phi_{+}} \text {]: }
\end{aligned}
\]

If Player 1 has the highest bid in the ..rst auction and he exercises the option, his pro.ts are \(\pm(\mathrm{v})\); \(2 \mathrm{~b}^{1} \mathrm{Y}^{1}\) : If Player 1 wins the \(e_{a}\).rst object and does not exercise the option, his pro..ts are \(\max v ; \pm v) ; Y^{1} ;{ }^{1}\left(Y^{1}\right)\). He will exercise his option if

However if \(\left.\left.v, \sharp(v) ; Y^{1}, k=\sharp v\right) i v i^{i} \sharp Y^{1}\right) ; Y^{1}{ }^{\dagger} \quad 0\) since from the positive synergy assumption we have that \(Y^{1}, \pm(v) i v, ~ v\). If \(\left.v<\sharp v\right) ; Y^{1}\), \(\left.\mathrm{K}=\mathrm{i}\left( \pm \mathrm{Y}^{1}\right) \mathrm{i} \mathrm{Y}^{1}\right)+\mathrm{Y}^{1} \quad\) 0. Therefore Player 1 never exercises his option. Thus the maximizing problem of Player 1 is the same as in theorem 1 . A nd the solution is therefore the same.

QED
In the presence of positive synergies we have \(\pm(v) i v, \frac{ \pm(v)}{2}\) : Therefore, a player who exercises the option to buy would pay too much for the object. (Recall that expected prices are decreasing in the equilibrium of the sequential auction when the option is not exercised).

\subsection*{3.2 The option to buy when the synergy is negative.}

It happens sometimes in auctions that the option to buy both objects is exercised. We have seen that if the synergy is always positive this will not happen in equilibrium. However Black and M eza have shown that if the average synergy is negative and constant then the option is sometimes exercised. In general, we cannot ..nd a closed form solution for the case of negative synergies. The di¢ culty in solving for equilibrium bids in this case is explained below.

We suppose that \(\sharp x\) ) i \(x \quad x\) for every \(x\). We look for a symmetric equilibrium \(b(\nsubseteq\) such that \(\sharp x) ; x \quad b(x) \quad x\). We follow the same procedure as in proof of theorem 1. Suppose Player 1 bids \(x=b(!)\). If \(x>b\left(Y^{1}\right)\) he will exercise his option \({ }^{2}\) if
\[
\left.k=\sharp v) ; 2 b\left(Y^{1}\right) ;\left[v_{i} b\left(Y^{1}\right)+(\# v) ; v_{i} Y^{1}\right)^{+}\right], 0:
\]

\footnotetext{
\({ }^{2}\) In this section we suppose that the option is exercised if the bidder is indixerent between exercising it or not.
}

If \(\left.Y^{1} \quad \sharp v\right) ; \quad v\) Player 1 exercises his option since \(k=Y^{1} ; \quad b\left(Y^{1}\right), ~ 0\) : If \(\left.\left.Y^{1}>\sharp v\right) ; v, k=\sharp v\right) ; v i b\left(Y^{1}\right)\). Since \(\left.\left.b(\sharp v) ; v\right) \sharp v\right) ; v\) the option will be exercised if and only if \(\# v) ; v, \quad b\left(Y^{1}\right)\). Thus symmetrically the Player with valuation \(Y^{1}\) does not exercise his option if and only if \(\sharp Y^{1}\) ) \(; Y^{1}<\) \(\operatorname{maxf} x ; b\left(Y^{2}\right) \mathrm{g}\). Therefore Player 1 expected utility is \(\mathrm{h}(!)=\)

Z!
\(\left.\left(n_{i} 1\right) \quad \operatorname{maxf} \sharp v\right) ; 2 b(y) ; \max f v ; \sharp(v) ; y g i b(y) g F^{n_{i}^{2}}(y) f(y) d y+\) \({ }^{0}\)


Where \(g\) is the inverse of \(\# y)\); \(y\). Given that \(b(y), ~ \# y)\); \(y\) we can show that there does not exist!, with ! >v; that maximizes \(\mathrm{h}(!)\). Naturally, the maximization of \(h\) is a cumbersome problem that we do not pursue here. See the remark below. Instead, we can show that if there is a symmetric equilibrium with \(b(x) \quad x\) then prices may go up or down. This is the content of the next lemma:

Lemma 5 If there is a symmetric equilibrium with \(b(x) \quad x\) for every \(x\) then in the negative synergy case equilibrium prices may either increase or decrease.

Proof. The ..rst period equilibrium price is \(P^{1}=b\left(X^{2}\right)\). If the option is not exercised then we have that \(\left.\Psi X^{1}\right) ; X^{1}<b\left(X^{2}\right)\). Since second period equilibrium price is \(\left.P^{2}=f X^{2} ; X^{3} ; \sharp X^{1}\right) ; X^{1} g^{2}\) and \(\left.\# X^{1}\right) ; X^{1}<X^{2}\); we have that \(\left.P^{2}=\operatorname{maxf} X^{3} ; \sharp X^{1}\right) ; X^{1} g\). Thus if \(X^{3} 2\left(b\left(X^{2}\right) ; X^{2}\right)\), then \(P^{2}>P^{1}\). Otherwise, the equilibrium prices decrease.

QED
Remark 3 Suppose \(\# x)=(1+)\),\(x with 0<,<1\). A ssuming \(b(\$\) is increasing in \(X\), denoting by \({ }^{3} Y^{1}\) and \(Y^{2}\) the two highest rival values, respectively, and by \(M\left(Y^{1}\right)\) the distribution of \(Y^{1}\), Black and de \(M\) eza provide the ..rst-order condition of a bidder's maximization problem as follows:
\[
\left(b(x) \text { i } E\left[\operatorname{maxf}, x ; Y^{2} g j Y^{1}=x\right]\right) \frac{M(x)}{b^{9}(x)}=
\]
\[
E\left[x i \max \left(Y^{2} ; b(x)\right) j, Y^{1}=b(x) ; Y^{2}<x\right] \operatorname{Pr}^{f} Y^{2}<x j, Y^{1}=b(x)^{x} M\left(\frac{b(x)}{}\right):
\]

For this case, they show that equilibrium prices may either increase or decrease under the option and that the seller's revenue may be larger under the option than under no option.

So far we examined price trends in sequential auctions with symmetric players. A feature of sequential auctions, however, is the existence of asymmetric players. In the sale of wine, for example, restaurant owners and collectors form two distinct groups of players. In the next two sections we provide examples with asymmetric players where again we are able to obtain expected prices that may be decreasing or increasing.

\footnotetext{
\({ }^{3}\) In their's notation \(x_{H}=Y^{1}\) and \(x_{L}=Y^{2}\)
}

\section*{4 A symmetric synergies}

There are two objects and two bidders. The objects are sold sequentially through second-price auctions. Bidder \(i, i=1 ; 2\), values one object at \(v_{i}\), and two objects at \(\pm v_{i}\) : We assume that \(\pm \pm, 2+\max f \pm ; ~ \pm 2 g\) : Suppose the \(v_{i}\) 's are determined by independent draws from the uniformi \([0 ; 1]\) distribution and that each bidder knows only his value. A s before, in the second auction it is a dominant strategy for player \(i\) to bid \(v_{i}\) if he did not win the ..rst object and \(\ddagger v_{i} i v_{i}\) if he won the ..rst object. We now assume that player 2 , who has a value \(y\), follows a strategy \(b_{2}(y)=k_{2} y\) in the ..rst auction and compute player 1's best response. Denote l's bid by \(x\) and his value by \(v\). His expected pro..ts are given by:

We need only to consider \(x \quad k_{2}\) : Taking the expected value we obtain
\[
\begin{aligned}
& 0 \\
& \frac{\times}{K_{2}}
\end{aligned}
\]

Player 1 chooses \(x\) to maximize \(H(x)\) yielding

It su申 ces to consider three cases.
(i) If the two expressions between parentheses are less than zero, then \(\mathrm{v}=\mathrm{x}\).

This will occur if \(k_{2} \quad \operatorname{minf} \frac{1}{\not \mathrm{~m}_{\mathrm{i}} 1} ; ~+2 \mathrm{i}\) :
(ii) If the ..rst expression between parentheses is greater than zero and the second less than zero, then \(x=\frac{\not{ }_{2} k_{2} v}{k_{2}+1}\) : This will occur if \(\frac{1}{\not m_{i} 1} \quad k_{2}\)丸 ( \(\ddagger\); 1) i 1:
(iii) If the two expressions between parentheses are positive, then \(x=\frac{( \pm i 1) k_{2} \mathrm{~V}}{2+\mathrm{k}_{2} \mathrm{i} \text { t2 }}\) :

This holds whenever \(k_{2}, ~ \pm( \pm ; 1) ; 1\) :
Thus the best response to \(b_{2}=k_{2} y\) is \(b_{1}=k_{1} v\) : Let us determine the equilibrium \(\left(k_{1} ; k_{2}\right)\) : Given that we are looking for an equilibrium, we will consider the case where both players are in case (ii) above. Therefore, we obtain \(k_{1}=\frac{\not k_{1}}{k_{2}+1}\)



Equilibrium prices are given by
\[
\begin{aligned}
& \left.p^{(2)}=\operatorname{minf}^{1 / 2} \pm i 1\right) v_{1} ; v_{2} g \text {; if P layer } 1 \text { wins the ..rst object } \\
& \min f( \pm 2 i 1) v_{2} ; v_{1} g \text {; if Player } 2 \text { wins the ..rst object }
\end{aligned}
\]

Now if Player 1 wins the ..rst object, we have
\[
p^{(1)}=\frac{\text { н七 } i 1}{\not \pm+1} v_{2} \text { and } p^{(2)}=\operatorname{minf}( \pm i 1) v_{1} ; v_{2} g
\]
 \(v_{2}\), prices decrease. If \(p^{(2)}=( \pm i 1) v_{1}<v_{2}<\frac{ \pm \nmid ~}{\ddagger+1} v_{2}=p^{(1)}\) : That is, prices also fall.

If player 2 wins the ..rst object we have

A similar analysis demonstrates that equilibrium prices will also fall as long as \(\pm_{2}( \pm\) i 1\()>2\) : It should not be di \(\phi\) cult to provide an example where player 2 has negative synergy, player 1 has positive synergy and expected prices increase.

\section*{5 A symmetric Demands}

The example in \(\mathrm{Branco}(1997)\) is such that Player 1 wants the ..rst object only, Player 2 wants the second object only and Player 3 wants both objects and has positive synergy. Player \(\mathrm{i} ; \mathrm{i}=1 ; 2 ; 3\), receives independently a signal \(\mathrm{x}_{\mathrm{i}}\) from a uniform distribution in the interval \([1 ; 2]\). The value of the object for player \(i, i=1 ; 2\), is simply \(\frac{x_{i}}{2}\). The value of the two objects for player 3 is equal to \(\mathrm{X}_{3}+{ }^{\circledR}\), where \({ }^{\circledR}>0\) is a constant that is known by all players.

As a result of the assumption that Player 1 wants only the ..rst object and that Player 2 wants only the second object, these two players behave as in a single-object second-price auction and bid their valuations. Prices then decrease because player 3 's bidding behavior in the ..rst auction re \(\ddagger\) ects the value of winning the ..rst object for the realization of the positive synergies. Branco argues that this intuition should carry out to a more general model

However, this may not hold in general. W hen players can buy either the ..rst or the second object, those players with single-unit demand do not bid their true valuations in the ..rst period because winning in the ..rst period precludes them from winning the second object for a price that may be inferior. This is demonstrated next.

Theorem 6 Suppose that Player 1 wants both objects and there are synergies from owning the two identical objects. A ssume that Players \(2 ;::: \mathrm{N}\) want only one object. If there is an equilibrium strategy \((b(\phi) ; c(\phi ;::: ; c(\phi)\); then \(c(y)\) y for every \(y\) :

Proof. Let us suppose bidders i = 3;:::; N play c( \(\Phi\) and Bidder 1 plays \(b(d)\) : De.ne \(Z=\max \mathrm{f}_{3} ;::: ; y_{N} g\) and \(Z^{(2)}\) as the second greatest among \(\mathrm{f}_{3} ;::: ; \mathrm{y}_{\mathrm{N}} \mathrm{g}\). If bidder 2 bid x his expected pro..t
\[
\begin{align*}
g(x)= & E\left[\hat{A}_{x, \operatorname{maxf} b\left(y_{1}\right) ; c(Z) g}\left(v_{i} \max f b\left(y_{1}\right) ; c(Z) g\right)+\right.  \tag{1}\\
& \hat{A}_{x<\operatorname{maxf} b\left(y_{1}\right) ; c(Z) g f_{3} \hat{A}_{b\left(y_{1}\right)>\operatorname{maxf} x ; c(Z) g}\left(v_{i}, \max f \pm\left(y_{1}\right) i y_{1} ; Z g\right)^{+}} \quad  \tag{+2}\\
& \left.\hat{A}_{b\left(y_{1}\right)<\operatorname{maxf} x ; c(Z) g} v i \max y_{1} ; Z^{(2)}{ }^{+}\right]: \tag{3}
\end{align*}
\]

Suppose now that \(x>v\) : Then \((2,3)\) are nonnegative and increase if \(x\) decreases. And (1) is negative in the region \(x, \quad \max f b\left(y_{1}\right) ; c(Z) g>v\). Therefore \(g(x)\) increases if \(x\) decreases until \(x=v\). Therefore the optimum \(x=c(v) \quad v: Q E D\)

One can also prove that \(b(v) \quad \pm(v) i v\). The proof will be omittad. Thus for example if \(b\left(y_{1}\right)>c^{i} Y^{1^{4}}\) : In this case \(P^{1}=b\left(v_{1}\right) ; c^{i} Y^{1^{4}} ; c^{i} Y^{2^{Q}}{ }^{2}=c^{i} Y^{1^{4}}\) : Since \(P^{2}= \pm\left(v_{1}\right) i v_{1} ; Y^{1} ; Y^{2}{ }^{2}, P^{1}\) we see that prices may be increasing \({ }^{4}\). Let us calculate as an example the best response of bidder 1 to \(c(y)=y\) : His expected pro..t is
\[
\begin{aligned}
h(x)= & E\left[\hat{A}_{x, \operatorname{maxf} y_{2} ; y_{3} g}{ }^{3} v i \operatorname{maxf} y_{2} ; y_{3} g+\left( \pm(v) i v_{i} \operatorname{maxf} y_{2} ; y_{3} g\right)^{+}+\right. \\
& \left.\hat{A}_{x<\operatorname{maxf} y_{2} ; y_{3} g}\left(v_{i} \min f y_{2} ; y_{3} g\right)^{+}\right]:
\end{aligned}
\]

Or
\[
\begin{aligned}
& Z_{x}{ }^{3} \quad \text { Z Z } \\
& h(x)=v_{i} z+( \pm(v) i v i z)^{+} 2 z d z+2 \quad\left(v i y_{3}\right)^{+} d y_{2} d y_{3}= \\
& Z_{x^{3}}^{0} \quad, \quad Z_{1}^{x<y_{2}} \quad y_{3}<y_{2} \\
& { }_{0} v_{i} z+\left( \pm(v) i_{i} v^{+} 2 z d z+2{ }_{x} G\left(y_{2}\right) d y_{2}:\right. \\
& \text { Here } G\left(y_{2}\right)={ }_{0}^{R_{y_{2}}}\left(\begin{array}{ll}
v_{i} & \left.y_{3}\right)^{+} d y_{3} \text { : Thus }, ~
\end{array}\right. \\
& h^{0}(x)=2 x \vee i x+( \pm(v) ; v i x)^{+} \quad \text { i } 2 G(x)=0 \text { : }
\end{aligned}
\]
\[
\begin{aligned}
& \text { If } x \quad v ; G(x)_{f}=v x ; x^{2}=2 \text { : Thus }
\end{aligned}
\]

Therefore, \(x^{\mathscr{a}}=\frac{2\left( \pm(v)_{i} v\right)}{3} \quad v\) (since \(\pm(v) \quad \frac{5 v}{2}\) ) and \(x^{\mathbb{}}\) is the maximum of \(h\) on [0; v]:

If \(x, \pm(v) ; v\); then \(h^{0}(x)<0\) : Suppose now that \(x 2_{p}(v ; \pm(v) ; v)\) : Then \(h^{0}(x)=2 x( \pm(v)\) i \(2 x)\) i \(v^{2}\) : The solution is now \(x^{0}=\frac{ \pm(v)+\frac{ \pm(v)^{2} ; 4 v^{2}}{4}}{4}\) However

\footnotetext{
\({ }^{4}\) Since \(c(y)=y\) is not an equilibrium in general.
}
\(x^{0}>v\) if and only if \({ }^{9} \overline{ \pm(v)^{2}{ }_{i} 4 v^{2}}>4 v i^{ \pm}(v)\) : This inequality is true if \(\pm(\mathrm{v}), 4 \mathrm{v}\). When \(\pm(\mathrm{v})<4 \mathrm{v}\); the inequality is true if and only if
\(\pm(v)^{2} ; 4 v^{2}>16 v^{2} ; \quad 8 v \pm(v)+ \pm(v)^{2}, \quad 8 v \pm(v)>20 v^{2}, \quad \pm(v)>\frac{5 v}{2}:\)
Thus the optimal bid of player 1 is \(\mathrm{b}(\mathrm{v})=\frac{2( \pm(\mathrm{v}) \text { i } \mathrm{v})}{3} \quad \mathrm{v}\) :
Branco's prediction that expected prices will decrease rely on the assumption that the bidder who only wants the ..rst object and the bidder who only wants the second object - despite the objects being ex-ante identical - will bid their true valuations in the ..rst and second auctions, respectively. At the same time, the bidder who wants the two objects will bid more aggressively in the ..rst auction. However, the above theorem demonstrates that players with singleunit demand will generally not bid their true valuations in the ..rst period when they are allowed to bid for any of the two identical objects. There are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the ..rst auction by the players with single unit demand. That is, there is no clear tendency for a declining price.

\section*{6 Conclusion}

In this paper we examine sequential auctions of identical objects where individuals demand more than one object and there are synergies. We show that the existence of positive synergies implies in declining expected prices. When two objects are worth more as a bundle than as separate objects, whoever buys the ..rst object has the opportunity to realize the synergy. Therefore, the price in the ..rst period includes a premium to rełect such opportunity. In addition, when the synergies are negative, we show that expected prices are increasing.

W hen synergies are positive, we show that 1) the seller's expected revenue is the same under both simultaneous or sequential auctions; 2) an option to buy the additional object at the price of the ..rst object is never exercised in the symmetric equilibrium.

M oreover, in the case of negative synergies, the revenue equivalence theorem does not hold as the individual with the highest signal, who wins the simultaneous auction when the two objects are sold as a bundle, always wins the ..rst auction but may not win the second auction when the objects are sold sequentially. We also examine the exects of introducing an option in the case of negative synergies. We show that if there is an equilibrium where the option is never exercised, then equilibrium prices may either increase or decrease and, therefore, the net exect on the seller's revenue of the introduction of an option is ambiguous. This conforms with the results of Black and de M eza for the case of constant negative synergies.

Finally, we present two examples with asymmetric players. In the ..rst example, players have distinct synergies. In this example, even if one player has positive synergies and the other has negative synergies, it is still possible for expected prices to decline. In the second example, one player wants two ob-
jects and the remaining players want one object each. For this example, we show that expected prices may not necessarily decrease as predicted by Branco (1997). The reason is that players with single-unit demand will generally bid less than their true valuations in the ..rst period. Therefore, there are two opposing forces; the reduction in the bid of the player with multiple-demand in the last auction and less aggressive bidding in the ..rst auction by the players with single-unit demand.

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\section*{A Appendix}

Proof of Lemma (1).
Proof. If \(n=2\) then \(b(x)= \pm(x) ; x\) is strictly increasing by assumption. Suppose now that \(n>2\) and \(x^{0}>x, 0\) : We want to prove that
\(b\left(x^{9}\right) ; b(x)=\)
\[
Z_{x^{0}} \operatorname{maxf} \pm\left(x^{9}\right) i x^{0} ; y g \frac{d^{3} F(y)^{n_{i} 2}}{F(x)^{n_{i} 2}} i Z_{x} \max f \pm(x) i x ; y g \frac{d^{3} F(y)^{n_{i} 2}}{F(x)^{n_{i} 2}}>0
\]

If \(\sharp x) ; x, x^{0}\) then \(\left.b\left(x^{9}\right) ; b(x)= \pm\left(x^{9}\right) ; x^{0} ;((\Psi x) ; x)\right)>0\) : Now suppose \(\left.x^{0}> \pm x\right)\) i \(x, x\) : Then
\[
\begin{aligned}
& \left.b\left(x^{9}\right) ; b(x)>Z_{x^{0}} \operatorname{maxf} \pm(x) i x ; y g \frac{d F(y)^{n_{i} 2}}{F\left(x^{9}\right)^{n_{i} 2}} i(\# x) ; x\right) \text {, } \\
& \left.\left.Z_{x^{0}} \#(x)^{x} y \frac{d F(y)^{n_{i} 2}}{F\left(x^{0}\right)^{n_{i} 2}}+(\# x) i x\right) \frac{F(\# x) i x)^{n_{i} 2}}{F\left(x 9^{n_{i} 2}\right.} i(\# x) i x\right)= \\
& \left.\frac{x^{0} F(x)^{n_{i} 2} i \begin{array}{l}
R_{x^{0}} \\
\#(x) i_{i}
\end{array} F(z)^{n_{i}{ }^{2} d z}}{F(x)^{n_{i} 2}} i(\# x) i x\right)=
\end{aligned}
\]

Finally we consider the case \(\sharp x)\); \(x<x\) : Thus
\[
\begin{aligned}
& b\left(x^{9}\right) ; b(x)>Z_{3}^{Z_{x^{0}}} \operatorname{maxf} \pm(x) ; x ; y g \frac{d F(y)^{n_{i} 2}}{F\left(x 9^{n_{i} 2}\right.} ; b(x), \\
& Z_{x^{0}} \quad y d F(y)^{n_{i} 2} \quad Z_{x} \quad \text { yd } F(y)^{n_{i} 2} \\
& \neq(x)_{i} x \frac{H_{A}}{F(x)^{n_{i} 2}} i x \frac{H^{n}(x)^{n_{i} 2}}{}+ \\
& \left.(\# x) i x) F^{n_{i}^{2}}(\# x) ; x\right) \frac{1}{F\left(x^{q^{n i 2}}\right.} i \frac{1}{F(x)^{n_{i} 2}}=
\end{aligned}
\]

Thus bis strictly increasing.```

