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Synthesis of multi-degree of freedom, parallel flexure system concepts via Freedom and Constraint Topology (FACT) – Part I: Principles

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ABSTRACT

In this paper we introduce a new design principle, and complementary geometric entities, that form the basis for a new approach to the synthesis of multi-degree of freedom, purely parallel precision flexure systems. This approach – Freedom and Constraint Topology (FACT) – is unique in that it is based upon sets of geometric entities that contain quantitative information about a flexure system's characteristics. A first set contains information about a flexure system's degrees of freedom (its freedom topology) and a second set contains information about the flexure system's topology (its constraint topology). These sets may be used to visualize the quantitative relationships between all possible flexure designs and all possible motions for a given design problem. We introduce a new principle – complementary topologies – that enables the unique mapping of freedom and constraint spaces. This mapping makes it possible to visualize and determine the general shape(s) that a viable parallel flexure system concept must have in order to permit specified motions. The shapes contain all of the relevant quantitative information that is needed to rapidly sketch early embodiments of complex parallel flexure system concepts. These shapes may then be used to rapidly synthesize a multiplicity of flexure system concepts that have (a) independent rotational and/or linear motions, (b) coupled linear and rotational motions, and (c) redundant constraints that permit the desired motions while improving stiffness, load capacity and thermal stability. This enables early-stage flexure system design via "paper and pencil sketches" without undue complications that arise when one focuses upon detailed mathematical treatments that are better-suited for optimization rather than visualization and synthesis.

1. Introduction

The intent of this paper is to introduce the principles behind a new method – Freedom and Constraint Topology (FACT) – that may be used to diagnose and synthesize multi-degree of freedom (multi-DOF) flexure system concepts. Flexure systems consist of a combination of rigid and flexural elements. These elements are arranged and interconnected in a way that their compliant directions permit specified motions and their stiff directions prevent motions in all other directions. Flexure systems have been used as precision machine elements for over a century [1] due to their excellent resolution characteristics, their low-cost characteristics, and the ease with which they may be fabricated.

Flexure systems continue to be important to conventional precision applications, for instance they are commonly used within optical manipulation stages, precision motion stages and as general purpose flexure bearings. More recently, flexure systems have become attractive for use in motion stages for nanomanufacturing equipment, instruments that are used in nano-scale research/manufacturing, and micro- and nano-manipulators. These instruments and devices typically require the capability to move in four to six axes with nanometer-level resolution [2–11]. This paper provides a means to synthesize flexure systems for these types of applications.

There are many ubiquitous one-, two- or three-axis flexure systems [12] that may be combined in series to solve some types of multi-DOF motion problems. Parallel flexure systems are usually preferred to serial systems because they have better dynamic characteristics and they do not suffer from stacked axis errors that are inherent in serial flexure systems.

The generation of multi-DOF parallel flexure system concepts is difficult because there are typically several flexural components that provide constraints in several directions while allowing motions in many other directions. It is necessary, and difficult, for designers to keep track of (a) the relative, three-dimensional orientations of the flexural constraints, (b) the orientation of the permitted motions, and (c) the three-dimensional relationships between each constraint and the permissible motions. Even if an expert designer is capable of the preceding, the existing body of published precision flexure system knowledge provides little

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information or guidance as to how they should deal with two important problems that are inherent to parallel flexure systems:

(a) The problem of coupled rotations–translations and,
(b) The problem of identifying when and where redundant constraints may be added to a flexure system. Redundant constraints are often needed to endow a flexure system with suitable stiffness, load capacity and thermal stability characteristics.

In general, parallel flexure systems, regardless of the number of degrees of freedom (DOFs), are often designed via iteration and many designers are fortunate if they are able to synthesize one or two new concepts that possess only the specified motions.

Fig. 1A provides a contrast of the conventional flexure design method, constraint-based design (CBD), and FACT. In CBD, a designer must use his visualization skills, pattern recognition and CBD's Rule of Complementary Patterns to guide a visual iteration process until a viable flexure system concept is identified. Additional concepts are generated via more iteration. In contrast, FACT uses sets of three-dimensional geometric entities, for example planes and spheres, to embody quantitative information about a flexure system's shape and its DOFs [13]. These types of geometric entities, for example those shown in Fig. 1Bii, are capable of displaying the general form of a flexure system design. All possible concepts are represented within the entities therefore a designer will know the general form of a flexure design for a desired set of DOFs.

In the FACT method, a first set of entities contains information about the flexure system's DOFs (its freedom topology) and a second set contains information about the flexure system's geometry (its constraint topology). The sets of shapes are made useful via a principle – the principle of complementary topologies – that provides a unique mapping of the first set (DOFs) to the second set (geometry). This is illustrated via the example that is shown in Fig. 1B. If a designer wanted to generate concepts for a flexure system with one rotation DOF, they would first specify the geometric entity (freedom topology) that represents this rotation. The designer then identifies which freedom space this topology belongs – the rotation line in Fig. 1Bi. The principle of complementary topologies would then be used to find the geometric entities (constraint space) – the intersecting planes in Fig. 1B ii – that represents concepts for flexure systems that permit only the specified DOF. The designer then selects constraints that lie within the constraint space to form various different concepts, for example the three concepts shown in Fig. 1Biii contain flexural constraints that lie within the planes of the constraint space. This is a simplified example that illustrates the basic approach. Sections of Part I of this paper, and Part II of this paper, will provide examples of increasing complexity.

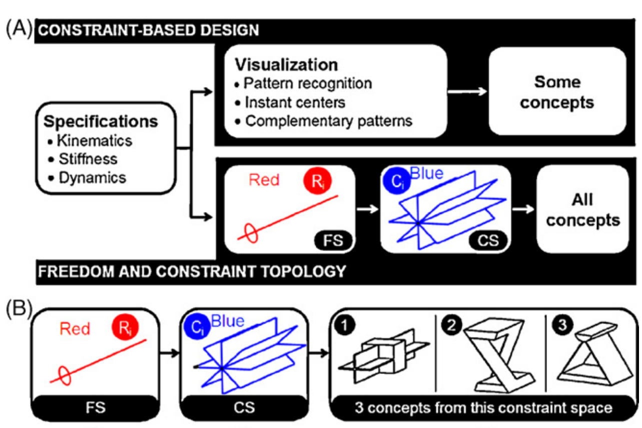

Fig. 1. Illustration of (A) the contrast between constraint-based design and FACT and (B) example geometric entities that represent the permissible motion – *Ri* – and the appropriate geometric entities that contain the constraints – *Ci* – that are used to generate several concepts for a flexure system that permits the desired motion. Note, color coding will be used in this paper to distinguish motions and flexure geometry. For example, the red and blue in this figure indicate a rotation and flexure constraints respectively. Further details regarding color coding will be in a later section. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

1.1. Overview of flexure system design history and sources for principles and best practices

It is a common misunderstanding that flexure systems could not be invented until the later 20th century. The engineering of flexure systems requires at a minimum, an understanding of principles that have existed for some time:

i. Hooke's linear stress–strain relationship from 1678 [14],
ii. Bernoulli and Euler's kinematic and elastomechanic beam behavior from 1744 [15], and
iii. Maxwell's rules that govern the relationship between constraints and DOFs c.a. 1890 [16].

This knowledge enables the engineering of precision flexure systems and compliant mechanisms. Given this knowledge, Clay and Roy [17], Jones [18] and others were able to generate, model and implement new flexure systems and early compliant mechanisms throughout the early 20th century. These flexure systems became ubiquitous precision machine elements and so there was a need to catalogue the concepts, principles and best practices that were used to engineer them. This was accomplished by Smith [12] c.a. 2000.

Blanding created a formal base of exact constraint principles for use in the design of flexures [19], c.a. 1999 and Hale augmented these principles [20] for precision flexure systems. The contributions of Maxwell, Blanding and Hale constitute the core of what is called constraint-based design. The fundamental premise of CBD is that all motions of a rigid body are determined by the position and orientation of the constraints, i.e. the topology of constraints, which act upon the body. In CBD, a designer arranges flexural and rigid elements into a geometric layout that endows a device with the ability to permit and forbid motions in specified directions. Constraint-based design principles are central to precision engineering as the layout of a device's constraints governs the device's DOFs, stiffness, repeatability, mode shapes, etc. Constraint-based design has been practiced by using a combination of visualization techniques, experience and rules of thumb. It is currently the primary synthesis method used to engineer precision flexure systems.

1.2. Scope

This paper focuses on improving the synthesis of precision flexure systems with a specific emphasis on the creation of parallel flexure system concepts to meet kinematic requirements. We limit the scope of this paper to include small-motion kinematics and linear elastic material properties. Parasitic errors that are associated with large motions are not addressed. These assumptions are appropriate for the early-stage synthesis of precision flexure systems. The content of this paper applies to systems wherein the guided component may be considered as a rigid body; therefore the number of DOFs is limited to six DOFs.

There are two types of flexure systems, they are systems wherein (a) chains or serial conjugated flexures link the mobile element to ground, i.e. serial flexure systems, and (b) all ground-to-stage links consist of a single flexural element, i.e. parallel flexure systems. The former requires the later to address the parallel combinations of the chained flexures and it also requires a means to deal with the serial nature within the chains. The solution of the later is a necessary first step; therefore this paper is focused upon parallel solutions wherein serial chains do not exist. The methods of this paper may be extended to address some practical serial flexure systems [21], but this requires the introduction of considerable material and so this topic will be the subject of a future paper.

It should also be noted that this approach may be used for rigid mechanisms that rely on sliding/rolling joints. The utility of these devices within precision applications is limited given the inherent problems with the accuracy and repeatability of the non-flexural joints. The utility to other fields, e.g. robotics, is presently limited to small-motion kinematics, although it should be possible to extend the work to large-motion kinematics.

2. Background knowledge that is used in FACT

2.1. Maxwell's principles of constraint

A rigid body has six DOFs and any non-redundant constraint upon the body removes a DOF. A constraint is idealized as providing resistance to motion along its line of action only. This may be expressed as:

$$R = 6 - C \quad (1)$$

where C is the number of non-redundant constraints and R is the number of DOFs. Non-redundant constraints are mathematically equivalent to constraints that possess lines of action, i.e. vectors, which are mathematically independent. Maxwell augmented this equation with observations that enable one to understand some of the DOFs that are permitted given the lines of action of a system of constraints [16].

2.2. Projective geometry

It will be useful for us to visualize geometric entities that possess a mix of finite dimensions and dimensions that approach infinity. The field of projective geometry [22] addresses these types of geometric entities. The first principle of import is that a line may be perceived as a circle with a radius of curvature that approaches infinity. The relevance of this principle is demonstrated in Fig. 2. For small motions, translations may be emulated by rotations about a circle whose radius approaches infinity. The circle is shown as a "hoop" in Fig. 2. The rotation of the stage could occur about points on this hoop such that the rotation yields a motion that emulates a translation in a direction that is perpendicular to the plane of the hoop. This principle is important because CBD and FACT treat all translations as though they are the result of a rotation about a hoop.

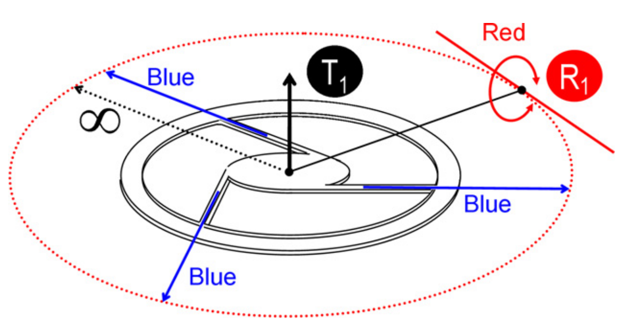

Fig. 2. Illustration showing how a hoop (red) represents a translation – *Ti* – shown as a black arrow. The lines of action of the flexure system's constraints (blue) intersect the hoop as the lines extend toward infinity. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

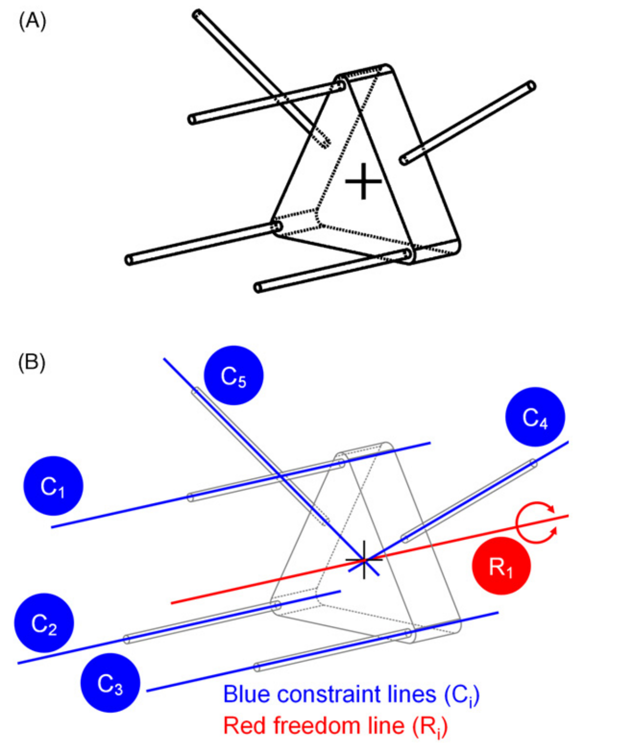

Fig. 3. Example that illustrates the Rule of Complementary Patterns between *Rs* and *Cs* for a rigid stage that is constrained with five non-redundant constraints. The ends of the constraints that are not attached to the rigid stage are considered to be grounded. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

The second principle from this field is that parallel lines may be considered to share an intersection as the lines approach infinity. This principle is used in CBD and FACT to identify the intersections between geometric entities that represent (i) constraints and (ii) DOFs.

2.3. The Rule of Complementary Patterns

Blanding [19] viewed constraints and DOFs via constraint lines and freedom lines, respectively. A constraint line is the line of action of an idealized constraint. DOFs are viewed as rotations about a freedom line. The hoop principle from the previous section is used to describe translations in terms of rotations. Blanding's Rule of Complementary Patterns [19] states that every freedom line intersects every constraint line. This is powerful because it enables a designer to visualize the relative relationships between a flexure system's constraints and the DOFs that these constraints permit. The Rule of Complementary Patterns has been used to design many mechanical devices, precision flexure systems and precision fixtures [19,20].

The principle is demonstrated via the flexure system in Fig. 3. The flexure system consists of a rigid stage and five independent constraints, C_1–C_5. The constraint lines in Fig. 3, and throughout the rest of the paper, are shown in blue. Eq. (1) predicts that the stage should move with one DOF and the Rule of Complementary Patterns may be used to find this DOF. The only line that intersects all of the constraint lines is the red freedom line, R_1, in Fig. 3. This freedom line, and every other freedom line throughout this paper, is shown in red. From projective geometry, we know that parallel lines intersect at infinity and so C_1–C_3 intersect the freedom line as the line approaches infinity. Constraints C_4 and C_5 intersect the freedom line at the centroid of the triangular stage. This is the only

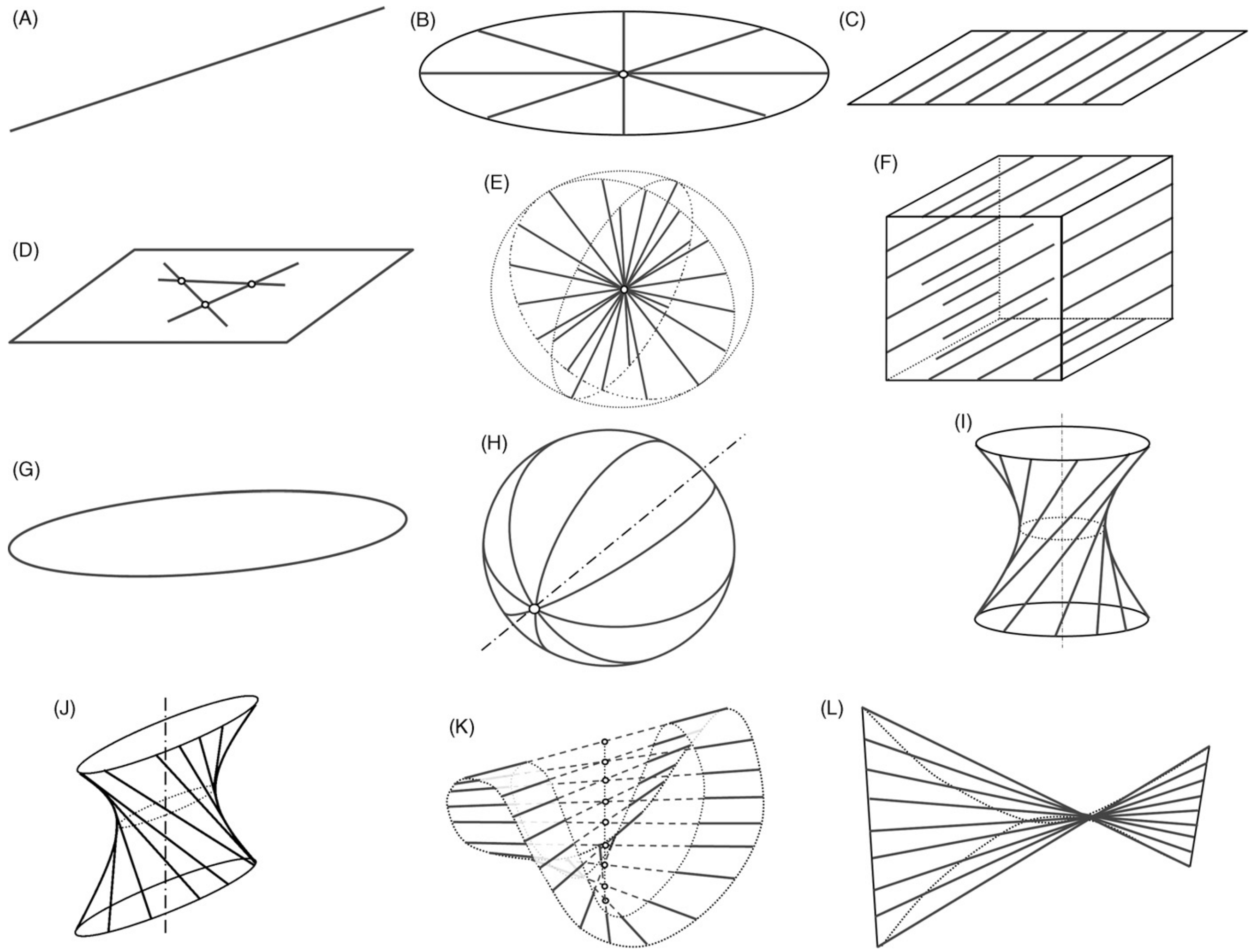

Fig. 4. Projective geometric entities that are used as freedom and constraint sets.

line that intersects all constraint lines and so it represents the only permissible DOF—a rotation about the freedom line.

The example in Fig. 2 illustrates how the Rule of Complementary Patterns applies to hoops and constraints. The lines of action of the three constraints intersect the hoop. As such, the hoop represents a permissible DOF and the associated translation, *T*, is one of the permitted DOFs.

3. Fundamentals of FACT

3.1. Freedom and constraint sets

It is difficult, even for the most experienced designer, to visualize how the constraint and freedom lines within a complex flexure system relate to each other. Visualization is made easier through the use of geometric entities – freedom sets and constraint sets – that represent a collection of freedom and constraint lines, respectively. Fig. 4 shows 12 shapes that are commonly used as freedom and constraint sets. The entities in Fig. 5A–H are of the most import to precision flexure system design. The entities in Fig. 5I–L correlate to flexure systems whose import to practical precision flexure systems have yet to be identified. They are provided in the interest of completeness.

Although the entities in Fig. 4 are readily described via equations from Euclidean geometry, the form they take is best described for introductory purposes via logical expressions:

- A: Line – A line of a given orientation
- B: Pencil – All co-planar lines that intersect at a common point
- C: P-plane – All co-planar, parallel lines of a given orientation
- D: A-plane – All lines on a given plane
- E: Sphere – All lines that intersect at a common point

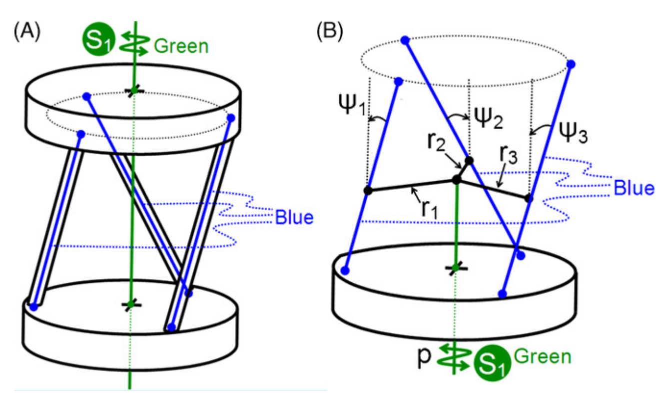

Fig. 5. (A) Example of a flexure system wherein the constraints are skew with respect to the permissible screw motion, S1. The result is a coupled, i.e. screw, DOF (green) and (B) the geometric parameters that govern the degree of coupling between translation and rotation. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

- F: Box – A box of infinite extent that contains all parallel lines of a given orientation
- G: Hoop – A circle that has a radius that approaches infinity
- H: Hoop surface – All hoops with normal vectors that are orthogonal to a given axis
- I: Circular hyperboloid – A ruling of lines that exist within the surface of a circular hyperboloid
- J: Elliptical hyperboloid – A ruling of lines that exist within the surface of an elliptical hyperboloid
- K: Cylindroid – All lines that exist within the surface of a cylindroid
- L: Hyperbolic paraboloid – All lines that exist within the surface of a hyperbolic paraboloid

These entities may be used by designers to visualize and understand the characteristics of a flexure system. More detail on the shapes of these geometric entities, the equations used to describe them, and their evolution in the context of FACT are provided by Hopkins [13]. Subsequent sections will provide an overview of how these sets relate to flexure systems and how they are used.

3.2. Treatment of coupled DOFs and constraints via screw theory

To this point we have only covered freedom sets that contain rotational freedom lines. It is possible for a freedom set to contain screw lines that represent coupled rotations and translations. Fig. 5A shows a flexure system that possesses this type of coupled DOF – a coupled rotation and translation – along a vertical axis shown as a green line. A downward displacement of the stage is accompanied by a proportional rotation of the stage. In cases such as this, the constraint lines (blue) do not intersect the line associated with the rotation or the translation. CBD principles are unable to diagnose or synthesize flexure systems with this type of coupling.

Screw theory may be used to create geometric entities that contain information about coupled DOFs, i.e. screws [23–25]. Each vector is in essence a line that corresponds to a motion or load. Others, including Bottema and Roth [26], Hunt [27], Merlet [28] and McCarthy [29,30] have extended this work to include the use of "Grassman Geometries," which are geometric shapes that represent independent lines. When used in this context, the shapes in Fig. 4 are Grassman Geometries. These shapes may be found within a number of kinematics texts as descriptors of either an existing mechanism's kinematic or a mechanism's constraint characteristics.

In FACT, the shapes are used for a different purpose and in a different way. Their purpose is to enable easy visualization of the possible flexure constraints that permit a given set of DOFs without loss of the quantitative link between motion and constraints. The way the shapes are used differs from screw theory and equation-based approaches. Screw theory and equation-based approaches are best-suited to analysis of existing concepts and optimization of existing designs. The shapes within FACT are primarily used for rapid early-stage concept synthesis via visualization/sketches, thereby preventing undue complications from screw theory's mathematic complexity that tends to camouflage practical design issues/characteristics and thereby form a barrier to a designer's ability to understand the essence of how/why the concept works. It is critical to understanding the practical issues and how/why a precision flexure system works prior to investing the time/resources to use screw theory, or an equation, or other simulations for detailed analysis/optimization.

The following equation provides the relationship between constraints and screws (i.e. coupled rotations–translations) [13,30]:

$$p \cos(\psi_i) = r_i \sin(\psi_i) \tag{2}$$

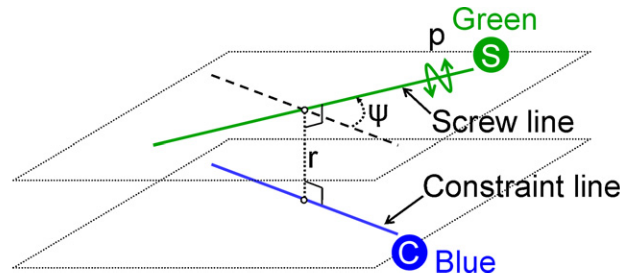

Fig. 6. Geometric parameters used to relate a screw and constraint line. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

In Eq. (2), r is the shortest distance between the constraint line and the screw, ψ is the skew angle between the screw and the constraint line, and p is the pitch, that is the translation per unit rotation. These parameters are defined in Fig. 6. Eq. (2) may be used to generate appropriate freedom sets that have the form of the geometric entities in Section 3.1. Three-dimensional geometric entities have been used with screw theory in the past to (a) illustrate the linear combinations of Plueker vectors [27] and (b) identify singularities in rigid, parallel mechanisms [29,30]. In FACT, the entities are used to visualize all types of permissible motions (rotations, translations and screws), non-redundant constraints and redundant constraints.

Some freedom sets may only contain screws while others may contain a combination of screws and rotational freedom lines. In this paper, the color green and/or an inscribed "S" are used to represent the line of action of a screw line along which the screw's translation occurs, and about which the screw's rotation occurs.

3.3. Freedom and constraint spaces

The superposition of several freedom or constraint sets – called a freedom or constraint space – is usually needed to capture the entirety of a flexure system's freedom or constraint characteristics. For instance, the constraint space (blue) in Fig. 7A is a combination of a sphere with an A-plane. The freedom space (red) in Fig. 7B is the combination of two pencils. The next section describes how spaces may be mapped to each other, thereby providing a link between a flexure system's DOFs and its constraints.

3.4. The principle of complementary topologies

The Rule of Complementary Patterns only covers lines, it does not explicitly cover all of the shapes that represent freedom sets and constraint sets in FACT. The Principle of Complementary Topologies was created to provide a mapping between freedom and constraint topologies that exist within freedom and constraint spaces. The principle states 'a freedom space and a constraint space contain complementary freedom and constraint topologies when all lines

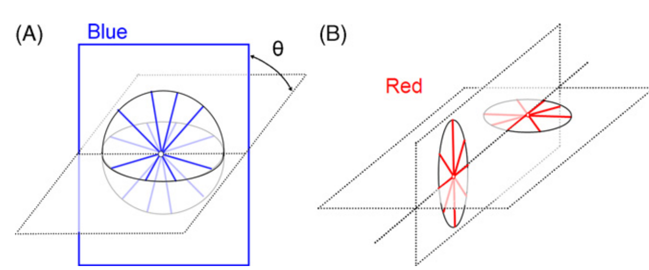

Fig. 7. Examples of freedom and constraint spaces. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

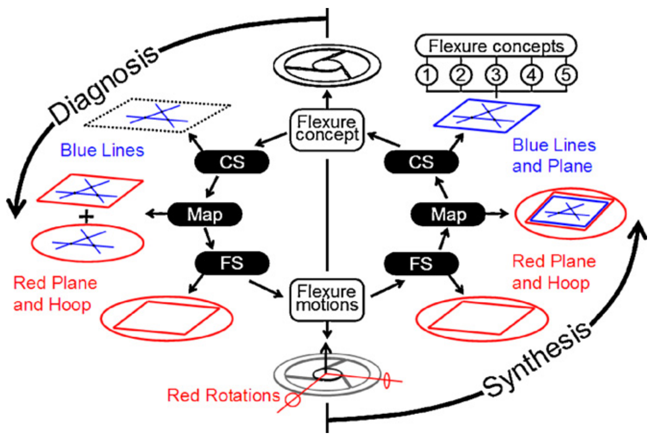

Fig. 8. The unique relationship between freedom and constraint spaces as shown during the diagnosis (constraints to freedoms) and synthesis (freedoms to constraints) cycles. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

in the constraint space are complementary to all lines in the freedom space.' Here, complementary adheres to the classical definition from mechanism kinematics. Although the definition is typically defined via mathematics, it is more appropriate for our purposes to phrase this in logical terms as a given layout of constraints and a given motion is complementary if the constraints permit the motions.

Appendix A contains several sets of complementary freedom and constraint spaces that were mapped to each other via the Principle of Complementary Topologies. There are two important corollaries that may be deduced from this principle:

(i) The freedom and constraint spaces are uniquely mapped to each other,
(ii) Any constraint line that is selected from the constraint space will be complementary to the rotational freedom lines and screw lines according to Eq. (2).

Corollary 1. *The first corollary means that a designer may use the catalogue of matching freedom-constraint spaces in Appendix A to immediately select the appropriate constraint spaces (design type) that may be used to synthesize concepts which possess a desired freedom space (DOF). The reverse may also be done. That is, a designer may diagnose the DOFs that a flexure permits by finding the constraint space that the flexure fits within, and then looking up the matching freedom space.*

The synthesis and diagnosis cycles for a simple flexure system are illustrated in Fig. 8. The sequence of steps within the figure should be read in a counter clockwise direction. We first describe the diagnosis of the permitted DOFs for the flexure that is shown at the 12 o'clock position. The flexure is broken down into its constraint lines and then all possible freedom sets that contain lines which intersect these constraint lines are added to form the flexure system's freedom space. In this case, only a hoop or the lines contained within an A-plane will obey Eq. (2) in that they intersect each of the constraint lines, so they are combined to form the freedom space of the flexure. The motions that correspond to the hoop and the A-plane, a translation and two independent rotations, are shown at the 6 o'clock position.

The synthesis of this flexure system concept is achieved using the steps shown on the right of Fig. 8. If the desired DOFs are two rotations and a translation that is orthogonal to the plane that contains the lines, the appropriate freedom space is selected from the catalogue in Appendix A. Eq. (1) is used to determine the minimum and necessary number of constraints that must be selected from the appropriate constraint space. There are many combinations of three co-planar lines that may be selected from the constraint space plane. We are interested in combinations that do not intersect at the same point.

After the minimum number of constraints has been selected via Eq. (1), additional constraints may be selected from within the constraint space because each additional constraint is guaranteed to permit the desired DOF. This would be done for instance if it was necessary to improve the load capacity, symmetry, thermal stability and/or the stiffness of the flexure system in some, or all, of the constrained directions. Another consequence of adding additional constraints is that the stiffness in the free directions will also increase. More information on these constraints, referred to as redundant constraints, is provided in Part II [31] of this paper.

There are many ways to select combinations of non-redundant and redundant constraints, so many viable topologies of constraint lines may be generated. Each topology is a different concept and so the constraint space may be used to synthesize several concepts that permit the desired DOFs. One of them could be the flexure that is shown at the 12 o'clock position.

Corollary 2. *The second corollary provides a necessary relationship between the constraint lines and the lines within the freedom space – rotational freedom lines and screw lines. The essence of this relationship, as embodied in Eq. (2), is that:*

(1) Rotational freedom lines and constraint lines must intersect in order to be complementary.
(2) Screw lines and constraint lines do not need to intersect in order to be complementary; they only need to satisfy Eq. (2).

It is important to note that the freedom space of a parallel flexure system is the intersection, not the union, of the freedom spaces of each individual flexible constraint within the system.

4. Examples that show how spaces correlate to flexure system concepts

It is important to realize that FACT is not a means to enable designers with little experience to become expert flexure designers. FACT enables designers that understand principles and best practices (content reviewed in Section 1.1) to more easily generate new flexure system concepts. For example, one must know the different types of flexure elements (e.g. blades, wires, hinges, etc.), understand their constraint characteristics (stiff directions, compliant directions), how they are emulated by constraint lines (e.g. three constraint lines emulate a blade flexure) and how to ascertain redundancy when many flexure elements are combined to create a system.

4.1. A flexure system with three independent permissible rotations (Case 3, Type 4)

Here we examine a situation wherein a flexure system is required to permit three independent rotations of an optic about its focal point. These permissible freedoms correlate to Case 3, Type 4 in Appendix A. This freedom space contains all freedom lines that intersect at a common point as shown in Fig. 9A. The complementary constraint space, shown in Fig. 9B, contains all lines that intersect every rotation line. There are no permissible screws. Eq. (1) tells us to select three non-redundant lines from the constraint space in order to form a concept constraint topology that permits three DOFs. The combination of the three constraint lines shown in Fig. 9C permits three independent rotations. The three independent rotation lines in Fig. 9C are the device's freedom topology. Fig. 9D

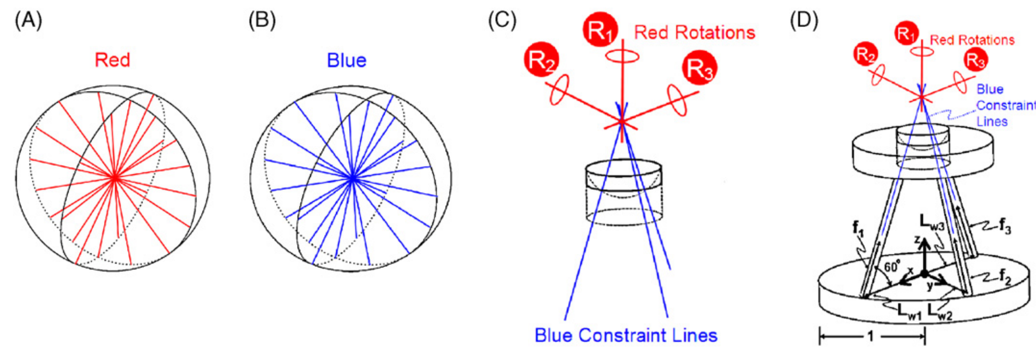

Fig. 9. (A) The freedom space for three independent rotations and (B) the complementary constraint space that yields (C) a viable concept topology of constraints. A possible embodiment of the concept is shown in (D). (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

shows a possible constraint topology embodied in a flexure system design.

For this flexure system, any three rotations about the focal point may occur simultaneously and independently. The rotation about specific axes may be obtained by using three actuators that are placed at prescribed locations and orientations. The constraints for this system shown in Fig. 9D may be modeled as wrenches [23,30] and described per Eq. (3) where L_{wi} is a vector that points from the origin to a location along its corresponding constraint line and f_i is a vector that points along its corresponding constraint line's axis.

$$[W]=\begin{bmatrix} f_1 & L_{w1}\times f_1 \\ f_2 & L_{w2}\times f_2 \\ \vdots & \vdots \\ f_n & L_{wn}\times f_n \end{bmatrix}=\begin{bmatrix} -1 & 0 & \sqrt{3} & 0 & -\sqrt{3} & 0 \\ 0 & -1 & \sqrt{3} & \sqrt{3} & 0 & 0 \\ 1 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 \end{bmatrix} \quad (3)$$

Standard mathematical techniques may be used to show that the wrench vectors that correspond to the constraints are mathematically independent. The expression of constraint lines as wrenches enables designers to convert graphical flexure concepts and formulate engineering models for detailed design and optimization. This process of optimization may also be performed in conjunction with FEA techniques.

4.2. *A flexure system with one permissible screw (Case 5, Type 2)*

Here we examine a situation wherein a stage must permit only a coupled rotation and translation. This permissible freedom correlates to Case 5, Type 2 (1 coupled DOF) in Appendix A. This freedom topology is the same as the freedom space in this case. It contains a single screw as shown in Fig. 10A. The complementary constraint space is shown in Fig. 10B. This space is comprised of five constraint lines that satisfy Eq. (2) with respect to the desired screw axis. Constraint 5 shows that an intersecting constraint would need to intersect at 90° in order to satisfy Eq. (2).

Fig. 10C shows a flexure system that was created from the freedom topology in Fig. 10A. Constraints C_1–C_4 intersect and are perpendicular to the line of the freedom topology and thus these constraints permit every collinear screw with every pitch value along that line. Constraint C_5 supports a screw possessing only one finite pitch value according to Eq. (2). The only screw that is permissible is one that is complementary to all constraints, and therefore the screw permitted by C_5 is the only DOF. The coupled motions, $\Delta\theta z$ and Δz, of this flexure system design are shown in Fig. 10 D and E.

5. Moving beyond early-stage synthesis

This paper helps designers get over the first hurdle, that is to generate a conceptual representation of the design. This generation

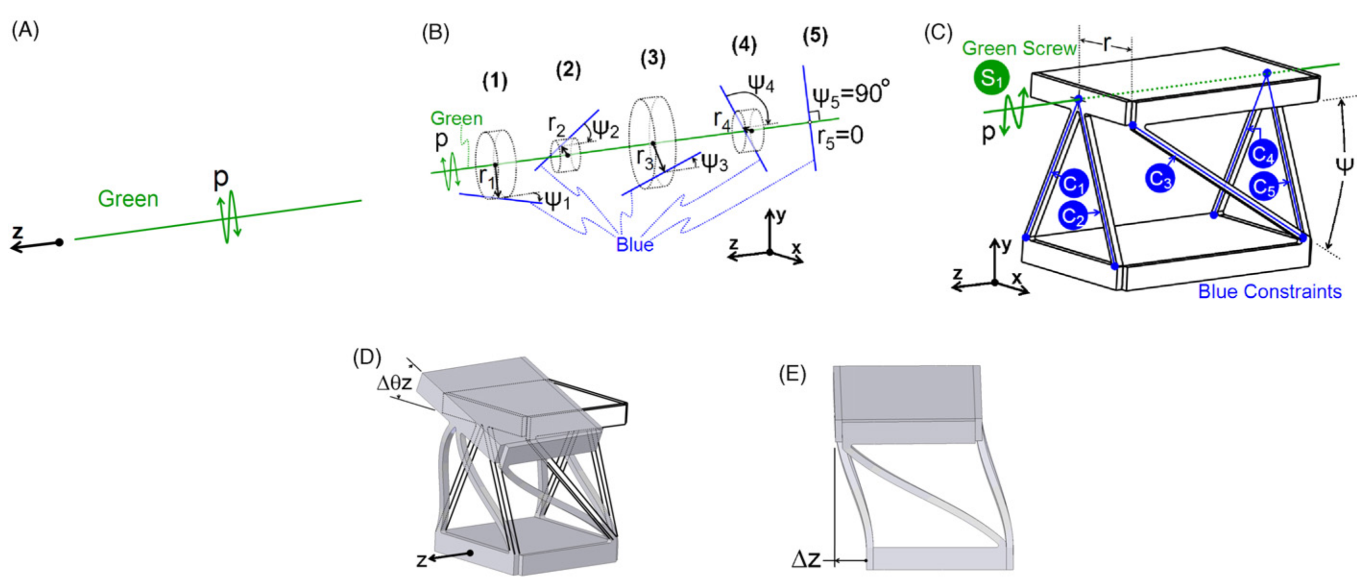

Fig. 10. (A) The freedom space for a coupled DOF and (B) the constraint space that yields (C) a viable topology of constraints. Images from FEA post-processing show the simultaneous rotation and translation via (D) isometric and (E) side views. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

phase is the front-end of the engineering process and its outcome (concepts) are readily ported into conventional refinement stage—modeling/simulation/optimization methods for subsequent stages. At this point, it is best if mathematical models take over from visualization because they more easily identify the specific/detailed outputs (dimensions, angles, etc.). The refinement stage involves the assignment of relevant dimensions (beam lengths, widths, orientation angles, etc.) and material properties. This representation may then be ported to a simulation or calculation. For example, given the concept, dimensions and material properties, one could generate several ways to simulate behavior: (1) beam equations and stiffness matrices, (2) screw theory representation of the constraints/stage, (3) pseudo-rigid body models, (4) FEA simulations, etc. The engineering process then moves onto the next stage—fabrication.

6. Summary

In this paper we have introduced a method, supporting principles and geometric entities that may be used to visualize the kinematics of parallel flexure systems. A catalogue of matching entities was made available for use in representing the freedom characteristics (freedom spaces) and the constraint characteristics (constraint space) of parallel flexure systems. The Principle of Complementary Topologies was introduced and then used to provide a unique mapping between the freedom and constraint spaces. The means to treat coupled DOFs and redundant constraints were provided. At present, we are working on extending the capabilities so that other types of geometric shapes may be used to represent the elastomechanics, dynamics (mode shapes and normalized natural frequencies), parasitic errors, and best actuator layout/connection points for parallel flexure systems. At present, we are working on modifying the approach so that it captures large motion kinematics and serial systems.

Acknowledgements

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Appendix A. Sets of matching freedom and constraint spaces

This appendix provides graphical and textual descriptions of 26 types that represent matched sets of freedom and constraint spaces for parallel flexure systems. These types are divided among six cases, where the case number represents the number of non-redundant constraints within the system and the type number represents a particular arrangement of those constraints. The freedom and constraint spaces are denoted as FS_{ij} and CS_{ij}. Where "*i*" represents the case and "*j*" represents the type.

Any flexure from Case "*C*" that consists of "*C*" non-redundant constraints, maps to a freedom space that contains 6-*C* DOFs, i.e. independent motions.

Types that are used in conventional flexure systems are marked with "●" and types with promise to provide new motions are marked with "○". Types that are marked with a "⊗" have yet to be linked to any practical application; however they are provided here in the interest of completeness. In some types, the screw sets are too complex to display in a useful graphical form and therefore they are denoted by an "S" that is inscribed within a green circle.

In Figs. 11–16, the constraint spaces are shown to the left of the thick arrows and consist of blue constraint lines. The freedom spaces are shown to the right of the thick arrows. The spaces to the immediate right of the thick arrows consist of red pure rotational freedom lines and the spaces to the right of the addition signs consist of green screw lines unless otherwise indicated in the figure.

Fig. 11. The only type for Case 1. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

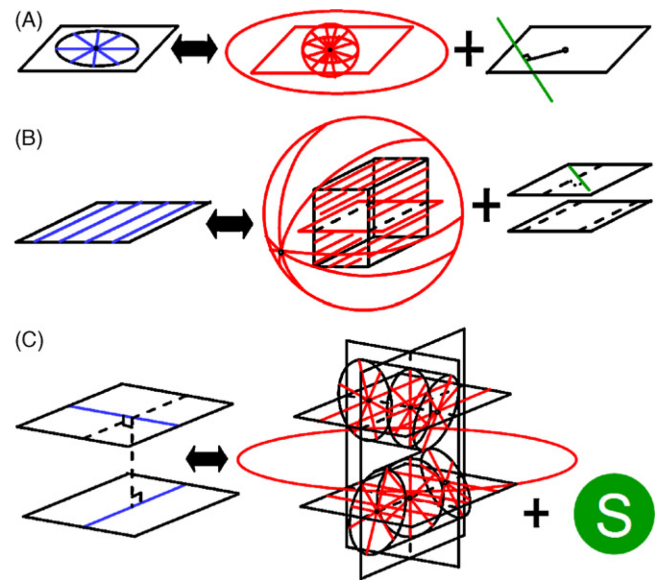

Fig. 12. Types of flexure system arrangements for Case 2. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

A.1. Case 1: Flexure systems with one constraint

In this case, every flexure system contains one non-redundant constraint and five DOFs.

A.1.1. Type 1

CS_{11}: A single constraint line.

FS_{11}: Any line, on every plane, that contains the constraint line is a permissible rotation. Any direction that is perpendicular to the constraint line is a permissible translation. Any line that satisfies Eq. (2) is a permissible screw.

A.2. Case 2: Flexure systems with two non-redundant constraints

In this case, every flexure system contains two non-redundant constraints and four DOFs.

A.2.1. Type 1

CS_{21}: A pencil of constraint lines.

FS_{21}: Every line within a sphere that intersects at the center of the constraint pencil is a permissible rotation. Any line that lies on the plane of the constraint pencil is a permissible rotation. A permissible translation points in the direction normal to the plane of the constraint pencil. Lines that are orthogonal to the constraint lines are permissible screws.

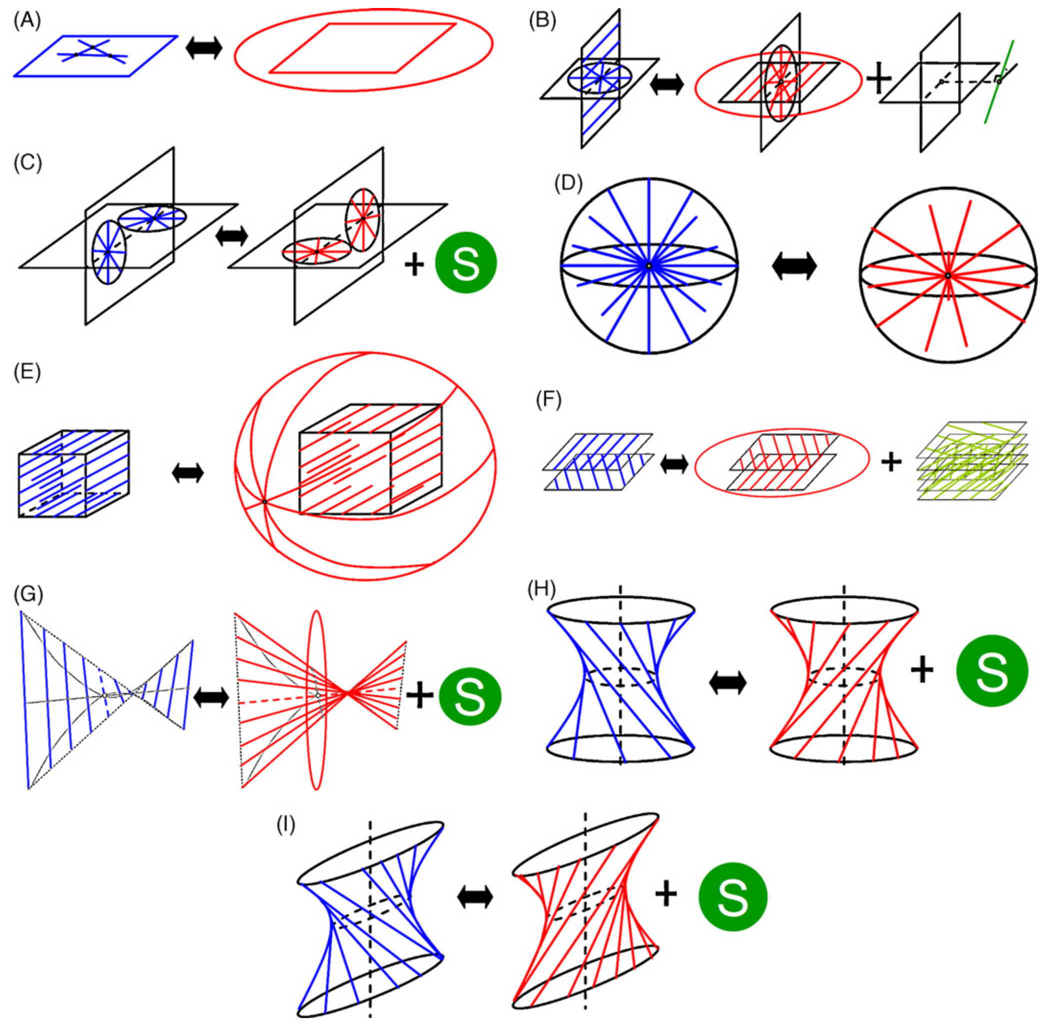

Fig. 13. Types of flexure system arrangements for Case 3. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

A.2.2. Type 2

CS_{22}: A P-plane of constraint lines.

FS_{22}: Every line within a box that (a) is parallel to the constraint lines or (b) lies on the plane of the constraint lines is a permissible rotation. Every line that is perpendicular to any constraint line is a permissible translation. Lines that exist within planes that are parallel to the plane of constraint lines are permissible screws.

A.2.3. Type 3

CS_{23}: Two skew constraint lines.

FS_{23}: Every line within any pencil that (a) is intersected at its center point by one of the constraint lines and (b) lies on a common plane with the other constraint line is a permissible rotation. A permissible translation exists in a direction that is normal to the parallel planes of the skew constraint lines. Permissible screws also exist.

A.3. Case 3: Flexure systems with three non-redundant constraints

In this case, every flexure system contains three non-redundant constraints and three DOFs.

A.3.1. Type 1

CS_{31}: An A-plane of constraint lines.

FS_{31}: Every line that lies on the plane of constraints is a permissible rotation. A permissible translation exists in a direction that is normal to this plane.

A.3.2. Type 2

CS_{32}: A pencil of constraint lines and a P-plane of constraint lines that intersect. The lines within the P-plane are parallel to this intersection line and this intersection line pierces the center of the pencil.

FS_{32}: Every line within a pencil that lies on the plane of parallel constraint lines is a permissible rotation. Every line that is (a) parallel to the constraint lines on the P-plane and (b) lies on the plane of the pencil of constraint lines is a permissible rotation. A permissible translation exists in a direction that is normal to the plane of the pencil of constraints. Permissible screws also exist.

A.3.3. Type 3

CS_{33}: Two pencils of constraint lines that exist within intersecting planes. The intersection line of the planes pierces the center of each pencil.

FS_{33}: Every line within two pencils that exist within the same planes as the pencils of constraint lines is a permissible rotation. The pencils of freedom lines share a common center point with a corresponding pencil of constraint lines. The plane that contains the pencil of freedom lines is orthogonal to the plane that contains

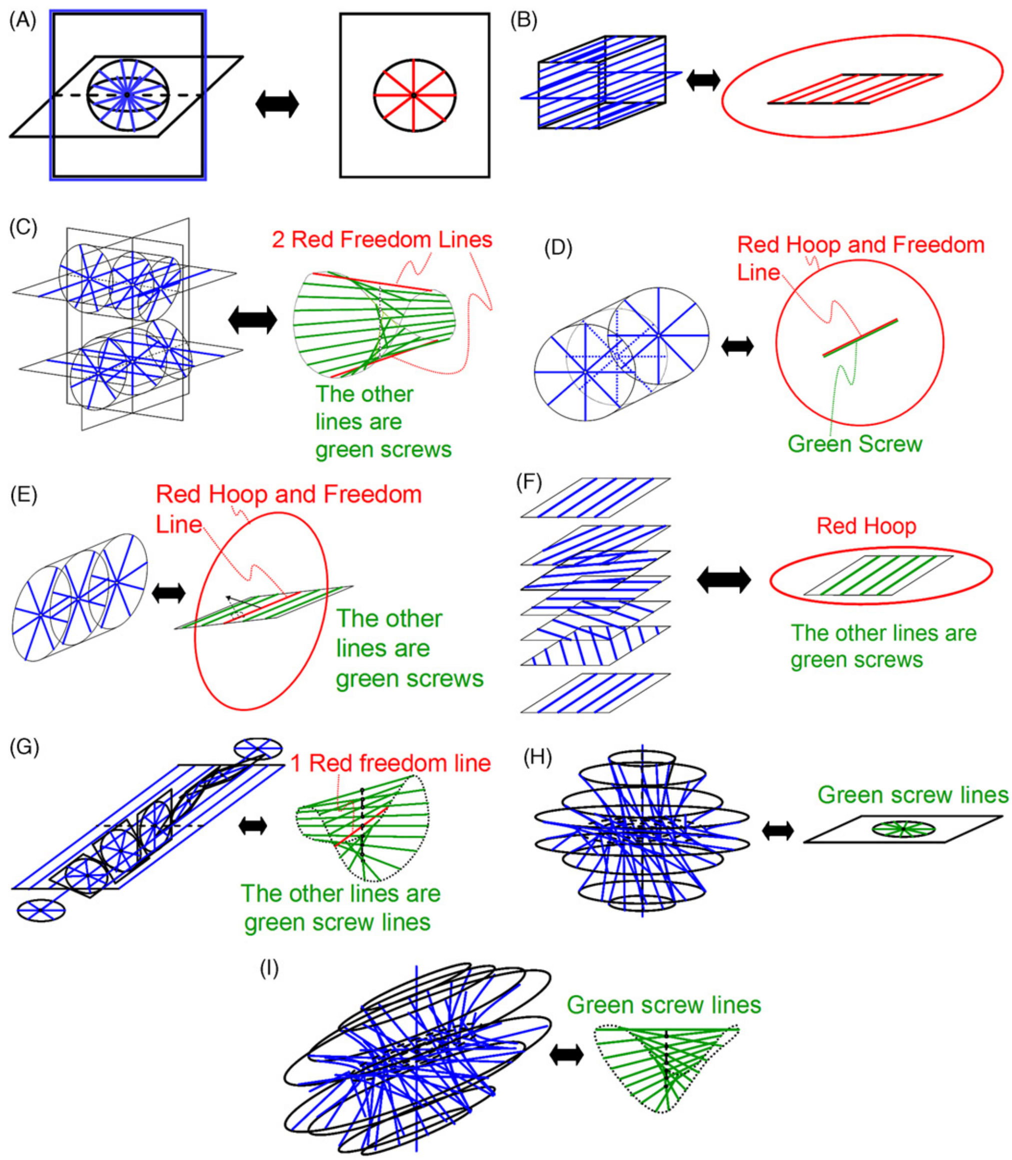

Fig. 14. Types of flexure system arrangements for Case 4. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

a corresponding pencil of constraint lines. Permissible screws also exist.

A.3.4. Type 4

CS_{34}: A sphere of constraint lines that represents all lines that intersect a common point.

FS_{34}: Every line within a sphere that intersects the sphere of constraint lines at its center point is a permissible rotation.

A.3.5. Type 5

CS_{35}: A box that contains every constraint line that is parallel to a specific direction.

FS_{35}: Every line that is parallel to the constraint lines is a permissible rotation. Every line that is perpendicular to a constraint line points in the direction of a permissible translation.

A.3.6. Type 6

CS_{36}: Two P-planes of constraint lines. The parallel constraint lines on one plane are skew with respect to the parallel constraint lines on the other plane.

FS_{36}: Every line that (a) lies on one of the planes and (b) is parallel to the constraint lines on the other plane is a permissible rotation. Other sets of parallel lines that exist within planes that are parallel to the two planes of constraint lines are permissible screws. A permissible translation points in the direction perpendicular to the two planes.

A.3.7. Type 7

CS_{37}: One of two rulings of lines that exist on the surface of a hyperbolic paraboloid consists of constraint lines.

FS_{37}: The other ruling of lines that exist on the surface of the same hyperbolic paraboloid consists of pure rotations. The direction that is orthogonal to every constraint line is a translation. Screws also exist.

A.3.8. Type 8

CS_{38}: One of two rulings of lines that exist on the surface of a circular hyperboloid consists of constraint lines.

FS_{38}: The other ruling of lines that exist on the surface of the same circular hyperboloid consists of pure rotations. Screws also exist.

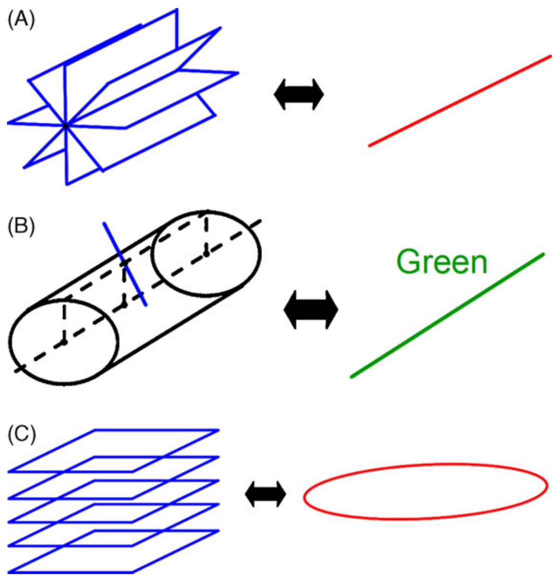

Fig. 15. Types of flexure system arrangements for Case 5. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

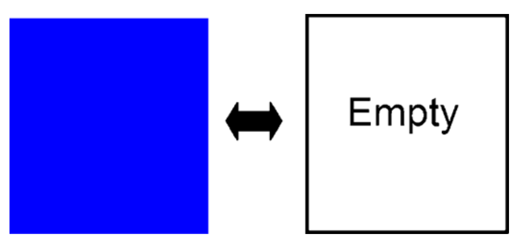

Fig. 16. The only type for Case 6. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of the article.)

A.3.9. Type 9

CS_{39}: One of two rulings of lines that exist on the surface of an elliptical hyperboloid consists of constraint lines.

FS_{39}: The other ruling of lines that exist on the surface of the same elliptical hyperboloid consists of pure rotations. Screws also exist.

A.4. Case 4: Flexure systems with four non-redundant constraints

In this case, every flexure system contains four non-redundant constraints and two DOFs.

A.4.1. Type 1

CS_{41}: Every line within a sphere that intersects a common point is a constraint line. Every line that lies on a plane that contains this point is also a constraint line.

FS_{41}: Every line within a pencil that (a) lies on the plane of constraints and (b) intersects the constraint sphere at its center point is a permissible rotation.

A.4.2. Type 2

CS_{42}: A box representing every constraint line that is parallel to a specific direction. Every line that lies on a plane that is parallel to, or coincident with, the parallel constraint lines is also a constraint line.

FS_{42}: Every line that is (a) parallel to the parallel constraint lines and (b) lies on the constraint plane is a permissible rotation. A permissible translation exists in a direction that is normal to the plane of constraints.

A.4.3. Type 3

CS_{43}: Every line within any pencil that (a) is intersected at its center point by a permissible pure rotation line and (b) lies on a common plane with another permissible rotation line that is skew to the first permissible rotation line is a constraint line.

FS_{43}: Two skew lines within a cylindroid are permissible rotations and every other line within the cylindroid is a permissible screw.

A.4.4. Type 4

CS_{44}: Every line within a pencil that is (a) perpendicular to a permissible rotation line and (b) intersected at its center point by the permissible rotation line, is a constraint line.

FS_{44}: A permissible rotation line and permissible screw lines that are coincident. A permissible translation exists in a direction that is collinear with the rotation line and the screw lines.

A.4.5. Type 5

CS_{45}: Every line within a pencil that is (a) not perpendicular to a permissible rotation line and (b) intersected at its center point by a permissible rotation line, is a constraint line. The pencils exist within parallel planes.

FS_{45}: A permissible rotation line that is (a) parallel to and (b) lies on a plane of parallel permissible screws. A permissible translation points in a direction that is normal to the planes that contain the constraint pencils.

A.4.6. Type 6

CS_{46}: A set of P-planes where each plane is parallel and contains constraint lines. The directions of the parallel constraint lines on each plane are different and are determined by Eq. (2).

FS_{46}: A P-plane of permissible screws that all have the same pitch values exists. This plane of permissible screws is coincident with the plane of parallel constraints that are orthogonal to the permissible screws. A permissible translation is normal to the plane of permissible screws.

A.4.7. Type 7

CS_{47}: Every line within certain pencils that rotate as they translate along a permissible rotation line is a constraint line. Every line on a plane that is parallel to that permissible rotation line is also a constraint line. The rate that the pencils rotate as they translate may be determined using Eq. (2) and the pitch value of a permissible screw that is orthogonal to, and intersects, the permissible rotation.

FS_{47}: A cylindroid of permissible screws with a principal generator that is a permissible rotation.

A.4.8. Type 8

CS_{48}: Every line from one of the two rulings of lines on the surface of an infinite number of nested circular hyperboloids is a constraint line.

FS_{48}: A pencil of permissible screws with the same pitch value.

A.4.9. Type 9

CS_{49}: Every line from one of the two rulings of lines on the surface of an infinite number of nested elliptical hyperboloids is a constraint line.

FS_{49}: A cylindroid of permissible screws.

A.5. Case 5: Flexure systems with five non-redundant constraints

In this case, every flexure system contains five non-redundant constraints and one DOF.

A.5.1. Type 1

CS_{51}: Every line that lies on any plane that intersects a single permissible rotation line is a constraint line.

FS_{51}: A single permissible rotation line.

A.5.2. Type 2

CS_{52}: Every line that is (a) tangent to the surface of a cylinder that possesses an axis that is a permissible screw and (b) satisfies Eq. (2), is a constraint line.

FS_{52}: A single permissible screw line.

A.5.3. Type 3

CS_{53}: Any line that lies within any plane that belongs to a set of parallel A-planes is a constraint line.

FS_{53}: A single permissible translation in a direction that is normal to the parallel planes of constraint lines.

A.6. Case 6: Flexure systems with six non-redundant constraints

This case has no DOFs, i.e. the six constraints exactly constrain the rigid body. This case is listed here in the interest of completeness.

A.6.1. Type 1

CS_{61}: Any line is a constraint line.

FS_{61}: No lines exist as there are no permissible motions.

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