SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY.

by

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ABSTRACT

SYNTHESIS OF A FINITE TWO-TERMINAL NETWORK WHOSE DRIVING-POINT IMPEDANCE IS A PRESCRIBED FUNCTION OF FREQUENCY.

by

O. Brune.

A general method is given of synthesizing a finite two-terminal network whose alternating current impedance is prescribed as a function of frequency, provided only that this function satisfies the necessary and sufficient conditions that it may be the impedance of a finite passive network with constant circuit parameters.

In place of the complex alternating current impedance it is often more convenient to deal with the scalar impedance function $Z(\lambda)$ from which the impedance may be derived by writing $\lambda = i\omega$, and conversely.

The necessary and sufficient conditions to be fulfilled by $Z(\lambda)$ are found to be

(i) $Z(\lambda)$ is a rational function (quotient of two polynomials) which is real for real values of λ;

(ii) the real part of $Z(\lambda)$ is positive when the real part of λ is positive; (or $Z(\lambda)$ lies in the right-half Z plane when λ lies in the right-half λ plane).

For the sake of brevity a function which satisfies these conditions is called a "positive real" function; the process of synthesizing a network of which the given "positive real" function is the impedance function is called "finding a network representation" of the function.

That it is <u>necessary</u> for $Z(\lambda)$ to be a "positive real"

function is readily seen from physical considerations; a contradiction of (i) would mean that a real voltage produces a complex current, which has no physical meaning, while a contradiction of (ii) means that the network can under certain conditions generate energy.

That it is sufficient for $Z(\lambda)$ to be a "positive real" function can be proved only by showing that a physically realizable network representation of every "positive real" function can be found.

It is evident that the properties of "positiveness" and "reality" must play an important part in the process of finding a network representation of a given function. For this reason a thorough investigation of the properties of "positive real" functions is desirable. Various theorems for such functions are proved, the proofs for the most part being based on well-known properties of functions of a complex variable.

Of especial interest are certain properties of duality and reciprocity in virtue of which $Z(\lambda)$ and its reciprocal satisfy exactly similar conditions, as do also $Z(\lambda)$ and $Z(\frac{1}{\lambda})$.

Very important too are the properties which can be deduced for $Z(\lambda)$ on the boundary of the right-half λ plane (i.e. on the axis of imaginaries). It follows that a "positive" function $Z(\lambda)$ can have no poles or zeros within the right-half λ plane (corresponding to a well-known physical condition for electrical transients) and that poles or zeros on the boundary must be simple, while at such poles or zeros the residue or differential coefficient (respectively) of $Z(\lambda)$ must be a real positive constant. It follows further since $Z(\lambda)$ is regular in the right-half plane that its real part assumes a minimum value for this region, on the boundary.

A very interesting conclusion, corresponding to the expression of $Z(\lambda)$ and λ in polar coordinates, is that the radius vector of a "positive real" function $Z(\lambda)$ always lies closer to the reference line than the radius vector of λ provided the latter lies within the angle $-\frac{\pi}{2} < \arg\lambda < \frac{\pi}{2}$ On the same line of reasoning it can also be shown that the magnitude of $Z(\lambda)$ and λ cannot coincide for more than one point on the line of reference (considered as a half-line).

Using the properties of "positive real" functions a general process for finding a network representation can be established. The general procedure adopted is to separate the given "positive real" function into two components, one of which shall correspond to a simple network element, or group of elements, while the other component will be a simpler "positive real" function. The method of combining the components shall correspond to a known method of connecting impedances.

It is found that such a procedure can be repeatedly applied until the entire function has been expanded into components corresponding to recognizable network elements. The main steps in this procedure are the following:

(a) Separate poles on the imaginary axis as partial fractions. Every partial fraction corresponding to a pole will be represented by a parallel combination of inductance and capacity (anti-resonant component) in series with the network representation of the remaining "positive real" function.

(b) Treat zeros on the imaginary axis as poles of the reciprocal of the function. Every partial fraction corresponding to a zero will be represented by a series combination of inductance and capacitance (resonant component) in parallel with the network representation of the remaining "positive real" function.

(c) When all the zeros are in the interior of the left-half plane, consider the real part of the function on the boundary. The minimum value of this real part can be separated and represented by a resistance in series with the network representation of the remaining "positive real" function.

(d) When (c) has been performed the function which remains is pure imaginary at some point on the boundary. Subtraction of a term corresponding to this pure imaginary value on the boundary leaves a function with a zero on the boundary (i.e. a resonant component in parallel with the remaining network representation) and a pole at infinity. Separation of these according to (b) and (a), together with the term subtracted to produce a zero on the boundary, corresponds to the calculation of a reactive four-terminal network of T structure, having capacitance in the pillar and two perfectly coupled inductances in the branch-arms; across one pair of terminals of this four-terminal network is connected the network representation of the remaining "positive real" function.

These four processes are sufficient to reduce any positive real function completely for network representation. Simple formulae for numerical application are also discussed.

The process lends itself also to the calculation of certain equivalent networks. Most important are such as are not derivable from each other by affine transformation of their quadratic forms. These equivalents arise as follows: If $Z(\lambda)$ be regarded as the sum of an even and an odd function, the even part becomes the real part of $Z(i\omega)$ by the substitution $-\lambda^2 = \omega^2$; let this even part of $Z(\lambda)$ be written as $R(-\lambda^2) = R(\Omega)$. Then the series resistance used in (c) corresponds to the minimum of $R(\Omega)$ for positive values of Ω. If, however, $R(\Omega)$ has stationary values for negative Ω which are less than (or equal to) the value used in (c), any one of these can be subtracted from $Z(\lambda)$ as in (c), and thereafter a procedure exactly corresponding to (d) applied to the remaining "positive real" function; this leads also to a physically realisable network.

The possibility of further extending the method to obtain other equivalents is pointed out.

Finally a brief discussion is given of the derivation of the impedance function from other prescribed characteristics. A purely algebraic procedure is outlined for obtaining the impedance function when the real part, imaginary part, squared modulus, or tangent of the argument of the alternating current impedance are given as rational functions of the frequency. The solution of this type of problem probably forms the most important link, still to be forged, in the application of the methods of network synthesis here submitted to practical problems.

C O N T E N T S

CONTENTS (contd.)

A C K N O W L E D G M E N T S.

The author wishes to express his appreciation of the keen interest taken in the progress of this research by Professors V. Bush and E.A. Guillemin of the Department of Electrical Engineering, and by Professor N. Wiener of the Department of Mathematics, all of the Massachusetts Institute of Technology. Their contributions and continued activity in the field of electric network theory were instrumental in the first place in interesting the author in this field, and their interest has been a constant source of inspiration in the pursuit of this investigation.

The author is also indebted to Dr. W. Cauer of the University of Göttingen, who is at present in Cambridge (Mass.) as International Research Fellow on the Rockefeller Foundation, and who suggested the subject of this research. His extensive knowledge and original work in the subject of network synthesis have been invaluable in orientating the author's ideas in the present research.

INTRODUCTION

(a) Statement of the Problem.

1. The discussion presented in this thesis is a contribution to the theory of synthesis of electrical networks. In network synthesis the behavior of a network for certain conditions is specified, and the structure of such a network is desired. This is the inverse of network analysis in which the structure of the network is given and its behavior under given conditions is to be found. The problem of network synthesis has assumed increasing importance in view of the present needs in electrical communications and sound recording for talking pictures.

2. The particular problem to be attacked here will be restricted as follows:-

(i) One pair of terminals only of the network is to be considered;

(ii) The network is assumed to be finite with constant dissipative lumped circuit parameters (i.e. it contains only ordinary ohmic resistances, condensers, and inductance coils which may or may not be mutually coupled);

(iii) The behavior of the network at its terminals is to be specified by the impedance function $Z(\lambda)$.

It will be seen that in (iii) we have assumed as done that part of the problem which derives the impedance function from the prescribed behavior of the network at its terminals. Some discussion of this will be found in the last chapter of this thesis.

3. The restriction (ii) means that $Z(\lambda)$ must be given as the quotient of two polynomials, i.e.

$$Z(\lambda) = \frac{f(\lambda)}{g(\lambda)} \tag{1}$$

where
$$f(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n \tag{1a}$$
$$g(\lambda) = b_0 + b_1\lambda + b_2\lambda^2 + \cdots \quad b_m\lambda^m \tag{1b}$$

Furthermore the coefficients of $f(\lambda)$ and $g(\lambda)$ are not entirely arbitrary since for a dissipative network we must have, for example,

(i) if α_r is any root of $f(\lambda) = 0$, β_r any root of $g(\lambda) = 0$ then

$$\mathcal{R}e\, \alpha_r \leq 0 \;* \tag{2a}$$
$$\mathcal{R}e\, \beta_r \leq 0 \tag{2b}$$

(ii) also
$$\mathcal{R}e\, Z(i\omega) \geq 0 \tag{3}$$

Equation (2a) corresponds to the physical fact that if the network of which $Z(\lambda)$ is the impedance function be disturbed electrically and then left to itself with its terminals short-circuited, it cannot generate an increasing amount of energy. A similar physical interpretation holds for equation (2b) except that for this case the network is disturbed and then left to itself with open terminals. Condition (3) is equivalent to the statement that the network cannot generate energy when a steady alternating electromotive force $E_0.\cos\omega t$ is applied at the terminals.

4. These conditions are necessary; the question arises as to whether they are sufficient. Furthermore they may not be independent. Evidently a solution of the problem of constructing a network corresponding to $Z(\lambda)$ will require

*The notation '$\mathcal{R}e$' means "real part of" throughout this thesis.

a clear understanding as to what constitutes an independent set of necessary conditions. The proof of the fact that such a set of conditions is sufficient can only be furnished by showing that a corresponding network can be built in every case when they are satisfied.

5. We may therefore state our problem thus:

To find the necessary and sufficient conditions that a quotient of two polynomials may be regarded as the driving-point impedance function of a finite network; this involves also giving a method of finding the structure of a network corresponding to any function satisfying these conditions.

(b) Contributions of Previous Investigators.

6. The first important contribution to the theory of synthesis of two-terminal networks was probably made by Foster, when in his "Reactance Theorem"(12) he gave the necessary and sufficient conditions which must be satisfied by the impedance function of a purely reactive network. These conditions are:

(i) poles and zeros of $Z(\lambda)$ are pure imaginary and occur in conjugate pairs;

(ii) the poles and zeros mutually separate each other on the imaginary axis; this together with (i) includes the necessity for a pole or zero at the origin and at infinity.

(12) For all references by number see Bibliography.

It will be seen that (i) is a limiting case of condition (i) in section 2; the second condition mentioned in section 2 also exists in a limiting form because the network is assumed to be purely reactive. It can be shown, however,* that the mutual separation of poles and zeros as specified in (ii) of this section can be obtained by considering $\mathcal{R}e\, Z(i\omega) \equiv 0$ as a limit of $\mathcal{R}e\, Z(i\omega) \geq 0$.

Foster gives two methods of constructing a network whose impedance function $Z(\lambda)$ satisfies the necessary and sufficient conditions for a pure reactive network; these methods correspond to a development in partial fractions of either $Z(\lambda)$ or its reciprocal. The networks so obtained are shown in Figs. 1 (a) and (b).

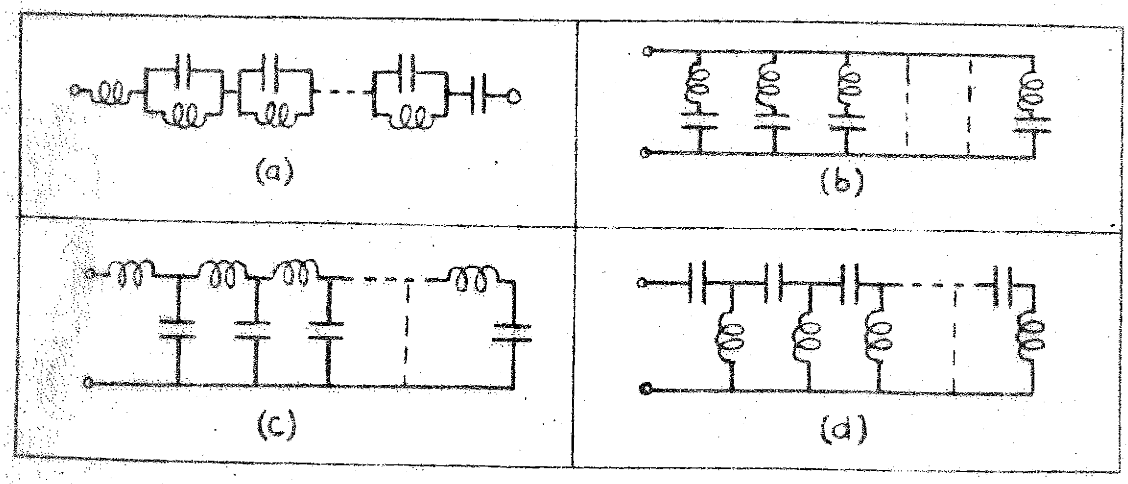

Fig. 1. Networks Representing a General Reactance

7. Cauer[14] has amplified these results of Foster by pointing out a continued fraction development for functions satisfying Foster's "reactance" conditions. This partial fraction development of the function corresponds to the ladder structures shown in Figs. 1(c) and (d). The

*See Bibliog. (31); c.f. also Chapter III of this thesis, Section 70.

structures (a) (b) (c) and (d) in Fig. 1 are equivalent networks.

Cauer next extends Foster's results to include all networks which contain only two kinds of elements, viz. networks containing only inductance and resistance, and networks containing only resistance and capacitance. For these networks Foster's conditions apply if

(i) for each pair of conjugate poles or zeros in Foster's case is substituted a single pole or zero (respectively) on the negative real axis; and if further

(ii) for <u>inductance-resistance</u> networks the order of the zeros $-\alpha_r$ and the poles $-\beta_r$ is

$$0<\beta_1<\alpha_1<\beta_2<\alpha_2<----<\alpha_{n-1}<\infty \quad (4a)$$

The zero $\alpha_0 = 0$ and the pole $\beta_n = \infty$ may or may not be present; for <u>resistance capacitance</u> networks the order is

$$0<\alpha_1<\beta_1<\alpha_2<\beta_2<--- <\beta_{n-1}<\infty \quad (4b)$$

and here the pole $\beta_0 = 0$ and the zero $\alpha_n = \infty$ may or may not be present.

The corresponding networks are obtained from those of Fig. 1 by replacing condensers or coils (respectively) with resistances.

The particular case of our problem in which the network contains only two kinds of elements has therefore been completely solved.

8. We turn next to such cases where the network may contain all three kinds of elements. Both Foster(13) and Cauer(14) have studied in considerable detail the function

$Z(\lambda)$ capable of representation by a two-mesh circuit. Such a function has the form

$$Z(\lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4}{b_1\lambda + b_2\lambda^2 + b_3\lambda^3} \tag{5}$$

In particular they have discussed the conditions which the a's and b's (coefficients in $Z(\lambda)$) must satisfy in order that (5) may correspond to a two-mesh circuit. These conditions are best visualized with the aid of a diagram used by Cauer (also suggested by Foster) and reproduced in Fig. 2(a). Here the ratios $\frac{b_2}{b_1}$, $\frac{b_3}{b_1}$ in (5) have been used for coordinate axes, while the ratios of the a's have been taken as parameters which fix the shape and position of the curves in this plane. In other words if the coefficients

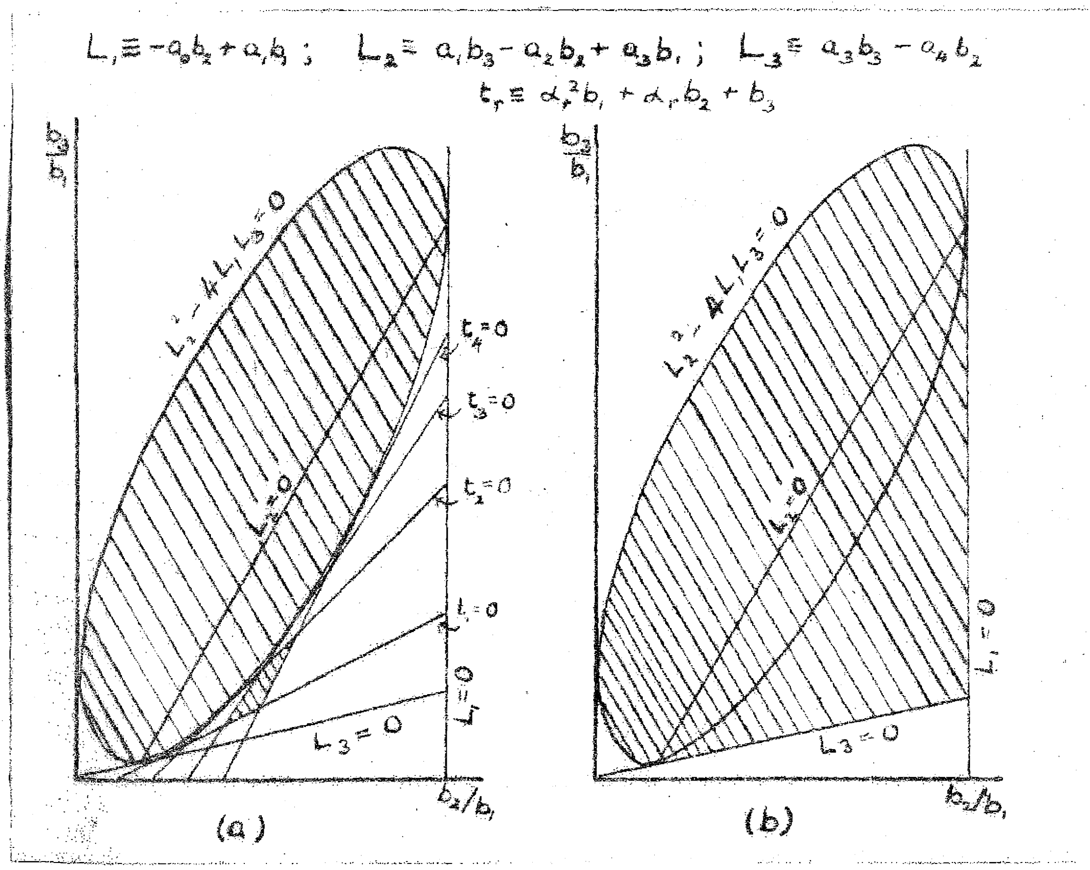

Fig.2 (a) Region in which (5) represents a 2-mesh network
(b) Region in which (5) satisfies condition (3).

of the numerator are fixed, a point representing the coefficients of the denominator (except for a constant factor) is restricted by certain inequalities to lie within a limited region. In Fig. 2(a) this region is shown shaded, for the case where all the roots of the numerator are real.

In Fig. 2(b) is shown the corresponding region when $Z(\lambda)$ in (5) satisfies condition (3). A comparison of Figs. 2(a) and 2(b) discloses the fact that two-mesh networks are not capable of representing all functions which satisfy condition (3). A first step in the solution of our problem would thus be to find the networks capable of representing (5) for the regions unshaded in Fig. 2(a) but shaded in Fig. 2(b)*.

9. T.C. Fry[15] has discussed in some detail the possibility of using the Stieltjes continued fraction as a means of expanding the impedance function in a form usable for the construction of a corresponding network. He is guided in this by the fact that a continued fraction development of the function is the counterpart of a ladder structure in the network. The functions which are capable of being expanded as Stieltjes continued fractions, however, must satisfy somewhat more severe conditions than those imposed by conditions (2) and (3). Fry overcomes this difficulty to some extent by several very ingenious transformations of

*This problem was pointed out to me by Dr. Cauer as a first step in the approach to the more general problem. Its solution is contained in Chapter III of this thesis.

the function but does not succeed in making the method general for all functions satisfying conditions (2) and (3). Moreover the Stieltjes continued fraction is an infinite expansion, i.e. for practical purposes it is an approximate rather than an exact representation of the function.

10. Yuk-Wing Lee[24] has treated a problem in the synthesis of networks in which the transfer admittance of the network is prescribed. The solution is carried through by a method of expanding the transfer admittance in a series of orthogonal functions, each of which represents the transfer admittance of a known network. The sum of such a series corresponds to a parallel combination of four-terminal networks; a change of sign in the coefficient of a term corresponds to a reversal of one pair of terminals in the corresponding network. This method may be termed approximate in the same sense as that of Fry since the expansion in orthogonal functions involves an infinite series.

Two important differences between driving point and transfer impedance come into play here. Firstly there is no restriction on the sign of the real part of a transfer impedance, since the product of input voltage and output current has no physical significance in terms of energy flow. Secondly driving point impedances can only be added in series or in parallel; they cannot be subtracted as is possible with transfer admittances. These factors constitute an essential difference between the methods applicable for building networks corresponding to a prescribed driving point impedance, and those which may be used when the transfer admittance is prescribed. There are, however, some points of resemblance and we shall have occasion to refer to Lee's work.

11. The question of equivalent networks is closely connected with our problem. Such networks already appear in the early work of Foster(12,13) and Cauer(14); in these cases only networks without superfluous elements are mentioned.

It has been shown by Howitt* that an infinite group of networks (including those with a superfluous number of elements) can in general be found by the application of an affine transformation to the quadratic forms** connected with this network; such a transformation results in a new set of quadratic forms from which the structure of the equivalent network can be derived.

In this connection it is important to note that the structure of the original network must be known; in other words, one network corresponding to the invariant $Z(\lambda)$ of the group is assumed to be given. It is the purpose of the present investigation to furnish at least one such network.

12. It may further be pointed out that in many cases this affine group of networks does not include all the equivalent networks corresponding to a given $Z(\lambda)$. This means that the totality of all equivalent networks may consist of a number

*Bibliog. (17); see also Cauer, Bibliog. (16) p.64 and Bibliog.(23)

**These quadratic forms corresponds to the stored dielectric energy, the stored magnetic energy, and the rate of dissipation of energy in heat in the network at any moment.

of distinct groups not derivable from each other by an affine transformation. It would be very desirable to have a method of finding at least one member of each group. We do not propose to solve this problem in its entirety here. Nevertheless the appearance of such distinct groups will be noted.

13. Lastly we wish to point out a connection of this problem, in which one pair of terminals only is considered, with the more general problem when any number of pairs of terminals are considered. For simplicity consider the case of two pairs of terminals. It can readily be shown that to specify the behavior of such a network under any given conditions at its terminals, three functions are required*. Cauer[22] has considered special cases of this where only two functions are required, viz. the case of symmetry with respect to the two pairs of terminals (network looks the same looking in from either end) and the special unsymmetrical case where, on short-circuiting one pair of terminals, the driving point impedance at the other pair is equal to the transfer impedance connecting voltage at the open pair and current in the short circuit. He has shown that such networks may always

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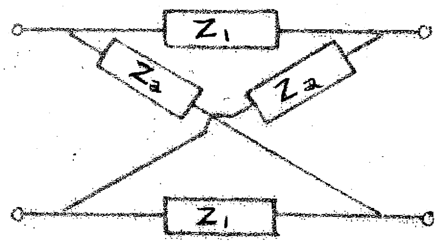
*To electrical engineers these are probably most familiar in the form of 4 "general circuit constants" in the equations

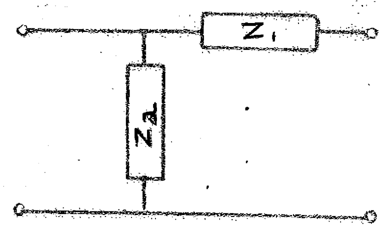
$$E_1 = AE_2 + BI_2$$
$$I_1 = CE_2 + DI_2$$

The four constants A,B,C,D being connected by the relation AD - BC = 1. They are constants for a fixed frequency but in general are functions at frequency.

be represented by the structures shown in Figs. 3(a) and (b)



(a) Representation of General Symmetrical Four-Terminal Network



(b) Representation of Special Unsymmetrical Four-Terminal Network

Fig. 3.

respectively. Here it is apparent that the networks are completely determined by two two-terminal networks Z_1 and Z_2, and the problem is reducible to the one stated in section 5 above. It is also probable that when the natural method of attack for the two-terminal problem has been found, it will indicate the natural method of attack for the more general case.

A solution of this problem is therefore highly desirable as a part of the general problem of network synthesis.

CHAPTER I.

The Impedance Function in Network Analysis.

14. It is a well known fact in electric circuit theory that the behavior of a passive network at a pair of terminals is completely determined by a scalar function $Z(\lambda)$. This function is known as the impedance function of the network; more specifically, if we wish to distinguish it from other functions which relate current and voltage in different parts of the network, it is called the "driving-point" impedance function of the network. The impedance function $Z(\lambda)$ is to be distinguished from the ordinary complex alternating current impedance, which is $Z(i\omega)$.

In this chapter we wish to emphasise certain steps in the process of deriving the impedance function from the electrical structure of the network, as well as certain results in its application to specific problems.

(a) Geometrical Structure of Networks.

15. An understanding of the geometrical properties of networks is valuable, since these properties play an important role in the mathematical formulation of the electrical problem. These properties are embodied in the theory of "linear graphs", a branch in the wider subject of Analysis Situs*. The first step in the theory of linear graphs was in fact made by G. Kirchhoff(2) in connection with the electrical problem. Very little that is of interest in the electrical problem is as yet to be found outside the results

*See for example Bibliog. (28) and (29).

of Kirchhoff's paper. The following is a summary of those results.

16. A geometrical network, or linear graph, may be considered as made up of a number of branches connecting together a number of branch-points. Every branch lies between two branch-points, and every branch-point is common to two or more branches. A mesh is determined by a closed circuit of branches, i.e. if by proceeding in a uniform direction along these branches, all branches (and no more) will have been traversed once when the starting point is re-encountered, these branches determine a mesh. The mesh may be regarded as belonging to this group of branches in exactly the same way as a branch-point is said to belong to the branches which meet in it.

Let b = the number of branches in the network

n = the number of branch-points,

then if m = the smallest number of branches which must be removed from the network so that no closed circuit (mesh) remains, it can be shown that

$$m = b - n + 1 \tag{7}$$

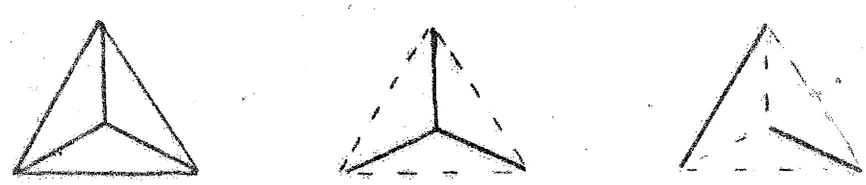
The figure that remains after these m branches have been removed is called a tree.

The number m can now be shown to represent the number of independent meshes that can be chosen in the network; (among mathematicians m is known as the "cyclomatic number" of the linear graph).

If equation (7) be written

$$m + n = b + 1 \tag{7b}$$

a duality between branch-points and meshes is indicated by the symmetry of this equation in m and n*. The number n may also be defined as the least number of branches which must be removed from the network so that no two of the remaining branches have a point in common (i.e. no branch-point remains). This is illustrated for a simple case in Fig. 4.



(a) Network (b) m branches removed (Tree) (c) n branches removed

Fig. 4. Linear graph: $m = 3$, $n = 4$, $b = 6$.

It may be added that there are several ways of choosing the m or n branches.

(b) Electrical Laws in a Network.

17. The two laws for the distribution of current and voltage in a network first stated by Kirchhoff[1] are the electrical analogs of the laws of conservation of energy and of matter applied to the network. They are

I. The sum of the potential drops encountered in going around a closed circuit (mesh) is zero.

II. The sum of the currents flowing towards a branch-point is zero.

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*Compare Euler's theorem on the relation between the number of edges, faces and vertices of a polyhedron.

Before we can apply these equations we need in addition Ohm's Law which expresses voltage in terms of current or vice versa. Consequently I and II give equations which may be expressed either in branch currents or branch voltages. The number given by Law II is $(n - 1)$ since the equation at the last branch-point can always be obtained as a linear combination of all previous ones. The total number of independent equations is thus $m + n - 1 = b$, which is exactly the number of unknowns.

18. It is usual to substitute for the b branch currents in terms of m circulating mesh currents, which automatically satisfy the $n - 1$ equations given by Law II. The dual procedure is also possible: instead of b branch voltages assume $n - 1$ branch-point potentials (to any one of the branch points the potential zero may be arbitrarily assigned; these will automatically satisfy the m equations given by Law I. The equations of Law II (expressed in terms of the branch-point potentials) will give the necessary $n - 1$ equations to solve for the potentials.

If in the case of a two-terminal network we solve this system of equation for the potential difference between the terminals in terms of the current entering a terminal we get an admittance expressible in determinant form (cf. section 19 following). The interesting point, however, is that every system of linear equations derived in one way (either in terms of mesh currents or branch-point potentials) can be interpreted also in the dual way. A mesh is replaced by a branch point and vice versa; branches between meshes become branches between branch points; impedances become admittances, and so on.

While this duality is perfect in broad principles, one dissymmetry in a matter of detail on the physical side should be noted. The dual of an inductive reactance is a capacitive susceptance (and vice versa) but there is no complete physical dual of mutual inductance. It is always possible, however, to state general results on the assumption that such a duality exists and then to remove any physical incongruities by substituting an equivalent which does not contain unrealizable mutual capacities. (An example of this will be found in Chapter IV, Sections 63, 64).

We shall not pursue this duality further here, although it is a fascinating and fruitful study. An example of its fruitfulness was the possibility of predicting Lee's theorem* from Thévenin's theorem(6).

A complete discussion of the principle of duality in electrical networks with its applications has not yet been given, but should result in a material broadening of our ideas.

(c) The Impedance Function.

19. The derivation of the impedance function from the Kirchhoff equations depends on a well known generalization of Ohm's Law in terms of operators. The usual resistances in the Kirchhoff equations (or admittances in the dual interpretation) are then replaced by operators. In our case

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* See Bibliog. 24, page 55.

where we are only interested in voltage and current at the terminals of a passive network the m equations in terms of the mesh currents then have the form*

$$
\begin{aligned}
E &= a_{11} I_1 + a_{12} I_2 + \cdots\cdots a_{1n} I_n \\
0 &= a_{21} I_1 + a_{22} I_2 + \cdots\cdots a_{2n} I_n \\
\vdots & \qquad \vdots \qquad\quad \vdots \qquad\qquad\qquad \vdots \\
0 &= a_{n1} I_1 + a_{n2} I_2 \quad \cdots\cdots a_{nn} I_n
\end{aligned} \tag{8}
$$

$$
\left.\begin{aligned}
&\text{where} \quad a_{rs} = a_{sr} = L_{rs}p + R_{rs} + D_{rs}p^{-1} \\
&\text{and} \qquad\qquad p = \frac{d}{dt}
\end{aligned}\right\} \tag{8a}
$$

and the relation between current and voltage at the terminals is given by

$$\frac{E}{I_1} = \frac{D}{M_{11}} = Z(p) \tag{9}$$

where D is the determinant of the coefficients in this equation and M_{11} is the complementary minor of a_{11} in this determinant.

20. It may be further noted that Z(p) need not necessarily be calculated by means of determinants. It is formed from the branch impedance operators by the same laws of combination applicable for a pure resistance network, e.g. series and parallel combinations, transformations from T to Δ connections, and vice versa. These laws may sometimes be inadequate on account of the complication of the network; the determinant rule on the other hand is general. Other laws for determining Z(p) without the aid of determinants may be found in papers by Kirchhoff[2], ~~and~~ Feussner[3], and P. Franklin[4].

*See for example Bibliog. (5).

21. It is of interest to enquire into the degree possible in the numerator and denominator of $Z(p)$ (when written as a quotient of polynomials). Let us multiply each equation in (8) by p (see equations 8a); then each element in the new determinant can at most be of the second degree while $pZ(p)$ will be the quotient of the new determinant divided by its corresponding minor. The numerator of $Z(p)$ is therefore at most of degree $2m$ in p, while the denominator is at most of degree $2m-1$ (with no constant term). This places a lower limit to the number of meshes necessary in a network corresponding to a given $Z(p)$. It is not possible, however, to place an upper limit on the number of meshes.

A similar discussion can be made for branch-points; in this case every junction point of two different kinds of elements must be considered as a branch point, for the $n-1$ equations in terms of branch-point potentials will not have coefficients reducible to the second degree in p unless this is done. Thus we have that in $Y(p)$ (which is the reciprocal of $Z(p)$) the degree of the numerator is at most of degree $2(n-1)$ while the denominator is at most of degree $2(n-2)+1$. This fixes a lower limit to the number of branch points possible in a network corresponding to any given admittance (or impedance) function.

(d) Application of the Impedance Function.

22. The impedance function $Z(p)$ as derived above is essentially an operator. It can be interpreted physically only after it has operated on something. In reality it is

merely a shorthand method of writing the differential equation that results on the elimination of the "internal" currents from the set of differential equations (8).

Now the solution of such a differential equation is the sum of a "particular integral" and a "complementary function" which in the usual language of electrical engineers are respectively "steady state" and "transient state" currents*. Let us express this by writing

$$I = \frac{E}{Z(p)} = I_s + I_t \tag{10}$$

where I_s and I_t are respectively the steady state and transient state currents. The transient current contains an arbitrary constant factor determined by the initial conditions. The transient current is independent of the voltage impressed except for this constant factor, being a solution of the equation $0 = Z(p)I$.

23. By Heaviside's expansion formula* for "unit" applied voltage

$$I_t = \sum_k \frac{\varepsilon^{\alpha_k t}}{\alpha_k \left(\frac{dZ(p)}{dp}\right)_{p=\alpha_k}} \tag{11}$$

the summation extending over all α_k, roots of $Z(p) = 0$.

From physical considerations, since transient currents are independent of impressed voltage, they must ultimately vanish unless a contradiction of the Law of Conservation of Energy is to be obtained. Consequently

$$\mathcal{R}e\, \alpha_k \leq 0 \tag{2a}$$

*See for example any one of the references: Bibliog. (7) to (11).

the equality sign applying only in the limit of a non-dissipative network.

We note further that the denominator of Z(p) is the determinant of a system of equations similar to (8) corresponding to the same network but having the mesh in which E occurs open circuited. Consequently we may write also for the poles β_K of Z(p)

$$\mathcal{R}e\, \beta_K \leq 0 \tag{2b}$$

24. If a voltage

$$E = E_m \cos \omega t \tag{12}$$

be applied we may write

$$E = E_m \mathcal{R}e\, \varepsilon^{i\omega t}$$

and

$$I_s = E_m \mathcal{R}e \frac{\varepsilon^{i\omega t}}{Z(i\omega)}$$

$$= \frac{E_m}{|Z(i\omega)|} \cos(\omega t - \varphi) \text{ where } \varphi = \arg Z(i\omega) \tag{12a}$$

After a sufficiently long time the transient current will be negligible in comparison with this "steady state" current (except in a purely reactive network). From the fact that the power taken by such a circuit must be positive we therefore conclude that $|\varphi| \leq \frac{\pi}{2}$ or

$$\mathcal{R}e\, Z(i\omega) \geq 0 \tag{3}$$

25. <u>"Pseudo-Steady State"</u>. Let us consider an applied voltage

$$E = \varepsilon^{\gamma t} \cos \omega t$$
$$= \mathcal{R}e\, \varepsilon^{(\gamma + i\omega)t} \tag{13}$$

The "steady state" current may be written

$$I_s = \mathcal{R}e \frac{\varepsilon^{(\gamma+i\omega)t}}{Z(\gamma+i\omega)}$$

$$= \frac{\varepsilon^{\gamma t}}{|Z(\gamma+i\omega)|}\cos(\omega t-\varphi) \qquad \text{where } \varphi = \arg Z(\gamma+i\omega) \tag{13a}$$

If γ is positive the transient term can, after a sufficiently long time, be neglected (even if the network is purely reactive). If, on the other hand, γ is negative the transient term may become more important than the steady state term. A very simple example illustrating this is given in Example 1 in the Appendix.

Returning to the case of $\gamma > 0$, let us determine the energy W taken by the network after the transient term has become negligible, and consider the voltage at the beginning of this time unity

$$E = \varepsilon^{\gamma t}\cos\omega t \quad ; \quad I = \frac{\varepsilon^{\gamma t}\cos(\omega t-\varphi)}{|Z(\gamma+i\omega)|}$$

$$W = \frac{1}{|Z(\gamma+i\omega)|}\int_0^T \varepsilon^{2\gamma t}\cos\omega t.\cos(\omega t-\varphi)\,dt$$

$$= \frac{1}{2|Z(\gamma+i\omega)|}\int_0^T \varepsilon^{2\gamma t}\left[\cos(2\omega t-\varphi)+\cos\varphi\right]dt$$

$$= \frac{1}{2|Z(\gamma+i\omega)|}\left\{\frac{\varepsilon^{2\gamma t}\cos(2\omega t-\varphi-\chi)}{2(\gamma^2+\omega^2)^{\frac{1}{2}}} + \frac{\varepsilon^{2\gamma t}}{2\gamma}\cos\varphi\right\}\Big/_0^T$$

$$\text{where } \chi = \arg(\gamma+i\omega)$$

$$= \frac{1}{4|\gamma+i\omega|.|Z(\gamma+i\omega)|}\Big\{\varepsilon^{2\gamma T}\cos(2\omega T-\varphi-\chi)-\cos(\varphi+\chi) + \frac{\cos\varphi}{\cos\chi}(\varepsilon^{2\gamma t}-1) \tag{14}$$

It is evident that if the network had no energy initially W can never be negative (i.e. the network can never return more energy than it received). Consequently $\frac{\cos\varphi}{\cos\chi} \geq 1$ (the equality sign holding only for the non-dissipative case). This requires besides $\mathcal{R}e\, Z(\lambda) \geq 0$ when $\mathcal{R}e\,\lambda \geq 0$ (15)

also $|\arg Z(\lambda)| \leq |\arg\lambda|$ when $|\arg\lambda| \leq \frac{\pi}{2}$ (15a)

26. In the foregoing we have noticed several properties of an impedance function derived from a physical network. These properties may all be termed necessary conditions. For reference we collect the following inequalities.

First we mention the obvious but nevertheless important condition that the coefficients in the numerator and denominator of $Z(\lambda)$ (equation (1) section 3) must be real, i.e.

$$Z(\lambda) \text{ real when } \lambda \text{ is real} \tag{1c}$$

From sec. 23

$$\mathcal{R}e\, \alpha_r \leq 0 \tag{2a}$$

$$\mathcal{R}e\, \beta_r \leq 0 \tag{2b}$$

From sec. 24

$$\mathcal{R}e\, Z(i\omega) \geq 0 \tag{3}$$

From sec. 25

$$\mathcal{R}e\, Z(\lambda) \geq 0 \quad \text{when} \quad \mathcal{R}e\, \lambda \geq 0 \tag{15}$$

also

$$|\arg Z(\lambda)| \leq |\arg \lambda| \quad \text{when} \quad |\arg \lambda| \leq \frac{\pi}{2} \tag{15a}$$

In the following chapter we shall discuss the interrelation of these conditions on a purely mathematical basis.

CHAPTER II

Theorems on "Positive Real" Functions

(a) Relations between Necessary Conditions; Preliminary Discussion.

27. It is clear that the conditions imposed on the zeros and on the poles of $Z(\lambda)$ in equations (2a) (2b) section 26 are independent of each other; the former involves a set of relations (inequalities) among the coefficients of the numerator of $Z(\lambda)$ while the latter involves a similar set of relations among the coefficients of the denominator. These conditions for a polynomial whose roots have non-positive real part, have been stated by Hurwitz(30) and are as follows:

If $Z(\lambda) = \dfrac{a_o + a_1\lambda + a_2\lambda^2 + \quad \cdots\cdots \quad + a_n\lambda^n}{b_o + b_1\lambda + b_2\lambda^2 + \quad \cdots\cdots \quad + b_m\lambda^m}$

the determinant
$$\begin{vmatrix} a_1 & a_3 & a_5 & \cdots & \cdots & a_r \\ a_o & a_2 & a_4 & \cdots & \cdots & \cdot \\ 0 & a_1 & a_3 & \cdots & \cdots & \cdot \\ 0 & a_o & a_2 & \cdots & \cdots & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \end{vmatrix}$$
together with

its principal minors must be 0 (16a)

Similarly
$$\begin{vmatrix} b_1 & b_3 & b_5 & \cdots & \cdots & b_s \\ b_o & b_2 & b_4 & \cdots & \cdots & \\ 0 & b_1 & b_3 & \cdots & \cdots & \\ 0 & b_o & b_2 & \cdots & \cdots & \\ \cdot & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & & & \cdot \end{vmatrix}$$
together with

its principal minors must be 0 (16b)

28. If, on the other hand, we write the condition for

$$\mathcal{R}e\ Z(i\omega) \geq 0 \qquad (3)$$

it will be seen that it involves the coefficients of both numerator and denominator of $Z(\lambda)$. There is thus a possibility that conditions (2a), (2b) and (3) are not entirely independent. It is readily seen that (3) is not a consequence of (2a) and (2b). This is best illustrated by the numerical case given in Example 2, Appendix.

Furthermore, condition (3) can be satisfied without (2a) and (2b) being satisfied, for by a mere substitution $\lambda' = -\lambda$ the poles and zeros can be shifted from the negative half-plane to the positive half-plane without affecting values on the axis of imaginaries.

It can, however, be shown that if $\mathcal{R}e\ Z(i\omega) \geq 0$ (together with a condition for poles or zeros <u>on</u> the axis of imaginaries) then poles and zeros must lie on the <u>same</u> side of the imaginary axis. A special case of this was proved by Cauer[31], and his method of proof, which is very interesting and instructive, can be extended to a more general case. The proof which will be found in Theorem V (section 36) below is, however, to be preferred from the point of view of elegance.

29. We come now to the condition

$$\mathcal{R}e\ Z(\lambda) \geq 0 \text{ when } \mathcal{R}e\ \lambda \geq 0 \qquad (15)$$

Clearly this implies the condition (3) and something more. We shall begin from this condition and show that all other conditions (except (1c) which is: $Z(\lambda)$ real on the real λ axis) are a consequence of this one condition. Condition

15(a) presents a similar possibility but is less natural from the usual view-point. For the sake of brevity we adopt the following definition:

Definition: A "positive" function $Z(\lambda)$ is a function whose real part is positive when the real part of λ is positive; the real part of $Z(\lambda)$ may be positive or zero when the real part of λ is zero. If in addition $Z(\lambda)$ is real for real values of λ it will be called a "positive real" function. (The quotes will be retained as a reminder of the special sense in which these terms are being used).

In general the notation $Z(\lambda)$ will imply that this function falls under the above definition. If a function does not, or is not known to, fall under this definition an asterisk will be added (e.g. $Z^*(\lambda)$) or another notation employed.

The proof that every "positive real" function with a finite number of poles and zeros may be regarded as the impedance function of a finite physical network is reserved for a later chapter. In the meantime we shall frequently anticipate this fact in interpreting mathematical properties of "positive real" functions:

Definition: If a function is the impedance function of a physical network the function will be said to have a "network representation". The process of finding such a network will be called "finding a network representation" of the function.

The discussions which follow are mostly discussions of well known properties of functions of a complex variable, but their application to this particular problem is new.

(b) Theorems on "Positive Real" Functions.

30. The idea of a "positive" function can be interpreted very conveniently in terms of the transformation of one complex plane into another. By the equation $Z = Z(\lambda)$ every point in the λ plane determines uniquely a point in the Z plane; (Z is called a uniform function of λ; on the other hand λ is in general a multiform function of Z unless the numerator and denominator of $Z(\lambda)$ are linear in λ). The property of "positiveness" as defined by (15a) means that if λ is chosen in the right half plane Z will fall in the right half plane; but not necessarily conversely. Consequently if $Z(\lambda)$ is a "positive" function, the transformation $Z = Z(\lambda)$ will transform the right half of the λ plane into a part of the right half of the Z plane (except in special cases where the right half λ plane goes into the whole right half Z plane). A simple example of this is shown in Fig. 5 where $Z = \dfrac{a_0 + a_1\lambda}{b_0 + b_1\lambda}$

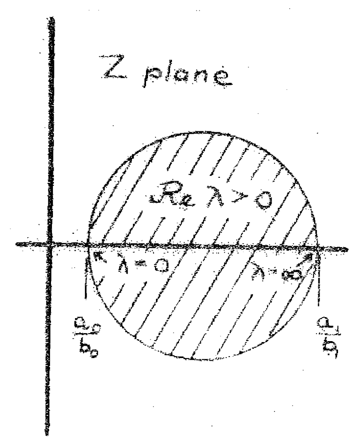


Fig. 5.

We thus have

Theorem I: If $Z = Z(\lambda)$

and $Z' = W(Z)$

Z and W both being "positive functions" then

$$W\left[Z(\lambda)\right]$$

will be a "positive" function of λ.

This theorem can obviously be applied in two ways illustrated by the two corollaries:-

Theor.I Coroll. 1 If $Z(\lambda)$ is a "positive" function, $Y(\lambda) = \frac{1}{Z(\lambda)}$ will be a "positive" function.

Theor.I Coroll. 2 If $Z(\lambda)$ is a "positive" function $Z_1(\lambda) = Z(\frac{1}{\lambda})$ will be a "positive" function.

31. The physical interpretation of Corollary 1 is: to every network corresponds a "reciprocal" network whose admittance characteristic is the impedance characteristic of the first network. In certain cases the second network is derivable from the first by a dual construction in the sense of Chapter I, section 18. This will not always be possible because of the difficulty met in the physical representation of "mutual capacities", but a physical network not dual in structure will nevertheless be possible.

On the other hand Coroll. 2 means that to every network corresponds another whose impedance characteristic is that of the first network but with the reciprocal frequency scale (e.g. frequency is replaced by wave-length).

Taken more generally Theorem I states that if in $Z(\lambda)$, the impedance function of a given network N, λ be replaced by the impedance function $z(\lambda)$ of another network n, the impedance function of a third network N' will result. In certain cases N' can be derived from N by replacing inductances by networks proportional to n, and capacities by networks proportional to the network $(\frac{1}{n})$, reciprocal to n. In general, however, the network N' cannot be constructed directly from N in this fashion because of the difficulty of replacing mutual inductances in an analogous fashion.

32. We next come to a discussion of the poles and zeros of a "positive" function. We prove first of all the

Lemma 1: A zero of multiplicity n is surrounded by 2n sectors of equal angles (Fig. 6) in which the real part of the function is alternately positive and negative. For at such a point $\lambda = c$ the Taylor's series for the function becomes

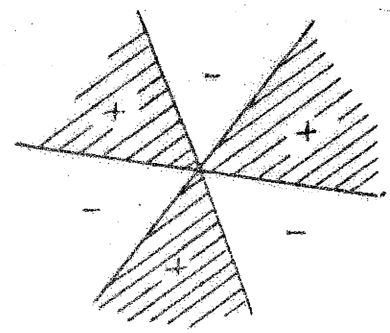


Fig. 6

$$Z(\lambda) = \left\{\frac{d^n}{d\lambda^n} \cdot \frac{Z(\lambda)}{n!}\right\}_{\lambda=c} (\lambda-c)^n + \text{terms of higher degree}$$

In the immediate neighbourhood of the point $\lambda = c$ the first term predominates.

Placing $\quad (\lambda - c) = \rho\varepsilon^{i\theta} \quad ; \quad \left\{\frac{d^n}{d\lambda^n} \cdot \frac{Z(\lambda)}{n!}\right\}_{\lambda=c} = k\varepsilon^{i\varphi} \quad$ (k, φ real constants)

we have
$$Z(\lambda) = Z(\rho,\theta) \backsimeq k\rho^n\varepsilon^{i(n\theta+\varphi)}$$
$$\mathcal{R}e\, Z(\lambda) \backsimeq k\rho^n \cos(n\theta + \varphi) \tag{17}$$

whence the truth of the lemma is evident.

A zero within the right-half plane would therefore obviously violate the condition of "positiveness". Moreover on the boundary of this region (i.e. on the imaginary axis) only two sectors are possible; the real part of the function must be positive on the one side of the boundary, negative on the other. This means that zeros on the imaginary axis must be simple and that φ in equation (17) must be zero, i.e. $\left[\frac{d}{d\lambda} Z(\lambda)\right]_{\lambda=i\sigma}$ is a real positive constant.

Corresponding conditions for the poles are immediately evident from Theorem I, Coroll.1. For a pole, however, the condition $\left(\frac{d}{d\lambda}\frac{1}{Z(\lambda)}\right)_{\lambda=i\sigma}=$ a real positive constant, can be expressed more simply by saying that the residue of the function at $\lambda = i\sigma$ must be real and positive. All these facts are summarized in

Theorem II. If $Z(\lambda)$ is a "positive" function then

(i) its zeros and poles must lie in the left half plane or on the boundary;

(ii) zeros and poles on the imaginary λ axis must be simple;

(iii) at a zero on the imaginary λ axis $\frac{dZ}{d\lambda}$ is a real positive (non-zero) constant while at a pole on the boundary the residue is a real positive constant.

33. This theorem is very important for a clear understanding of the properties of positive functions. It is interesting to note that parts (ii) and (iii) contain more than is immediately evident physically, but are nevertheless consequences of necessary physical conditions.

Of especial interest are the zeros and poles at $\lambda = 0$ and $\lambda = \infty$. These points lie on the boundary; consequently at them only simple poles and zeros can exist.

If $\lambda = 0$ is a simple zero or a pole, λ will be a factor in the numerator or denominator respectively. If $\lambda = \infty$ is a simple pole or zero, the degree of the numerator or denominator will be greater by one than the degree of the denominator or numerator respectively. This consequence of Theorem II (ii) can therefore be summarized as

Theorem II Coroll. 1. The degree in λ or in $\frac{1}{\lambda}$ of numerator and denominator of $Z(\lambda)$ cannot differ by more than 1.

34. We interpose here a well known theorem in function theory, which finds special application in the theory of potentials. This theorem states that a function $R(x,y)$ which is regular within a certain region attains its maximum and minimum value on the boundary of that region*. Applying this to our case we have

Theorem III. If $Z(\lambda)$ is a "positive" function the real part of $Z(\lambda)$ in the right half plane attains its minimum value on the axis of imaginaries.

Theorem III. Coroll. 1 If R_1 is equal to or less than the minimum value of the real part of $Z(i\omega)$ then

$$Z'(\lambda) = Z(\lambda) - R_1$$

is a "positive" function.

35. We shall apply this theorem immediately to the separation of poles on the axis of imaginaries.

Consider first a pole at $\lambda = \infty$. The numerator is of degree one higher than the denominator, i.e.

$$Z(\lambda) = \frac{a_{n+1}\lambda^{n+1} + a_n\lambda^n + --- + a_1\lambda + a_0}{b_n\lambda^n + --- + b_1\lambda + b_0}$$

By ordinary division of numerator into denominator

$$Z(\lambda) = L\lambda + \frac{a'_n\lambda^n + --- + a'_1\lambda + a'_0}{b_n\lambda^n + --- + b_1\lambda + b_0}$$

where $L = \frac{a_{n+1}}{b_n}$, $a'_r = a_r - L b_{r-1}$

Note that the real part of $L\lambda$ on the axis of imaginaries

*See for example Bibliog. (33), Vol. II.

is zero, consequently $\mathcal{Re}\, Z'(i\omega) = \mathcal{Re}\, Z(i\omega) \geq 0$ (where $Z'(\lambda) = Z(\lambda) - L\lambda$) Since $\mathcal{Re}\, Z'(\lambda)$ is regular in the right half plane however (i.e. has no poles there) it is everywhere within this region greater than the minimum value on the boundary, i.e. > 0; hence $Z'(\lambda)$ is a "positive" function.

The case when $\lambda = 0$ is a pole follows from the above by making the substitution $\lambda' = \frac{1}{\lambda}$ everywhere.

Then
$$Z(\lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_n\lambda^n}{b_1\lambda + b_2\lambda^2 + \cdots + b_m\lambda^m}$$
$$= \frac{D}{\lambda} + \frac{a_1' + a_2'\lambda + \cdots + a_n'\lambda^{n-1}}{b_1 + b_2\lambda + \cdots + b_m\lambda^{m-1}}$$
$$= \frac{D}{\lambda} + Z'(\lambda)$$

and $Z'(\lambda)$ is a "positive real" function.

If $Z(\lambda)$ has a pole on the boundary at $\lambda = i\sigma$ (say) the condition of reality on the real axis necessitates the presence of the conjugate pole. These poles can always be removed in the following way, which is merely an application of partial fractions.

Let
$$Z(\lambda) = \frac{f(\lambda)}{g(\lambda)} = \frac{f(\lambda)}{(\lambda^2+\sigma^2)g_1(\lambda)}$$

Then
$$Z(\lambda) = \frac{k}{\lambda + i\sigma} + \frac{k}{\lambda - i\sigma} + \frac{f_1(\lambda)}{g_1(\lambda)}$$

(k being the real positive residue of $Z(\lambda)$ at $\lambda = \pm i\sigma$)

$$Z(\lambda) = \frac{K\lambda}{\lambda^2+\sigma^2} + Z'(\lambda) \qquad K = 2k \qquad (18)$$

The important difference between equation (18) and the ordinary partial fraction development is that the numerator of the first partial fraction would ordinarily have been $2(k\lambda + h\sigma)$; here however, because the residue at $\lambda = i\sigma$ is real, $h = 0$. This fact enables us to conclude that the real part of the first partial fraction in equation (18) is zero on the axis of imaginaries whence it follows as before that $Z'(\lambda)$ is a "positive" function. Summing up we have

Theorem IV. If $Z(\lambda)$ is a "positive real" function which has poles (residue $= \frac{K_r}{2}$) at $\lambda = \pm i\sigma_r$ $(r=1,\dots n)$, a pole residue K_0 at $\lambda = 0$, and a pole, residue L, at $\lambda = \infty$ $\sigma_0 = 0$
Then

$$Z(\lambda) = L\lambda + \sum_{r=0}^{n} \frac{K_r \lambda}{\lambda^2 + \sigma_r^2} + Z'(\lambda) \tag{19}$$

where each term is a "positive real" function and $\mathcal{Re}\, Z'(i\omega) = \mathcal{Re}\, Z(i\omega)$.

Theorem IV. Coroll. 1. If $Z(\lambda)$ is a "positive real" function which has zeros at $\lambda = 0, \pm i\sigma_1 \dots \pm i\sigma_r \dots \infty$ then

$$\frac{1}{Z(\lambda)} = C\lambda + \sum_{r=0}^{n} \frac{K_r \lambda}{\lambda^2 + \sigma_r^2} + \frac{1}{Z'(\lambda)} \tag{19a}$$

where each term is a "positive real" function and $\mathcal{Re} \frac{1}{Z'(i\omega)} = \mathcal{Re} \frac{1}{Z(i\omega)}$

Theorem IV. Coroll. 2. If $Z(\lambda)$ is a "positive real" function all of whose poles lie on the imaginary axis then $Z'(\lambda)$ in (19) is a "real positive" constant. A similar statement holds when all the zeros lie on the axis of imaginaries by applying Corollary 1.

The interpretation of each term corresponding to a pole or zero on the axis of imaginaries in equations (19) (19a) as part of a network is evident.

36. Corresponding to Theorem III above is the well known fact that a function $R(x,y)$ which is regular within a certain region is completely specified within that region by the values which it takes on the boundary*. The full

*Bibliog. 33, Vol. II, pp. 67 et seq.

implications of this will be discussed at greater length in a later chapter (Ch. VI). At present we wish to note only the connection between this fact and the possibility of stating the necessary and sufficient conditions in terms of conditions on the boundary.

We note first of all that the condition of regularity in the right-half plane excludes the possibility of poles within this region, i.e. is synonymous with equation (2b) section 26. We have seen further that poles on the imaginary axis are independent of the real part of $Z(i\omega)$ (Theorem IV). Consequently the condition to be satisfied by poles on the boundary must be stated separately. If in addition to this we state that the real part of $Z(i\omega)$ is positive, all conditions for a "positive" function are fulfilled. We thus have

Theorem V. If $Z(\lambda)$ is a function such that

(i) No poles lie in the right half plane

(ii) Poles on the boundary have positive real (finite) residues

(iii) $\mathcal{R}e\, Z(i\omega) \geq 0$

then $Z(\lambda)$ is a "positive" function. In particular the zeros $\lambda = \alpha_r$ will satisfy the condition 2(a).

By Theorem I, Coroll. 1 we have also

Theorem V. Coroll. 1. If $Z(\lambda)$ satisfies the conditions

(i) No zeros lie within the right half plane

(ii) At zeros on the boundary $\frac{dZ}{d\lambda}$ = a positive real (non-zero) constant.

(iii) $\mathcal{R}e\, Z(i\omega) \geq 0$

then $Z(\lambda)$ is a "positive" function.

In particular the poles $\lambda = \beta_r$ will satisfy the condition (2b).

It is always possible to reduce a given function to one in which the second restriction in Theorem V is not necessary by the procedure of Theorem IV.

37. So far we have dealt entirely in rectangular coordinates in the λ and Z planes. In Chapter I, section 25 the question of an inequality between λ and $Z(\lambda)$ in polar coordinates was raised. This is essentially a question of conformality, and like most questions of this nature can be made to depend on the following fundamental lemma:-

Schwarzian Lemma:* If W is a function of γ which is regular in the interior of the unit circle ($|\gamma| < 1$), and which has $|W| < 1$ when $|\gamma| < 1$, and $W(o) = 0$, then it also satisfies $\left|\frac{W}{\gamma}\right| < 1$ for $|\gamma| < 1$. (except when $W = \gamma\varepsilon^{i\varphi}$).

In geometrical language this means that if a transformation $W(\gamma)$ transforms the interior of the unit circle into a part of itself, and leaves the origin fixed, then the distances from the origin of all points within the unit circle are shortened by this transformation.

By the application of a homographic transformation to γ and W the Schwarzian lemma has been generalized to the following:

Theorem due to Pick:(32) If the function W of γ has no essential singularities for values of γ within the circle K_γ, and takes on values which lie only in the interior of another circle K_W, then all non-euclidean distances, elements of arcs and arcs are shortened in the conformal

*H.A.Schwarz 1869. See also Bibliog. (32) & (33) Vol.II,p.114
I am indebted to Dr. Cauer for these references.

mapping by $W(\gamma)$. If one such measure remains unchanged, all remain unchanged and W is a linear function of γ.

In this theorem a non-euclidean distance between two points γ_1, γ_2 is defined by the logarithm of a cross-ratio in the usual manner of projective measurement* as follows:- Through the points γ_1, γ_2 construct a circle orthogonal to the circle K_γ and cutting it in the points h and k; the non-euclidean distance is then

$$\rho(\gamma_1, \gamma_2) = \log \frac{\gamma_1 - h}{\gamma_1 - k} \cdot \frac{\gamma_2 - k}{\gamma_2 - h} \tag{20}$$

This theorem of Pick also holds for the limiting case when the interior of the circle K_γ or K_w becomes a half-plane.

Now W will be a "positive" function of γ if K_γ and K_w become the right half of the γ and W planes respectively. The further restriction that W be real for real values of γ makes the correspondence between W and γ on the one hand and a "positive real" function $Z(\lambda)$ and λ on the other hand complete.

All that remains is to interpret the non-euclidean distances. We shall do this only for the more interesting cases:-

(i) Two conjugate points $\lambda, \bar{\lambda}$ will transform into two other conjugate points $Z, \bar{Z}$ because of the condition of "reality". In Fig.7 let $P_1(\lambda)$, $P_2(\bar{\lambda})$ be a pair of conjugate points and let

$$\lambda = \nu + i\sigma \quad , |\lambda| = \rho$$

The circle through $\lambda, \bar{\lambda}$ orthognal to the boundary of the right half plane will cut this boundary in the points

*See for example Bibliog. (34).

$Q_1(0+i\rho)$ and $Q_2(0-i\rho)$

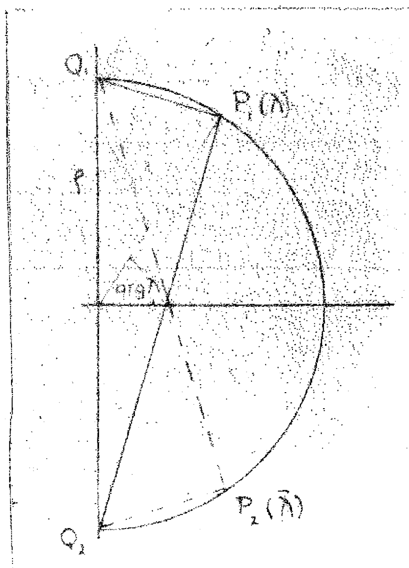


Fig. 7.

$$\rho(\lambda\bar{\lambda}) = \log\frac{(\nu + i\overline{\sigma-\rho})}{(\nu + i\overline{\sigma+\rho})}\cdot\frac{(\nu - i\overline{\sigma-\rho})}{(\nu - i\overline{\sigma+\rho})}$$

$$= \log\frac{\nu^2+(\sigma-\rho)^2}{\nu^2+(\sigma+\rho)^2}$$

$$= 2\log\frac{P_1Q_1}{P_1Q_2} \qquad (21)$$

As $\rho(\lambda,\bar{\lambda})$ decreases towards zero (in absolute magnitude since sign does not enter into the idea of distance) the ratio $\frac{P_1Q_1}{P_1Q_2}$ must approach unity. A ratio of lengths will involve only angles and in Fig. 7 an approach of the ratio $\frac{P_1Q_1}{P_1Q_2}$ to unity is readily seen to mean a decrease in the angle P_1OP_2 towards zero, i.e. $|\arg\lambda|$ decreases

(ii) Consider two points $\lambda_1\lambda_2$ on the real axis. The orthogonal circle in this case is the real axis itself. The non-euclidean distance, obtained by considering a limit, becomes

$$\rho(\lambda_1,\lambda_2) = \log\frac{\lambda_1}{\lambda_2} \qquad (22)$$

From this we see that if for one value of λ on the positive real axis we have $\lambda_1 = Z(\lambda_1) = Z_1$ this will be the only point for which such a relation holds unless $\lambda \equiv Z(\lambda)$. For if there be another point where $\lambda_2 = Z(\lambda_2) = Z_2$ we shall have

$$\rho(\lambda_1,\lambda_2) = \log\frac{\lambda_1}{\lambda_2} = \log\frac{Z_1}{Z_2} = \rho(Z_1\,Z_2)$$

which by the theorem of Pick is possible only if this is true for all values of λ. The same is obviously true if we substitute $L\lambda$ for λ (L a real positive constant).

Hence we have

Theorem VI (i) If $Z(\lambda)$ is a "positive real" function $|\arg Z(\lambda)| \leq |\arg \lambda|$ for all values of λ satisfying $0 < |\arg \lambda| \leq \frac{\pi}{2}$. The equality signs can only hold simultaneously, unless they hold identically.

(ii) On the positive real axis (where $\arg \lambda = 0$) the equation $L\lambda = Z(\lambda)$ cannot be satisfied at more than one internal point (L a real positive constant).

Theor. VI. Coroll.1. The function $(Z(\lambda) - L\lambda)$, (which is not necessarily a positive function) cannot have more than one zero within the right-half plane.

Theorem VI may be interpreted in the light of the physical interpretation given to Theorem I. Thus if λ be replaced by a "positive" function of λ the resulting network will have a power factor closer to unity than the original network, for corresponding frequencies.

38. It is now apparent that if $Z(\lambda)$ satisfies the conditions

$$Z(\lambda) \text{ real when } \lambda \text{ is real} \tag{1c}$$

and
$$\left.\begin{array}{l} \mathcal{R}e\, Z(\lambda) \geq 0 \\ \text{when } \mathcal{R}e\, \lambda \geq 0 \end{array}\right\} \tag{15}$$

it will satisfy all the necessary conditions listed in section 26; for conditions2(a) and 2(b) follow from Theorem II; condition (3) is merely (15) when $\mathcal{R}e\, \lambda = 0$; and condition (15a) is fulfilled by Theorem VI.

Theorem V gives two sets of conditions equivalent to (15), which are easier to handle numerically and more readily interpreted electrically.

It is further evident that if $Z(\lambda)$ satisfies conditions 15(a), it must satisfy condition (15) as well as condition (1c). Hence 15(a) is a necessary and sufficient condition for $Z(\lambda)$ which may replace (1c) and (15).

It is possible to add to the necessary conditions listed in section 26 by the application of special sets of physical conditions. All further necessary conditions would at once be disposed of if we can show that every "positive real" function has a network representation and must therefore satisfy every physical condition that can be imposed on the network.

For this purpose the Theorems just discussed will prove particularly useful.

CHAPTER III.

Functions with not more than Two Poles (or Zeros) in the Interior of the Left Half Plane.

39. With the more thorough understanding of the properties of "positive" functions obtained in the preceding chapter we proceed to consider some of the simpler cases of positive functions. These have been treated in part by Foster, Cauer and Fry, but the greater power of the methods now at our disposal will be abundantly clear even for these cases.

For convenience in calculation we use the conditions for a "positive real" function as stated in Theorem V (previous chapter). We may restrict ourselves to functions which have no poles or zeros on the axis of imaginaries for if such poles or zeros are present they may be removed in the manner demonstrated in Theorem IV; they correspond to pure reactance elements in series or parallel with each other and with the circuit.

(a) Functions with One Pole (or Zero)

40. Consider the function*

$$Z(\lambda) = \frac{a_0 + a_1\lambda}{b_0 + b_1\lambda} \tag{23}$$

$$\mathcal{R}e\, Z(i\omega) = \frac{a_0 b_0 + a_1 b_1 \omega^2}{b_0^2 + b_1^2\omega^2} \tag{23a}$$

We may arbitrarily assume a_0 positive without imposing any restrictions on $Z(\lambda)$. For $\mathcal{R}e\, Z(i\omega) \geq 0$ we must have $b_0 > 0$ and $a_1 b_1 \geq 0$. The condition for the pole requires $b_1 \geq 0$ hence also $a_1 \geq 0$. (In this particular case the condition

*This function was also discussed by Fry, Bibliog. (15)

for poles and zeros includes the condition for a "positive" function).

Let us now plot $\mathcal{R}e\, Z(i\omega)$ as a function of $\Omega=\omega^2$(Negative values of ω^2 are included in the figure for the sake of completeness although they do not enter into the present discussion). The curve is a hyperbola which takes one of

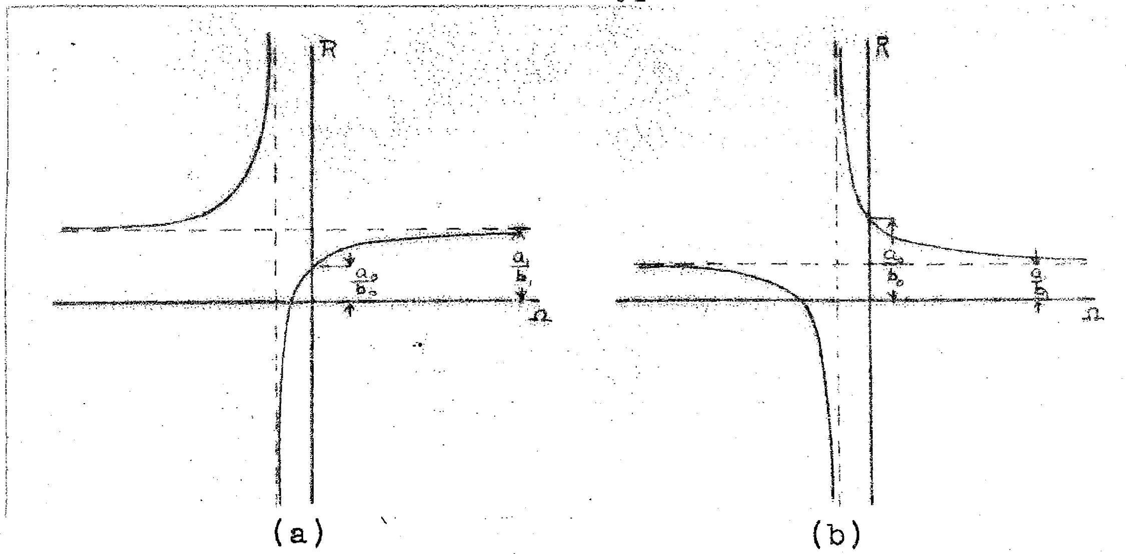


Fig. 8

two forms depending on whether $\frac{a_0}{b_0} <$ or $> \frac{a_1}{b_1}$ (see Fig. 8). In the case $\frac{a_0}{b_0} < \frac{a_1}{b_1}$ (Fig. 8a) the smallest value of $\mathcal{R}e\, Z(i\omega)$ in the right half plane occurs at the origin. This value may be subtracted from $Z(\lambda)$ and still leave a "positive" function (Theorem III, Coroll. 1). Then

$$Z(\lambda) = \frac{a_0}{b_0} + \frac{(a_1 b_0 - a_0 b_1)\lambda}{b_0^2 + b_0 b_1 \lambda}$$

$$= R_1 + Z'(\lambda) \qquad (23b)$$

$Z'(\lambda)$ may be immediately recognized as the impedance function of a network, or we may proceed by noting that it has a zero at the origin. Hence applying Theorem IV

$$\frac{1}{Z'(\lambda)} = \frac{b_0^2}{(a_1 b_0 - a_0 b_1)\lambda} + \frac{b_0 b_1}{(a_1 b_0 - a_0 b_1)} \qquad (23c)$$

The corresponding network is shown in Fig. 9a (next page).

41. An exactly similar procedure is possible when $\frac{a_0}{b_0} > \frac{a_1}{b_1}$ (see Fig. 8b).

Then $$Z(\lambda) = \frac{a_1}{b_1} + \frac{(a_0 b_1 - a_1 b_0)}{b_0 b_1 + b_1^2 \lambda}$$

---and so on.

(The results are obtainable from those of section 40 by the substitution $\lambda' = \frac{1}{\lambda}$ or by an interchange of the subscripts 0 and 1). The corresponding network is shown in Fig. 9b.

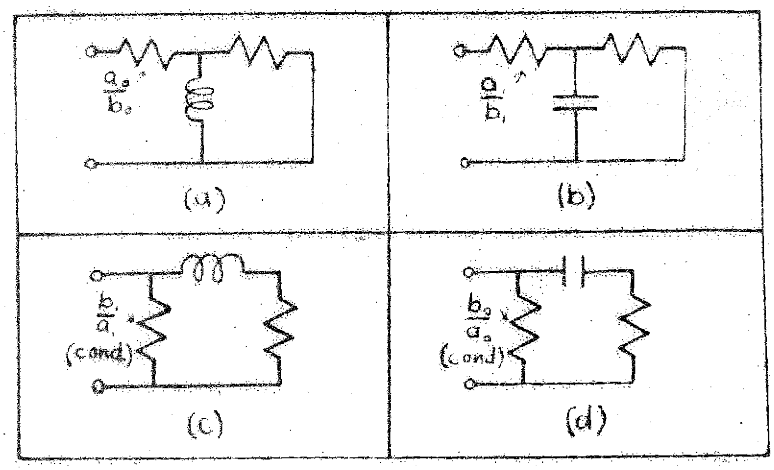


Fig. 9.

The network representation of 23 thus contains resistance and inductance or resistance and capacitance according as

$$\frac{a_0}{b_0} < \text{or} > \frac{a_1}{b_1} \quad (23d)$$

When $\frac{a_0}{b_0} = \frac{a_1}{b_1}$, $Z(\lambda)$ obviously reduces to $R = \frac{a_0}{b_0} = \frac{a_1}{b_1}$.

42. A corresponding discussion which is dual to the above in the sense of Theorem I, Coroll. 1 can be carried through with $\frac{1}{Z(\lambda)}$ giving rise to the circuits in Fig. 9(c)(d). The networks of Figs. 9(c) and (d) are <u>equivalent</u> to the networks of Figs. 9(a) and (b) respectively.

It is interesting to note that a complete set of equivalent circuits is obtained by combining the processes leading to Figs. 9(a) and 9(c) in different ways. For

For example if $\frac{a_0}{b_0} < \frac{a_1}{b_1}$, any value less than $\frac{a_0}{b_0}$ can be separated as a series resistance (Fig. 9a) and thereafter the remaining network constructed as in Fig.9c.

43. The above discussion disposes immediately of such forms as

$$\begin{array}{ll|ll}
\text{(i)} & Z(\lambda) = \dfrac{a_0 + a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda} & \text{(iii)} & Z(\lambda) = \dfrac{a_0 + a_1\lambda + a_2\lambda^2}{b_1\lambda + b_2\lambda^2} \\
\text{(ii)} & Z(\lambda) = \dfrac{a_0 + a_1\lambda}{b_0 + b_1\lambda + b_2\lambda^2} & \text{(iv)} & Z(\lambda) = \dfrac{a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2}
\end{array} \tag{23c}$$

for which the corresponding networks are shown in Fig.10. The coefficients in these functions may not be arbitrary

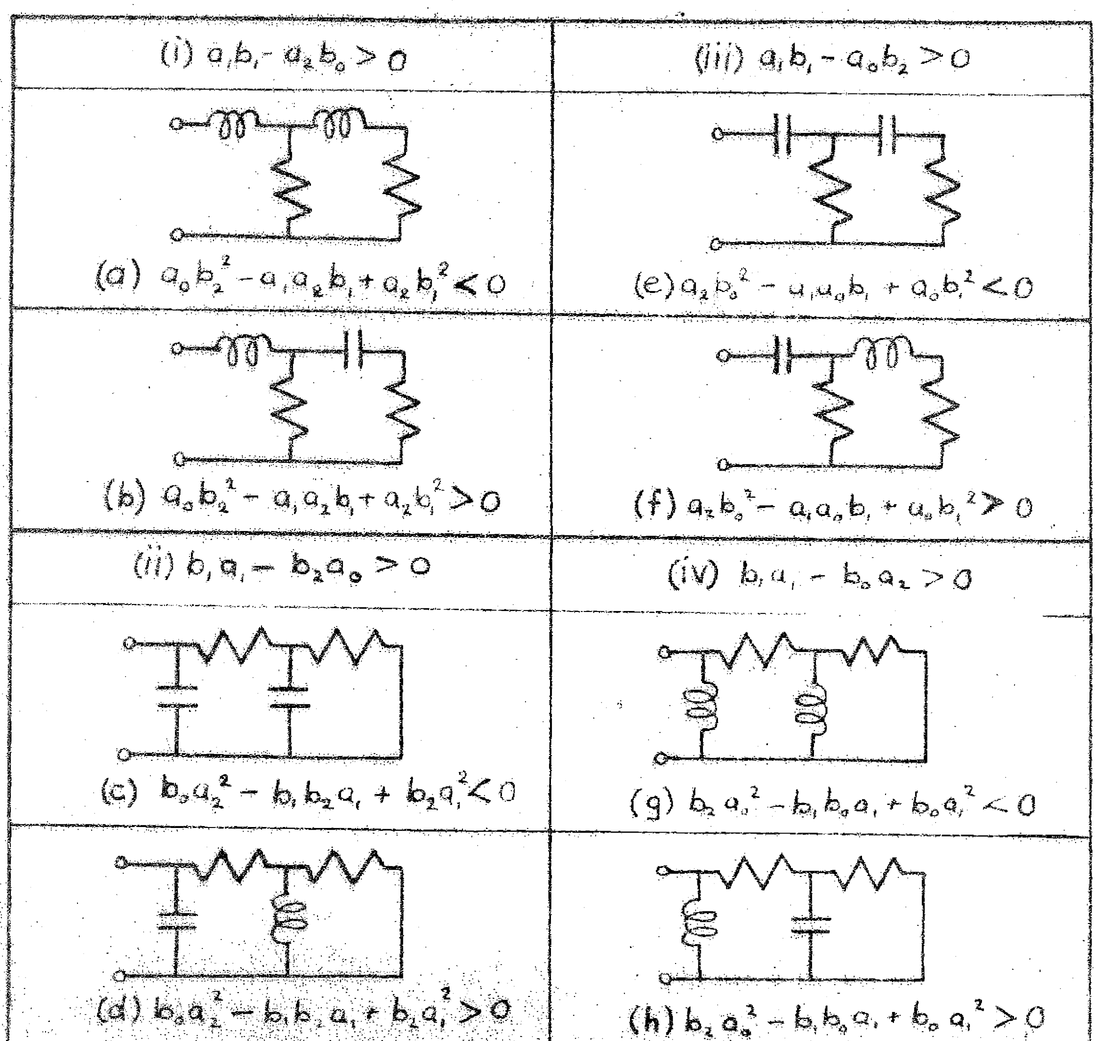


Fig 10.

positive constants as in (23) the additional conditions which they must fulfill being shown in the Figure for each case. These conditions are readily obtained by applying the conditions for (23) to the reduced functions which result on separating the pole (or zero) on the imaginary axis from the forms (23c). The poles and zeros on the imaginary axis in the forms (23c) are situated at $\lambda = 0$ or $\lambda = \infty$, but they may obviously lie at any point on this axis. The removal of such poles by Theorem IV supplemented by the procedure of sections 40 and 41 will always lead to a network representation if only one pole and one zero lies in the interior of the left-half plane.

It is interesting to note the dualities in Fig. 10. The forms (ii) and (iv) are derivable from (i) and (iii) respectively by the substitution $Z' = \frac{1}{Z}$ while (iii) and (iv) are derivable from (i) and (ii) respectively by the substitution $\lambda' = \frac{1}{\lambda}$ (compare Theor.I Coroll. 1 and 2). A discussion of any one of them is therefore readily adaptable to all the others.

Equivalents to the networks in Fig. 10 similar to those in Fig. 9 are also possible.

(b) Two Poles and Two Zeros not on the Boundary.

44. Consider next the case

$$Z(\lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2} \qquad (24)$$

$$\mathcal{R}e\, Z(i\omega) = \frac{a_0 b_0 + (-a_0 b_2 + a_1 b_1 - a_2 b_0)\omega^2 + a_2 b_2 \omega^4}{b_0^2 + (b_1^2 - 2b_0 b_2)\omega^2 + b_2^2\omega^4} \qquad (24a)$$

The condition for the poles to have negative real part is given by $b_0, b_1, b_2 > 0$ (24b)

In (24a) the denominator cannot be hegative for real values of ω. (This is best seen by expressing it as the sum of squares). Hence the numerator must always be positive, i.e. it must have no real positive roots when considered as a function of $\tilde{\omega}$. Hence we must have

$a_0 > 0$, and either

$$a_2 > 0, \quad a_0 b_2 - a_1 b_1 + a_2 b_0 \leq 0 \qquad (24c)$$

or

$$(a_0 b_2 - a_1 b_1 + a_2 b_0)^2 - 4 a_0 a_2 b_0 b_2 \leq 0 \qquad (24d)$$

The coefficients are thus restricted in their range of values over and above the Hurwitz requirements (16a and 16b, section 28) by the equations (24c) or (24d). These restrictions can be given a more tangible geometrical meaning in the following manner*.

Consider the ratios $a_0:a_1:a_2$ to be fixed, the only restriction being that they be positive; thereafter the ratios $b_0:b_1:b_2$ are restricted to lie within certain limits by the inequalities (24c) or (24d). The latter may be chosen as coordinates of a point in a plane. The inequalities (24c) then state that the point $(\frac{b_1}{b_0}, \frac{b_2}{b_0})$ may lie on one side of a certain straight line, while the inequality (24d) states that it may lie within a certain parabola. To make this more clear let us write

$$\left.\begin{array}{ll} x = \frac{b_1}{b_0}, & y = \frac{b_2}{b_0} \\ \frac{a_1}{a_0} = \alpha, & \frac{a_2}{a_0} = \beta \end{array}\right\} \qquad (25)$$

*Compare Fig. 2 Introduction, section 8.

Then (24c) and (24d) become

$$y - \alpha x + \beta \leq 0 \qquad (24c')$$
$$(y - \alpha x + \beta)^2 - 4\beta y \leq 0 \qquad (24d')$$

The straight line is a diameter of the parabola. The location of these curves is shown in Fig. 11 and permissible values of ($x = \frac{b_1}{b_0}$, $y = \frac{b_2}{b_0}$) for the given values of ($\frac{a_1}{a_0}$, $\frac{a_2}{a_0}$) lie within the shaded region (here the inequalities (24b) have also been considered). The curves themselves are completely determined by the two parameters (α, β) which have a simple geometrical meaning as coordinates of the two tangent points T_1, T_2. (An exactly similar figure is of course possible for restricting the a's when the b's are fixed).

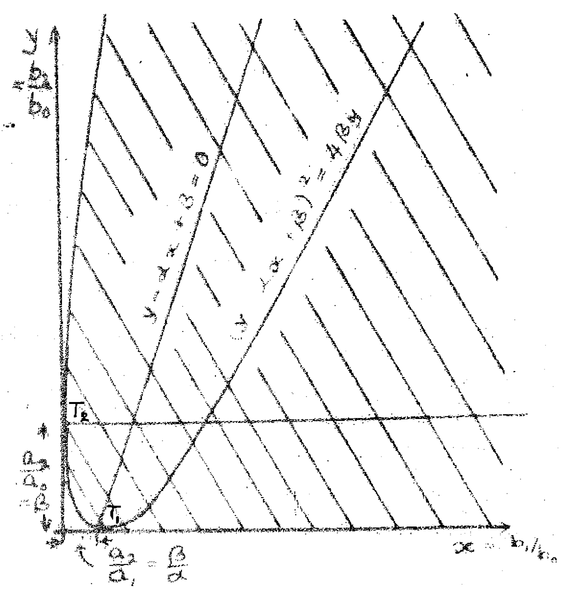


Fig. 11

45. Assuming that the coefficients satisfy the conditions (24b,c,d) we return to a discussion of the possible manners of variation of $\mathcal{Re}\, Z(i\omega)$ according to equation (24a). Let us write

$$\mathcal{Re}\, Z(i\omega) = R = \frac{A_0 + A_1\Omega + A_2\Omega^2}{B_0 + B_1\Omega + B_2\Omega^2} \quad \text{where } \Omega = \omega^2 \qquad (26)$$

The discussion of such functions is an elementary problem in algebra, and is usually carried through in the following way. From equation (26)

$$(RB_o - A_o) + (RB_1 - A_1)\Omega + (RB_2 - A_2)\Omega^2 = 0 \tag{26a}$$

For real values of Ω we must have

$$(RB_1 - A_1)^2 - 4(RB_o - A_o)(RB_2 - A_2) \geq 0$$

i.e. $R^2(B_1 - 4B_oB_2) - 2R(A_1B_1 - 2A_oB_2 + A_2B_o) + (A_1 - 4A_oA_2) \geq 0$ (27)

This means that R can or cannot (for real values of Ω) take on values lying between the roots of the expression (27) according as the sign of $(B_1 - 4B_oB_2)$ is negative or positive. (Note that this is identical with the sign of $b_1 - 4b_ob_2$, i.e. is determined by the poles of $Z(\lambda)$ being real or complex).

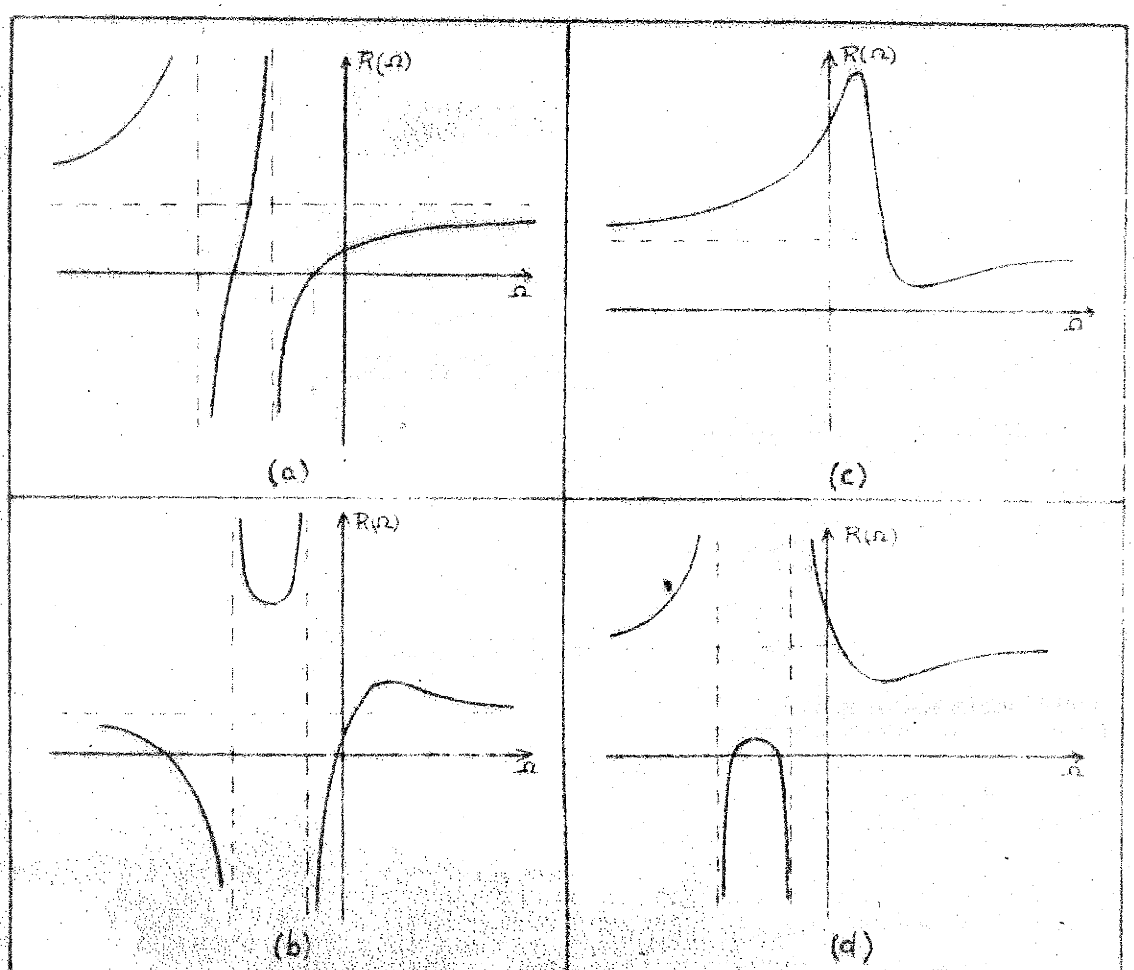


Fig 12

Only in the case where the roots of this expression are equal or pure imaginary can R take on all values. Typical curves for R are shown in Fig. 12. In these figures the axes can be shifted parallel to themselves and reverse direction in any manner provided only R remains positive for positive values of Ω.

We shall show first that one network can always be constructed to correspond to an impedance function of the given form (24), if this condition is fulfilled.

46. As in sections 40, 41 the procedure depends on the frequency at which the least value of 'resistance' occurs. For this purpose we need consider only positive values of Ω in Fig. 12 since these alone correspond to points on the boundary of the right-half λ plane (or to real frequencies).

If this least value occurs for $\Omega = 0$ or ∞, as for example in Figs. 12(a) and (b) the procedure is essentially the same as that described in sections 41, 42. To illustrate, let the minimum value of R occur for $\Omega = 0$, i.e.

$$\frac{a_0}{b_0} \leq R(\Omega) \text{ for } \Omega \geq 0 \tag{28}$$

Then

$$\begin{aligned} Z(\lambda) &= \frac{a_0 + a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2} \\ &= \frac{a_0}{b_0} + \frac{(a_1b_0 - a_0b_1)\lambda + (a_2b_0 - a_0b_2)\lambda^2}{b_0(b_0 + b_1\lambda + b_2\lambda^2)} \\ &= \frac{a_0}{b_0} + Z'(\lambda) \end{aligned} \tag{28a}$$

in which the function $Z'(\lambda)$ is a "positive" function under the assumption (28) by Theorem III, Coroll. 1. Furthermore this form of $Z'(\lambda)$ has already been discussed (Fig. 10 (iv), page 42). Consequently $Z(\lambda)$ can be represented by such a network together with a series resistance, as shown in Fig. 13(i).

The case when $R(\omega)_{min}$ occurs for $\omega = \infty$ needs no further discussion, the corresponding networks being shown in Fig. 13(ii)

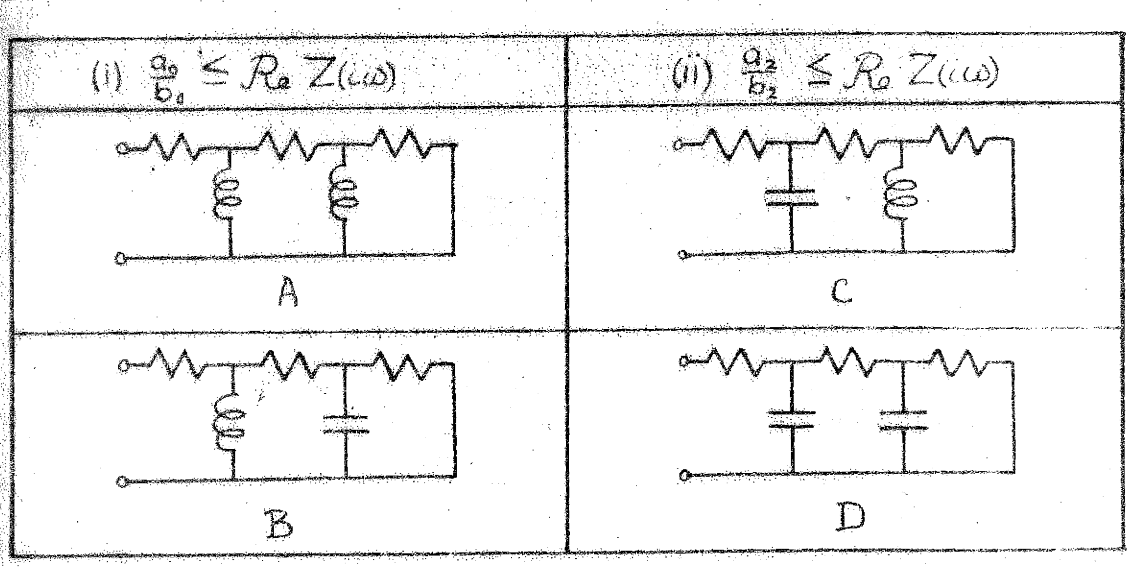


Fig. 13

47. There remains the case when $R(\omega)_{min}$ occurs for some finite value of $\omega = \sigma$. Proceeding by analogy, let this value R_σ be separated out. It will be a solution of (27) (with the equality sign). Let us write

$$Z(\lambda) = R_\sigma + \frac{a_0' + a_1'\lambda + a_2'\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2} = R_\sigma + Z'(\lambda) \qquad (29)$$

The function $Z'(\lambda)$ now does not have a zero on the axes of imaginaries, as in Equation (28a), but will be characterized by the fact that for some value of frequency ($\lambda = i\sigma$) $Z'(\lambda)$ will be pure imaginary. Physically this means that the corresponding network behaves like a pure reactance at this frequency. This can only mean that that part of the network which contains resistance is short circuited by (i.e. is in parallel with) a reactive branch which is in resonance for this frequency; in series with this part is a pure reactance which may be either positive or negative, since the pure imaginary part may have either sign. The only type of element which can meet this requirement is inductance, and in fact mutual inductance. Accordingly Fig. 15 (page 50) is a possible representation of $Z'(\lambda)$. Let us first consider this kind of connection in greater detail.

48. Equivalent Representation of a Mutual Inductance. It is a well known fact that two mutually coupled inductance coils (L_a, L_b, M_{ab}) connected at one point have an equivalent representation in terms of three inductances L_1, L_2, L_3 connected in T as shown in Fig. 14. The necessary relations are

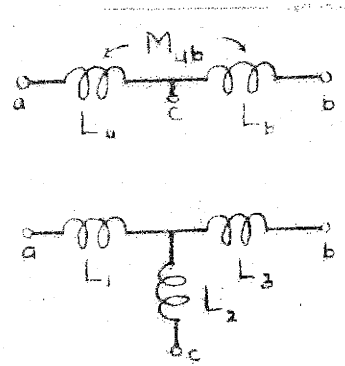


Fig. 14

$$\begin{aligned} -L_2 &= M_{ab} \\ L_1+L_2 &= L_a \qquad (30) \\ L_3+L_2 &= L_b \end{aligned}$$

To the condition $\quad L_aL_b - M_{ab}^2 \geq 0 \qquad (30a)$

corresponds (by substitution) $L_1L_2 + L_2L_3 + L_3L_1 \geq 0 \qquad (30b)$

(In this relation a positive sign for M_{ab} means that the coils are connected with fluxes aiding). It thus appears that L_1, L_2, L_3 must satisfy the inequality (30b) to be physically realisable by two mutually coupled inductances. It is further necessary that at least two of them be positive if a positive L_a, L_b are to result from them. This is also sufficient, for if the positive values be denoted by L_1, L_3, and L_2 is negative and equal to L_2', (1) can be written

$$\frac{1}{L_2'} \geq \frac{1}{L_1} + \frac{1}{L_3}$$

i.e. $$L_2' < \text{either } L_1 \text{ or } L_3 \tag{30c}$$

It also appears that it is immaterial which of L_1, L_2, L_3 be chosen equal to M_{ab} in (30).

49. <u>A Useful Four Terminal Network.</u> Consider now the type of network shown in Fig. 15 which was suggested by its property of being purely reactive at a given frequency $\sigma^2 = \frac{D}{L_2}$

$$Z(\lambda) = L_1\lambda + \frac{(L_2\lambda^2 + D)(L_3\lambda + \mathfrak{z})}{(L_2\lambda^2 + D) + \lambda(L_3\lambda + \mathfrak{z})}$$

$$= \frac{(L_1L_2 + L_2L_3 + L_1L_3)\lambda^3 + (L_1 + L_2)\mathfrak{z}\cdot\lambda^2 + (L_1 + L_3)D\lambda + D\mathfrak{z}}{(L_2 + L_3)\lambda^2 + \mathfrak{z}\lambda + D} \tag{31}$$

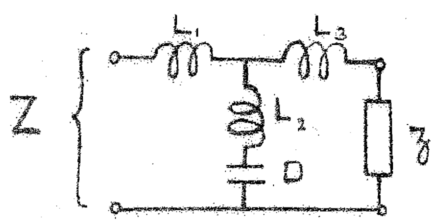


Fig.15

It now becomes evident that if this network is to represent $Z'(\lambda)$ in the case under discussion (29) we must have

$$L_1L_2 + L_2L_3 + L_3L_1 = 0 \tag{31a}$$

and $\mathfrak{z}$ a pure resistance. (= r)

Relation (31a) states that L_1, L_2, L_3 are equivalent to two perfectly coupled inductances ($L_aL_b - M_{ab}^2 = 0$)*.

---

*Note: This is not a limitation of the method of representation but of the given form of impedance function just as $L\lambda$ is an ideal inductance without resistance).

50. We are now in a position to construct a network corresponding to $Z(\lambda)$ when the minimum value R_σ of $\mathcal{R}_e Z(i\omega)$ occurs for a finite frequency $\omega = \sigma$

We have
$$Z(\lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2}$$
$$= R_\sigma + Z'(\lambda) \tag{32}$$

where $\mathcal{R}_e Z'(i\omega) = 0$ has a double root at $\omega = \pm\sigma$

i.e. if $\mathcal{R}_e Z'(i\omega) = \dfrac{A_0' + A_1\omega^2 + A_2\omega^4}{B_0 + B_1\omega^2 + B_2\omega^4}$ (cf. eqn.26)

we have
$$0 = A_1'^2 - 4A_0A_2 \tag{32a}$$

Further let $Z'(\lambda) = \dfrac{a_2'\lambda^2 + a_1'\lambda + a_0'}{b_2\lambda^2 + b_1\lambda + b_0}$ $\quad (a_r' = a_r - R_\sigma b_r)$
$$\equiv \frac{(L_1+L_2)r\lambda^2 + (L_3+L_1)D\lambda + Dr}{(L_2+L_3)\lambda^2 + r\lambda + D} \tag{32b}$$

whence by a comparison of the ratio of coefficients

$$\left.\begin{aligned}
r &= \frac{a_0'}{b_0} \;;\quad D = \frac{a_0'}{b_1}\left(= b_0 \times \frac{a_0'}{b_1 b_0}\right)\\
L_1 + L_2 &= \frac{a_2'}{b_1}\\
L_2 + L_3 &= b_2 \times \frac{a_0'}{b_1 b_0}\\
L_1 + L_3 &= \frac{a_1'}{b_0}\\
\text{giving}\quad L_1 &= \frac{-a_0'b_2 + a_1'b_1 + a_2'b_0}{2b_1b_0}\\
L_2 &= \frac{a_0'b_2 - a_1'b_1 + a_2'b_0}{2b_1b_0}\\
L_3 &= \frac{a_0'b_2 + a_1'b_1 - a_2'b_0}{2b_1b_0}
\end{aligned}\right\} \tag{32c}$$

with the condition
$$0 = L_1L_2 + L_2L_3 + L_3L_1$$
$$\text{i.e}\quad 0 = (a_0'b_2 - a_1'b_1 + a_2'b_0)(a_0'b_2 + a_1'b_1 - a_2'b_0) + 2a_0'b_2(-a_0'b_2 + a_1'b_1 + a_2'b_0)$$
$$= 4a_0'a_2'b_0b_2 - (a_0'b_2 - a_1'b_1 + a_2'b_0)^2$$
$$\doteq 4A_0'A_2' - A_1'^2 \quad \text{[cf. eqns.(24a) and (26)]} \tag{32d}$$

which is satisfied by equation (32a).

51. A "positive" function $Z(\lambda) = \dfrac{a_0 + a_1\lambda + a_2\lambda^2}{b_0 + b_1\lambda + b_2\lambda^2}$ can thus

always be represented by one of the networks of Fig. 13 or the network of Fig. 16.

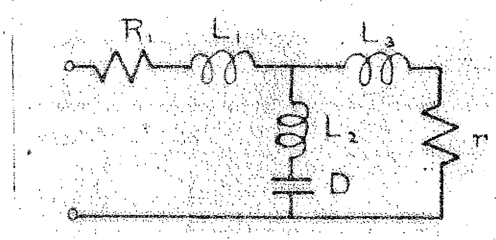


Fig. 16

All these networks have the minimum number of independent elements in the sense that this number is equal to the number of independent coefficients in $Z(\lambda)$. This number is 5. In Fig. 16 we have apparently six elements but three of them (L_1, L_2, L_3) are related by equation (30b), in fact they correspond to only two inductance coils wound in a special ideal way.

52. Negative Values of Ω (Equivalent two-mesh networks). The foregoing discussion was confined entirely to the consideration of positive values of $\Omega (=\omega^2)$. Negative values of Ω do not correspond to real frequencies, so that their physical significance is not so evident. A complete physical interpretation is, however, not essantial to the process of constructing a network, the mathematical condition of "positiveness" being a sufficient guide.

In identifying the expressions in (32b) a first condition to be satisfied is (31a) which is accomplished by having a double zero in the numerator of $\mathcal{R}e\, Z'(i\omega)$ (equation 32a). This will happen if either of the roots of (27) be subtracted from $Z(\lambda)$. In this there is no reference to the value of Ω for which the double zero of $\mathcal{R}e\, Z'(i\omega)$ occurs, and it may be negative. A second condition, however, is $\mathcal{R}e\, Z'(i\omega) \geq 0$ when $\Omega \geq 0$.

Consequently a root R_v of equation (27) can only be subtracted provided

$$R_v \leq \mathcal{R}\, Z(i\omega) \text{ for all positive values of } \Omega\ (=\omega^2) \tag{33}$$

It is readily seen by referring to Figs. 12 and considering different positions of the axes that in many cases this can be satisfied for an R_v which occurs for negative Ω. Fig. 12(d) is such a case, while a translation of the axes to the left in Fig. 12(c) would also give it. This therefore presents a possibility of obtaining equivalent networks.

(c) Equivalences between 2,3 and 4 Mesh Networks having no Superfluous Elements.

53. Whether it be possible to construct one or more of the networks of Figs. 13 and 16 to represent a given impedance function is determined by the nature of the variation of $\mathcal{R}\, Z(i\omega)$ as exemplified in Fig. 12. The nature of this variation is determined by the relation between the coefficients of $Z(\lambda)$ and this relation in turn is most readily interpreted in terms of Fig. 11, page 45.

54. Two-Mesh Representation of $Z(\lambda)$ in Equation (24). In order that a two-mesh network may be constructed to represent $Z(\lambda)$ it is necessary that at least one of the critical values R_v of $R(\Omega)$ (maximum or minimum) be positive and $<$ $R(\Omega)$ for positive values of Ω. Thus, for example, a curve of the type Fig. 12(a) can never correspond to a two-mesh circuit while curves of the type Fig. 12(b) and (c) may or may not, depending on the

location and direction of the axes. A rough plot of $R(\Omega)$ will readily show if this be possible.

The conditions may, however, also be expressed as inequalities in terms of the coefficients $a_0 a_1 a_2$, $b_0 b_1 b_2$. This has been done by Foster and Cauer - without reference to the variation of $R(\Omega)$ with Ω however, the reasoning being entirely algebraic, and the criterion of realizability being the positiveness of the resulting network elements. Cauer(14) has also interpreted these inequalities in terms of a diagram like Fig. 11 (cf. Fig. 2(a) page 6), the only difference between Cauer's diagram and Fig. 11 being that it refers to the impedance function of a general two-mesh. It may immediately be reduced to our case by making the two additional zeros in the numerator occur at 0 and ∞ respectively. Two cases are to be distinguished according as the remaining zeros are real or complex. The corresponding regions for which two-mesh networks can be constructed are shown in Fig. 17(e) and (f), the latter being the case of real zeros (page 59).

55. Three-Mesh Representations of $Z(\lambda)$ in Equation (24).
In order that a three-mesh network of Fig. 13 may be realizable it is necessary that either $Z(\lambda) - R(0)$ or $Z(\lambda) - R(\infty)$ be "positive" functions. The conditions for this are readily obtainable from the discussion in Section 43.

Thus for the two networks Figs. 13(i) we have

$$Z(\lambda) = \frac{a_0}{b_0} + \frac{(a_1 b_0 - a_0 b_1)\lambda + (a_2 b_0 - a_0 b_2)\lambda^2}{b_0(b_0 + b_1\lambda + b_2\lambda^2)} \qquad (34)$$

The conditions for the second term to be a "positive" function are (see section 43)

$$a_2b_0 - a_0b_2 \geq 0 \tag{34a}$$

and $$(a_1b_0 - b_1a_0)b_1^2 - b_1b_0(a_2b_0 - a_0b_2) \geq 0$$

i.e. $$-a_0b_1^2 + a_0b_0b_2 + a_1b_0b_1 - a_2b_0^2 \geq 0 \tag{34b}$$

With the notation $x = \frac{b_1}{b_0}$, $y = \frac{b_2}{b_0}$

$$\frac{a_1}{a_0} = \alpha \ , \ \frac{a_2}{a_0} = \beta \tag{25}$$

these inequalities may be written

$$y \leq \beta \tag{34a'}$$

$$y \geq x^2 - \alpha x + \beta \tag{34b'}$$

In terms of Fig. 17 the inequality (34a') confines the point (x , y) on the lower side of the straight line $y = \beta$ while the inequality (34b') confines the point to the interior of a parabola. The corresponding region is shown in Figs. 17(a) and (b). Note that the parabola (34b) passes through the points $(0,\beta)$ and (α,β). The latter point lies inside or outside the main parabola (24d) according as $\alpha^2 <$ or $> 4\beta$, i.e. $a_1^2 - 4a_0a_2 <$ or >0, i.e. according as $Z(\lambda)$ has complex or real zeros. Note also that when the zeros are real, the point (α,β) is the point of intersection of the real tangents in Fig. 17(f).

The conditions for the networks of Fig. 13(ii) are derivable from the above by a simple interchange of the subscripts 0 and 2 in (34a) and (34b) giving

$$a_0b_2 - a_2b_0 \geq 0 \tag{34c}$$

and $$-a_2b_1^2 + a_2b_0b_2 + a_1b_1b_2 - a_0b_2^2 \geq 0 \tag{34d}$$

Using the notation (25) these become

$$y \geq \beta \tag{34c'}$$

$$\beta x^2 - \alpha x y + y^2 - \beta y \gtrless 0 \qquad (34d')$$

(34d') represents the interior of an ellipse or hyperbola according as $\alpha^2 <$ or $> 4\beta$. The corresponding region is indicated in Figs. 17(c) and (d).

We can proceed further in our analysis and distinguish between the networks A and B or C and D of Fig. 13 in each of the above cases. The criteria for these networks are deducible from those given in Section 43, Fig. 10. They are nothing more than condition (23d) applied at the corresponding stage in the construction of the three-mesh network. Thus the criterion for Fig. 13 A results from comparison of the form (34) with the form (iv) in (23e), section 43. Referring to Fig. 10(g) it is

$$(a_1 b_0 - a_0 b_1)^2 b_2 - (a_1 b_0 - a_0 b_1)(a_2 b_0 - a_0 b_2) b_1 + (a_2 b_0 - a_0 b_2)^2 b_0 \leq 0$$

which, with a little manipulation, may be rewritten

$$a_0(a_0 b_2 - a_2 b_0)^2 - a_1(a_0 b_1 - a_1 b_0)(a_0 b_2 - a_2 b_0) + a_2(a_0 b_1 - a_1 b_0)^2 \leq 0 \qquad (34e)$$

Substituting in terms of the notation of equation (25)

$$(y-\beta)^2 - \alpha(x-\alpha)(y-\beta) + \beta(x-\alpha)^2 \leq 0 \qquad (34e')$$

The point must lie in a pair of alternate angles between the pair of straight lines (34e'). It is readily shown that these two straight lines are the tangents through the point (α, β) to the parabola (24d). The region is shown by the cross-hatched area in Fig. 17(b), one of the alternate angles being excluded by (34a). (The lines are imaginary in Fig. 17(a), i.e. in this case network A is not possible).

An exactly similar discussion holds for the networks C and D, the condition being derivable from the above by a change of suffixes 0 and 2. The condition (34e) is in fact symmetrical with respect to such a change of suffixes, so that we merely get the other of the alternate angles given by (34e') and (34c'). This is shown by the cross-hatched area in Fig. 17(d).

We note that condition (34e) is exactly the condition for the mutual separation of poles and zeros, which was given by Cauer(14) in determinant form for networks with only two kinds of elements. We note also that for this special case inductance-resistance and capacitance-resistance networks are distinguished by the criterion $a_0b_2 - a_2b_0 < $ or > 0 respectively. This is equivalent to specifying whether the poles $-\beta_r$ and zeros $-\alpha_r$ lie in the order $\beta_1 < \alpha_1 < \beta_2 < \alpha_2$ or $\alpha_1 < \beta_1 < \alpha_2 < \beta_2$ respectively.

Another set of three-mesh networks which may be used for representing $Z(\lambda)$ in (24) under certain conditions arises when we consider $\frac{1}{Z(\lambda)}$ as an admittance function and consider its real part in a manner similar to that of section 46. The resulting three-mesh networks are shown in Fig. 17 F,G,H,I (compare Fig. 10 (i) and (iii)); they have the reciprocal structure of the networks in Fig. 13 (or 17) A,B,C,D respectively. The corresponding conditions giving the region in Fig. 17 (g) and (h) are derivable immediately from 34 (a) (b) (c) (d) above by an interchange of a's and b's. We shall therefore simply write them down together with their equivalents in the notation (25)

For Fig. 17 F and G

$$\left.\begin{array}{r} a_0b_2 - a_2b_0 \geq 0 \\ -a_0^2 + a_0a_1b_1 + a_0a_2b_0 - a_1^2b_0 \geq \end{array}\right\} \text{i.e.} \left\{\begin{array}{ll} y \geq \beta & (34f) \\ y \leq \alpha x + \beta - \alpha^2 & (34g) \end{array}\right.$$

For Fig. 17 H and I

$$\left.\begin{array}{r} a_0 b_2 - a_2 b_0 \gtrless 0 \\ -a_1^2 b_2 + a_0 a_2 b_2 + a_1 a_2 b_1 - a_2^2 b_0 \geqq 0 \end{array}\right\} \text{i.e.} \left\{\begin{array}{ll} y \leq \beta & (34h) \\ (\alpha^2-\beta)y \leq \alpha\beta x - \beta^2 & (34k) \end{array}\right.$$

The corresponding regions appear in Figs. 17 (g) and (h). The cross-hatched areas for networks F and I follow from 34(e) which is again unchanged. It appears that the regions given by 34 f,g,h,k have simpler boundaries than those given by 34 a,b,c,d. This suggests that a figure in which the poles (or $\frac{b_1}{b_0}$, $\frac{b_2}{b_0}$) are considered fixed and the zeros (or $\frac{a_1}{a_0}$, $\frac{a_2}{a_0}$) as free would be more natural for discussing the complete covering of the permissible area (evidently accomplished by networks A,B,C,D,E, but not by networks E,F,G,H,I). We have, however, adhered to the system used by Cauer for a more ready comparison of results.

An examination of Fig.17 now makes it very easy to see the equivalences which exist between the two and three-mesh networks discussed in this chapter. It also brings out very conveniently the difference between the results of Foster[13] and Cauer[14] and those of this chapter. The inequalities represented geometrically in Fig.17 are collected below for more ready reference.

$$P \equiv (y - \alpha x + \beta)^2 - 4\beta y \quad \leq 0 \qquad (24d')$$

$$D \equiv y - \alpha x + \beta \quad \leq 0 \qquad (24c')$$

$$L \equiv y - \beta \quad \lesseqgtr 0 \qquad (34a,c,f,h)$$

$$C_1 \equiv y - x^2 + \alpha x - \beta \quad \geq 0 \qquad (34b)$$

$$C_2 \equiv y^2 - \alpha x y + \beta x^2 - \beta y \quad \leq 0 \qquad (34d')$$

$$T \equiv (y-\beta)^2 - \alpha(x-\alpha)(y-\beta) + \beta(x-\alpha)^2 \leq 0 \qquad (34e')$$

$$S_1 \equiv y - \alpha x - \beta + \alpha^2 \quad \leq 0 \qquad (34g)$$

$$S_2 \equiv (\alpha^2 - \beta)y - \alpha\beta x + \beta^2 \quad \leq 0 \qquad (34k)$$

where $x = \frac{b_1}{b_0}$, $y = \frac{b_2}{b_0}$; $\alpha = \frac{a_1}{a_0}$, $\beta = \frac{b_2}{b_0}$ (25)

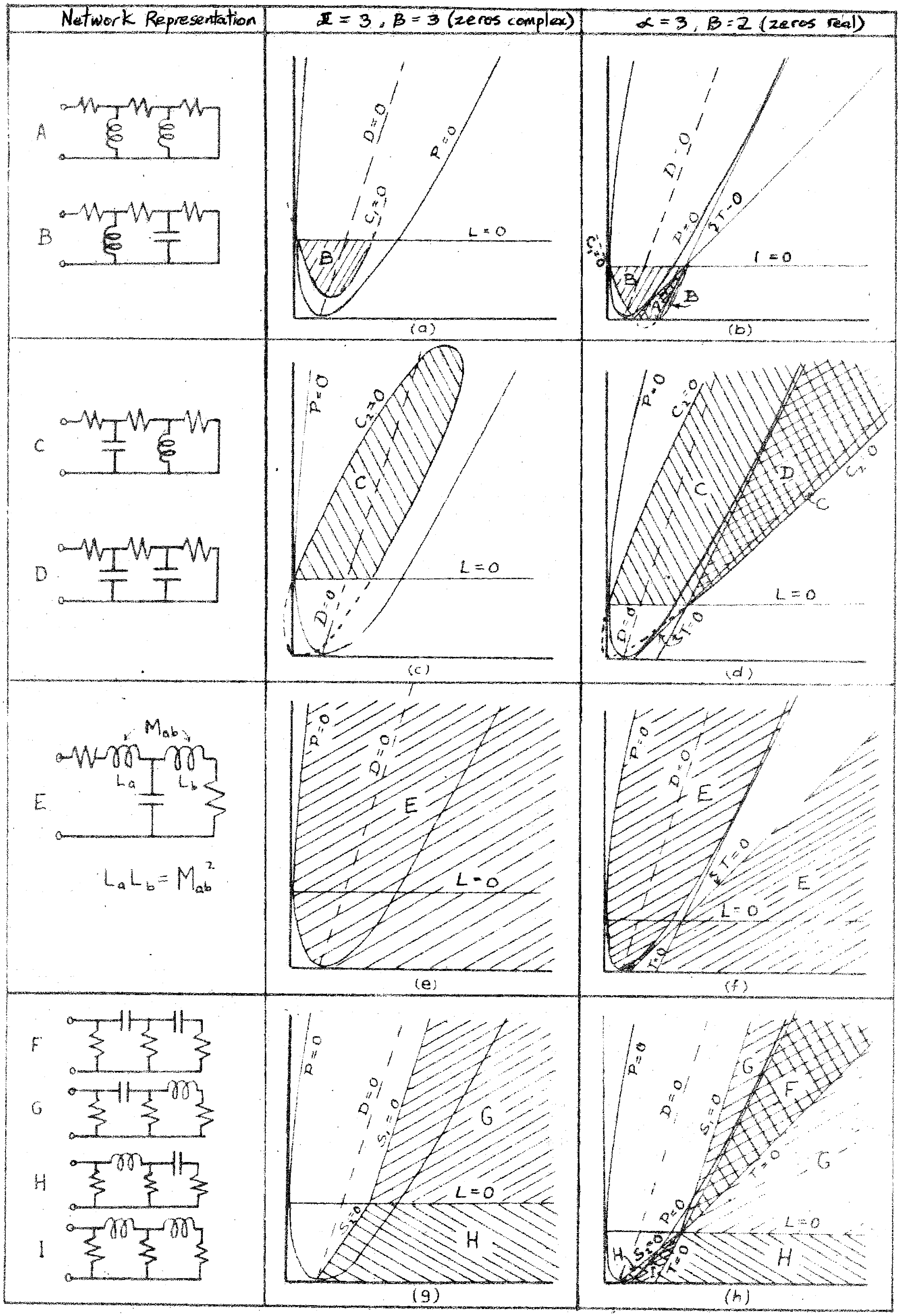


Fig 17. Network Representations of (24) and the corresponding regions in which the point ($x = \frac{b_1}{b_0}$, $y = \frac{b_2}{b_0}$) must lie for fixed values of $\frac{a_1}{a_0} = \alpha$, $\frac{a_2}{a_0} = \beta$

56. Four-Mesh Representations of $Z(\lambda)$ in (24). The question of using a structure reciprocal to the two-mesh networks is suggested by the foregoing discussion. We note first that in order to use reciprocals for equivalent networks the reciprocal function must be of the same form, i.e. the degree of numerator and denominator must be the same in λ and in $\frac{1}{\lambda}$ Furthermore a reciprocal structure is not realizable if negative mutual inductances are involved. This rules out the procedure for two-mesh circuits in sections 50 and 52 for use in a dual sense*.

Foster has shown, however, that in certain regions of Fig. 11 (or 17) a network representation of (24) having the structure shown in Fig. 18(a) is possible**. Now such a structure has the realizable reciprocal structure shown in Fig. 18(b). We need thus only interchange a's and b's in Foster's condition to obtain a region in which a four-mesh representation of (24) with the minimum number of elements will be possible. These conditions are rather complicated and will not be given here.

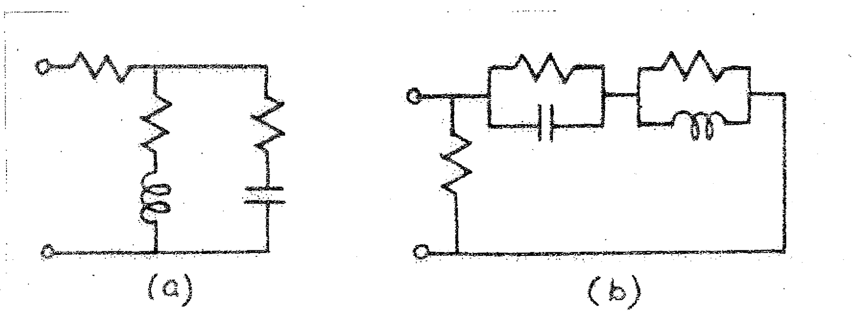


Fig. 18

*It is, however, possible to remove a minimum conductance occurring for a frequency $\omega = \sigma$ and then to proceed with the reciprocal of the remaining function as an impedance function in the usual manner; this is possible because the condition (32d) is symmetrical in a and b and therefore holds also for the reciprocal. The resulting network is a three-mesh (see Ex. 8 at end of chapter).

**Foster's networks Bibliog. 13, Circuits 16,17,18,19 reduce to the network in Fig. 20(a) when the general two-mesh impedance function takes the form (24).

(A numerical example is given in Ex. 9 in the Appendix). We note, however, that the four-mesh network of Fig. 18(b) will always be equivalent to either a two-mesh or a three-mesh network.

The subject of equivalent circuits will be taken up again in Chapter V. The very interesting fact is evident, however, that the number of meshes is not necessarily invariant for a network representation of a given impedance function, even if this network has the minimum number of elements.

CHAPTER IV.

Network Representation of a General "Positive Real" Function with a Finite Number of Poles and Zeros.

57. The methods exemplified in sections 46-50 of the last chapter for constructing a network representation of a "positive real" function with two poles and two zeros in the interior of the left-half plane, can be applied with very little modification to the general "positive real" function with a finite number of poles and zeros.

In this chapter we shall content ourselves with demonstrating a procedure which will lead uniquely to one network for any given "positive real" function. We do this for the sake of simplicity and not because this procedure is the only one possible. On the subject of equivalent networks more will be said in the next chapter. The present discussion suffices, however, to establish a complete equivalence between "positive real" functions with a finite number of poles and zeros and finite passive electrical networks with constant circuit parameters.

(a) General Procedure.

58. The general procedure by which it is possible to find a network corresponding to every given "positive real" function is a step by step process; at each step the function is decomposed into a part which can immediately be represented by an element of the network, and another part which is

again a "positive real" function. The new function should be simpler than the previous function if any progress is to be made towards complete representation; this will show itself in a decrease in the number of poles and zeros (the number of poles being always equal to the number of zeros).

59. Let the given function be $Z(\lambda)$. Poles and zeros on the axis of imaginaries can immediately be removed by Theorem IV and corresponding network elements constructed. When this has been repeated often enough we are left with a "positive real" function which is either a constant or has all its poles and zeros in the interior of the left-half plane. Let this function be $Z_1(\lambda)$.

Next examine the real part of $Z_1(\lambda)$ on the axis of imaginaries.

We remark in passing that $\mathcal{R}_e Z(i\omega)$ is an even function of ω while $\mathcal{I}_m Z(i\omega)$ is odd, in other words we may always write

$$Z(i\omega) = R(\omega^2) + i\omega N(\omega^2) \tag{35}$$

(cf. equation (43) section 74).

Since $\mathcal{R}_e Z(i\omega)$ cannot be negative it must have a lower limit ≥ 0. Let this minimum value be R_1 and occur when $\omega = \sigma$ (σ real). Then clearly $Z_2(\lambda) = Z_1(\lambda) - R_1$ will have its real part ≥ 0 on the boundary and regular in the right-half plane, i.e. $Z_2(\lambda)$ will be a "positive real" function. Moreover $Z_2(i\sigma)$ will be pure imaginary. Let

$$Z_2(i\sigma) = iX \tag{36}$$

60. Now if $\sigma = 0$ or ∞, $X = -X = 0$; hence we can immediately conclude that $Z_2(\lambda)$ has a zero at $\lambda = 0$ or ∞ respectively. We could therefore write

$$\frac{1}{Z_2(\lambda)} = \frac{D}{\lambda} + \frac{1}{Z_3(\lambda)} \qquad (36a)$$

or

$$\frac{1}{Z_2(\lambda)} = L\lambda + \frac{1}{Z_3(\lambda)} \qquad (36b)$$

respectively; in either case $Z_3(\lambda)$ will have one less zero and pole than $Z_2(\lambda)$ (or $Z_1(\lambda)$)

61. Consider next the case when σ is not zero or infinite. We proceed to make the points $\lambda = \pm i\sigma$ zeros of a new function simply derivable from $Z_2(\lambda)$ by a step corresponding to the realization of an element in the network representation. Note that if

$$Z_2(i\sigma) = i\,X$$

then

$$\frac{1}{Z_2(i\sigma)} = -i\,\frac{1}{X}$$

Two conditions may arise; $\frac{X}{\sigma} = N(\sigma^2)$ (equation 35) may be either positive or negative. We consider these two possibilities separately.

62. (i) Let $\frac{X}{\sigma} = N(\sigma^2) = L_1$ be negative.

Then we can write

$$Z_2(\lambda) = L_1\lambda + W(\lambda) \qquad (37a)$$

where $W(\lambda)$ is a "positive real" function and has zeros at $\lambda = \pm i\sigma$. For $-L_1\lambda$ is a "positive real" function and $W(\lambda) = Z_2(\lambda) - L_1\lambda$ is thus the sum of two "positive" functions; also $W(i\sigma) = Z_2(i\sigma) - iX = 0$. (Note: $W(\lambda)$ has a pole at infinity). Hence we may write

$$\frac{1}{W(\lambda)} = \frac{K\lambda}{\lambda^2+\sigma^2} + \frac{1}{W'(\lambda)} \tag{37b}$$

where $W'(\lambda)$ is a "positive real" function which, like $W(\lambda)$, has a pole at infinity.

Let
$$W'(\lambda) = L_3\lambda + Z_3(\lambda) \tag{37c}$$

If we further write

$$L_2 = \frac{1}{K}, \quad D = \frac{\sigma^2}{K} \tag{37d}$$

the relation between $Z_2(\lambda)$ and $Z_3(\lambda)$ in terms of a network representation is shown in Fig. 19. Comparing this with Fig. 15 and formula (31), section 49, and remembering that $Z_2(\lambda)$ has no pole at $\lambda = \infty$ we have as the algebraic relation

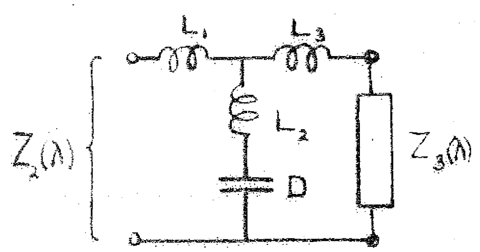


Fig.19

$$Z_2(\lambda) = \frac{(L_1+L_2)\lambda^2 Z_3(\lambda) + (L_3+L_1)D\lambda + DZ_3(\lambda)}{(L_2+L_3)\lambda^2 + \lambda Z_3(\lambda) + D} \tag{37e}$$

Also L_2 L_3 are positive by 37b, c and d, hence (see section 48) the network is realizable.

63. (ii) Let $\frac{X}{\sigma} = N(\sigma^2)$ be positive.

Then we may consider $Y_2(\lambda) = \frac{1}{Z_2(\lambda)}$ where $Y_2(i\sigma) = -i\frac{1}{X} = -i\sigma\frac{1}{\sigma^2 N(\sigma^2)}$ and $C_1 = -\frac{1}{\sigma^2 N(\sigma^2)}$ is negative, and proceed in an exactly analogous manner as with the preceding case. Translating the results into network elements, however, the network will be reciprocal to that in Fig. 19. ~~This~~

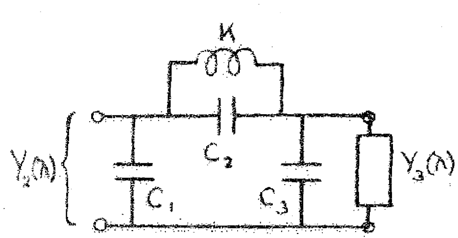


Fig.20

This is shown in Fig. 20, L_1 being replaced by C_1, L_2 by C_2, D by K and L_3 by C_3. The relation between $Y_2(\lambda)$ and $Y_3(\lambda)$ will be

$$Y_2(\lambda) = \frac{(C_1+C_2)\lambda^2 Y_3(\lambda) + (C_3+C_1)K\lambda + KY_3(\lambda)}{(C_2+C_3)\lambda^2 + \lambda Y_3(\lambda) + K} \tag{37f}$$

64. The network elements in Fig. 20 are not directly realizable since they contain the negative capacity C_1. Equation (37f) may, however, be rewritten

$$Z_2(\lambda) = \frac{(C_3+C_2)\lambda^2 Z_3'(\lambda) + \lambda + KZ_3'(\lambda)}{(C_1+C_2)\lambda^2 + (C_3+C_1)K\lambda Z_3'(\lambda) + K} \tag{37g}$$

[where $Z_3'(\lambda) = \frac{1}{Y_3(\lambda)}$]

which is of exactly the same form as (37e). Let us write down the necessary relations between the coefficients of (37e) and (37g) in order that Z_3 and Z'_3 may be identical. They are

$$D = \frac{1}{C_3+C_1} \tag{38a}$$

$$\left.\begin{aligned} L_1 + L_2 &= \frac{1}{K}\cdot\frac{C_2+C_3}{C_3+C_1} \\ L_2 + L_3 &= \frac{1}{K} \\ L_1 + L_3 &= \frac{1}{K}\cdot\frac{C_1+C_2}{C_3+C_1} \end{aligned}\right\} \tag{38b}$$

we have also (analogous to (37d) & (31a))

and

$$\left.\begin{aligned} \frac{K}{C_2} &= \sigma^2 \\ C_1C_2 + C_2C_3 + C_3C_1 &= 0 \end{aligned}\right\} \tag{38c}$$

From 38a

$$\left.\begin{aligned} L_1 &= \frac{1}{K}\cdot\frac{C_3}{C_3+C_1} \\ L_2 &= \frac{1}{K}\cdot\frac{C_2}{C_3+C_1} \\ L_3 &= \frac{1}{K}\cdot\frac{C_1}{C_3+C_1} \end{aligned}\right\} \tag{38d}$$

with the relations from (38c)

$$\left.\begin{aligned} \frac{D}{L_2} &= \frac{K}{C_2} = \sigma^2 \\ L_1L_2 + L_2L_3 + L_3L_1 &= \frac{C_3C_2 + C_2C_1 + C_1C_3}{K^2(C_3+C_1)^2} = 0 \end{aligned}\right\} \tag{38e}$$

It follows that the fictitious network of Fig. 20 can be made equivalent to the physically realizable network of Fig. 19 by means of the equations (38a) and (38d). Note that the signs of L_1 L_3 and C_1 C_3 are interchanged. Hence in this case L_1 and L_2 are positive.

Furthermore, since we now know the structure of the network it is evident that $Z_2(i\sigma) - iL_1\sigma$ must be zero. Consequently $L_1 = N(\sigma^2)$, and the procedure of Case (1) may be followed irrespective of the sign of $N(\sigma^2)$. (At one stage a non-"positive" function will be encountered but the final $Z_3(\lambda)$ will be a "positive" function).

65. Examining the reduction of poles and zeros by this process, we note that the first step (equation 37a) increases the number by 1. The next step (equation 37b) decreases the number by 2, while the last step (equation 37c) effects a further reduction of 1. In all, thus, if R_1 occurs for $\lambda = \pm i\sigma \neq 0$ or ∞, $Z_3(\lambda)$ will have two poles and two zeros less than $Z_1(\lambda)$.

66. This procedure can now be applied to all cases and repeated until the final function has no poles and no zeros, i.e. is a positive constant corresponding to a pure resistance. It leads uniquely to a network representation of the original "positive real" function.

The calculation of the network is made entirely in terms of the values of successive impedance functions on the axis of imaginaries. Each reduction of the function is effected by the removal of a zero (or pole) of the function on the axis of imaginaries. If a function does not have such a zero on the imaginary axis it is shown always to be possible to derive from it a function which will have such a zero; the removal of this zero together with the related steps also corresponds to the calculation of realizable elements of the network.

For more ready reference we summarize the different steps with the corresponding network interpretations in Table I page 69, and state the conclusion as

Theorem VIII. To every "positive real" function with a finite number of poles and zeros corresponds a finite physically realizable network. Hence the terms "positive real" function with a finite number of poles and zeros and impedance function of a finite network are synonymous.

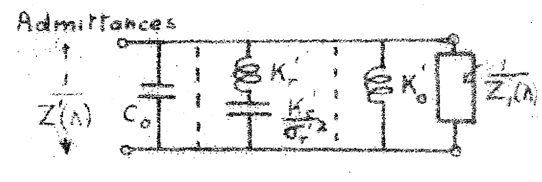
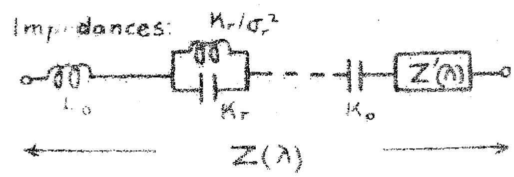
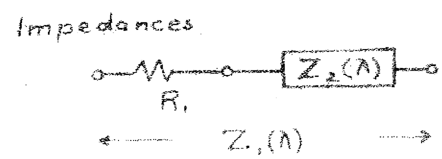
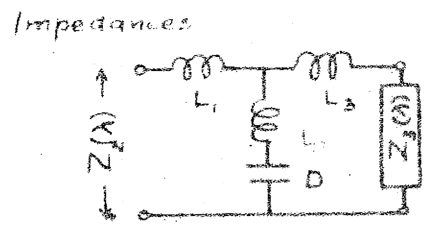
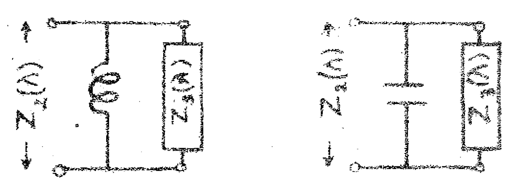
(b) Discussion of Networks Obtained.

67. A discussion of the general type and properties of the network obtained by the process summarized in Table I, as well as of certain special cases, is of interest.

It will be noticed first of all that the network always has a ladder structure. The ladder structure is thus capable of realizing the most general form of driving point impedance.

In general mutual inductances will be involved but this mutual inductance if present occurs always between self-inductances in adjacent meshes; in certain cases

TABLE I. General Procedure for constructing a Network Representation of any given "Positive Real" Function.

(a) Remove all poles on the axis of imaginaries:- $Z(\lambda) = L_0\lambda + \sum_r \frac{K_r\lambda}{\lambda^2+\sigma_r^2} + Z'(\lambda)$ (Theor.IV)	
(b) Remove all zeros on the axis of imaginaries:- $\frac{1}{Z'(\lambda)} = C_0\lambda + \sum_r \frac{K'_r\lambda}{\lambda^2+\sigma_r'^2} + \frac{1}{Z_1(\lambda)}$ (Theor.IV Cor. 1)	
Repeat (a) and (b) until no poles or zeros of $Z_1(\lambda)$ lie on the axis of imaginaries.	
(c) Study $R(\omega^2)$ in $Z_1(i\omega) = R(\omega^2) + i\omega N(\omega^2)$ Let R_1 be the least value of $R(\omega^2)$ for positive values of ω^2 and let $R_1 = R(\sigma^2)$ $Z_1(\lambda) = R_1 + Z_2(\lambda)$	
(d) Write $L_1 = N(\sigma^2)$ (σ found in (c)) $Z_2(\lambda) = L_1\lambda + W(\lambda)$ $W(\lambda)$ has a zero at $\lambda = \pm i\sigma$ and hence $\frac{1}{W(\lambda)} = \frac{\lambda}{L_2\lambda^2 + D} + \frac{1}{L_3\lambda + Z_3(\lambda)}$ As special cases when $\sigma = 0$ or ∞, this reduces to $L_1 = 0$, $L_3 = 0$ $\frac{1}{Z_2(\lambda)} = \frac{1}{L_2\lambda} + \frac{1}{Z_3(\lambda)}$ $\frac{1}{Z_2(\lambda)} = \frac{D_2}{\lambda} + \frac{1}{Z_3(\lambda)}$	 

mutual inductance may be entirely absent, viz. when procedure c Table I p.69 involves only minimum values R_1 at $\lambda = 0$ or ∞ throughout.

The procedure is carried out in such a way that mutual inductances, when they occur, do so with a coefficient of coupling unity; this apparently involves an ideal physical condition not generally inherent in the given impedance function. If, however, the function $Z(\lambda)$ has a pole at $\lambda = \infty$ the necessity for perfect coupling is only apparent since a pole at $\lambda = \infty$ corresponds to a self-inductance L_o in series with the perfectly coupled inductance L_a in Fig. 21. But this separation of L_o and L_a is quite arbitrary, the only limitation being that L_o is the largest inductance which can be so separated and still leave the remaining network physically realizable.

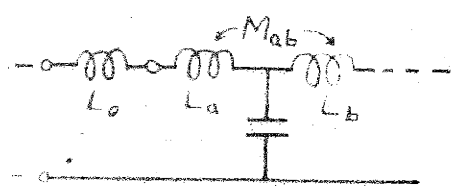


Fig. 21

If L_o (or any part of it) be combined with L_a into an inductance L'_a, coupled with the inductance L_b then obviously $L_a' . L_b > M_{ab}^2$ and the coupling will be "loose". This condition of "loosening" will simultaneously extend throughout the remaining network. Only in such cases where the condition of "perfect" coupling is indeed inherent in the given impedance function because of the absence of a pole at $\lambda = \infty$ will this condition be forced on the corresponding network (cf. footnote to Section 49).

In fact a network structure like that indicated in Fig. 21 would under certain conditions be eminently suitable for practical construction. The series resistances are in the proper position to be combined with the inductances, while in the parallel branches only capacities, whose series resistance is negligible, occur.

The whole question of practical usability, however, depends on the special conditions to be met in the particular application. Its full discussion would of necessity involve the consideration of equivalent networks.

In this connection it may also be pointed out that the dual circuit met with in Fig. 20 p. 55 could sometimes be used, namely in such cases where the negative capacity occurs in parallel with a larger positive capacity (corresponding to a zero at $\lambda = \infty$) so that the combination will be positive.

68. It appears very definitely that the number of meshes in a network is not predetermined by the form of the impedance function. The fact that the number of meshes in an electrical network corresponds to the number of degrees of freedom in a mechanical system has given rise to the idea that this number is contained also in the impedance function.* It is evident that the impedance function determines the behavior of the network only at its terminals and that its internal behavior is still, to a certain extent, free. The true relation between "degrees of freedom" internally and response externally is as yet imperfectly understood.

*This is possibly one reason why no advance beyond general two-mesh networks has previously been made in the present problem.

69. At each stage in the process described in sections 54-66 the determination of a given number of independent elements in the network is accompanied by a corresponding reduction in the number of independent constants in the function which remains. The resulting network therefore does not contain any superfluous elements. Let us consider this property of the network in greater detail.

By an independent network element is meant one whose value does not depend on the values of other elements; thus in Fig. 19 the group of elements L_1 L_2 L_3 are not independent since their values are connected by the relation

$$L_1L_2 + L_2L_3 + L_1L_3 = 0$$

Similarly by the number of independent constants in the function is meant the number of such constants necessary to determine the function.

A function is completely determined by prescribing its poles, its zeros, and an arbitrary factor . Poles and zeros in the interior of the left-half plane are either real or occur in conjugate complex pairs. Hence to each such pole or zero in the interior of the left-half plane corresponds one independent constant. A pair of poles (or zeros) on the imaginary axis involves only one constant, while poles or zeros at the origin or infinity introduce no new constants. The number of independent constants in the function is thus equal to

$$= (n + m + h + l) \qquad (38.1)$$

where

n = number of poles in the interior of left-half plane

m = " " zeros " " " " " " "

h = number of pairs of poles and zeros on the imaginary axis (excluding 0 and ∞)

We have stated this result in this way because a mere count of the number of coefficients in numerator and denominator is not always sufficient. To every complex pair of poles or zeros on the imaginary axis corresponds in general a relation among the coefficients. In the extreme case where all zeros (or poles) lie on the imaginary axis, these relations are simply expressed by stating that the coefficients of either the even or odd powers of λ in the numerator (or denominator) are zero. In intermediate cases the relations are not so easily seen* and therefore likely to be missed unless a detailed examination of the poles and zeros is made.

With these facts in mind it is easy to trace the reduction of the independent constants in the function brought about by the removal of the various groups of elements in Table I, page 69, from the network.

In special cases additional relations besides those mentioned above may exist among the coefficients, making possible a still further reduction in the number of elements in the circuit. The possibility of making this reduction will not always be evident from the process of Sections 59-63. A simple illustration of this is given in Example 10 in the

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* See for example section 73, equation (41).

Appendix. This, however, takes us into the consideration of equivalent networks, a complete discussion of which will not be attempted in this thesis.

70. Foster's "Reactance Theorem"(12) is readily seen to be a particular case of the application of Theorem IV in our process; it occurs when all the poles and zeros lie on the axis of imaginaries and the impedance function is completely represented (for example) by the terms representing the poles in Table I (a). It is also evident that every function which is purely imaginary on the imaginary axis must be of this form.

The property of mutual separation of poles and zeros follows as a consequence of the partial fraction development* but it can also be shown in a very interesting geometrical manner in terms of the complex plane and our Lemma 1 (section 32). By this lemma and Theorem IV (ii) each simple zero on the imaginary axis corresponds to a point around which real and imaginary parts change sign as shown in Fig. 22(a) page 75. Each simple pole on the other hand has the direction of rotation reversed (corresponding to the fact that $(\lambda^2 + \sigma^2)$ appears in the denominator) and real and imaginary parts consequently change sign as shown in Fig. 22(b). If we superimpose a number of

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* See also Bibliog. (18).

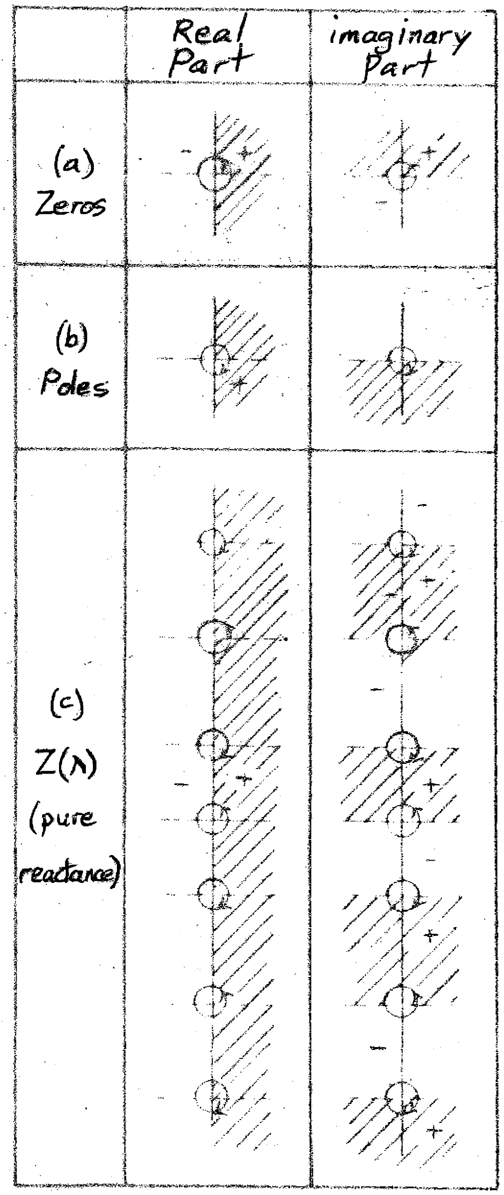

Fig. 22

such regions corresponding to the various poles (as in Table I (a) when $Z'(\lambda) = 0$) it is evident that a reasonable sequence in the change of sign of the imaginary part cannot be obtained unless an intermediate region corresponding to a zero occurs between each pair of poles as shown in Fig.22 (c). This includes the necessity of either a pole or a zero at $\lambda = 0$ and $\lambda = \infty$ as is also obvious algebraically.

71. The extension of Foster's Reactance theorem by Cauer to all cases when two kinds of elements only are present follows at once by a suitable transformation of $Z(\lambda)$. This transformation corresponds to a substitution of resistance for either inductance or capacitance in the purely reactive network, e.g., if $pZ(p) = Z_1(p^2)$ then $Z_1(\lambda)$

has a network representation of inductances and resistances only. On the other hand if $\frac{Z(p)}{p} = Z_2(p^2)$, then $Z_2(\lambda)$ has a network representation of resistances and capacitances only.

The condition of mutual separation of poles and zeros is however not a general condition for cases where all poles and zeros lie on the real axis; it ceases to be necessary as soon as three kinds of elements (resistance, inductance and capacitance) are permitted (cf. example 3, Appendix). The difference lies in the fact that there is no general theorem like Theorem IV for poles on the real axis. Physically it is also obvious that the impedance function of a network with three kinds of elements can have all its poles and zeros on the real axis which must then have a different law of distribution from that which is necessary and sufficient for a network with only two kinds of elements.

(c) Formulas and Artifices to simplify Calculation.

72. We mention first a very simple method of determining poles and zeros on the imaginary axis

Consider
$$Z(\lambda) = \frac{f(\lambda)}{g(\lambda)} \qquad (1)$$

If we write the numerator (say) as

$$f(\lambda) = U(-\lambda^2) + \lambda V(-\lambda^2) \qquad (39)$$

where U and V represent the even and odd parts of $f(\lambda)$ respectively it is evident that any factor $(\lambda^2 + \sigma_r^2)$ of $f(\lambda)$ must be a factor of both U and V. Consequently the H.C.F.

of U and V will contain all such factors. Furthermore, if $f(\lambda)$ is the numerator of a "positive" function all the roots of this H.C.F. (considered as a function of λ^2) must be real and negative (i.e. σ_r^2 positive) for otherwise one of the corresponding zeros would fall within the right-half plane. Having found all these roots by any one of the known methods*, all the zeros on the axis of imaginaries will be known. The same can be done for the poles.

73. We next proceed to discuss a method of determining the residue at a pole on the axis of imaginaries. In general if $\lambda = \beta$ is a simple pole of $Z(\lambda)$ the residue $\frac{K}{2}$ (cf. section 35 equation (18)) at this pole is given by

$$\frac{K}{2} = \left[(\lambda-\beta)Z(\lambda)\right]_{\lambda=\beta}$$

Now take $\beta = i\sigma$ and let us write

$$Z(\lambda) = \frac{f(\lambda)}{(\lambda^2+\sigma^2)\,h(\lambda)}$$

Then $$(\lambda - i\sigma)\,Z(\lambda) = \frac{f(\lambda)}{(\lambda+i\sigma)\,h(\lambda)}$$

and $$\frac{K}{2} = \frac{1}{2i\sigma}\cdot\frac{f(i\sigma)}{h(i\sigma)}$$

If now we write similarly to (39)

$$\lambda h(\lambda) = P(-\lambda^2) + \lambda Q(-\lambda^2) \tag{39a}$$

we shall have

$$K = \frac{U(\sigma^2) + i\sigma V(\sigma^2)}{P(\sigma^2) + i\sigma Q(\sigma^2)} \tag{40}$$

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*See for example Bibliog. (35) and (36)

It has been shown, however, (Theorem IV (iii)) that K must be a "positive real" quantity. Consequently

$$K = \frac{U(\sigma^2)}{P(\sigma^2)} = \frac{V(\sigma^2)}{Q(\sigma^2)} \tag{41}$$

and these equations as well as the fact that K must be positive are a check on whether we are indeed dealing with a "positive real" function. The simplification introduced in calculating the residue $\frac{K}{2}$ by (41) as compared with (40) is obvious. The additional equation in (41) at the same time furnishes a check on the necessary condition which the pole at $\lambda = \pm i\sigma$ must fulfill in order that $Z(\lambda)$ may be a "positive" function.

74. We turn next to the calculation of real and imaginary parts of $Z(\lambda)$. For this purpose let us consider

$$Z(\lambda) = \frac{f(\lambda)}{g(\lambda)} \tag{1}$$

For reference write also $f(\lambda) = U(-\lambda^2) + \lambda V(-\lambda^2)$ (39)

$$g(\lambda) = S(-\lambda^2) + \lambda T(-\lambda^2) \tag{39b}$$

and resolve $Z(\lambda)$ into even and odd functions of λ as follows :

$$Z(\lambda) = \frac{f(\lambda)\ g(-\lambda)}{g(\lambda)\ g(-\lambda)} \tag{42}$$

The expression $g(\lambda) \cdot g(-\lambda)$ will contain only even powers of λ since it may be written (equation 39b) $S^2(-\lambda^2) - \lambda^2 T^2(-\lambda^2)$ The numerator will contain both even and odd powers; let us separate these and write

$$Z(\lambda) = \frac{H(-\lambda^2)}{G(-\lambda^2)} + \lambda \frac{I(-\lambda^2)}{G(-\lambda^2)} \tag{43}$$

If $\lambda = i\omega$, we have $-\lambda^2 = \omega^2$ and the first and second terms on the right-hand side of equation (42a) become the real and imaginary parts of $Z(i\omega)$ respectively. Let us write $-\lambda^2 = \omega^2 = \Omega$ and

$$G(\Omega) = B_0 + B_1\Omega + B_2\Omega^2 + \text{-----} + B_m\Omega^m \tag{43a}$$

$$H(\Omega) = C_0 + C_1\Omega + C_2\Omega^2 + \text{-----} + C_p\Omega^p \tag{43b}$$

$$I(\Omega) = D_0 + D_1\Omega + D_2\Omega^2 + \text{-----} + D_q\Omega^q \tag{43c}$$

and for subsequent reference

$$F(\Omega) = A_0 + A_1\Omega + A_2\Omega^2 + \text{-----} + A_n\Omega^n \tag{43d}$$

where $f(\lambda)\, f(-\lambda) = F(-\lambda^2) = F(\Omega)$

Then it can readily be shown that

$$\left.\begin{aligned} B_0 &= b_0^2 \\ B_1 &= b_1^2 - 2b_0b_2 \\ B_2 &= b_2^2 - 2b_1b_3 + 2b_0b_4 \\ &\\ B_m &= b_m^2 \end{aligned}\right\} B_r = \sum_{s=-\infty}^{+\infty} b_{r+s}\, b_{r-s}(-1)^s \tag{44a}$$

with similar relations between the A's and a's.

Further

$$\left.\begin{aligned} C_0 &= a_0b_0 \\ C_1 &= a_1b_1 - (a_0b_2 + a_2b_0) \\ &\\ C_p &= a_pb_p \end{aligned}\right\} C_r = \sum_{s=-\infty}^{+\infty} a_{r+s}\, b_{r-s}(-1)^s \tag{44b}$$

(i.e. the C's are the polar forms of the A's or B's)

and

$$\left.\begin{aligned} D_0 &= -a_0b_1 + a_1b_0 \\ D_1 &= a_0b_3 - a_1b_2 + a_2b_1 - a_3b_0 \\ D_3 &= -a_0b_5 + a_1b_4 - a_2b_3 + a_3b_2 \\ &\qquad - a_4b_1 + a_5b_0 \\ &\vdots \end{aligned}\right\} D_r = \sum_{s=-\infty}^{+\infty} a_{r+s}\, b_{r\, s+1}(-1)^{s+1} \tag{44c}$$

75. <u>Minimum Value of $R(\Omega)$.</u> For the purposes of the general procedure summarized in Table I, page 69, it is necessary only to calculate the coefficients of $G(\Omega)$ and

$H(\Omega)$; we have given the formulas of the other coefficients, however, for the sake of completeness.

It now becomes necessary to find the minimum value of

$$R(\Omega) = \frac{H(\Omega)}{G(\Omega)} \tag{45}$$

$$= \frac{C_0 + C_1\Omega + C_2\Omega^2 + \dots + C_n\Omega^n}{B_0 + B_1\Omega + B_2\Omega^2 + \dots + B_n\Omega^n}$$

This is a perfectly straightforward problem in differential calculus, but actually it involves the most troublesome numerical computation of the whole procedure since it requires the solution of an algebraic equation of high degree. The following appears to be the best method of attack:

(i) Calculate the numerator of $\frac{dR}{d\Omega}$; it is

$$(B_0C_1 - B_1C_0) + 2(B_0C_2 - B_2C_0)\Omega + \{(B_1C_2 - B_2C_1) + 3(B_0C_3 - B_3C_0)\}\Omega^2$$
$$+ \{2(B_1C_3 - B_3C_1) + 4(B_0C_4 - B_4C_0)\}\Omega^3 + \dots$$
$$\dots + \left\{\sum_{s=-\infty}^{+\infty} s\, B_{\frac{r-s}{2}} C_{\frac{r+s}{2}}\right\}\Omega^{r-1} + \dots \tag{46}$$

(ii) Make a rough plot of $R(\Omega)$ calculating the numerator and denominator for different values of Ω by Horner's process.

(iii) Select the value of Ω which appears to give the minimum value of $R(\Omega)$ and substitute in expression (46); this should give a value not greatly different from zero. By a method of successive linear interpolation find accurately the root of (46) giving the minimum $R(\Omega)$ and from it calculate accurately R_1 (Table I (c)).

(iv) If, by some mischance, the value R_1 thus found were not the actual minimum but merely a minimum value ($\frac{dR}{d\Omega} = 0$) this will very soon make itself evident in the subsequent work. An immediate check could, however, be made as follows:

Calculate the numerator of $R(\Omega) - R_1$ and apply Sturm's theorem* to determine the number of real roots between 0 and ∞. This should be just one double root.

76. To determine the imaginary part of $Z(i\sigma)$ when or $L_1 = N(\sigma^2)$ in Table I (d) p. 69, note that

$$Z(i\sigma) = \frac{U(\sigma^2) + i\sigma V(\sigma^2)}{S(\sigma^2) + i\sigma T(\sigma^2)}$$

(cf. equations (39) and (39b) section 74).

Since $Z(i\sigma)$ is pure imaginary we have

$$Z(i\sigma) = \frac{U(\sigma^2)}{i\sigma T(\sigma^2)} = \frac{i\sigma V(\sigma^2)}{S(\sigma^2)}$$

(condition for vectors in numerator and denominator to be at right angles),

i.e. $$\frac{Z(i\sigma)}{i\sigma} = \frac{U(\sigma^2)}{-\sigma^2 T(\sigma^2)} = \frac{V(\sigma^2)}{S(\sigma^2)}$$

We therefore have to calculate

$$\begin{aligned} L_1 &= \frac{U(\sigma^2)}{-\sigma^2 T(\sigma^2)} = \frac{V(\sigma^2)}{S(\sigma^2)} \\ &= \frac{a_0 - a_2\sigma^2 + a_4\sigma^4 - \;---\; (-1)^n a_{2n}\sigma^{2n}}{-b_1\sigma^2 + b_3\sigma^4 \;----\; (-1)^n b_{2n-1}\sigma^{2n}} \qquad (47) \\ &= \frac{a_1 - a_3\sigma^2 + a_5\sigma^4 \;---\; (-1)^n a_{2n+1}\sigma^{2n}}{b_0 - b_2\sigma^2 + b_4\sigma^4 \;---\; (-1)^n b_{2n}\sigma^{2n}} \end{aligned}$$

which is a comparatively simple calculation and at once affords a check on itself, and on the correctness of the determination of $R(\sigma^2)$ and σ in the previous step.

77. After determining L_1 the next step corresponds to a separation of the pole in the reciprocal of the new impedance function $W(\lambda)$ (see Table I, p. 69). The calculation of the residue at such a pole has been

*See for example Bibliog. (35) Part I, pp. 198-204.

discussed in section 73. Knowing this residue the separation of the pole is no more complicated than simple algebraic division, for let

$$\frac{1}{W(\lambda)} = \frac{g(\lambda)}{f_1(\lambda)} = \frac{K\lambda}{\lambda^2+\sigma^2} + \frac{g_2(\lambda)}{f_2(\lambda)}$$

where $f_1(\lambda) = (\lambda^2+\sigma^2)\, f_2(\lambda)$, and $\frac{K}{2}$ is the residue of the pole at $\lambda = \pm i\sigma$, calculated by equation (41) section 73; then

$$g_2(\lambda) = g(\lambda) - K\, f_2(\lambda) \;/\; (\lambda^2+\sigma^2) \qquad (47.1)$$

This eliminates the necessity for solving a system of linear equations as in the ordinary partial fraction method.

78. There remains only the separation of the pole at infinity to complete process (d) in Table I, page 69; this presents no difficulty whatever, being a simple process of division. The numerical case in Ex. 11 Appendix illustrates all the methods discussed above. It should be noted, however, that in many cases the process can be carried through entirely in terms of values of $Z(\lambda)$ at 0 and ∞.

CHAPTER V.

Extension of Method to the Determination of Certain Equivalent Networks.

(a) Change in the Order of Procedure.

79. Very slight consideration will show that the order of the steps (a), (b), (c) and (d) specified in Table I, p. 69 is not essantial. Thus, for example, in the simple case where all the poles and zeros lie on the axis of imaginaries any pole or any zero may be removed first and thereafter again any pole or any zero. The networks shown in Fig. 1 (page 4) are special cases when a definite order is adopted; Fig. 1(a) results when all poles of $Z(\lambda)$ are removed in succession, i.e. the network given by Table I; (b) results when all zeros are removed in succession; (c) results when we remove first the pole at infinity, then the zero at infinity and so on alternately; and (d) when poles and zeros are removed alternately at the origin. The order may, however, be entirely haphazard. In fact only "part of a pole", i.e. a term representing a pole at the same point but with residue which is a positive proper fraction of the corresponding residue of $Z(\lambda)$, may be separated; the remaining positive function can thereafter be represented in any of the ways shown in Fig. 1 (b) (c) or (d); this would give a network with superfluous elements.

80. When we are dealing with a more general type of function,

the order of removing poles and zeros on the imaginary axis is still arbitrary until the procedure of Table I(c) becomes necessary. Obviously if there are any zeros on the axis of imaginaries these will also be points at which $\mathcal{R}e\, Z(i\omega)$ is zero; consequently step (d) becomes equivalent to (b) until all zeros on the axis of imaginaries have been removed. The presence of a pole in $Z_1(\lambda)$ on the imaginary axis however will not alter the value of R_1 in step (c). In the next step (d) therefore $Z_2(\lambda)$ will be a "positive" function and nothing will be altered in the argument concerning the validity of the procedure (d). In the event of a pole at infinity having been left in $Z_2(\lambda)$ we shall obtain $L_1.L_2 + L_2.L_3 + L_3.L_1 > 0$ instead of $= 0$, which obviously still fulfills the physical requirements. In fact in this case L_1 is exactly equal to $(L_0 + L_1)$ Table I (a) and (d). Poles on the imaginary axis can therefore be left in the function or separated at will at any stage in the procedure (see Example 12 Appendix).

81. Another alternative is to treat $\frac{1}{Z(\lambda)}$ as an admittance; step (c) will correspond to the separation of a parallel conductance from $Z(\lambda)$; in this case zeros of $Z(\lambda)$ do not necessarily have to be first removed.

(b) <u>Consideration of Negative Values of Ω</u>

82. It was seen in Chapter III section 52 that for the simple case of two poles and zeros in the interior of the

left-half plane, a stationary value of $R(\Omega)$ occurring at a negative value of Ω could be used for obtaining a physically realizable network, provided it satisfied the condition

$$R_\gamma \leq R(\Omega) \quad \text{when } \Omega \text{ is positive} \tag{48}$$

It will now be shown that this is true for the general case.

83. Negative values of Ω correspond to imaginary values of ω which in turn means real values of λ. If therefore we write

$$Z_1(\lambda) = R(-\lambda^2) + \lambda N(-\lambda^2) \tag{49}$$

(cf. equation (43) section 74) the interpretation in terms of real values of λ can be carried over from the discussion for pure imaginary values (sections 61 to 64) without any difficulty.

Let a stationary value of $R(-\lambda^2)$ satisfying (48) occur when $\lambda^2 = \gamma^2$, and let $R_\gamma = R(-\gamma^2)$

Then if $$Z_1(\lambda) = R_\gamma + Z_2(\lambda) \tag{50}$$

it is clear that $Z_2(\lambda)$ will be a "positive real" function in virtue of (48).

Now if $$Z_2(\lambda) = \left\{R(-\lambda^2) - R_\gamma\right\} + \lambda N(-\lambda^2)$$

we know that $$Z_2(\gamma) = \gamma N(-\gamma^2)$$

Let us write $$L_1 = N(-\gamma^2)$$

and $$Z_2(\lambda) = L_1\lambda + W(\lambda) \tag{51}$$

Then it follows from equations 50 and 51 that

$$W(\lambda) = \left\{R(-\lambda^2) - R_\gamma\right\} + \lambda\left\{N(-\lambda^2) - L_1\right\} \tag{51a}$$

$$\text{i.e.} \quad W(\gamma) = 0 \quad \text{and} \quad L_1 = \frac{Z_2(\gamma)}{\gamma} \tag{51b}$$

It is further clear that equations (51b) hold for $\lambda = \pm\nu$ Hence $W(\lambda)$ will not be a "positive" function, since it has the zero $\lambda = \nu$ in the right-half plane. By Theorem VI, Coroll. 1 (section 37) however we know that this is the only zero of $W(\lambda)$ within the right-half plane. Note further that

$$\mathcal{R}e\, W(i\omega) \geq 0 \tag{51c}$$

in virtue of (48).

If therefore we separate the zeros at $\lambda = \pm\nu$ from $W(\lambda)$ by a partial fraction expansion of $\frac{1}{W(\lambda)}$, the remaining function may be a "positive real" function.

For this purpose we examine the residues of the poles of $\frac{1}{W(\lambda)}$ at $\lambda = \pm\nu$. We may rewrite (51a)

$$W(\lambda) = R'(-\lambda^2) + \lambda N'(-\lambda^2)$$

which, in virtue of (51b) and the fact that R_ν is a stationary value of $R(-\lambda^2)$, can again be written

$$W(\lambda) = (\lambda^2-\nu^2)^2 r(-\lambda^2) + \lambda(\lambda^2-\nu^2)n(-\lambda^2)$$

Hence the residue at $\lambda = \nu$ is

$$\left[\frac{\lambda-\nu}{W(\lambda)}\right]_{\lambda=\nu} = \frac{1}{2\nu^2 n(-\nu^2)} = -\frac{K}{2} \text{ (say)} \tag{51d}$$

The residue at $\lambda = -\nu$ obviously has the same value*

Consequently we may write

$$\frac{1}{W(\lambda)} = \frac{-\frac{K}{2}}{\lambda+\nu} + \frac{-\frac{K}{2}}{\lambda-\nu} + \frac{1}{W_1(\lambda)}$$

$$= \frac{K\lambda}{-\lambda^2+\nu^2} + \frac{1}{W_1(\lambda)} \tag{51e}$$

---

*This property is the exact analogy to the property that for a "positive real" function the residue at a pole on the imaginary axis is a positive real quantity.

Since $\mathcal{R}e \frac{1}{W(i\omega)} \geq 0$ and the first term of the right-hand side of (51e) is pure imaginary for $\lambda = i\omega$, it follows from (51c) that $\mathcal{R}e \frac{1}{W_1(i\omega)} \geq 0$. But $W_1(\lambda)$ has no zeros in the right-half plane and no zeros on the imaginary axis. Consequently by Theorem V, Coroll. 1, $W_1(\lambda)$ is a "positive real" function. The process is completed by writing

$$W_1(\lambda) = L_3\lambda + Z_3(\lambda) \tag{51f}$$

Also let

$$L_2 = -\frac{1}{K}, \quad D = \frac{\gamma^2}{K} \tag{51c}$$

It is now only necessary to show that the reduction of $Z_2(\lambda)$ to $Z_3(\lambda)$ corresponds to the calculation of elements in a physically realizable circuit.

Referring to equation (51b), we have

$$L_1 = \frac{Z_2(\gamma)}{\gamma} \tag{51b}$$

where, if γ is positive (and real), $Z(\gamma)$ will be positive and real, since $Z_2(\lambda)$ is a "positive real" function. Consequently L_1 is positive.

Furthermore L_3, equation (51f), is positive since Z_3 is a "positive real" function with a pole at infinity; and the relation

$$L_1L_2 + L_2L_3 + L_3L_1 = 0 \tag{51g}$$

can be shown to hold in exactly the same manner as in Section 62; consequently L_2 is negative and D is positive. (K could also be proved to be positive by adapting equation (41), sec. 73).

The process therefore corresponds to a determination of the elements L_1, L_2, L_3, D, of a realizable network connected in the manner of Fig. 19 (section 62).

(c) Relation to Affine Transformation of Quadratic Forms.

84. The application of an affine transformation to the quadratic forms connected with a network mentioned in section 11 is equivalent to the following matrix multiplication(17)

$$A' = C' \cdot A \cdot C \tag{52}$$

where A is the matrix $\| a_{rs} \|$ of the coefficients of a system of equations corresponding to a given network similar to (8) section 19, A' is a similar matrix $\| a_{rs}' \|$ corresponding to an equivalent network; C is the matrix of the affine transformation. The matrix C may be written $\| c_{rs} \|$ where

$$c_{11} = 1 , \quad c_{1s} = 0 \ (s \neq 1) \tag{52a}$$

the elements c_{rs} are real constants, and $C' = \| c_{sr} \|$. If we denote the determinant of A by D, the determinant of A' by D', and the determinant of C by Δ we have

$$D' = \Delta^2 D \tag{53a}$$

and in virtue of (52a) the first minor

$$\frac{\partial D'}{\partial a_{11}'} = \Delta^2 \frac{\partial D}{\partial a_{11}} \tag{53b}$$

It is on account of this property that the impedance function is invariant since

$$Z'(p) = \frac{D'}{\frac{\partial D'}{\partial a_{11}'}} = \frac{D}{\frac{\partial D}{\partial a_{11}}} = Z(p) \tag{53c}$$

This property is however equally true of the determinants $|L_{rs}|$, $|R_{rs}|$, $|D_{rs}|$

and their first minors with respect to L_{11}, R_{11}, D_{11} respectively where

$$a_{rs} = L_{rs} p + R_{rs} + D_{rs} p^{-1} \tag{8a}$$

(cf. section 19).

85. Turning now to the networks obtained by the procedure outlined in sections 61-64 and 83 we note that disregarding the purely reactive part resulting from poles and zeros on

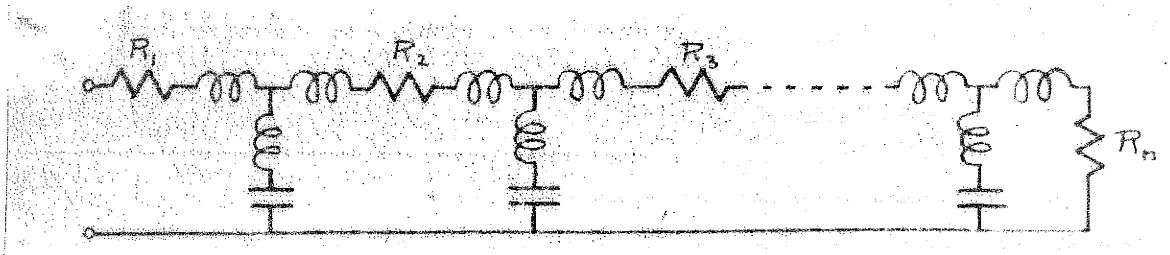

Fig. 23

the imaginary axis they always have a ladder structure with no resistance in the parallel branches (see Fig. 23) consequently

$$|R_{rs}| = \begin{vmatrix} R_1 & 0 & 0 & \dots & 0 \\ 0 & R_2 & 0 & \dots & 0 \\ 0 & 0 & R_3 & \dots & 0 \\ \vdots & & & & \\ 0 & \dots & & & R_n \end{vmatrix} \qquad (54a)$$

and the first minor

$$\frac{\partial}{\partial R_{11}}|R_{rs}| = \begin{vmatrix} R_2 & 0 & \dots & 0 \\ 0 & R_3 & \dots & 0 \\ \vdots & & & \\ 0 & \dots & & R_n \end{vmatrix} \qquad (54b)$$

The quotient of (54a) and (54b) is simply R_1 and this quotient is invariant under the transformation (52). Thus if according to section 83 several values of R_1 are possible (e.g. R_1, R_2, R_3, in Fig. 24) the corresponding networks

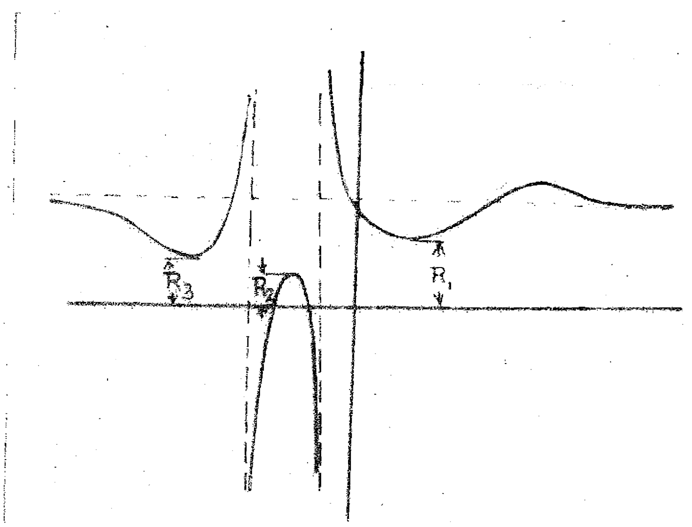

Fig. 24

cannot be derivable from each other by a transformation like

(52). Such networks are therefore representatives of separate groups in the sense of the affine transformation 52 (cf. section 12). This is true also when one of the usable values of $R(\Omega)$ occurs for $\lambda = 0$ or ∞. In such a case a network with a greater number of meshes results.

86. The following considerations seem to indicate that all equivalent networks are not exhausted by the separate groups pointed out in section 85 and the affine transformation within each group: It is possible in a three-mesh network to assign 18 absolutely arbitrary values to 18 independent elements in the network (see Fig. 25). These are represented by six elements in each of the matrices $\|L_{rs}\|, \|R_{rs}\|, \|D_{rs}\|$.

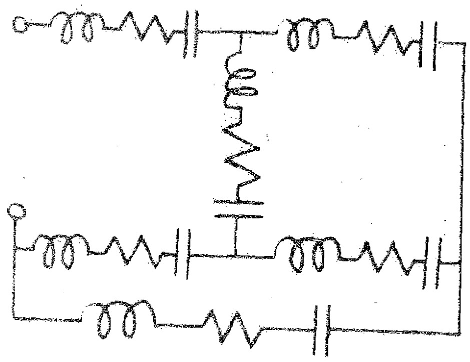

Fig. 25

The affine transformation matrix C corresponding to this case has six independent constants; presumably these may be so chosen that in the matrix A' six of the eighteen independent elements in A will have disappeared, leaving a total of twelve. Now the impedance function of the general three-mesh network is

$$Z(\lambda) = \frac{a_0 + a_1\lambda + a_2\lambda^2 + a_3\lambda^3 + a_4\lambda^4 + a_5\lambda^5 + a_6\lambda^6}{b_1\lambda + b_2\lambda^2 + b_3\lambda^3 + b_4\lambda^4 + b_5\lambda^5}$$

and has at most four poles and six zeros in the interior of the left-half plane. By equation (38.1) section 69 the minimum number of elements (given by the procedure of Table I) is (6+4+1) = 11. The affine transformation (52) therefore does not appear to give a means of proceeding from the general circuit in Fig. 25 to the circuit with the minimum number of elements in Fig. 23 (closing at R_3) or vice versa.

87. In this connection Ex. 13 in the Appendix is also of interest. Here we have gone out from a three-mesh network like that shown in Fig. 25; the value of

$$|R_{rs}| \Big/ \frac{\partial}{\partial R_{11}} |R_{rs}| \qquad (53d)$$

for this network does not, however, appear as a stationary value of $\mathcal{R}e\, Z(i\omega)$ for this network; in other words this method apparently does not give representatives of all groups. The mathematical proof depends on whether or not it can be shown that expression (53d) is necessarily a minimum of the real part of $Z(i\omega)$. Such an investigation would probably also disclose the true relation between (53d) and $\mathcal{R}e\, Z(i\omega)$.

88. In the methods developed thus far we were able to separate poles and zeros only if they occur on the axis of imaginaries; in certain special cases this could also be done when they occur on the real axis. It is evident that under suitable conditions a zero (for example) in the interior of the left-half plane can be separated, and corresponds to a branch with series resistance, capacitance and inductance, in parallel with the network representation of the remaining function. Such conditions can readily be

found*; it is then only necessary to manipulate a given function in such a manner that a remaining "positive real" function satisfies these conditions, to obtain further equivalent networks**.

The consideration of structures other than those of a ladder type may also lead to interesting results.

From these remarks and those of the preceding section it is evident that a considerable gap still remains to be bridged in the theory of equivalent networks. On the other hand the results of bridging this gap should be of interest from the mathematical as well as the physical point of view.

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*If $\lambda = -\nu \pm i\sigma$ are poles and the corresponding residues are $\frac{1}{2}(k \pm ih)$ the conditions are $k\nu + h\sigma = 0$, k positive. It is interesting to note that this expresses the fact that the vectors $\nu + i\sigma$ and $k + ih$ are at right angles, and that poles on the imaginary axis with positive real residue are a particular case of the general condition.

**Compare for example Foster Bibliog. (13) circuits 1,8,12.

CHAPTER VI.

Derivation of the Impedance Function from Prescribed Characteristics.

89. In the foregoing discussions we have assumed that the characteristics of a required network were prescribed by giving its impedance function. This is only one way in which these characteristics may be prescribed, and, for our methods, it is the best adapted. When the characteristics are given in a different manner the problem of deriving the impedance function from the given characteristics arises.

In the theory of network analysis it is shown that the complete characteristics can be determined by two methods of measurement*.

(i) Apply "unit" voltage (i.e. a voltage which is zero when $t < 0$ and unity when $t > 0$) and measure the corresponding current. This current is usually denoted by $A(t)$ and called the indicial admittance.

(ii) Apply a steady alternating voltage and measure the corresponding steady alternating current in magnitude and phase for all frequencies. The measured current can be denoted by $Y(i\omega)$**. (In the dual interpretation the applied quantity would be current and the measured quantity

*See for example Bibliog. 9 p. 78 and pp. 176-181.
**There is a superfluity of data in $Y(i\omega)$ which will be discussed later.

voltage).

Both forms of measurement are ideal in that (i) requires an infinite time and (ii) requires an infinite range of frequencies.

90. If A(t) is given, then by a well known theorem*

$$\frac{1}{\lambda Z(\lambda)} = \int_0^\infty \varepsilon^{-\lambda t} A(t)\,dt \tag{55}$$

If A(t) is given as an analytical function which actually does correspond to a finite network, performance of the integration will lead to an algebraic function satisfying the condition of "positiveness".

Assuming, however, that A(t) is given as a curve and that the infinite integration can be performed mechanically, $\frac{1}{\lambda Z(\lambda)}$ will be given as a curve. It will now be necessary to fit a quotient of polynomials to this curve in such a manner that $Z(\lambda)$ will be a "positive" function. It will probably be simpler to apply the condition of "positiveness" directly to the real part $P(\omega) = \mathcal{R}e\,\frac{1}{Z(i\omega)}$ which must be positive for real ω; $P(\omega)$ can be calculated from

$$\frac{P(\omega)}{\omega} = \int_0^\infty A(t) \sin \omega t \, dt. \tag{55a}$$

This equation results from (55) by placing $\lambda = i\omega$ and separating real and imaginary parts.

We do not propose to enter into the details of calculating $Z(\lambda)$ or $P(\omega)$ as a rational function to fit the given curve. This problem is complicated somewhat by the fact that the function must not only approximate certain points in a finite range of the variable but must ~~also~~ satisfy also certain

*See for example Bibliog.(8) p.19 or Bibliog.(9) p. 78.

conditions of positiveness for an infinite range of values of the variable.

91. Consider next the case where $Z(i\omega)$ is given. Again we shall waive the question of deriving suitable algebraic functions from given curves, with no intention of minimizing its practical importance however.

Now $Z(i\omega)$ contains essentially two quantities, e.g. a real and imaginary part, or alternatively a modulus and an argument (corresponding to amplitude and phase). It is a well known fact that the specification of any <u>one</u> of these determines the others (except perhaps for some arbitrary terms or factors). This is related to the Riemann conditions for the real and imaginary part of an analytic function and also to the Poisson integral* connecting the value of harmonic functions in a closed region with the value of one of them on the boundary. Lee(24) has discussed the problem from the standpoint of electrical networks and has given the results in terms of Hilbert transforms. These involve integrals and are perfectly general, but for even comparatively simple cases, present computations quite formidable in detail though simple in principle.

92. We wish to point out here an algebraic procedure which is applicable when the given functions are rational. This

*See for example Bibliog. 33 Vol. 1 p. 67 et seq.

procedure may in such cases be preferable to the evaluation of integrals. But the algebraic procedure is particularly interesting in that it brings out the inherent simplicity of the relations, at the same time making clear why the results are not always unique.

The relations given in section 74 are taken as the starting point; these relations express the coefficients of $R(\Omega)$, $N(\Omega)$, etc. in terms of the coefficients of $Z(\lambda)$ while in the present discussion we wish to proceed in the opposite direction. These equations, however, also point the way for this opposite procedure.

Having obtained $Z(\lambda)$ from such a characteristic, the question arises if it will be a "positive" function. It would be very desirable to be able to predict this from the given characteristic without first going through the procedure of finding $Z(\lambda)$. If the characteristic given be $R(\omega^2)$ (the real part of the impedance) this presents no difficulty since the conditions for "positiveness" of $Z(\lambda)$ have already been stated in terms of $R(\omega^2)$, Theorem V. In some of the other cases, however, it is not so easy to determine beforehand whether $Z(\lambda)$ will be a "positive" function. An extension of the properties discussed in Theorem VI may be of value for this purpose. For the present it is not essential to do this, since it is enough to be able to derive a $Z^*(\lambda)$ which will generate the given characteristic and then decide whether $Z^*(\lambda)$ is "positive" or not.

93. Real Part of Impedance given. Let

$$R(\Omega) = \frac{H(\Omega)}{G(\Omega)} = \frac{C_0 + C_1\Omega + C_2\Omega^2 + \_\_\_ + C_n\Omega^n}{B_0 + B_1\Omega + B_2\Omega^2 + \_\_\_ + B_n\Omega^n} \qquad (45)$$

be given.

We note that if $R(\Omega)$ is to be the real part of the driving point impedance of a two-terminal network, $R(\Omega)$ must be positive for real positive values of Ω; further it can have no poles on the positive real Ω axis (Theorem IV).

Now referring to equations (43), (44a), (44b) in section 74 which we rewrite here

$$Z(\lambda) = \frac{f(\lambda)\ g(-\lambda)}{g(\lambda)\ g(-\lambda)} = \frac{H(-\lambda^2)}{G(-\lambda^2)} + \lambda\frac{I(-\lambda^2)}{G(-\lambda^2)} \qquad (43)$$

$$B_r = \sum_{s=-\infty}^{+\infty} b_{r+s}\, b_{r-s}\, (-1)^s \qquad (44a)$$

$$C_r = \sum_{s=-\infty}^{+\infty} a_{r+s}\, b_{r-s}\, (-1)^s \qquad (44b)$$

we notice first of all that we may write

$$G(-\lambda^2) = g(\lambda).g(-\lambda)$$

in other words, the roots of $G(\Omega)$ are the negative squares of the roots of $g(\lambda)$. If the roots of $G(\Omega)$ are determined* and are denoted by γ_r then the roots β_r of $g(\lambda)$ will immediately be given by

$$\beta_r = -\sqrt{-\gamma_r} \qquad (56)$$

where the negative sign before the root-sign means that the square root which lies in the left-half plane must be taken. Since positive real values of γ_r are not permitted because of the

*A method of finding these roots, which has an interesting connection with the present problem, is that of raising them to the 2^nth power. A detailed discussion of this method will be found for example in Bibliog. (10) Ch. XXVIII or Bibliog. (36) pp. 106-118.

exclusion of poles of $R(\Omega)$ for positive Ω the question of square roots falling on the boundary does not arise. Note that complex roots can be handled in conjugate pairs as follows: Let a pair of conjugate complex roots of $G(\Omega)$ be represented by the factor

$$A'_2\Omega^2 + A'_1\Omega + A'_0$$

then the corresponding pair in $g(\lambda)$ will be represented by

$$+\sqrt{A'_2}\,\lambda^2 + \sqrt{A'_1 + 2\sqrt{A'_0 A'_2}}\,\lambda + \sqrt{A'_0} \tag{56a}$$

The roots of $g(\lambda)$ being known, $g(\lambda)$ itself is determined but for an arbitrary constant; but this is fixed by a comparison of the coefficients of the highest (or lowest) power in $G(\Omega)$ and $g(\lambda)$, in equation (44a).

Now all the coefficients b_r of $g(\lambda)$ are known. Equation (44b) thus becomes a linear system of equations with known coefficients for determining the a_r.

We thus have a definite method of finding a $Z(\lambda)$ which will generate the given $R(\Omega)$. Reference to Theorem IV however will show that this is not the only $Z(\lambda)$ which will generate this $R(\Omega)$ since an arbitrary number of terms representing poles with arbitrary positive real residues can be added to this $Z(\lambda)$ without affecting $R(\Omega)$. If we wish to fix $Z(\lambda)$ we must specify besides its real part on the axis of imaginaries, also its poles on this axis and the residue ~~of~~ at these poles.

It is not difficult to check this in equation (44b) by substituting, e.g. $a_r = a'_r + L\, b_{r-1}$ and noting that a'_r satisfies exactly the same equations as a_r (see Ex.14 Appendix).

94. Imaginary Part of Impedance given. The procedure here is very similar to that described in the preceding section. The relevant equations are, besides (43) (44a) (45)

$$N(\Omega) = \frac{D_0+D_1\Omega+D_2\Omega^2+ - - + D_n\Omega^n}{B_0+B_1\Omega+B_2\Omega^2+ - - +B_n\Omega^n} \tag{45a}$$

$$D_r = \sum_{s=-\infty}^{\infty} a_{r+s}\, b_{r-s+1} (-1)^{s+1} \tag{44c}$$

We note that a necessary condition for $N(\Omega)$ is that poles on the positive real Ω axis shall be simple and have negative residue (poles at infinity are excluded). These poles may be removed if desired as

$$N(\Omega) = \sum_r \frac{K}{-\Omega+\sigma_r^2} + N_1(\Omega) \tag{57a}$$

the terms corresponding to poles under the summation being immediately recognizable as belonging to the imaginary part of $\sum_r \frac{K\lambda}{\lambda^2+\sigma_r^2}$ when $\lambda = i\omega$. Moreover the degree of the numerator of $N(\Omega)$ can be made lower by unity than the degree of the denominator by a simple process of division

$$N(\Omega) = L + \frac{D_0'+D_1'\Omega+ \quad - \quad - \quad +D_{n-1}\Omega^{n-1}}{B_0+B_1\Omega+ - - - - +B_n\Omega^n} \tag{57b}$$

where we recognize the first term as belonging to the imaginary part of $L\lambda$ when $\lambda = i\omega$; L must be positive. We may thus reduce the problem to that of dealing with a function $N(\Omega)$ which has no poles on the real positive Ω axis, and whose numerator is of degree one lower than that of the denominator.

The coefficients b_r of the denominator of ~~the~~ a corresponding $Z^*(\lambda)$ may be found from the denominator of $N(\Omega)$ as in the preceding section.

In examining the linear equations (44c) for the a_r

it is at once evident that the number of equations is one less than the number of unknowns. But this system of equations has the property that if one set of values a_r satisfies them, any other set

$$a_r' = a_r + Rb_r \tag{58}$$

will also satisfy them, for this merely results in adding

$$-R \sum_{s=-\infty}^{\infty} b_{r+s} b_{r-s+1} (-1)^s \equiv 0$$

to the left-hand member of each equation (44c). We may therefore arbitrarily place a_0 (say) = 0 and solve for the remaining a_r, thus determining a $Z^*(\lambda)$. The equation (58), however, corresponds to adding an arbitrary cônstant R to $Z^*(\lambda)$. The value of R can obviously be chosen so that

$$Z(\lambda) = Z^*(\lambda) + R \tag{59}$$

will be a "positive" function provided $R \geq R_m$, where R_m is the least value which will make $\mathcal{R}e\, Z^*(i\omega) + R_m \geq 0$. R_m is finite because of the restrictions imposed on the poles of $I(\Omega)$. These restrictions are thus sufficient in order that a "positive" function $Z(\lambda)$ may be found to generate the given $I(\Omega)$. The function $Z(\lambda)$ contains a semi-arbitrary additive constant (see Ex. 15 Appendix).

95. <u>Modulus of Impedance given.</u> If we write

$$Z(i\omega) = \frac{f(i\omega)}{g(i\omega)}$$

then obviously

$$|Z(i\omega)|^2 = \frac{f(i\omega)\ f(-i\omega)}{g(i\omega)\ g(-i\omega)} \tag{60}$$

or writing $M(\Omega)$ for $|Z(i\omega)|^2$

$$M(\Omega) = \frac{F(\Omega)}{G(\Omega)} = \frac{A_0 + A_1\Omega + A_2\Omega^2 + \cdots + A_n\Omega^n}{B_0 + B_1\Omega + B_2\Omega^2 + \cdots + B_m\Omega^m} \tag{60a}$$

where $F(-\lambda^2) = f(\lambda).f(-\lambda)$ and the A_r are defined by equations similar to (44a) section 74.

It is now readily seen that $g(\lambda)$ is obtained from $G(\Omega)$ as in the two preceding sections, while $f(\lambda)$ is found from $F(\Omega)$ by an exactly similar process; the two calculations are quite independent of each other.

Note that $Z^*(\lambda)$ is completely determined by $M(\Omega)$, there being no arbitrary terms or factors. The function $M(\Omega)$ may have either poles or zeros on the positive real Ω axis, but examination of equation (60) in the light of Theorem II shows that such poles or zeros must be double, since a single pair of conjugate poles $\lambda = \pm i\sigma$ gives rise to two poles $\Omega = \sigma^2$; similarly for zeros. $M(\Omega)$ is thus always positive for positive values of Ω. This is not yet sufficient however for $Z^*(\lambda)$ to be a positive function, for while the poles and zeros are properly located there is nothing to indicate that the condition $\mathcal{R}_e\, Z(i\omega) \geq 0$ will be satisfied. This must therefore be tested in the resulting $Z^*(\lambda)$ before any conclusion can be drawn regarding the possibility of a network having this characteristic.

96. Argument of Impedance given. From the equation

$$Z(i\omega) = \frac{H(\Omega)}{G(\Omega)} + i\omega\, \frac{I(\Omega)}{G(\Omega)} \tag{61}$$

we readily deduce

$$\frac{\tan(\arg Z(i\omega))}{i\omega} = \frac{I(\Omega)}{H(\Omega)} = J(\Omega) \ \text{(say)} \tag{62}$$

But comparing with

$$Z(\lambda) = \frac{f(\lambda).g(-\lambda)}{g(\lambda).g(-\lambda)} = \frac{\{H(-\lambda^2)+\lambda I(-\lambda^2)\}K}{G(-\lambda^2).K}$$

we have $$K.f(\lambda).g(-\lambda) = H(-\lambda^2) + \lambda I(-\lambda^2)$$

It is therefore only necessary to find the roots of the equation

$$H(-\lambda^2) + \lambda I(-\lambda^2) = 0 \tag{63}$$

and assign them among $f(\lambda)$ and $g(-\lambda)$. If the resulting function

$$Z^*(\lambda) = K\,\frac{f(\lambda)}{g(\lambda)} \tag{64}$$

is to have its poles and zeros in the left-half plane, those roots of (63) which have negative real part must be assigned to $f(\lambda)$ while those which have positive real part must be assigned to $g(\lambda)$. It is evident that pure imaginary roots need not be considered at this stage since a factor $(\lambda^2 + \sigma^2)$ of $f(\lambda).g(-\lambda)$ must be a factor of both $H(-\lambda^2)$ and $I(-\lambda^2)$; in prescribing the function $J(\Omega) = \frac{I(\Omega)}{H(\Omega)}$, it can therefore be cancelled.

It is clear, however, that such factors can be introduced without affecting the prescribed $J(\Omega)$. This is just what we need if $Z^*(\lambda)$ in (64) is not a "positive" function. For let

$$\mathcal{R}e\; Z^*(i\omega) = R^*(\Omega) \leq 0$$

when Ω has values in the intervals

$$\left.\begin{array}{l} 0 < \gamma_1 \leq \Omega \leq \gamma_2 \\ \gamma_2 < \gamma_3 \leq \Omega \leq \gamma_4 \\ \text{- - - - - - - -} \\ \text{- - - - - -} \end{array}\right\} \tag{64a}$$

(see Fig.26)

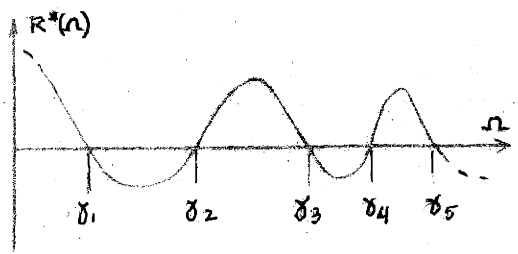


Fig. 26

(Note: a change of sign in the whole range is accomplished by merely changing the sign of K in (64))

If we now write

$$\left.\begin{array}{l} R(\Omega) = R^*(\Omega)\prod_{\tau=1}^{n}(\gamma_\tau-\Omega)^{\delta_\tau} \\ \text{where} \quad \delta_\tau = \pm 1 \text{ and } \gamma_n \text{ is the greatest root of } R^*(\Omega) \end{array}\right\} \quad (64b)$$

we shall have made $R(\Omega) \geq 0$ when $\Omega \geq 0$; for $R^*(\Omega)$ and the product $\prod_{\tau=1}^{n}(\gamma_\tau-\Omega)$ have the same sign for positive values of Ω, (irrespective of the sign chosen for δ_τ in (64b)).

Moreover the relation between $R^*(\Omega)$ and $R(\Omega)$ expressed in (64b) is given exactly by

$$\left.\begin{array}{l} Z(\lambda) = Z^*(\lambda)\prod_{\tau=1}^{n}(\lambda^2+\sigma_\tau^2)^{\delta_\tau} \\ \delta_\tau = \pm 1 \\ \text{and } \mathcal{R}e\, Z(i\omega) = R(\omega^2), \quad \sigma_\tau^2 = \gamma_\tau \end{array}\right\} \quad (64c)$$

$Z(\lambda)$ in (64c) will now satisfy the conditions: (i) No poles within the right-half plane; (iii) $\mathcal{R}e\, Z(i\omega) \geq 0$; of Theorem V. It must still be made to satisfy the condition: (ii) Poles on the boundary must have positive real residues. For this purpose we have at our disposal the sign of δ_τ in (64c): If any particular δ_τ be given the value -1, $\lambda = \pm i\sigma_\tau$ becomes a pole. The choice of the magnitude of σ_τ (Fig. 27) ensures that the residue at this pole will be real (cf. section 73). If the sign of the residue be negative $\lambda = \pm i\sigma_\tau$ is not permissible as a pole, and consequently δ_τ must be assigned the value +1. If on the other hand the sign of the residue is positive, δ_τ must be assigned the value -1*.

In this way a "positive real" function may be obtained**

*It can readily be checked in section 73 that if $Z_1(\lambda) = \frac{f(\lambda)}{g(\lambda)(\lambda^2+\sigma^2)}$ and $Z_2(\lambda) = \frac{f(\lambda)(\lambda^2+\sigma^2)}{g(\lambda)}$, then the residues at $\lambda = \pm i\sigma$ for Z_1 and $\frac{1}{Z_2}$ are opposite in sign.

**Note the very drastic restriction on amplitude in (64c) which may be necessary to accomplish this.

to generate the given 'phase' function $J(\Omega)$. $Z(\lambda)$ still contains in it the arbitrary positive factor K in (64), this arbitrariness being obvious also from physical considerations.

97. It would be very interesting to make a closer comparison between the above algebraic procedures and Lee's Hilbert transform methods. Lee did not impose the restriction of "positive" functions since, in his problem, a proper location of poles and zeros was sufficient. It would be possible to do the same in the foregoing discussion.

A further possibility which is of practical importance is that of prescribing network characteristics over a finite frequency range, instead of the infinite range of frequencies as has been done above. In such cases it may be possible to prescribe both real and imaginary parts within certain limits*.

It is evident that some improvement and amplification in our present methods of deriving an impedance function from prescribed characteristics is still desirable.

*Taking this point of view Dr. Cauer has already made some progress. Publication of this work in the "Göttinger Nachrichten" is planned in the near future.

APPENDIX.

Example 1.

Let $\quad Z(\lambda) = \lambda + 1 \;, \quad E = \varepsilon^{-2t}$

Then $\quad I_s = \dfrac{\varepsilon^{-2t}}{-2+1} = -\,\varepsilon^{-2t}$

$$I_t = C\,\varepsilon^{-t}$$

If $I = 0$ when $t = 0$, $C = 1$ and $\underline{I = \varepsilon^{-t} - \varepsilon^{-2t}}$

The corresponding currents together with the total current are shown in the diagram.

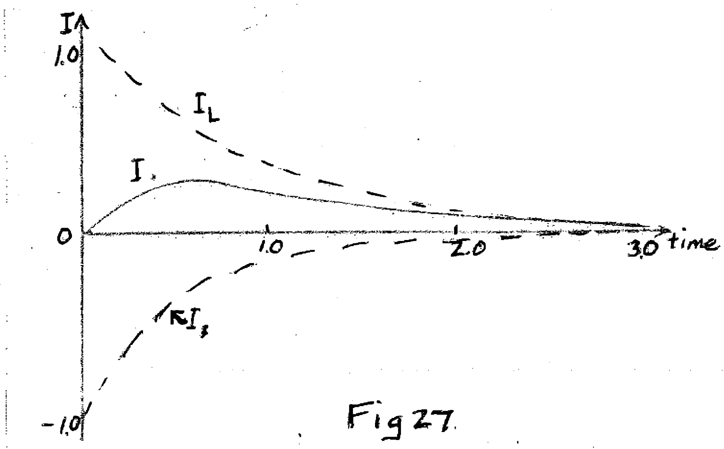


Fig 27.

Example 2.

Consider $\quad Z^*(\lambda) = \dfrac{2+3\lambda+\lambda^2}{1+\lambda+4\lambda^2}$, zeros are $=-1,-2$; poles are $= -\frac{1}{8} \pm i\frac{\sqrt{15}}{8}$

$$\mathcal{R}e\, Z^*(i\omega) = \frac{2-6\omega^2+4\omega^4}{1-7\omega^2+16\omega^4}$$

This satisfies conditions (2a) and (2b) but is negative for values of ω^2 between $\frac{1}{2}$ and 1.

Example 3.

Consider the function

$$Z(\lambda) = \frac{1 + 3\lambda + 2\lambda^2}{1 + 7\lambda + 12\lambda^2}$$

We find $\mathcal{R}e\, Z(i\omega) = R(\Omega) = \dfrac{1 + 7\Omega + 24\Omega^2}{1 + 25\Omega + 144\Omega^2}$ (where $\Omega = \omega^2$)

Whence $\quad (R-1) + (25R-7)\Omega + (144R-24)\Omega^2 = 0$

For real Ω $\quad (25R-7)^2 - 4(R-1)(144R-24) \geq 0$

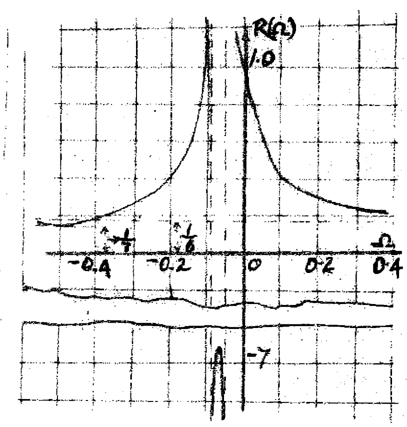


Fig. 28.

i.e. $49R^2 - 322R - 47 \geq 0$

i.e. $(7R + 47)(7R - 1) \geq 0$

$$R \leq -\frac{47}{7} \text{ or } \frac{1}{7} \leq R$$

When $R = -\dfrac{47}{7}, \Omega = -\dfrac{3}{34}$

$\qquad R = \dfrac{1}{7}, \Omega = -\dfrac{1}{2}$

The general variation of R with Ω is shown in Fig. 28. From this it is evident that $R(\infty) = \frac{1}{6}$ can be taken out. Proceeding in this manner

$$Z(\lambda) = \frac{1}{6} + \frac{\frac{5}{6} + \frac{11}{6}\lambda}{1 + 7\lambda + 12\lambda^2}$$

$$= \frac{1}{6} + 1\Big/\left(\frac{72\lambda}{11} + \frac{1 + \frac{17}{11}\lambda}{\frac{5}{6} + \frac{11}{6}\lambda}\right)$$

$$= \frac{1}{6} + 1\Big/\left(\frac{72\lambda}{11} + 1\Big/\left\{\frac{5}{6} + 1\Big/\left[\frac{11}{6\lambda} + \frac{17}{6}\right]\right\}\right)$$

$$= \frac{1}{6} + \frac{1}{\frac{72\lambda}{11} +}\;\frac{1}{\frac{5}{6} +}\;\frac{1}{\frac{11}{6\lambda} +}\;\frac{1}{\frac{6}{17}}$$

The corresponding network is shown in Fig. 29 (cf. also Fig. 17d).

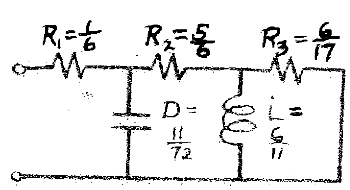


Fig. 29.

Example 4.

Consider next

$$Z(\lambda) = \frac{1 + 3\lambda + 2\lambda^2}{3 + \lambda + 2\lambda^2}$$

$$\mathcal{R}e\, Z(i\omega) = R(\Omega) = \frac{3 - 5\Omega + 4\Omega^2}{9 - 11\Omega + 4\Omega^2}$$

whence $(9R-3) - (11R-5)\Omega + (4R-4)\Omega^2 = 0$

For real Ω $(11R-5)^2 - 4(9R-3)(4R-4) \geq 0$

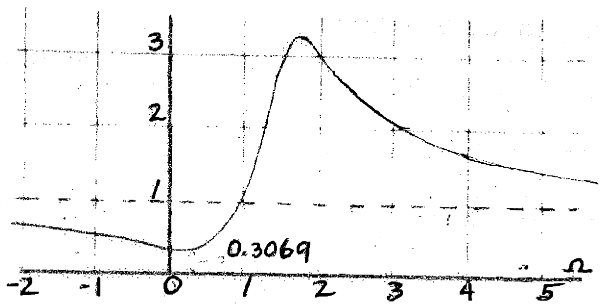


Fig 30

i.e. $-23R^2 + 82R - 23 \geq 0$

or $0.3069 \leq R \leq 3.258$

when $R = 0.3069, \Omega = 0.294$

$R = 3.258, \Omega = 1.71$

The variation of $R(\Omega)$ is shown in Fig. 30

The largest value of R that can be removed is .3069

Then

$$Z(\lambda) = .3069 + \frac{.0793 + 2.693\lambda + 1.386\lambda^2}{3 + \lambda + 2\lambda^2}$$

Using equations (32c) (page 51)

$r = .0264$ $D = .0793$

$$\begin{aligned} L_1 + L_2 &= 1.386 \\ L_2 + L_3 &= .0529 \\ L_1 + L_3 &= .898 \end{aligned} \qquad \begin{aligned} L_1 &= 1.116 \\ L_2 &= .270 \\ L_3 &= .218 \end{aligned}$$

Check: $\frac{D}{L_2} = 0.294$
$= \Omega$ when $R(\Omega) = .307$

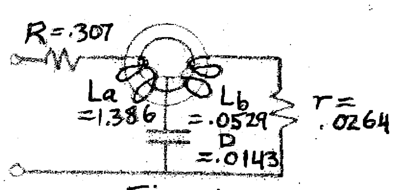


Fig.31

The network representation is shown in Fig. 31. L_1, L_2, L_3 represent two inductances $L_a = 1.386$, $L_b = .0529$ perfectly coupled with fluxes opposing.

Example 5.

Let
$$Z(\lambda) = \frac{6\lambda^2 + 15\lambda + 4}{9\lambda^2 + 23\lambda + 8}$$
$$\mathcal{R}e\, Z(i\omega) = R(\Omega) = \frac{54\Omega^2 + 261\Omega + 32}{81\Omega^2 + 385\Omega + 64}$$

Treating this in exactly the same way as Examples 3 and 4 we find

$$R(\Omega) = 0 \quad \text{when } \Omega = -.12 \text{ or } -4.70$$
$$R(\Omega) = \infty \quad \text{when } \Omega = -.175 \text{ or } -4.57$$
$$R(\Omega) \leq .669 \text{ or } .716 \leq R(\Omega)$$
$$R(\Omega) = .669 \text{ when } \Omega = 7.3; \; R(\Omega) = .716 \text{ when } \Omega = -1.85$$

The variation of $R(\Omega)$ appears in Fig. 32.

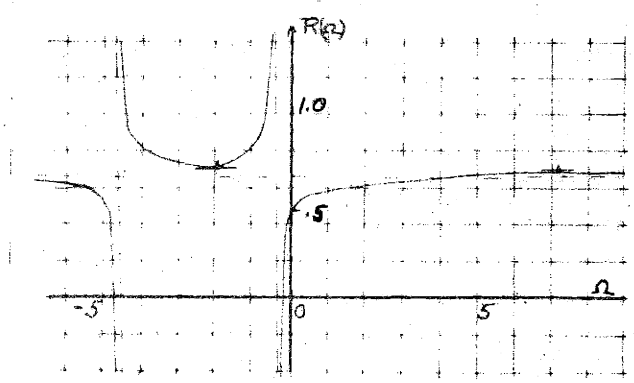


Fig 32.

$$Z(\lambda) = \frac{1}{2} + \frac{3.5\lambda + 1.5\lambda^2}{8 + 23\lambda + 9\lambda^2}$$
$$= \frac{1}{2} + \cfrac{1}{\cfrac{16}{7\lambda} + \cfrac{\frac{137}{7} + 9\lambda}{3.5 + 1.5\lambda}}$$
$$= \frac{1}{2} + \cfrac{1}{\cfrac{16}{7\lambda} + \cfrac{1}{\cfrac{1}{6} + \cfrac{\frac{5}{21}}{\frac{137}{7} + 9\lambda}}}$$
$$= \frac{1}{2} + \cfrac{1}{\cfrac{16}{7\lambda} + \cfrac{1}{\cfrac{1}{6} + \cfrac{1}{\frac{189}{5}\lambda + \cfrac{1}{\frac{5}{411}}}}}$$

The network representation is shown in Fig. 33.

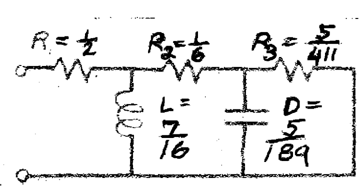


Fig. 33

Example 6.

It is evident from Fig. 28 that the function

$$Z(\lambda) = \frac{1 + 3\lambda + 2\lambda^2}{1 + 7\lambda + 12\lambda^2}$$

has an equivalent two-mesh representation (cf. also Fig.17f) since $R(\Omega) = \frac{1}{7}$ is a stationary value and $< R(\Omega)$ for positive Ω. Accordingly

$$Z(\lambda) = \frac{1}{7} + \frac{6 + 14\lambda + 2\lambda^2}{7 + 49\lambda + 84\lambda^2}$$

Using equations (32c) page 51.

$$r = \frac{6}{7} \qquad D = \frac{6}{49}$$

$$L_1 + L_2 = \frac{2}{49} \qquad L_1 = \frac{14}{49}$$

$$L_2 + L_3 = \frac{72}{49} \qquad L_2 = -\frac{12}{49}$$

$$L_1 + L_3 = \frac{98}{49} \qquad L_3 = \frac{84}{49}$$

Check: $\frac{D}{L_2} = -\frac{1}{2}$; $R(-\frac{1}{2}) = \frac{1}{7}$

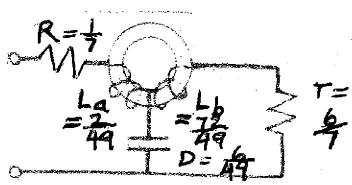


Fig.34

In Fig. 34 the group of inductances L_1, L_2, L_3 is shown as two coils L_a, L_b perfectly coupled with fluxes aiding.

Example 7.

The function $Z(\lambda)$ of Examples 2 and 5 permits of a further equivalent representation in terms of Fig. 17G as will be seen by locating the point (7,12) in Fig. 17(h) or by examining the graph of $\mathcal{R}e \frac{1}{Z(i\omega)}$

The procedure for such a representation is

$$Z(\lambda) = \frac{1}{\dfrac{1+7\lambda+12\lambda^2}{1+3\lambda+\ 2\lambda^2}}$$

$$= \frac{1}{1+}\ \frac{4\lambda+10\lambda^2}{1+3\lambda+2\lambda^2}$$

$$= \frac{1}{1+}\ \frac{1}{\frac{1}{4\lambda}+}\ \frac{\frac{1}{2}+2\lambda}{4+10\lambda}$$

$$= \frac{1}{1+}\ \frac{1}{\frac{1}{4\lambda}+}\ \frac{1}{5+}\ \frac{\frac{3}{2}}{\frac{1}{2}+2\lambda}$$

$$= \frac{1}{1+}\ \frac{1}{\frac{1}{4\lambda}+}\ \frac{1}{5+}\ \frac{1}{\frac{4\lambda}{3}+}\ \frac{1}{3}$$

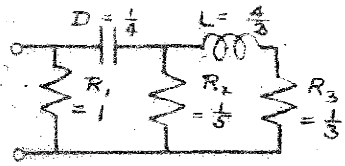


Fig. 35.

The corresponding network is shown in Fig. 35.

Example 8.

Let us consider the function $Z(\lambda)$ of Ex. 4 in the reciprocal

$$\frac{1}{Z(\lambda)} = \frac{3+\lambda+2\lambda^2}{1+3\lambda+2\lambda^2}$$

$$\mathcal{R}e\,\frac{1}{Z(\omega)} = \frac{3-5\Omega+4\Omega^2}{1+5\Omega+4\Omega^2}$$

i.e. $(R-3) + 5(R+1)\Omega + 4(R-1)\Omega^2 = 0$

For real Ω $\quad 25(R+1)^2 - 16(R-1)(R-3) \geq 0$

i.e. $9R^2 + 114R - 23 \geq 0$

i.e. $R \leq -12.9$ or $0.1980 \leq R$

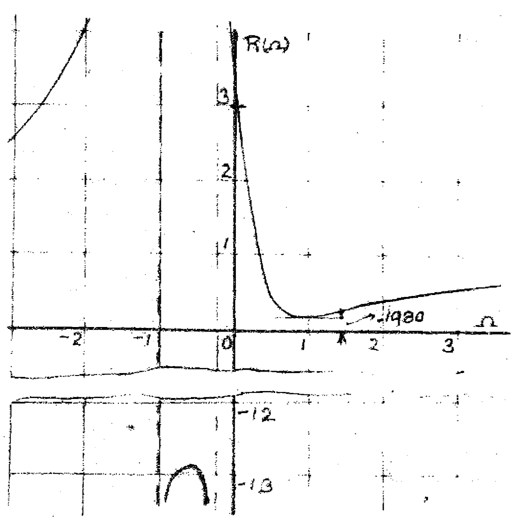


Fig. 36.

when $R = -12.9$, $\Omega = -.535$,

when $R = .1980$, $\Omega = 0.935$.

This variation is shown in Fig. 36.

Now $\dfrac{1}{Z(\lambda)} = .1980 + \dfrac{1}{Z'(\lambda)}$

$$= \frac{1}{5.05} + \frac{1}{Z'(\lambda)}$$

where $Z'(\lambda) = \dfrac{1+3\lambda+2\lambda^2}{2.802+.406\lambda+1.604\lambda^2}$

Applying equation 32c page 51

$$r = \frac{1}{2.802} = .356; \quad D = \frac{1}{.406} = 2.465$$

$$\begin{array}{lll} L_1+L_2 = 4.93 & L_1 = 2.29 & L_a = L_1+L_2 = 4.93 \\ L_2+L_3 = 1.410 & L_2 = 2.64 & L_b = L_3+L_2 = 1.41 \\ L_1+L_3 = 1.07 & L_3 = -1.23 & \end{array}$$

Check: $\frac{D}{L_2} = .935$

$R(.935) = .1980.$

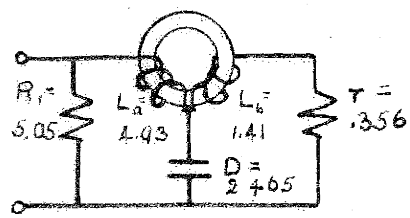


Fig. 37

The network is shown in Fig. 37.

Example 9.

Consider the network in Fig. 38. The impedance function of this network is

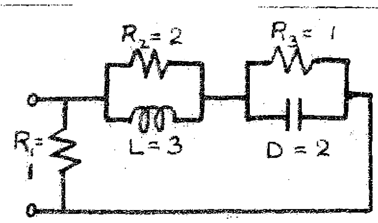


Fig 38

$$Z(\lambda) = \cfrac{1}{1+\cfrac{1}{\frac{6\lambda}{3\lambda+2}+\frac{2}{\lambda+2}}}$$

$$= \cfrac{1}{1+\cfrac{3\lambda^2+8\lambda+4}{6\lambda^2+15\lambda+4}}$$

$$= \frac{6\lambda^2+15\lambda+4}{9\lambda^2+23\lambda+8}$$

This function has been discussed in Ex. ~~6~~ 5 and found to have the network representation shown in Fig. 33. The four-mesh network of Fig. 38 is therefore equivalent to the three-mesh network of Fig. 33.

Example 10.

The network shown in Fig. 39 has the impedance function

$$Z(\ ) = \frac{1}{\frac{6\lambda}{1+\lambda} + \frac{1}{1+2\lambda}}$$

$$= \frac{1+3\lambda+2\lambda^2}{1+7\lambda+12\lambda^2}$$

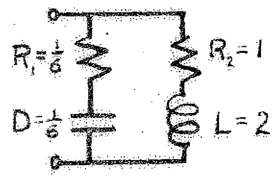


Fig. 39.

This network is thus equivalent to the networks discussed in Ex. 3, 6 and 7. We note, however, that it has only four elements while the networks previously discussed had (apparently) five independent elements. This is a special case where, in virtue of a "hidden" relation among the coefficients the general process does not give the absolute minimum number of elements which will realize the impedance function.

The "hidden" relation in this case is

$$R = (a_1b_1 - a_ob_2 - a_2b_o)/b_1{}^2$$

where R is a usable stationary value of $R(\Omega)\left(= \frac{1}{7}\right)$.

It may further be pointed out that the circuit of Fig. 39 is derivable from that of Fig. 34 by an affine transformation of the quadratic forms (section 84)

For $\quad |R_{rs}| \Big/ \frac{\partial}{\partial R_{11}}|R_{rs}| = \begin{vmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{1}{6} & \frac{7}{6} \end{vmatrix} \Big/ \frac{7}{6} = \frac{1}{7}$

Example 11.

To illustrate the general methods described in Chapter IV let us consider

$$Z(\lambda) = \frac{37\lambda^6 + 74.4\lambda^5 + 279.2\lambda^4 + 369.56\lambda^3 + 570.36\lambda^2 + 390.4\lambda + 254}{26\lambda^5 + 39.8\lambda^4 + 108.36\lambda^3 + 74.2\lambda^2 + 79\lambda}$$

After separating the poles at $\lambda = 0$ and $\lambda = \infty$ we have

$$Z(\lambda) = 1.4231\lambda + \frac{3.2152}{\lambda} + Z_1(\lambda)$$

where

$$Z_1(\lambda) = \frac{17.7615\lambda^4 + 41.400\lambda^3 + 136.003\lambda^2 + 109.539\lambda + 151.833}{26\lambda^4 + 39.8\lambda^3 + 108.36\lambda^2 + 74.2\lambda + 79}$$

Using formulae (44a) and (44b) section 74 we have

$$\mathcal{R}e\, Z(i\omega) = R(\Omega) = \frac{.46180\Omega^4 - 3.8130\Omega^3 + 12.6566\Omega^2 - 19.0691\Omega + 11.9948}{.6760\Omega^4 - 4.0507\Omega^3 + 9.9436\Omega^2 - 11.6152\Omega + 6.241}$$

and from formula (46) $\frac{dR}{d\Omega} = 0$ when

$$0 = \Omega^6 - 11.2137\Omega^5 + 50.8270\Omega^4 - 122.7961\Omega^3 + 165.4610\Omega^2 - 113.9550\Omega + 28.7315$$

A rough plot of $R(\Omega)$ for various values of Ω is shown in Fig. 40

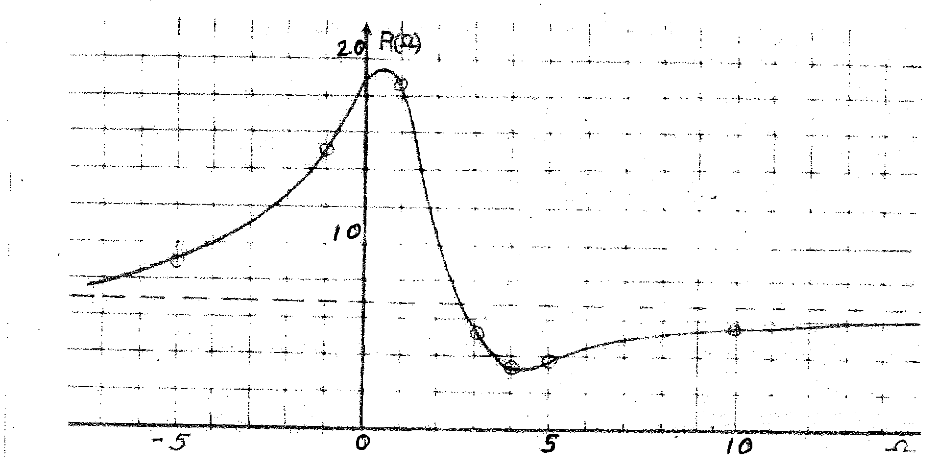


Fig 40

The minimum value of $R(\Omega)$ is indicated between $\Omega = 4$ and $\Omega = 5$. By successive approximation in the equation for $\frac{dR}{d\Omega} = 0$, the point is located at

$$\Omega = 4.097; \; R(\Omega) = .37930$$

$$Z_1(\lambda) = .3793 + Z_2(\lambda)$$

where

$$Z_2(\lambda) = \frac{7.8997\lambda^4 + 26.3043\lambda^3 + 94.8992\lambda^2 + 81.3948\lambda + 121.8681}{26\lambda^4 + 39.8\lambda^3 + 108.36\lambda^2 + 74.2\lambda + 79}$$

Next from equations (47) we find

$$L_1 = Z_2(\iota\sigma) \qquad (\sigma^2 = 4.097)$$

$$= \frac{7.8997\sigma^4 - 94.8992\sigma^2 + 121.8681}{-39.8\sigma^2 + 74.2} = -.36905$$

$$= \frac{-26.3043\sigma^2 + 81.3948}{26\sigma^4 - 108.36\sigma^2 + 79} = -.36902$$

$$Z_2(\lambda) = -.36904\lambda + W(\lambda)$$

where

$$W(\lambda) = \frac{9.5950\lambda^5 + 22.5875\lambda^4 + 66.2934\lambda^3 + 122.2820\lambda^2 + 110.5489\lambda + 121.86810}{26\lambda^4 + 39.8\lambda^3 + 108.36\lambda^2 + 74.2\lambda + 79}$$

$$\frac{1}{W(\ )} = \frac{26\lambda^4 + 39.8\lambda^3 + 108.36\lambda^2 + 74.2\lambda + 79}{(\lambda^2 + 4.097)(9.5950\lambda^3 + 22.5875\lambda^2 + 26.984\lambda + 29.746)}$$

Using equations (41) the residue $\frac{K}{2}$ at $\lambda = \pm\iota\sigma$ is given by

$$K = \frac{26\sigma^4 - 108.36\sigma^2 + 79}{9.5950\sigma^4 - 26.984\sigma^2} = 1.415$$

$$= \frac{-29.8\sigma^2 + 74.2}{-22.5875\sigma^2 + 29.746} = 1.415$$

$$\text{i.e.} \frac{1}{W(\lambda)} = \frac{1.415\lambda}{\lambda^2 + 4.097} + \frac{12.422\lambda^2 + 7.8369\lambda + 19.284}{9.5950\lambda^3 + 22.5875\lambda^2 + 26.984\lambda + 29.746}$$

$$= \frac{1.415}{\lambda^2 + 4.097} + \frac{1}{W'(\lambda)}$$

where

$$W'(\lambda) = .7724\lambda + \frac{16.534\lambda^2 + 12.089\lambda + 29.746}{12.422\lambda^2 + 7.8369\lambda + 19.28}$$

$$= .7724\lambda + Z_3(\lambda)$$

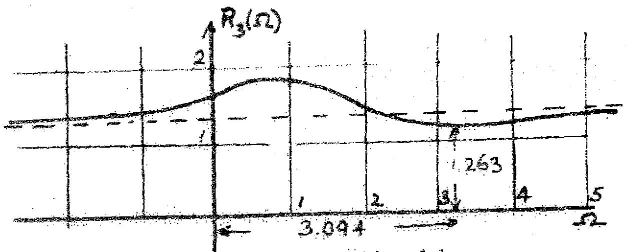


Fig. 41.

Again $\mathcal{R}e\ Z_3(i\omega) = \frac{2.0539\Omega^2 - 5.9362\Omega + 5.7362}{1.5431\Omega^2 - 4.1694\Omega + 3.718}$

The variation of this may be discussed as in Exs. 2, 3 etc; the plot is shown in Fig. 41.

$$Z_3(\lambda) = 1.263 + Z_4(\lambda)$$

$$Z_4(\lambda) = \frac{.8465\lambda^2 + 2.1922\lambda + 5.3930}{12.422\lambda^2 + 7.8369\lambda + 19.284}$$

Using equations 32c p. 51

$$R_3 = .2797 \qquad D\tfrac{1}{2} = .688$$

$$L_1 + L_2 = .1080 \qquad L_1 = -(.222) = -.111$$

$$L_3 + L_2 = .444 \qquad L_2 = .219$$

$$L_3 \quad L_1 = .1137 \qquad L_3 = .225$$

The network representation of the complete function is accordingly shown in Fig. 42

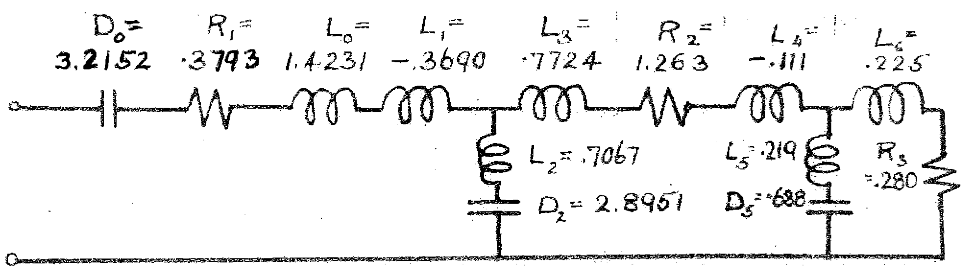


Fig. 42.

N.B. In this particular case the network may be realized without mutual inductance since the positive and negative inductances which appear in series in Fig. 42 may be combined into positive inductances.

Example 12

Consider the function

$$Z(\lambda) = \frac{(1+\lambda^2)(1+3\lambda+4\lambda^2)}{(1+2\lambda^2)(2+\lambda+\lambda^2)}$$

This has both a pole and a zero on the axis of imaginaries.

(i) If according to Table I we separate the pole first we get

$$Z(\lambda) = \frac{\lambda}{1+2\lambda^2} + \frac{1+\lambda+2\lambda^2}{2+\lambda+\lambda^2}$$

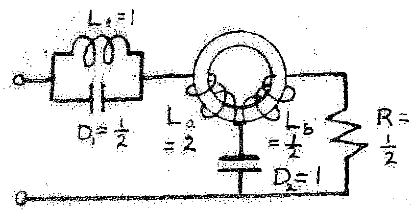


Fig. 43.

and the network representation which results when equations (32c) are applied is shown in Fig. 43 (we know that $\mathcal{R}e\, Z(i\omega)$ is zero when $\omega^2 = 1$).

(ii) We may, however, separate the zero first. Then

$$\frac{1}{Z(\lambda)} = \frac{\lambda}{3(1+\lambda^2)} + \frac{2(3+\lambda+3\lambda^2)}{3(1+3\lambda+4\lambda^2)}$$

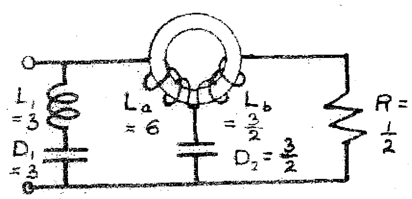
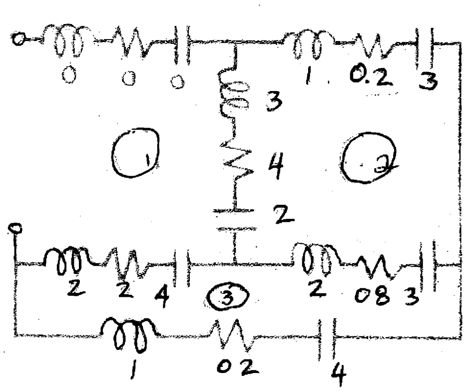


Fig. 44.

and the network representation of Fig. 44 results on applying equations 32c to the reciprocal of the second term. (We know that its real part = 0 when $\lambda = i\omega$ and $\omega^2 = \frac{1}{2}$).

This is a very simple example of the equivalent representations pointed out in section 80.

<u>Example 13.</u>



<u>Fig. 45</u>

The network of Fig. 45 has the impedance function discussed in Ex. 11. (In fact this function was originally derived from the network of Fig. 44).

The matrix of the resistances in this network is $\begin{vmatrix} 6 & -4 & -2 \\ -4 & 5 & -0.8 \\ -2 & -0.8 & 3.0 \end{vmatrix}$ and

$$\begin{vmatrix} 6 & -4 & -2 \\ -4 & 5 & -0.8 \\ 2 & -0.8 & 3.0 \end{vmatrix} \div \begin{vmatrix} 5 & -0.8 \\ -0.8 & 3 \end{vmatrix} = \frac{5.36}{14.36} = .3733$$

In Ex. 11 the minimum of $\mathcal{R}e\, Z(i\omega)$ was found to be = .3793 Moreover this is the only stationary value of R( ) occurring for real Ω besides the maximum shown in Fig. 40.

Although the difference between these values is comparatively small, it is considerably greater than the limits of error. An attempt to use the former value for the procedure outlined in this thesis readily discloses the unsuitability of this value for the procedure.

Example 14.

Let the real part of an a.c. impedance be specified as a rational function of the square of the frequency by

$$R(\Omega) = \frac{(\Omega+2)(\Omega^2+2\Omega+3)}{(\Omega+1)(\Omega^2-3\Omega+4)}$$
$$= \frac{\Omega^3+4\Omega^2+7\Omega+6}{\Omega^3-2\Omega^2+\Omega+4}$$

The former of these forms makes it clear that $R(\Omega)$ is positive for positive values of Ω and consequently a generating $Z(\lambda)$ can be found which will be a "positive" function.

We have immediately by equations (56) and (56a)

$$g(\lambda) = (\lambda+1)(\lambda^2+\lambda+2)$$
$$= \lambda^3 + 2\lambda^2 + 3\lambda + 2$$

Hence equations (44b) become

$$\left.\begin{array}{l} 1 = a_3 \\ 4 = -3a_3 + 2a_2 - a_1 \\ 7 = \quad -2a_2 + 3a_1 - 2a_o \\ 6 = \quad 2a_o \end{array}\right\} \quad \begin{array}{l} 7 = 2a_2 - a_1 \\ 13 = -2a_2 + 3a_1 \end{array} \quad \left\{\begin{array}{l} a_3 = 1 \\ a_2 = \frac{17}{2} \\ a_1 = 10 \\ a_o = 3 \end{array}\right.$$

Hence a generating function is

$$Z_1(\lambda) = \frac{2\lambda^3 + 17\lambda^2 + 20\lambda + 6}{2(\lambda^3 + 2\lambda^2 + 3\lambda + 2)}$$

If in addition we specify: a pole at $\lambda = \infty$ residue 1; a pole at $\lambda = 0$, residue $\frac{1}{2}$; and a pole at $\lambda = \pm i$ residue $\frac{1}{3}$; we have

$$Z(\lambda) = \lambda + \frac{1}{2\lambda} + \frac{2\lambda}{3(\lambda^2+1)} + \frac{2\lambda^3+17\lambda^2+20\lambda+6}{2(\lambda^3+2\lambda^2+3\lambda+2)}$$

<u>uniquely</u> determined.

<u>Example 15.</u>

Let the imaginary part of an a.c. impedance be given as

$$iX = i\,\frac{5\omega + 4\omega^3 + \omega^5}{4 - 3\omega^2 + \omega^4}$$

$$\text{i.e.}\, N(\Omega) = \frac{5 + 4\Omega + \Omega^2}{4 - 3\Omega + \Omega^2}$$

$$= 1 + \frac{1 + \Omega}{4 - 3\Omega + \Omega^2}$$

Considering the second term only, we have

$$g(\lambda) = 2 + \lambda + \lambda^2$$

and equations (44c) become

$$\left.\begin{array}{l} 1 = -a_o + 2a_1 \\ 1 = \quad - a_1 + a_2 \end{array}\right\} \qquad \text{Assume } a_o = 0 \text{ then } a_1 = \tfrac{1}{2}, \quad a_2 = \tfrac{3}{2}$$

We thus have

$$Z^*(\lambda) = \frac{1}{2}\cdot\frac{\lambda + 3\lambda^2}{2 + \lambda + \lambda^2}$$

$$\mathcal{R}e\; Z^*(i\omega) = \frac{1}{2}\cdot\frac{-5\Omega + 3\Omega^2}{4 - 3\Omega + \Omega^2}$$

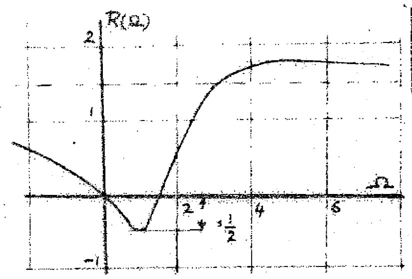


Fig. 46

The variation of this function is shown in Fig. 46. Evidently a constant of minimum value $\frac{1}{2}$ must be added to $Z^*(\lambda)$ to give a "positive" function. Hence, including the constant term in $N(\Omega)$

$$Z(\lambda) = r + \tfrac{1}{2} + \lambda + \frac{-\lambda + 3\lambda^2}{2 + \lambda + \lambda^2}$$
$$= r + \frac{1 + 3\lambda + 3\lambda^2 + \lambda^3}{2 + \lambda + \lambda^2}$$

where r is any positive constant.

Example 16.

Let the phase angle φ of an a.c. impedance be prescribed by

$$\tan\varphi = \omega\,\frac{5 + \Omega}{2 - 6\Omega + 4\Omega^2}$$

Forming the equation (63) section 94

$$H(-\lambda^2) = 2 + 6\lambda^2 + 4\lambda^4$$
$$I(-\lambda^2) = 5 - \lambda^2$$

$$H(-\lambda^2) + \lambda I(-\lambda^2) = 2 + 5\lambda + 6\lambda^2 - \lambda^3 + 4\lambda^4$$
$$= (1 + 3\lambda + 4\lambda^2)(2 - \lambda + \lambda^2) = 0$$

Distributing the roots of this equation between $f(\lambda)$ and $g(-\lambda)$ according as their real parts are positive or negative respectively we obtain

$$Z^*(\lambda) = C\,\frac{f(\lambda)}{g(\lambda)}$$
$$= C\,\frac{1 + 3\lambda + 4\lambda^2}{2 + \lambda + \lambda^2}$$

[Over

Further $\mathcal{Re}\, Z^*(i\omega) = R^*(\Omega) = C\dfrac{2 - 6\Omega + 4\Omega^2}{4 - 3\Omega + \Omega^2}$

It is readily seen that

$R^*(\Omega)$ is positive except when $1 \geq \Omega \geq \frac{1}{2}$

Let $Z(\lambda) = C\,\dfrac{1 + 3\lambda + 4\lambda^2}{2 + \lambda + \lambda^2}\,(\lambda^2 + 1)^{\delta_1}\,(2\lambda^2 + 1)^{\delta_2}$

Using equation 41 section 73 to calculate residues

if $\delta_1 = -1$, $K_1 = \left(\dfrac{1-4}{-1}\right) \times (-1) = -3$ (not permissible)

if $\delta_2 = -1$ $K_2 = \left(\dfrac{1-2}{-\frac{1}{2}}\right) \times \left(\frac{1}{2}\right) = 1$ (permissible)

Hence $Z(\lambda) = C\,\dfrac{(1 + 3\lambda + 4\lambda^2)(1 + \lambda^2)}{(2 + \lambda + \lambda^2)(1 + 2\lambda^2)}$

(cf. Example 12).

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BIOGRAPHICAL NOTE

Otto Brune

Born Jan. 10, 1901 in Bloemfontein, Orange Free State, South Africa. Received elementary and High School education in Kimberley, South Africa, matriculating from the Boys' High School there in Dec. 1917.

1918(Feb.) - 1921(Dec.) at the University of Stellenbosch, taking the degree of Bachelor of Science in 1920 and the degree of Master of Science in 1921.

1922 (Feb-Dec) master first of German, then of Mathematics and Science at the Potchefstroom Gymnasium, Transvaal.

1923-25 Lecturer in Mathematics at the Transvaal University College, Pretoria.

1926-29 at the Mass. Institute of Technology taking a co-operative course in electrical engineering (Course VI-A) in cooperation with the General Electric Co. at the Lynn and Pittsfield works; received degrees of S.B. and S.M. in Electrical Engineering June 1929.

1929 (Summer) Took part in field tests with artificial lightning on the S-19 power transmission line of the Consumers Power Co. at Croton Dam, Mich. under the direction of the Lightning Arrester Engineering Dept., General Electric Co., Pittsfield, Mass.

1929-30. Research Assistant in the Electrical Engineering Dept. Mass. Institute of Technology developing and applying the M.I.T. Network Analyser in the study of power networks.

1930 (Summer) Continued the field tests with artificial lightning at Croton Dam, Michigan.

<u>1930-31</u> Fellow in Electrical Engineering at M.I.T. holding the Austin Research Fellowship.

<u>Papers:</u>

"Reflection of Transmission Line Surges at a Terminal Impedance" G.E. Review May 1929.

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