Synthesis of a Class of n-Port Networks

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Abstract—The properties of a class of $2n$-node networks, called K-networks, are discussed. The characteristic of a K-network is that when any one of its ports is connected to a voltage source keeping all the other ports short circuited, then all the short-circuited ports are at the same potential. The $2n$-node network with a pair of equal conductances joining any two ports, as obtained by the presently known procedure for the realization of a dominant conductance matrix, is shown to be a special structure belonging to this general class. It is shown that the realization of a real dominant matrix as the short-circuit conductance matrix Y of an n-port network can be conveniently carried out using K-networks. Further, the "modified cut-set matrix" of a K-network is of a special form, independent of edge conductances. This property can be made use of in generating a range of equivalent $2n$-node n-port networks for a given Y. Examples illustrating the realization procedures are included.

I. Introduction

This paper considers the problem of realization of the short-circuit conductance matrix of a resistive n-port network with $2n$ nodes. The graph of the network is assumed to be complete and edges with zero conductance are permitted. The pair of nodes numbered $2i - 1$ and $2i$ constitutes the ith port.

Given a real dominant matrix $Y = [y_{ij}]$ of order n, it can be realized as the short-circuit admittance matrix of a resistive n-port network with $2n$ nodes by a well-known method.[1] In this realization the network configuration between any two ports i and j is as shown in Fig. 1, with the conductances g given by (1).

$$\begin{aligned}
g_{2i-1,2j} = g_{2i,2j-1} &= 0, && \text{if } y_{ij} \le 0\\
&= 2y_{ij}, && \text{if } y_{ij} > 0\\
g_{2i,2j} = g_{2i-1,2j-1} &= 0, && \text{if } y_{ij} \ge 0\\
&= 2\,|y_{ij}|, && \text{if } y_{ij} < 0\\
g_{2i-1,2i} &= y_{ii} - \sum_{\substack{k=1\\k\ne i}}^{n} |y_{ik}|\\
g_{2j-1,2j} &= y_{jj} - \sum_{\substack{k=1\\k\ne j}}^{n} |y_{jk}|.
\end{aligned} \tag{1}$$

The important features of this realization are as follows.

1) When any port i is excited with a voltage V and all the other ports short circuited, the short-circuited ports are all at the same potential, viz., at a potential of $\frac{1}{2}V$ with respect to the terminal $2i - 1$.

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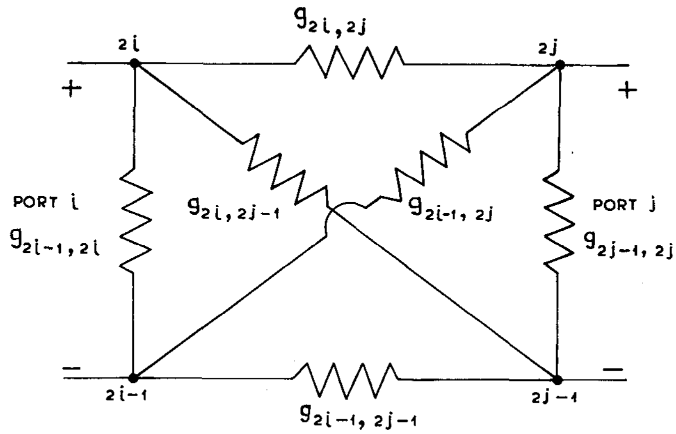

Fig. 1. Circuit used for the standard $2n$-node realization of a dominant matrix.

2) If two real dominant matrices Y_1 and Y_2 are realized as the short-circuit admittance matrices of two networks N_1 and N_2 according to this procedure, then the short-circuit admittance matrix of the parallel combination of N_1 and N_2 is given by $Y_1 + Y_2$. (It is well known that an arbitrary pair of n-port networks may not have this property.)

3) The transfer admittance y_{ij} between ports i and j is dependent only on the conductances of the edges directly joining the terminals of ports i and j.

4) The modified cut-set matrix of the network is independent of the edge conductances.[2],[3]

In this paper, a general class of networks, called K-networks, having the above properties is studied. The generalization consists in stipulating that the potential of all the short-circuited ports under the conditions indicated in 1) be KV where K is an arbitrary constant. The important properties of K-networks are discussed in Section II. The methods of synthesis are included in Section III. Finally, the generation of equivalent resistive networks using Cederbaum's modified cut-set matrix is discussed in Section IV.

II. K-Networks and Their Properties

Consider an n-port network with $2n$ nodes. Let port i be excited with a voltage V and all the other ports be short circuited. If the short-circuited port j is at a potential $K_{ij}V$ with respect to the terminal $2i - 1$, then K_{ij} is referred to as the *potential factor* of port i with respect to port j.[4]

Definition 1

An n-port network with $2n$ nodes, in which each port i ($i = 1, 2, \cdots, n$) is associated with a common potential

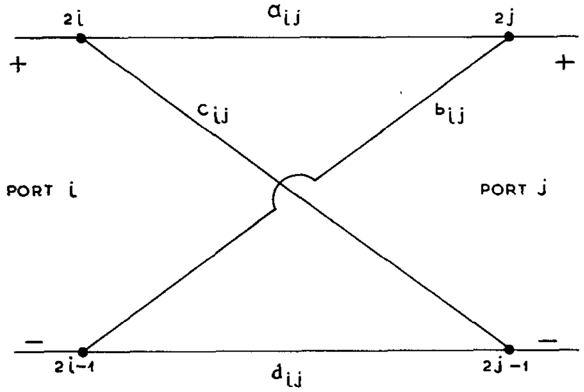

Fig. 2. Conductance values of the edges interconnecting any two ports.

factor K_i such that $K_{ij} = K_i$ for all $j \neq i$, is referred to as a K-network.

Let the network configuration between any two ports i and j be as shown in Fig. 2, where a_{ij}, b_{ij}, c_{ij}, and d_{ij} refer to the conductances of the respective edges. These conductances are finite and assumed to be non-negative. However, the edges shunting the ports are permitted to have conductances of either sign unless the K-network is specified to contain no negative elements.

Theorem 1

The necessary and sufficient condition that a given n-port network with $2n$ nodes be a K-network is that (2) is satisfied for every i and every j not equal to i.

$$K_i = \frac{a_{ij} + c_{ij}}{a_{ij} + b_{ij} + c_{ij} + d_{ij}}. \tag{2}$$

Proof

Necessity: Let port i be excited with voltage V and all the other ports be short circuited. In a K-network, all the short-circuited ports are at the same potential, and hence, the edges interconnecting the terminals of the short-circuited ports do not carry any current. From this it follows that the potential of the short-circuited port j with respect to the terminal $2i - 1$ can be calculated from the circuit in Fig. 2 with nodes $2j$ and $2j - 1$ short circuited. This is easily shown to be $V(a_{ij} + c_{ij})/(a_{ij} + b_{ij} + c_{ij} + d_{ij})$. Hence, $K_i = (a_{ij} + c_{ij})/(a_{ij} + b_{ij} + c_{ij} + d_{ij})$ for $j = 1, 2, \cdots n; j \neq i$. Obviously, such a relation should be valid for every i in a K-network.

Sufficiency: With port i excited with voltage V and all the other ports shorted, remove all the edges interconnecting the short-circuited ports. Then the potential of any short-circuited port j with respect to terminal $2i - 1$ is obtained as $V(a_{ij} + c_{ij})/(a_{ij} + b_{ij} + c_{ij} + d_{ij})$. From hypothesis it follows that all the short-circuited ports, $j = 1, 2, \cdots n; j \neq i$, are at the same potential, viz., $K_i V$ under these conditions. Now let an edge interconnecting any two short-circuited ports j and k be restored to its position. Using Thevenin's theorem it can be seen that no current passes through this edge. (It may be recalled that in the $2n$-node network considered, only edges with finite conductances are permitted.) Hence, the conditions in the rest of the network remain undisturbed and the potentials of the short-circuited ports after the introduction of this edge remain as before, i.e., at the common potential of $K_i V$ with respect to terminal $2i - 1$. Continuing this process, all the edges removed originally can be restored and all the short-circuited ports shown to remain at the same potential $K_i V$. The final stage corresponds to the given network, and hence, the sufficiency condition follows.

Since in a K-network with port i excited and all the other ports shorted the edges interconnecting the shorted ports do not carry any current, it is easy to show that

$$\begin{aligned} y_{ij} &= \frac{b_{ij}c_{ij} - a_{ij}d_{ij}}{a_{ij} + b_{ij} + c_{ij} + d_{ij}} \\ &= c_{ij}(1 - K_i) - K_i d_{ij} \\ &= K_i b_{ij} - a_{ij}(1 - K_i) \end{aligned} \tag{3}$$

and

$$y_{ii} = \sum_{\substack{j=1 \\ j \neq i}}^{n} (y_{ii})_j + g_{2i-1,2i} \tag{4}$$

where

$$\begin{aligned} (y_{ii})_j &= \frac{(a_{ij} + c_{ij})(b_{ij} + d_{ij})}{a_{ij} + b_{ij} + c_{ij} + d_{ij}} \\ &= K_i(b_{ij} + d_{ij}) \\ &= (a_{ij} + c_{ij})(1 - K_i). \end{aligned} \tag{5}$$

$(y_{ii})_j$ may be considered as the contribution to y_{ii} due to the conductances interconnecting the terminals of ports i and j.

Some of the important properties of K-networks are now considered.

Property 1

The potential factor K_i satisfies the inequality, $0 < K_i < 1$.

This follows from the consideration that there can be no voltage magnification in a resistive network with non-negative element values. Further, if $K = 0$ or $K = 1$, one or more conductances should be infinite and this possibility is precluded in the $2n$-node n-port network considered.

Property 2

Consider any two $2n$-node K-networks N_1 and N_2 with the same set of potential factors (i.e., the potential factors of port i in the two networks are the same for every i) and having Y_1 and Y_2 as the short-circuit admittance matrices. When connected in parallel, N_1 and N_2 yield a resulting network N_3 having the following properties.

1) N_3 is a K-network and the potential factor of any one of its ports is the same as that of the corresponding port in N_1 and N_2.

2) The short-circuit admittance matrix of N_3 is $Y_1 + Y_2$.

Proof: When the networks are paralleled, each edge of N_1 is in parallel with the corresponding edge of N_2. Let the unprimed quantities in the following refer to N_1 and the primed quantities to N_2.

1) For the common potential factor K_i of port i in N_1 and N_2, we have

$$K_i = \frac{a_{ij} + c_{ij}}{a_{ij} + b_{ij} + c_{ij} + d_{ij}} = \frac{a'_{ij} + c'_{ij}}{a'_{ij} + b'_{ij} + c'_{ij} + d'_{ij}}; \quad \begin{array}{l} j = 1, \cdots, n \\ j \neq i. \end{array} \tag{6}$$

For the parallel combination,

$$\begin{aligned} &\frac{(a_{ij} + a'_{ij}) + (c_{ij} + c'_{ij})}{(a_{ij} + a'_{ij}) + (b_{ij} + b'_{ij}) + (c_{ij} + c'_{ij}) + (d_{ij} + d'_{ij})} \\ &= \frac{K_i(a_{ij} + b_{ij} + c_{ij} + d_{ij}) + K_i(a'_{ij} + b'_{ij} + c'_{ij} + d'_{ij})}{(a_{ij} + b_{ij} + c_{ij} + d_{ij}) + (a'_{ij} + b'_{ij} + c'_{ij} + d'_{ij})} \\ &= K_i \text{ for all } i \text{ and every } j \text{ not equal to } i. \end{aligned} \tag{7}$$

By Theorem 1, this is necessary and sufficient for N_3 to be a K-network. Furthermore, the potential factor of port i in N_3 is seen to be K_i.

2) From (3), the transfer admittance y''_{ij} between ports i and j of the resulting K-network is given by

$$\begin{aligned} y''_{ij} &= (c_{ij} + c'_{ij})(1 - K_i) - K_i(d_{ij} + d'_{ij}) \\ &= c_{ij}(1 - K_i) - K_i d_{ij} + c'_{ij}(1 - K_i) - K_i d'_{ij} \\ &= y_{ij} + y'_{ij}. \end{aligned} \tag{8}$$

The driving-point admittance y''_{ii} of port i of the parallel combination is given from (4) and (5) by

$$\begin{aligned} y''_{ii} &= g_{2i-1,2i} \\ &\quad + g'_{2i-1,2i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} (a_{ij} + c_{ij} + a'_{ij} + c'_{ij})(1 - K_i) \\ &= g_{2i-1,2i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} (a_{ij} + c_{ij})(1 - K_i) \\ &\quad + g'_{2i-1,2i} + \sum_{\substack{j=1 \\ j\neq i}}^{n} (a'_{ij} + c'_{ij})(1 - K_i) \\ &= y_{ii} + y'_{ii}. \end{aligned} \tag{9}$$

From (8) and (9), the second part of Property 2 follows.

Property 3

The short-circuit admittance matrix of a $2n$-node K-network with non-negative conductances is dominant.

Proof: It is enough if it is proved that $(y_{ii})_j \geq |y_{ij}|$ for all i and j, since it would then follow from (4) that

$$y_{ii} \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |y_{ij}|.$$

Case 1: $y_{ij} > 0$. From (3) and (5),

$$\begin{aligned} (y_{ii})_j - |y_{ij}| &= (y_{ii})_j - y_{ij} \\ &= (a_{ij} + c_{ij})(1 - K_i) - c_{ij}(1 - K_i) + K_i d_{ij} \\ &= a_{ij}(1 - K_i) + K_i d_{ij} \geq 0, \\ &\qquad \text{since} \quad 0 < K_i < 1. \end{aligned} \tag{10}$$

Case 2: $y_{ij} < 0$. From (3) and (5),

$$\begin{aligned} (y_{ii})_j - |y_{ij}| &= (y_{ii})_j + y_{ij} \\ &= (a_{ij} + c_{ij})(1 - K_i) - a_{ij}(1 - K_i) + K_i b_{ij} \\ &= K_i b_{ij} + c_{ij}(1 - K_i) \geq 0, \\ &\qquad \text{since} \quad 0 < K_i < 1. \end{aligned} \tag{11}$$

From (10) and (11), Property 3 follows.

Definition 2

A K-network in which all the ports are associated with the same potential factor K is called a *constant-K network*.

Lemma 1

In a constant-K network, $b_{ij} = c_{ij}$ for all i and j.

Proof: Since $K_i = K_j$ in a constant-K network, it follows from (2) that $b_{ij} = c_{ij}$.

III. Realization of Short-Circuit Admittance Matrices by K-Networks

Definition 3

A real matrix $Y = [y_{ij}]$ is *marginally dominant* if

$$y_{ii} = \sum_{\substack{j=1 \\ j\neq i}}^{n} |y_{ij}| \qquad \text{for all} \quad i.$$

Theorem 2

If a real marginally dominant short-circuit admittance matrix Y with all off-diagonal entries positive is to be realized by a K-network with non-negative elements, the realization is possible only with a constant-K network with $K = \frac{1}{2}$.

Proof: Consider any two ports i and j. Since Y is marginally dominant and since from (10), the quantity $[(y_{ii})_j - y_{ij}]$ can not be negative, it is required that

$$\begin{aligned} (y_{ii})_j - y_{ij} &= 0 \\ \text{i.e.,} \quad a_{ij}(1 - K_i) + K_i d_{ij} &= 0. \end{aligned} \tag{12}$$

Equation (12) is satisfied for the following combinations of values:

i) $a_{ij} = d_{ij} = 0$;

ii) $K_i = 1, \quad d_{ij} = 0$;

iii) $K_i = 0, \quad a_{ij} = 0$.

However, combinations ii) and iii) require that $y_{ij} = 0$. Hence, for $y_{ij} > 0$, (12) is satisfied only for combination i). Therefore,

$$K_i = \frac{c_{ij}}{b_{ij} + c_{ij}} \quad \text{and} \quad K_j = \frac{b_{ij}}{b_{ij} + c_{ij}}. \tag{13}$$

It therefore follows that $K_i + K_j = 1$. Considering any other port m, it can be similarly shown that

$$K_m + K_j = 1 \quad \text{and} \quad K_m + K_i = 1. \tag{14}$$

Equations (13) and (14) lead to $K_i = K_j = K_m = \frac{1}{2}$. Generalizing this result,

$$K_i = \tfrac{1}{2} \quad \text{for all} \quad i = 1, 2, \cdots, n. \tag{15}$$

Thus, a realization is possible only with constant-K networks with $K = \frac{1}{2}$. It is seen that this network is the same as that obtained by the standard $2n$-node realization procedure previously given.[1]

Theorem 3

If a real marginally dominant short-circuit admittance matrix Y with all off-diagonal entries negative is to be realized by a K-network with non-negative elements, a realization is possible only with a constant-K network but with any value of K such that $0 < K < 1$.

Proof: A marginally dominant matrix Y with negative y_{ij}'s requires that

$$\begin{aligned} (y_{ii})_j + y_{ij} &= 0 \qquad \text{for all} \quad i \quad \text{and} \quad \text{all} \quad j \neq i \\ \text{i.e.,} \quad c_{ij}(1 - K_i) + K_i b_{ij} &= 0. \end{aligned} \tag{16}$$

Equation (16) is satisfied only for the following combinations:

i) $b_{ij} = c_{ij} = 0$;

ii) $K_i = 1, \quad b_{ij} = 0$;

iii) $K_i = 0, \quad c_{ij} = 0$.

However, combinations ii) and iii) require that $y_{ij} = 0$. Hence, for $y_{ij} < 0$, (16) is satisfied only with $b_{ij} = c_{ij} = 0$. We then have

$$K_i = \frac{a_{ij}}{a_{ij} + d_{ij}} = K_j \qquad \text{for all} \quad i \quad \text{and} \quad j \neq i. \tag{17}$$

Thus, $K_1 = K_2 = \cdots = K_n = K$. Any value of K such that $0 < K < 1$ may be chosen for the constant-K network realization. From (3) and combination i), it follows that

$$a_{ij} = \frac{|y_{ij}|}{1 - K} \quad \text{and} \quad d_{ij} = \frac{|y_{ij}|}{K}. \tag{18}$$

It is interesting to note that the choice of $K = 0$ or $K = 1$ under the conditions of Theorem 3 leads to the standard realization of a $n + 1$ node n-port network with a common terminal for all the ports.

In a general case, if a marginally dominant matrix can be made to have the following sign pattern by rearranging the rows and columns, then it can be shown that a realization is possible choosing any K, $0 < K < 1$, as the common potential factor for the first n_1 ports and $1 - K$ as the potential factor for the remaining n_2 ports. Where this is not possible and where a constant-K network is required, the choice is limited to $K = \frac{1}{2}$ for all the ports.

$$\begin{array}{cc} & \begin{array}{cc} \leftarrow n_1 \text{ columns} \rightarrow & \leftarrow n_2 \text{ columns} \rightarrow \end{array} \\ \begin{array}{c} \uparrow \\ \\ n_1 \text{ rows} \\ \\ \downarrow \\ \uparrow \\ n_2 \text{ rows} \\ \\ \downarrow \end{array} & \left[\begin{array}{c|c} + - - \cdots - - & + + + \cdots + + \\ - + - \cdots - - & + + + \cdots + + \\ - - + \cdots - - & + + + \cdots + + \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ - - - \cdots + - & + + + \cdots + + \\ - - - \cdots - + & + + + \cdots + + \\ \hline + + + \cdots + + & + - - \cdots - - \\ + + + \cdots + + & - + - \cdots - - \\ \cdots\cdots\cdots\cdots & \cdots\cdots\cdots\cdots \\ + + + \cdots + + & - - - \cdots + - \\ + + + \cdots + + & - - - \cdots - + \end{array}\right] \end{array}$$

From the foregoing discussion it is seen that for a marginally dominant matrix, the network between any two ports has essentially the same form as in Fig. 1 in that either b_{ij} and c_{ij} or a_{ij} and d_{ij} are zero. However, new types of realization are possible when the Y matrix is not marginally dominant.

Definition 4

A matrix Y is said to be superdominant if

$$y_{ii} > \sum_{\substack{j=1 \\ j \neq i}}^{n} |y_{ij}| \text{ for all } i.$$

Theorem 4

A superdominant matrix Y with any arbitrary sign pattern for the off-diagonal entries can be realized by a constant-K network with any K within a certain range of values.

Proof: For a constant-K network, $b_{ij} = c_{ij}$ for all i and j (Lemma 1). Let K be the common potential factor of the network. We wish to prove the theorem by giving a realization procedure and identifying the permissible range of values for K.

If y_{ij} is negative, then b_{ij} and c_{ij} are taken as zero so that $(y_{ii})_j - |y_{ij}| = 0$ for all values of K. a_{ij} and d_{ij} are then given by (18).

If y_{ij} is positive, then from (10)

$$\begin{aligned} (y_{ii})_j - y_{ij} &= a_{ij}(1 - K) + d_{ij}K \\ &= b_{ij}(2K - 1) + 2K\, d_{ij} \end{aligned} \tag{19}$$

for a constant-K network.

Type A Realization: Choose $d_{ij} = 0$ when $y_{ij} > 0$.

Then

$$(y_{ii})_j - y_{ij} = (1-K)a_{ij} = (2K-1)b_{ij} \tag{20}$$

and from (3),

$$y_{ij} = b_{ij}(1-K) \quad \text{since} \quad b_{ij} = c_{ij}. \tag{21}$$

Equations (20) and (21) lead to

$$\begin{aligned} (y_{ii})_j - y_{ij} &= y_{ij}\frac{(2K-1)}{(1-K)}; \\ a_{ij} = y_{ij}\frac{(2K-1)}{(1-K)^2}; &\quad b_{ij} = c_{ij} = \frac{y_{ij}}{(1-K)}. \end{aligned} \tag{22}$$

It is seen that $K \geq \frac{1}{2}$ for a proper realization. Let

$$\Delta_{ii} = y_{ii} - \sum_{\substack{j=1\\ j\neq i}}^{n} |y_{ij}|. \tag{23}$$

Now consider the realization of the admittances in the ith row of Y. If K is the potential factor of the n-port, then

$$\sum_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} [(y_{ii})_j - y_{ij}] = \frac{(2K-1)}{(1-K)} \sum_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} y_{ij} \leq \Delta_{ii},$$

in order that $g_{2i-1,2i}$ be non-negative. Therefore,

$$\frac{\Delta_{ii}}{\sum\limits_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} y_{ij}} = E_i \geq \frac{(2K-1)}{(1-K)}$$

or

$$K \leq \frac{E_i + 1}{E_i + 2}. \tag{24}$$

Hence, for the realization of admittances in this row,

$$\tfrac{1}{2} \leq K \leq \frac{E_i+1}{E_i+2}.$$

Similarly, the permissible range of K can be calculated for all the other rows. If the minimum value of $\{E_i\}$, $i = 1, 2, \cdots, n$, is $E_{\min}$, then K can have any value in the following interval.

$$\tfrac{1}{2} \leq K \leq \frac{E_{\min}+1}{E_{\min}+2} = K_{\max}. \tag{25}$$

Type B Realization: This corresponds to the choice of $a_{ij} = 0$ when $y_{ij} > 0$ and is applicable for values of K less than $\frac{1}{2}$. Proceeding as before, it can be shown that

$$\begin{aligned} (y_{ii})_j - y_{ij} &= y_{ij}\frac{(1-2K)}{K}; \\ b_{ij} = c_{ij} = \frac{y_{ij}}{K}; &\quad d_{ij} = y_{ij}\frac{(1-2K)}{K^2} \end{aligned} \tag{26}$$

and that for a proper realization of the admittances in the ith row,

$$\frac{(1-2K)}{K} \sum_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} y_{ij} \leq \Delta_{ii}$$

leading to the requirement

$$E_i = \frac{\Delta_{ii}}{\sum\limits_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} y_{ij}} \geq \frac{(1-2K)}{K}$$

or

$$K \geq \frac{1}{E_i + 2}. \tag{27}$$

If $E_{\min}$ is the smallest value of $\{E_i\}$, $i = 1, 2, \cdots, n$, then K can have any value in the interval $[\frac{1}{2}, (1/E_{\min}+2)]$ for proper realization.

It is thus demonstrated that for a superdominant matrix with any sign pattern for the off-diagonal entries, a proper realization with a constant-K network is possible with a value of K satisfying the relation,

$$K_{\max} = \frac{E_{\min}+1}{E_{\min}+2} \geq K \geq \frac{1}{E_{\min}+2} = K_{\min}. \tag{28}$$

Realization Procedure for a Superdominant Matrix

1) Determine $E_{\min}$ and choose any value of K in the interval

$$\left[\frac{1}{E_{\min}+2}, \frac{E_{\min}+1}{E_{\min}+2}\right].$$

2) If y_{ij} is negative, then

$$b_{ij} = c_{ij} = 0; \qquad a_{ij} = \frac{|y_{ij}|}{1-K}; \qquad d_{ij} = \frac{|y_{ij}|}{K}.$$

3) If y_{ij} is positive, use Type A realization if $K \geq \frac{1}{2}$ and Type B realization if $K \leq \frac{1}{2}$. Adopt the following values.

	Type A	Type B
a_{ij}	$\frac{(2K-1)}{(1-K)^2} y_{ij}$	0
$b_{ij} = c_{ij}$	$\frac{y_{ij}}{(1-K)}$	$\frac{y_{ij}}{K}$
d_{ij}	0	$\frac{(1-2K)}{K^2} y_{ij}$

4) Conductances shunting the ports:

$$g_{2i-1,2i} = \begin{cases} \Delta_{ii} - \dfrac{(2K-1)}{(1-K)} \sum\limits_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} y_{ij} & \text{for Type A realization} \\ \Delta_{ii} - \dfrac{(1-2K)}{K} \sum\limits_{\substack{\text{all } j\\ \text{where}\\ y_{ij}>0}} y_{ij} & \text{for Type B realization.} \end{cases}$$

It may be noted that for every potential factor K in the interval $[\frac{1}{2}, K_{\max}]$ there exists a potential factor $(1-K)$ in the interval $[K_{\min}, \frac{1}{2}]$ such that a Type A realization with a potential factor K is the same as the Type B realization with the potential factor $(1 - K)$ except for a reversal of the polarities of the ports.

Example 1

The following short-circuit conductance matrix of a 4-port network is to be realized.

$$\begin{bmatrix} 10 & -4 & 2 & 1 \\ -4 & 12 & -3 & 4 \\ 2 & -3 & 8 & -2 \\ 1 & 4 & -2 & 9 \end{bmatrix}.$$

For the foregoing matrix, $E_1 = 1$, $E_2 = \frac{1}{4}$, $E_3 = \frac{1}{2}$, and $E_4 = \frac{2}{5}$. Therefore, $E_{\min} = \frac{1}{4}$, and the permissible range of K is given by

$$\frac{1}{2 + E_{\min}} = \frac{4}{9} \leq K \leq \frac{E_{\min} + 1}{E_{\min} + 2} = \frac{5}{9}.$$

Choose $K = \frac{5}{9}$ for Type A realization. Using (18) and (22), we obtain the network shown in Fig. 3, where the conductances in mhos are marked near the respective edges.

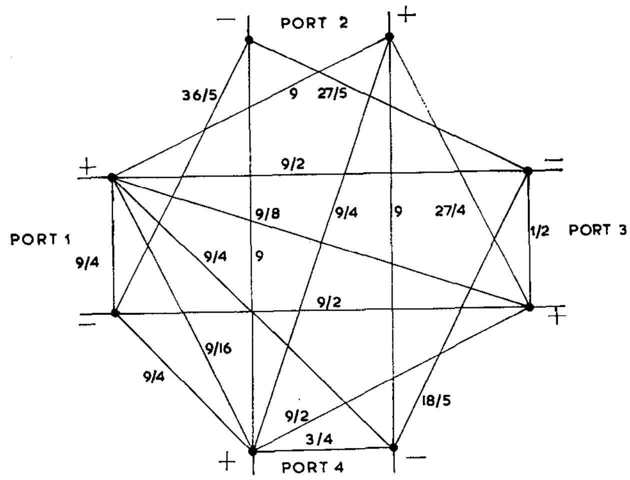

Fig. 3. Circuit realized in Example 1.

Example of Synthesis of a Nonconstant Type K-Network

In case the given short-circuit admittance matrix is neither marginally dominant nor superdominant, the realizations indicated by Theorems 2, 3, and 4 can not be directly used. However, the results obtained in the proofs of these theorems can be made use of in obtaining a K-network realization (not necessarily a constant-K type) of such matrices. The following example illustrates the techniques that may be used.

Example 2

$$Y = \begin{bmatrix} 7 & -3 & 2 & 1 \\ -3 & 6 & 2 & 1 \\ 2 & 2 & 6 & 1 \\ 1 & 1 & 1 & 6 \end{bmatrix}.$$

This matrix is not superdominant, since the second row is marginally dominant. In this row, y_{23} and y_{24} are positive but y_{21} is negative. Equations (13) and (17) can be applied here and the following relations obtained.

$$K_2 + K_3 = 1; \quad K_2 + K_4 = 1; \quad K_1 = K_2.$$

We may therefore choose a potential factor K for ports 1 and 2 and a potential factor $(1 - K)$ for ports 3 and 4. The realization of the transfer admittances in the second row may be done under the constraint $(y_{22})_i - |y_{2i}| = 0$, making use of the following relations:

$$a_{2i} = d_{2i} = 0; \quad b_{2i} = y_{2i}/K;$$

$$c_{2i} = y_{2i}/(1 - K) \quad \text{for} \quad y_{2i} > 0$$

and

$$b_{2i} = c_{2i} = 0; \quad a_{2i} = |y_{2i}|/(1 - K);$$

$$d_{2i} = |y_{2i}|/K \quad \text{for} \quad y_{2i} < 0.$$

The transfer admittances in row 1 can also be similarly realized.

It is then seen that for port 3,

$$(y_{33})_1 = y_{31} \quad \text{and} \quad (y_{33})_2 = y_{32}.$$

Similarly for port 4,

$$(y_{44})_1 = y_{41} \quad \text{and} \quad (y_{44})_2 = y_{42}.$$

The choice of the common potential factor of ports 3 and 4 may now be made so as to eliminate the shunt conductance at one of these ports. Since $(\Delta_{33}/y_{34}) < (\Delta_{44}/y_{34})$, the choice of

$$K_3 = K_4 = \frac{(\Delta_{33}/y_{34}) + 1}{(\Delta_{33}/y_{34}) + 2}$$

is appropriate. Therefore,

$$K_3 = K_4 = \tfrac{2}{3}; \quad K_1 = K_2 = \tfrac{1}{3}.$$

Using the above sets of potential factors, the K-network shown in Fig. 4 is obtained. It may be noted that the network has the same number of elements as the standard realization (constant-K network with $K = \frac{1}{2}$) according to the scheme in Fig. 1.

IV. Generation of Equivalent Networks

Cederbaum has given a procedure for generating equivalent n-port networks from a given realization, making use of the modified cut-set matrix.[2] It was also reported by the authors[3] that this method can be readily used only when the modified cut-set matrices of the original and the equivalent networks are the same. In this section it is shown that all K-networks having a specified set of potential factors have the same modified cut-set matrix independent of edge conductances. Hence, Cederbaum's

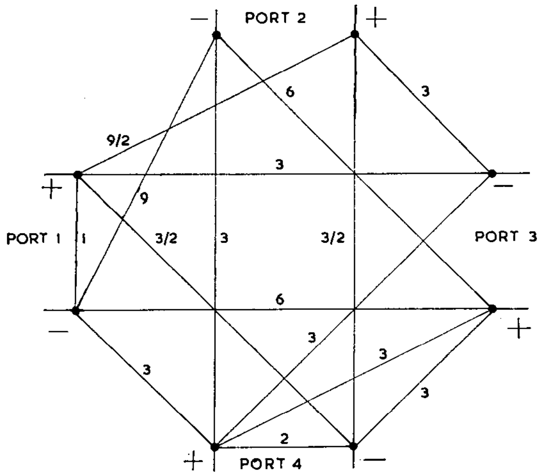

Fig. 4. Circuit realized in Example 2.

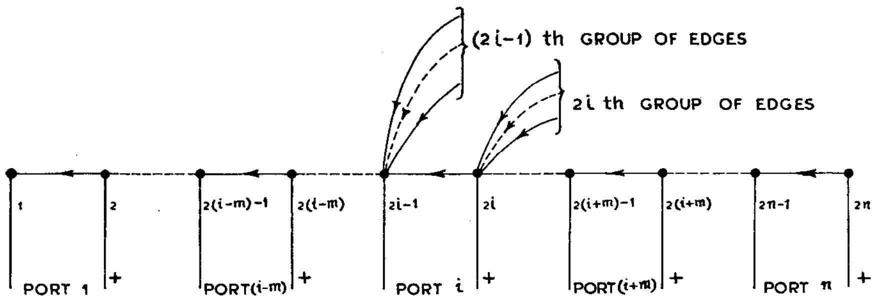

Fig. 5. Edge and port orientations.

procedure may be conveniently used to generate equivalent K-networks.

Consider a linear tree of the $2n$-node network with the nodes numbered in serial order starting from one end vertex of the linear tree so that the nodes numbered $2i - 1$ and $2i$ constitute the port i (see Fig. 5). Let any edge e_{ij} (with $j > i$) joining nodes i and j and having conductance g_{ij} be oriented away from j. Let the edges be so grouped that the ith group consists of all the edges e_{ij} with $j > i$, the mth edge in the ith group being $e_{i,i+m}$. Consider next the fundamental cut-set matrix C_0 of the network with respect to the foregoing tree. The rows are arranged so that the first n rows correspond to the branches of the tree identified as the ports, i.e., $e_{12}, e_{34}, \cdots, e_{2n-1,2n}$ in that order, and the next $(n - 1)$ rows correspond to the remaining branches of the tree, i.e., $e_{23}, e_{45}, \cdots, e_{2n-2,2n-1}$ in that order. The columns are arranged so that the first $(2n - 1)$ columns correspond to the first group of edges, i.e., $e_{12}, e_{13}, \cdots e_{1,2n}$; the next $(2n - 2)$ columns correspond to the second group of edges, i.e., $e_{23}, e_{24}, \cdots e_{2,2n}$ and so on. Then C_0 can be partitioned as follows.

Let G be the diagonal matrix of edge conductances with identical column ordering as that of the fundamental cut-set matrix C_0. Let

$$C_0\, G\, C_0' = \left[\begin{array}{c|c} C_1\, G\, C_1' & C_1\, G\, C_2' \\ \hline C_2\, G\, C_1' & C_2\, G\, C_2' \end{array}\right] = \left[\begin{array}{c|c} Y_{11} & Y_{12} \\ \hline Y_{21} & Y_{22} \end{array}\right]. \tag{30}$$

The short-circuit admittance matrix Y of the n-port network is then given by

$$\begin{aligned} Y &= Y_{11} - Y_{12}Y_{22}^{-1}Y_{21} \\ &= (C_1 - Y_{12}Y_{22}^{-1}C_2)G(C_1 - Y_{12}Y_{22}^{-1}C_2)' \\ &= C\, G\, C' \end{aligned} \tag{31}$$

where

$$C = C_1 - Y_{12}Y_{22}^{-1}C_2 \tag{32}$$

and is termed as the modified cut-set matrix with respect to the n accessible ports; it is, in general, dependent on both the network geometry and the edge conductances g_{ij}.

$$C_0 = \begin{bmatrix} C_1 \\ \hline C_2 \end{bmatrix} = \begin{array}{c} \begin{array}{cccc} \leftarrow 2n-1 \text{ col.} \rightarrow & \leftarrow 2n-2 \text{ cols.} \rightarrow & & \leftarrow 1 \text{ col.} \rightarrow \end{array} \\ \left[\begin{array}{c|c|c|c} C_{1,1} & C_{1,2} & \cdots & C_{1,2n-1} \\ \hline C_{2,1} & C_{2,2} & \cdots & C_{2,2n-1} \end{array}\right] \begin{array}{l} \updownarrow n \text{ rows} \\ \updownarrow n-1 \text{ rows.} \end{array} \end{array} \tag{29}$$

Now let the columns of the modified cut-set matrix C be partitioned in the same way as for C_0, so that

$$C = [C^1 \mid C^2 \mid \cdots \mid C^{2i-1} \mid C^{2i} \mid \cdots \mid C^{2n-1}] \tag{33}$$

where the submatrices C^{2i-1} and C^{2i} correspond to the $(2i - 1)$th and $2i$th groups of edges.

Theorem 5

The necessary and sufficient condition that a $2n$-node n-port network be a K-network is that its modified cut-set matrix be of the form specified by (34).

From the foregoing results and the interpretation of the entries of C as the pertinent potentials according to (35), it follows that the submatrices C^{2i-1} and C^{2i} should have the form in (34) for a K-network.

Sufficiency: We consider the entries in the ith row of the modified cut-set matrix, which has the form given by (34) and observe the following.

i) The entry corresponding to the edge $e_{2(i-m)-1,2i-1}$, i.e., in the $2m$th column of the $(2i - 2m - 1)$th group is $-K_i$ for $m = 1, 2, \cdots, i - 1$.

ii) The entry corresponding to the edge $e_{2i-1,2(i+m)-1}$ is K_i for $m = 1, 2, \cdots, (n - i)$.

$$[C^{2i-1} \mid C^{2i}]$$
$$= \left[\begin{array}{cccccc|cccccc} 0 & 0 & 0 & 0 \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 1 & K_i & K_i & K_i \cdots & K_i & K_i & K_i - 1 & K_i - 1 & K_i - 1 & \cdots & K_i - 1 & K_i - 1 \\ 0 & -K_{i+1} & 1 - K_{i+1} & 0 \cdots & 0 & 0 & -K_{i+1} & 1 - K_{i+1} & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 \cdots & -K_n & 1 - K_n & 0 & 0 & 0 & \cdots & -K_n & 1 - K_n \end{array}\right] \begin{array}{l} \uparrow \\ (i-1) \text{ rows} \\ \downarrow \\ \uparrow \\ (n-i+1) \\ \text{rows.} \\ \downarrow \end{array} \tag{34}$$

Proof

Necessity: Cederbaum[2] has shown that

$$V_e = C'V_p \tag{35}$$

where V_e represents the column vector of edge voltages and V_p represents the column vector of port voltages. It is clear from (35) that the entry in C in the rth row and the column corresponding to the edge e_{pq} is equal to the voltage appearing across the edge e_{pq} when port r is excited with a source of unit voltage and all the other ports short circuited. In a K-network the potential across any edge under these conditions can be determined readily, as all nodes in the network except $(2r - 1)$ and $2r$ assume a common potential of K_r with respect to the node $(2r - 1)$. We now wish to determine the voltages across the $(2i - 1)$th and the $2i$th groups of edges for these conditions. We consider three cases separately.

Case 1: $r = i - m$; $m = 1, 2, \cdots, i - 1$. Upon reference to Fig. 5, it is clear that the voltages across every edge of the $(2i - 1)$th group and the $2i$th groups is zero.

Case 2: $r = i$. The port of excitation coincides with port i. The voltage across the edge $e_{2i-1,2i}$ shunting the port is unity; all other edges in the $(2i - 1)$th group have an equal voltage of K_i. Every edge in the $2i$th group has a voltage of $-(1 - K_i)$ taking its orientation into account.

Case 3: $r = i + m$; $m = 1, 2, \cdots, n - i$. It is evident from Fig. 5 that the voltage across the edges $e_{2i-1,2i+2m-1}$ and $e_{2i-1,2i+2m}$ are $-K_{i+m}$ and $(1 - K_{i+m})$, respectively. The voltage across every other edge in the $(2i - 1)$th group is zero. Similarly, the voltages across the edges $e_{2i,2i+2m-1}$ and $e_{2i,2i+2m}$ are $-K_{i+m}$ and $(1 - K_{i+m})$, respectively. Every other edge in the $2i$th group has zero potential across it.

Taking the edge orientations into account, the interpretation of statement i) is that when port i is excited with a source of unit voltage and all the other ports short circuited, the potential of port j, $j < i$, is K_i with respect to the terminal $2i - 1$. Similarly, statement ii) implies that the potential of every port j, $j > i$, for the same conditions is K_i with respect to the terminal $2i - 1$. Thus, all the short-circuited ports are at a common potential. Since this is true for a general index i, an n-port network having the modified cut-set matrix according to (34) is a K-network.

The generation of an equivalent network using the modified cut-set matrix C is next considered. Let G_1 be the diagonal matrix of edge conductances of an n-port K-network, N_1, which has Y as its short-circuit conductance matrix. Then

$$CG_1C' = Y. \tag{36}$$

Now consider a second K-network N_2 with identical potential factors and therefore having the same modified cut-set matrix C. Let G_2, the diagonal conductance matrix of N_2, satisfy the relation,

$$CG_2C' = 0. \tag{37}$$

If the two networks N_1 and N_2 are connected in parallel, then the resulting network is also a similar K-network as a consequence of Property 2 of Section II. That this combined network is a new realization of Y can be seen by adding (36) and (37), and recognizing that C continues to be the modified cut-set matrix of the combined K-network. In finding a suitable N_2 it should be ensured that its edge conductances not only satisfy (37) but also meet the requirement that N_2 is a K-network having the same set of potential factors as N_1. These requirements, in essence, imply that the network N_2 has the edge conductances between any pair of ports i and j as given in (38), where x_{ij} is an arbitrary parameter. This parameter may be chosen differently for each pair of ports and may be zero for some.

$$\begin{aligned}(g_2)_{2i,2j} &= x_{ij}\\ (g_2)_{2i-1,2j} &= x_{ij}(1 - K_i)/K_i\\ (g_2)_{2i,2j-1} &= x_{ij}(1 - K_j)/K_j\\ (g_2)_{2i-1,2j-1} &= x_{ij}(1 - K_i)(1 - K_j)/K_iK_j\\ (g_2)_{2i-1,2i} &= -\sum_{\substack{j=1\\ j\neq i}}^{n} x_{ij}(1 - K_i)/K_i\\ (g_2)_{2j-1,2j} &= -\sum_{\substack{i=1\\ i\neq j}}^{n} x_{ij}(1 - K_j)/K_j.\end{aligned} \tag{38}$$

Using the foregoing set of equations, a variety of equivalent networks can be readily obtained.

Example 3

Let an equivalent network realizing the matrix Y of Example 2 be required, avoiding the conductance shunting the port 1.

This can be done by connecting in parallel to the network in Fig. 4 a network N_2 interconnecting ports 1 and 4 only (i.e., all x_{ij} except x_{14} are zero). The value of x_{14} is obtained by the equation, $-x_{14}(1 - K_1)/K_4 = -1$, which yields $x_{14} = 1$. The new network realizing Y is shown in Fig. 6.

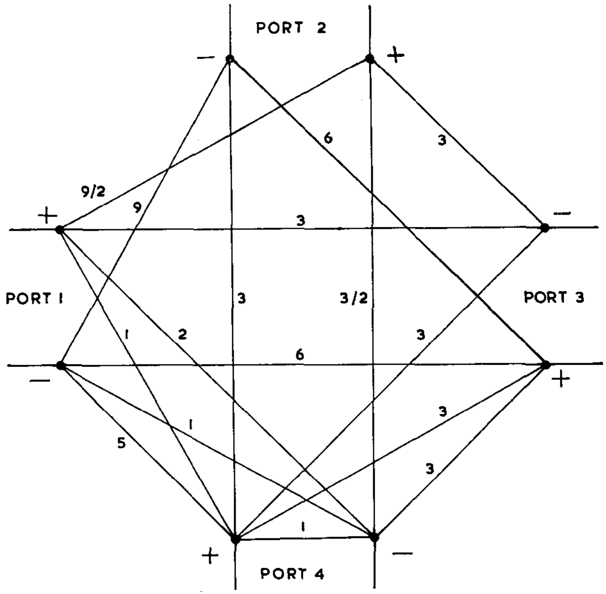

Fig. 6. Circuit realized in Example 3.

The above example is intended to point out the inherent facility available with K-networks to force certain edge conductances to preassigned values, without altering the port-conductance matrix. Extending the technique used in this example, one can always reduce the conductances shunting $(n - 1)$ ports to zero. We, however, wish to illustrate this property in a more general way. Suppose we are given a K-network. A new value not greater than the one existing in this K-network is specified for each conductance shunting the n-ports and it is desired to find an equivalent network in which as many edges shunting the ports as possible have these preassigned conductance values. We proceed to show that an equivalent network can always be found satisfying $(n - 1)$ of these specifications.

Consider first any pair of ports. Using (38), the conductance across one of them can always be reduced to its specified value keeping the conductance across the other port at a value not less than the corresponding specified value. In this new equivalent scheme, apart from the two conductances shunting the ports, only those of the edges interconnecting the two particular ports get altered. This process of reduction is repeated, considering a pair of ports at a time, the conductance across neither of which has been previously reduced by the required amount. At each stage one specification is complied with. Finally, only one port remains having a shunt conductance value higher than specified. This conductance can not be further reduced without reducing one of the other shunt conductances to less than the specified value. Hence, it is seen that only some $(n - 1)$ of the shunt conductances can be reduced to arbitrarily specified values.

Example 4

Consider a 4-port K-network whose edge conductance values are specified in the third column of Table I. The potential factors of the ports are $K_1 = 0.5$, $K_2 = 0.6$, $K_3 = 0.4$, and $K_4 = 0.3$. It is desired that the conductances shunting the ports have the following values: $g_{12} = 6$, $g_{34} = 12$, $g_{56} = 3$, and $g_{78} = 12$.

The calculations are summarized in Table I. In Step 1, the conductances of the edges related to ports 1 and 2 are altered to force the value of g_{12} to 6. In Step 2, ports 2 and 3 are considered and g_{56} made equal to 3. Finally, in Step 3, the value of the conductance shunting port 2 is reduced to 12. g_{78} is left at the value 14.235 as it can not be further reduced. It can be verified that the 4-port network with the original set of edge conductances or the new set has the following short-circuit conductance matrix.

$$Y = \begin{bmatrix} 81 & 14.4 & 7.2 & -7.2\\ 14.4 & 126 & -9 & 5.4\\ 7.2 & -9 & 126 & -12.6\\ -7.2 & 5.4 & -12.6 & 108 \end{bmatrix}.$$

Similarly, starting from a given K-network it may be possible to increase the values of the conductances shunting the ports to preassigned values, but the general applicability of this procedure is restricted as the conductances interconnecting the ports may turn out to be negative during this process in certain cases.

TABLE I

CALCULATIONS FOR EXAMPLE 4: $K_1 = 0.5$, $K_2 = 0.6$, $K_3 = 0.4$, $K_4 = 0.3$

			Step 1		Step 2		Step 3	
Ports	Conductance	Original values	Incremental value	Final value	Incremental value	Final value	Incremental value	Final value
1	g_{12}	7.50	−1.50	6.00				
2	g_{34}	16.02	−1.44	14.58	−1.74	12.84	−0.84	12.00
3	g_{56}	4.74			−1.74	3.00		
4	g_{78}	14.97					−0.735	14.235
1,2	a_{12}	7.20	1.80	9.00				
	b_{12}	36.00	1.80	37.80				
	c_{12}	28.80	1.20	30.00				
	d_{12}	—	1.20	1.20				
1,3	a_{13}	3.60						
	b_{13}	18.00						
	c_{13}	23.40						
	d_{13}	9.00						
1,4	a_{14}	32.40						
	b_{14}	18.00						
	c_{14}	51.60						
	d_{14}	66.00						
2,3	a_{23}	76.50			1.74	78.24		
	b_{23}	36.00			1.16	37.16		
	c_{23}	92.25			2.61	94.86		
	d_{23}	76.50			1.74	78.24		
2,4	a_{24}	13.50					0.63	14.13
	b_{24}	18.00					0.42	18.42
	c_{24}	49.50					1.47	50.97
	d_{24}	24.00					0.98	24.98
3,4	a_{34}	33.00						
	b_{34}	18.00						
	c_{34}	35.00						
	d_{34}	84.00						

V. CONCLUSIONS

A class of $2n$-node networks called K-networks which have interesting properties from the point of view of n-port realization of dominant matrices has been identified. New realization procedures using constant-K networks have been discussed. The presently known $2n$-node realization has been shown to be a special case belonging to this class. The advantage of these realization procedures is that for a given matrix Y, a network, choosing any value of K over a continuous range, can be found. In particular cases this may lead to a realization having a number of elements less than in the conventional realization.

A valuable feature of K-networks is that for a given network, a range of continuously equivalent networks can be obtained in a direct and easy manner. This has particular advantages when the conductance values of one or more edges of the network to be realized have been arbitrarily specified beforehand. It is shown in particular that $(n - 1)$ conductances shunting the ports can always be reduced to zero in the $2n$-node realization of a dominant matrix.

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