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SYNTHESIS OF MULTI-BODY SYSTEMS FOR DESIRED EIGENFREQUENCIES

Ibrahim I Esat, Shahzad Akbar
Dynamic Systems Research Group
Mechanical Engineering Department
Brunel University, Uxbridge
Middlesex, England

ABSTRACT

The eigenvalues in the modal co-ordinate frame are varied and the corresponding changes in the stiffness matrix is investigated. To represent the modified eigenvalues as a function of the stiffness matrix is the main focus of this paper. It is shown that the change in eigenvalue is proportional to the change in the stiffness matrix. This approach may be applied to shift certain natural frequencies of a structure away from the critical operating frequency by structural modification.

NOMENCLATURE

Φ = Mass Normalised Modal Matrix
Φ_i =Mass Normalised eigenvector
θ = Modal Matrix
θ_i = ith eigenvector
Λ = Diagonalised eigenvalue matrix
λ_i = ith Eigenvalue
$\partial\lambda_i$ = Modification of ith Eigenvalue
K = Stiffness Matrix
m_i = ith generalised modal mass

LITERATURE REVIEW

Much work has been done on structural modification using mass and stiffness matrices. Normally for large systems modal analysis results do not contain description of the complete system. It is therefore difficult to find the exact modification parameters which yield the desired spectrum (Berman 1984). This has been overcome by using truncated sets of modal data. Optimisation is performed instead of solving the exact solution. Baldwin and Hutton (1985) investigated structural modification and Zhang et al (1988) investigated the use of mass matrix modifications to achieve desired natural frequencies. Ram & Braun (1990, 1991) published many papers on determining the bounds for the natural frequencies and mode shapes of modified structures based on truncated modal analysis results. Tsuei (1991) has presented a method of shifting the desired eigenvalues using the forced response of the system. The method is based on modification of either the mass or stiffness matrix by treating the modification of the system matrices as an external forced response. Sivan & Ram (1996) presented an optimisation algorithm dealing with truncated modal analysis data to shift natural frequency to desired values.

THEORY

Evaluation of the structure

The dynamic equation of motion may be solved either by direct integration or by using modal superposition. Due to the presence of off-diagonal terms in the matrices [M] and [K] the equations of motion may be coupled. Direct integration uses step by step numerical integration such as Runge-Kutta. For mode superposition the matrix equation is transformed into the modal frame by pre and post multiplying by the mode shape matrix obtained by the free vibration solution. The transformation uncouples the

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equation of motion. The resultant uncoupled equations are then solved using numerical integration procedure like Newmark -Beta.

The procedure in brief

a) Formulate equation of motion in time variable at nodal points with assumed displacement function
b) Solve for natural modes
c) Uncouple terms in the equations of motion by using natural modes
d) Solve the uncoupled sets of equation one by one for the generalised co-ordinates
e) compute the displacement response

Designing a Two degree of Freedom System For Desired Eigenfrequency

For a two degree of Freedom system with two masses, it is easy to obtain the equation for the eigenvalues in terms of the mass and stiffness.

The equation of motion is shown in matrix form by equation (1). Cartesian co-ordinates x_1 and x_2 represent the displacement function in the time domain, f_1 and f_2 are the forcing functions

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix}+\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}=\begin{Bmatrix} f_1 \\ f_2 \end{Bmatrix} \quad (1)$$

For eigenvalue analysis

$$\begin{vmatrix} \dfrac{k_1+k_2}{m_1}-\omega^2 & \dfrac{-k_2}{m_1} \\ \dfrac{-k_2}{m_2} & \dfrac{k_2}{m_2}-\omega^2 \end{vmatrix}=0 \quad (2)$$

The determinant of the frequency domain solution may be obtained and equated to zero to obtain a quadratic equation in w^2 as shown in equation (3). Using this equation the sum and the product of the square of the roots respectively may be obtained in terms of the mass and stiffness matrix as shown in equation (4) and (5). The change in stiffness due to modification of the eigenvalues may be calculated easily by solving these equations.

$$\omega^4-\left(\frac{k_2}{m_2}+\frac{k_1+k_2}{m_1}\right)\omega^2+\frac{k_2(k_1+k_2)}{m_1m_2}-\frac{k_2^2}{m_1m_2}=0 \quad (3)$$

$$\omega_1^2+\omega_2^2=-\left(\frac{k_2}{m_2}+\frac{k_1+k_2}{m_1}\right) \quad (4)$$

$$\omega_1^2\omega_2^2=\frac{k_1k_2}{m_1m_2} \quad (5)$$

For the two degree of freedom as shown above the stiffness is proportional to w^2. However for a more complex system involving more masses the above equations cannot thus be represented and would be very difficult to solve analytically therefore alternative method needs to be devised.

Eigenvalue Analysis

Orthogonal transforms are usually of two type, the iterative and non-iterative solutions. The non-iterative solutions include Givens and Householder and the iterative methods usually adopted in vibration analysis are the Jacobi method and the LR, QR and QL algorithms.

As stated previously the equations of motion are coupled in the physical cartesian axes frame and need to be uncoupled.

This is done through direct integration or Modal superposition through the use of Orthogonal transforms. Here we concentrate on the modal method. By using a periodic displacement function the equation may be converted to the frequency domain through an appropriate function representing the displacement response. The frequency analysis of the equation of motion for a free system is

$$\left(K-M\omega^2\right)\theta=0 \quad (6)$$

The equation is solved by setting the determinant of equation (6) to zero and solving for the eigenvalues w^2 and obtaining the modal matrix θ. The modal matrix showing the modes of vibration is purely arbitrary and needs to be normalised. The orthogonal matrix here is the mass normalised modal matrix obtained from the equation (7).

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$$\Phi_i = \frac{\theta_i}{\sqrt{m_i}} \qquad (7)$$

where

$$\theta_i^T [M] \theta_i = m_i \qquad (8)$$

and m_i is the ith generalised modal mass and in equation (9) below Φ is the mass normalised modal matrix.

$$\Phi = \left[(\Phi_{11}\Phi_{12}..\Phi_{1n})^T (\Phi_{11}\Phi_{12}..\Phi_{1n})^T (\Phi_{11}\Phi_{12}...\Phi_{1n})^T \right] \qquad (9)$$

The mass normalised eigenvector matrix exhibits properties of the orthogonal transformation matrix with respect to the mass and stiffness matrix as shown below. The pre and post multiplying with the mass matrix yields the identity matrix I and it yields the diagonalised eigenvalues Λ for the stiffness matrix.

$$\Phi^T M \Phi = \mathrm{I} \qquad (10)$$

$$\Phi^T K \Phi = \Lambda \qquad (11)$$

Applying this transformation, the frequency response becomes.

$$\theta = \left(\Lambda - \omega^2 \mathrm{I}\right)^{-1} f \qquad (12)$$

In the Cartesian axes frame this is shown to be

$$X = \Phi \left(\Lambda - \omega^2 \mathrm{I}\right)^{-1} \Phi^T f \qquad (13)$$

Modification of the Eigenvalues

When a transformation is applied to the structure the modal co-ordinates obtained are de-coupled however there is no simple reference to the original physical co-ordinate frame. For this reason a reverse transformation needs to be applied. The diagonal matrix of eigenvalues obtained from application of the transformation function signifies a shift from the cartesian frame to the modal frame and may be manipulated to desired values before the reverse transformation changes the axis frame to cartesian frame.

The output of the reverse transform can be assessed so as to identify the associated modifications on the mass and stiffness matrices required to enable the change in the eigenvalue to be implemented.

Implementation of the Method

The figure 1.0 shows the procedure adopted for the analysis in a graphical form around the simple equation that the change in the eigenvalues is given by subtracting the old value from the new one.

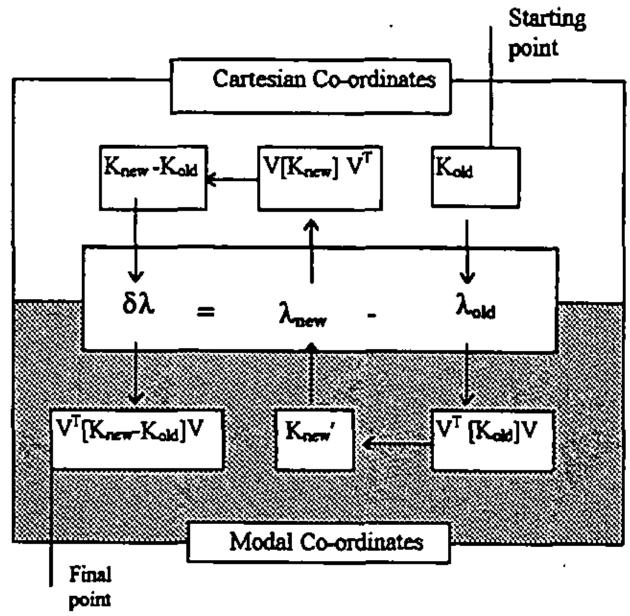

Figure 1. Diagram illustrating the procedure adopted

An initial stiffness matrix K_{old} for a given structure is pre and post multiplied (the mass matrix is unity for this case) by the normalised modal matrix to obtain the initial eigenvalues for the free system λ_{old} as shown in equation (14).

$$\Phi^T \mathbf{K}_{orig} \Phi = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \qquad (14)$$

Assuming no external force is existent in the system equation (15) shows how the eigenvalues are modified in the modal plane.

$$\delta\lambda = \begin{bmatrix} (\lambda_1 + \delta\lambda) & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \qquad (15)$$

The result may be termed K_{new}' which represents the new stiffness matrix in the modal axes. The reverse transformation procedure is used to obtain the new stiffness matrix. The new stiffness matrix K_{new} is subtracted from old stiffness matrix. The result is put

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through the system again to verify its compliance with the equation given in figure 1.

The iterative equation for further small modifications and subsequent change in stiffness matrix is shown in equation (16). It will be noted that the reverse transformation gives the value $M^{-1}K$. This is due to the fact that the modal matrix was divided by the modal mass to obtain the mass normalised modal matrix. The change for each iteration is given in equation (17).

$$M^{-1} K_{new} = \Phi(\Lambda + \delta\Lambda)\Phi^{-1} \quad (16)$$

$$K_{diff} = K_{n+1} - K_n \quad (17)$$

The associated stiffness with new eigenvalues

In order to obtain the new stiffness matrix, the output of the reverse transformation is multiplied by the mass matrix. The stiffness matrix values will have physical constraints and may not be able to be implemented in a real system.

The requirement for modification is that the stiffness values obtained on the non-diagonal elements be less than zero as shown in equation (18). The additional constraint imposed on spring mass system of the type examined in the paper is that for a n x n system, equation (19) applies except where there is a ground connection.

$$K_{ij} \leq 0 \qquad \forall\, i,j \quad \text{where } i \neq j \quad (18)$$

$$\sum_i \sum_j K_{ij} = 0 \quad (19)$$

Case Example

The following numerical example is used to show how a real physical system may be modified by varying one of the natural frequencies. The Mass, Stiffness and Diagonalised Eigenvalue matrix for this system are as given respectively.

$$K = \begin{bmatrix} 30000 & -20000 & 0 \\ -20000 & 70000 & -50000 \\ 0 & -50000 & 60000 \end{bmatrix}$$

$$M = \begin{bmatrix} 100 & 0 & 0 \\ 0 & 200 & 0 \\ 0 & 0 & 150 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 40.8 & 0 & 0 \\ 0 & 319.8 & 0 \\ 0 & 0 & 689.3 \end{bmatrix}$$

The 2nd eigenvalue was altered and the changes in the stiffness matrix plotted in Figure 2. Table 1.0 shows how the stiffness elements would change for a proportional increase in the eigenvalue. In the modifications carried out to the eigenvalue matrix it was found that the increase in the stiffness matrix was proportional. This was true for small as well as large increases in the eigenvalue as seen in figure 2.0.

K(1,1) orig	K(2,2) orig	k(3,3) orig	λ 2 orig
30000	70000	60000	319.8
K(1,1) last	**k(2,2) last**	**k(3,3) last**	**λ 2 last**
65316	71387	73486	769.8

Table 1.0. The initial and final values of the stiffness elements and the second eigenvalue.

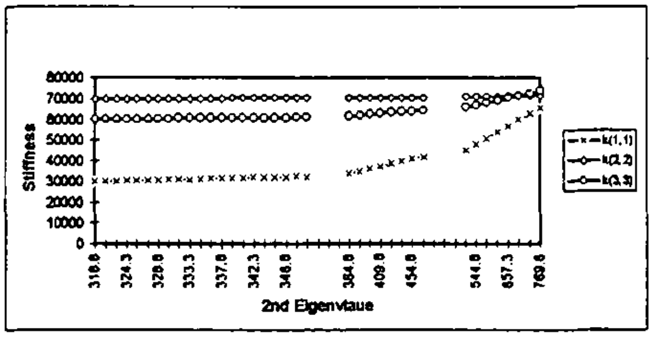

Figure 2.0 Graph showing the change in various stiffness elements for proportional change in eigenvalue

DISCUSSION

The example of a two degree of freedom system highlighted how the change of the stiffness varied linearly with the eigenvalues. Such an example however was not suitable for a more complex problem. In the case study example a real physical system was transformed to the modal co-ordinate frame and the value of the eigenvalue modified. The reverse transformation yielded the associated stiffness matrix for the modified system. In the

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numerical example, small modifications to the second eigenvalue were plotted against the changes in the stiffness matrix. It was shown that the stiffness elements increased linearly with the change in eigenvalue. Higher frequencies also show the same linearity.

There may not be a practical solution to the output presented by the technique to physically implement the necessary changes. A more complete solution to the problem will involve an algorithm to make modifications which take these factors into account. Many authors have written on the ways of shifting the natural frequencies of a structure by way of an optimisation procedure to modify the stiffness matrix. Authors such as Tsuei use a method whereby the solution is in a complex plane. The required natural frequency is shifted and the structural modification obtained. The optimisation algorithm arrives at the real solution after a few iterations.

Finally it is worth stating that showing the existence of a linear relationship between the change of eigenvalue and the resulting change in the stiffness matrix indicates that to shift eigenfrequencies one may not need complicated optimisation algorithms as developed by many researchers working on this problem.

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