3-1-1975

Synthesis for Spectrum Shaping Digital Filters of Recursive Design

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***Abstract*—Several techniques are presented, both iterative and noniterative, for synthesizing a recursive digital filter whose amplitude spectrum approximates a desired shape. The synthesis is carried out by implementing known algorithms for rational function approximation of a continuous function on a finite interval. In particular, techniques for synthesizing the best rational function approximation in the Chebyshev sense and in the least squares sense are treated. Examples are given which illustrate the quality of approximation achievable through each technique.**

I. Amplitude Design of Recursive Digital Filters

THERE ARE filtering applications which require the synthesis of a digital filter whose amplitude spectrum (henceforth referred to as a spectrum) approximates a given continuous shape. Such an application occurs in data transmission systems where signal shaping is often done at both the transmitter and receiver.[1]

Motivated by recent work concerning the synthesis of nonrecursive digital filters [1], we examine the problem of synthesizing a recursive digital filter whose spectrum best (in the Chebyshev sense) approximates a desired spectral waveshape. We reduce this problem to one of a rational function fit to a desired continuous function on a finite interval. Just as the polynomial approximation work of Remez provided a base for the nonrecursive design of digital filters, it is our intention here to make the filter designer aware of the relationship between the recursive filter problem and the vast amount of work done on rational approximation in a real variable. We do not intend to survey the current state of the art in rational function approximation theory, but rather demonstrate its equivalence to the digital filter problem. Specifically, there exist many algorithms which provide the rational function which approximates a desired function in a given norm. Of special interest is that approximation which is best in the Chebyshev sense.

We note in passing that a sizable amount of rational function approximation theory has been applied to s-plane filter design problems [2]. However, the digital filter designer encounters a great deal of difficulty in applying much of this work to his problems on the z plane. At best, some similarity in design procedures can be observed but the two design problems are by no means equivalent.

In this paper, we describe general purpose algorithms for iteratively finding rational function approximations of a given order. In exercising some of these algorithms, we have experienced that algorithmic convergence can sometimes become a problem. However, a linear programming design procedure we have also considered does not suffer from this convergence problem. We also considered the question of initial guesses to the iterative routines. Toward this end, we review two noniterative techniques of rational function approximation. As is illustrated by example, these latter techniques become interesting since they can provide filters whose amplitude characteristics are only slightly inferior to that given by an approximation which is best in a given norm.

II. Preliminaries

A digital filter of recursive design of order M has the following form for its transfer function[2]:

$$H(z^{-1}) = \frac{\alpha_0 + \alpha_1 z^{-1} + \cdots + \alpha_M z^{-M}}{\beta_0 + \beta_1 z^{-1} + \cdots + \beta_M z^{-M}} = \frac{A(z^{-1})}{B(z^{-1})}. \quad (1)$$

Consider the case where M is even and the α and β parameters are assumed to have the symmetry $\alpha_k = \alpha_{M-k}$, $\beta_k = \beta_{M-k}$, $k = 0,1,2,\cdots,M/2$. This symmetry causes zeros and poles to be located outside the unit z circle but let us set aside this point for the moment. By extracting a z^{-N} factor from the numerator and denominator, we have

$$H(z^{-1}) = \frac{a_N(z^N + z^{-N}) + a_{N-1}(z^{N-1} + z^{-(N-1)}) + \cdots + a_1(z + z^{-1}) + 2a_0}{b_N(z^N + z^{-N}) + b_{N-1}(z^{N-1} + z^{-(N-1)}) + \cdots + b_1(z + z^{-1}) + 2b_0} \quad (2)$$

where $N \triangleq M/2$ and where we have defined $a_k \triangleq 2\alpha_{N-k}$ except for $a_0 = \alpha_0$. We evaluate $H(z^{-1})$ on the unit circle for its frequency response

$$H(z^{-1})|_{z=\exp(j2\pi f)} = \frac{a_N \cos 2\pi Nf + a_{N-1} \cos 2\pi(N-1)f + \cdots + a_1 \cos 2\pi f + a_0}{b_N \cos 2\pi Nf + b_{N-1} \cos 2\pi(N-1)f + \cdots + a_1 \cos 2\pi f + b_0} \qquad |f| \le \tfrac{1}{2}.$$

Transforming the multiple angle cosine functions to powers of cosines, we have with $x \triangleq \cos 2\pi f$

$$H(x) = \frac{p_0 + p_1 x + \cdots + p_N x^N}{q_0 + q_1 x + \cdots + q_N x^N} = \frac{P(x)}{Q(x)} \qquad |x| \le 1 \quad (3)$$

Manuscript received March 18, 1974; revised July 1, 1974.
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[1] Phase equalization is a problem in this particular application as well. Such equalization is often accomplished through all-pass networks.

[2] We chose the numerator and denominator order to be identical for convenience only.

where the p and q parameters relate in a straightforward manner to the a's and b's. Now $H(x)$ is a real function and provides the workable form of the filter transfer function which we can now use to approximate a given spectrum $D(x)$ on the finite interval $|x| \leq 1$.

Specifically, we can apply known algorithms for rational function approximation of a continuous function on a finite interval. Upon obtaining the $\{p_k\}_0^N$ and $\{q_k\}_0^N$ of a rational approximation to $D(x)$, it is a simple matter to reverse the transformations of variable made to return to the form given in (1). We recall that symmetry was imposed on the α and β parameters of (1) in order to achieve the form (3). It is clear at this point that such a restriction on the α and β parameters does lead to instability when (3) is transformed back to (1). However, when instability does occur it is a simple matter to reflect the unstable roots back into the unit circle and not affect the filter's amplitude characteristic. For example, if $Q_0(z^{-1}) = r_1z^{-2} + r_2z^{-1} + r_3$ represents a quadratic factor of the resultant solution $B(z^{-1})$ with both its roots outside the unit z circle, then $Q_1(z^{-1}) = r_3z^{-2} + r_2z^{-1} + r_1$ has its roots inside the circle and $|Q_0(z^{-1})| = |Q_1(z^{-1})|$, $|z^{-1}| = 1$. Similarly, when a quadratic factor produces two real roots, one of which lies outside the unit circle, there exists another quadratic factor with identical magnitude which has both real roots inside the unit circle.

To prevent real singleton zeros from occurring in (3), the positivity constraint $p_0 + p_1x + \cdots + p_Nx^N \geq 0$ can be included in at least one of the design procedures we will discuss. This constraint forces a magnitude squared approximation to be made. In this case, $D(x)$ would then represent the square of the desired magnitude function.

At this point, it is beneficial to note that the rational function approximation can be made on disjoint closed subintervals of $|x| \leq 1$. The danger with doing this is that unwanted poles (i.e., $Q(x) = 0$) may appear in the transition regions (or "don't care" regions).

III. L_p, $p = 2, \infty$ Approximation in Recursive Digital Filter Design

Equation (3) provides the key for launching a direct synthesis procedure for recursive filters in the real variable x. We have tacitly assumed that any phase constraints on the filter's frequency specification will be handled by equalization networks. In the following we will discuss the characterization of L_p ($p = 2,\infty$) approximation of a continuous function on a finite interval with rational functions. A large number of known algorithms exists for achieving this type of approximation.[3]

A. Best Approximation in the Chebyshev Sense (L_∞)

It is well known [3] that to every continuous function $D(x)$ on an interval $[a,b]$ there corresponds a rational function approximation $R(x)$ of degree N (we have assumed that the numerator and denominator are both polynomials of degree N) which is the best rational approximation in the Chebyshev sense (or simply best). Without loss of generality the denominator polynomial is taken to be positive on $[a,b]$. To characterize the best approximation, an alternation theorem [3, p. 161] states that the error function, defined as the difference between the desired continuous function and a rational function of degree N (i.e., $D - R$), has at least one more alternation than the dimension of the polynomial subspace spanned by $\{P_N(x) + RQ_N(x)\}$ where $P_N(x)$ and $Q_N(x)$ are arbitrary polynomials of, at most, Nth degree. It is easily shown that the maximum dimension of this subspace is $2N + 1$. Hence a necessary and sufficient condition that R be the best rational approximation to D on $[a,b]$ is that the error have at least $2N + 2$ alternations. There exist algorithms which search for the best rational approximation of a continuous function [8]–[12]. Since the search is iterative and amounts to a nonlinear programming problem, the question of initial guesses arises. That is, convergence may depend upon the initial values assigned to the rational function's parameters. We will have more to say about this matter later.

1) Remez Techniques: One particular method of determining the best rational approximation to a continuous function D is the extension of Remez's second algorithm from polynomials to rational functions. An interesting feature of this algorithm for *polynomial* approximation is that it always converges [13] for any initial data. Further, if D is differentiable the convergence rate is known to be quadratic [14]. Shenitzer extended the algorithm to rational function approximation where the denominator polynomial is fixed [9]. Later, Fraser and Hart developed a simple technique which released the denominator polynomial in order to achieve the full extent of the approximation [8]. (While this algorithm may be pedagogically appealing to those digital filter theorists familiar with Remez-like algorithms for designing finite response filters, there is no indication that it is more suited for recursive filter design than any of the other methods available.) In Appendix III we review the steps traditionally required for rational function approximation using Remez techniques.

2) Differential Correction Algorithms: The advantage Remez-like algorithms have over other rational function approximation techniques is that convergence is at least quadratic. It has been known [3] that linear programming can be used to solve approximation problems on a finite set. However, convergence speeds have been known to be only at least linear.

Among those approximation techniques employing linear programming is a differential correction type algorithm attributed to Cheney and Loeb [15]. It has been recently shown by Barrodale *et al.* [12] that this algorithm enjoys at least quadratic convergence. An additional advantage of the algorithm is that convergence is guaranteed from any initial approximation with no poles (i.e., $Q(x) \neq 0$, $|x| \leq 1$). Finally, the condition $Q(x) > 0$ is maintained through every iteration automatically.

We have implemented this algorithm for the magnitude only design of recursive digital filters. We summarize the

[3] See, for example, [3]–[12]. Reference [4] provides a particularly good survey.

procedure in the following steps. In the notation of Section II, we wish to minimize

$$\max_{|x|\le 1} \left| D(x) - \frac{P(x)}{Q(x)} \right| \tag{4}$$

where $D(x)$ is our desired function while $P(x)$ and $Q(x)$ are the polynomials of degree N of (3). On a discrete point set of $\{x: |x| \le 1\}$, we have the problem

$$\inf_{P,Q} \max_{n} \left| D(x_n) - \frac{P(x_n)}{Q(x_n)} \right|, \qquad n = 1,2,\cdots,K. \tag{5}$$

Define the quantity Δ to be an upper bound of the error function $|D(x_n) - P(x_n)/Q(x_n)|$ for all x_n:

$$\Delta \ge \left| D(x_n) - \frac{P(x_n)}{Q(x_n)} \right|. \tag{6}$$

Since we always assume $Q(x_n) > 0$, we can rewrite (6) to read

$$|D(x_n) \cdot Q(x_n) - P(x_n)| \le \Delta Q(x_n). \tag{7}$$

At this point we decide on the iteration procedure towards finding $\Delta^* = \inf_{P,Q} \Delta$. Let the subscript k denote the kth iteration step. The product $\Delta Q(x_n)$ of the right-hand side of (7) is expanded in a Taylor series expansion traditional of differential correction techniques [3]. Truncating the series after the linear terms, we have

$$\Delta Q(x_n) = \Delta_k Q_k(x_n) + (\Delta - \Delta_k) Q_k(x_n) + (Q(x_n) - Q_k(x_n))\Delta_k. \tag{8}$$

Combining (7) and (8), we arrive at

$$|D(x_n)Q(x_n) - P(x_n)| - \Delta_k Q(x_n) - (\Delta - \Delta_k) Q_k(x_n) \le 0. \tag{9}$$

By expanding the absolute value sign, we have

$$(D(x_n) + \Delta_k)Q(x_n) - P(x_n) + (\Delta - \Delta_k)Q_k(x_n) \ge 0 \tag{10}$$

$$(-D(x_n) + \Delta_k)Q(x_n) + P(x_n) + (\Delta - \Delta_k)Q_k(x_n) \ge 0. \tag{11}$$

The objective at the $k + 1$th step then is to minimize $\Delta - \Delta_k$ which will become a negative quantity since Barrodale *et al.* [12] have shown $\Delta_{k+1} < \Delta_k$. Defining $\omega = \Delta - \Delta_k$, the linear programming problem is complete. We wish to minimize ω subject to (10) and (11) and a normalization constraint on $Q(x)$, notably either $q_0 \equiv 1$ or $|q_i| \le 1$, $i = 0,1,2,\cdots,N$.

At this juncture two important observations can be made concerning the digital filter design problem. First, the condition $P(x) \ge 0$ can also be added to the constraint set very easily. As we recall from our discussion in Section II, this condition allows a magnitude squared approximation problem to be solved without undue complications. Second, we note that the linear programming approach also allows the approximation to be made over disjoint closed intervals of $|x| \le 1$. However, in this case, unlike the Remez algorithms discussed in III-A1, it is possible to constrain $Q(x) > 0$ over the *entire* interval by simply adding linear constraints to those of (10) and (11). This means that appearance of poles in the transition region can be prevented.

B. Least-Squares Approximation (L_2)

The approximation objective here is to find a rational function of a fixed order which approximates (in the least squares sense) a desired function on a sufficiently dense finite point set. That is,

$$\mathrm{E} = \sum_{i=1}^{K} w(x_i) \left(f(x_i) - \frac{P(x_i)}{Q(x_i)} \right)^2, \qquad x_i \in X$$

is minimized over polynomials $P(x)$, $Q(x)$ of Nth order over a given point set $X = \{x_i, i = 1,2,\cdots,K\}$. The given function, $w(x_i)$ serves as a weighting function over X. A convenient computer algorithm for accomplishing the minimization of E is available in [16].

IV. Initial Guesses and Approximation Without Norm

An important feature of many iterative routines is the specification of an initial guess. Convergence may depend heavily on the closeness of the initial guess to the solution. Intuitively, it is felt that a good initial guess provides a better environment for convergence than any random initialization procedure. Hence, we draw attention to known techniques providing rational function approximations without iteration. These latter techniques, dubbed "approximations without norm," require only the solution to a linear set of equations. In addition to the simplicity and speed of obtaining these approximations, we will provide examples to show that they provide excellent rational function fits to continuous functions in their own right.

However, we should warn that there is no guarantee that these approximations lead to rational functions which have no poles in $|x| \le 1$. Of course, the approximation is worthless as an initial guess for an iterative routine should this occur.

A. Padé Approximant Technique

We obtain a rational function approximation to a series $\sum_{n=0}^{L} c_n x^n$ which can be the truncated Taylor series expansion of the desired function D, the best polynomial approximation to D, or perhaps a weighted least-squares approximation to D. In any case, a two-stage fitting procedure is implied. First, a polynomial estimate of D is obtained and then a rational function approximation is made to that polynomial. It is easily shown that only a linear set of equations needs to be solved for obtaining the rational function parameters after the polynomial expansion is found (see Appendixes II and III).

B. Rational Function Generalized Expansions

The idea in the Padé approximate technique can be generalized to orthonormal sets of polynomials $\{\phi_n\}$. Specifically, we can obtain a rational function approximation

$$R(x) = \frac{\sum_{n=0}^{N} p_n \phi_n(x)}{\sum_{n=0}^{N} q_n \phi_n(x)} \qquad |x| \le 1$$

by equating the coefficients of the $\{\phi_n(x)\}$ expansion of $R(x)$ to the coefficients of the desired function's expansion

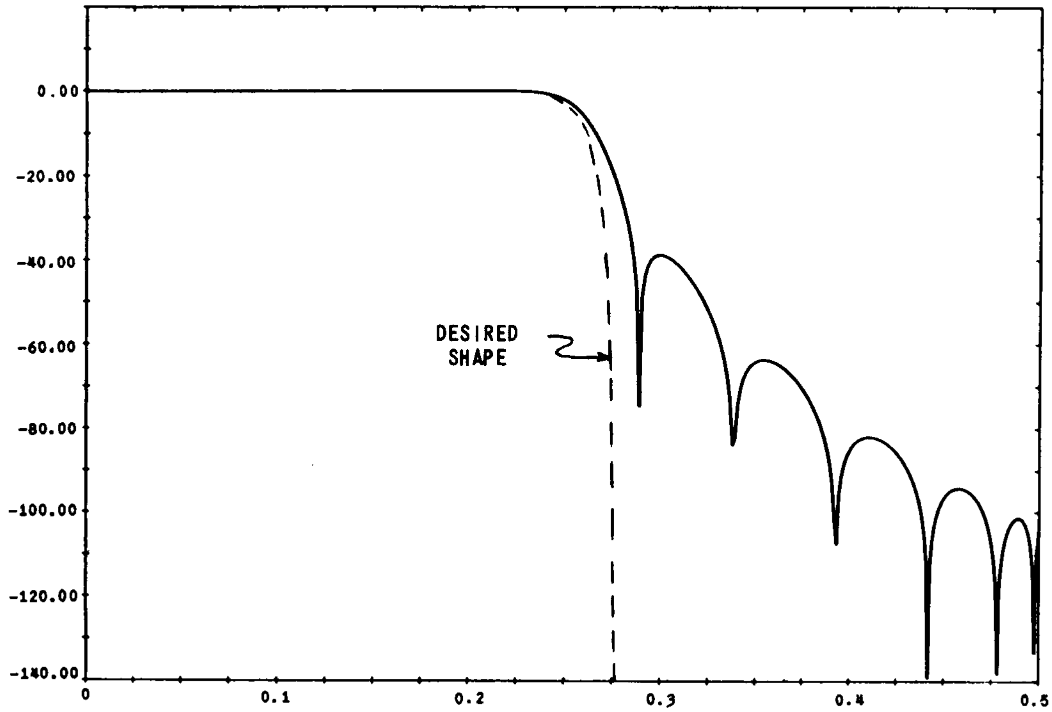

Fig. 1. 12th-order recursive filter design with rational Chebyshev approximation.

in $\{\phi_n(x)\}$, i.e., $f(x) \approx \sum_{n=0}^{M_0} c_n\phi_n(x)$ for a sufficiently high M_0. A case in point is that of the Chebyshev polynomial expansion. We choose to elaborate on this type of expansion because it is well known that the weighted least-squares approximation the expansion provides is "close" to the best polynomial approximation of the same order [3, p. 126]. As we convert from $f(x) \approx \sum c_n T_n(x)$ to $f(x) \approx \sum p_n T_n(x)/\sum q_n T_n(x)$, it is conceivable that this closeness may be preserved.[4] In any case, techniques exist for restricting the loss of approximation in going from a series expansion to rational function form [7]. As in the Padé case, only a set of linear equations needs to be solved to complete this generalized rational function approximation ([17, p. 80].

V. Examples

Iterative routines such as those discussed in Sections III-A1 and III-B often require initial starting points which themselves should be excellent approximations to the desired function. Hence we begin by illustrating one method of obtaining an initial guess for the parameter sets $\{p_k\}_0^N$, $\{q_k\}_0^N$.

Fig. 1 depicts the desired function shape for the amplitude spectrum of a recursive digital filter. As a first example, we find the Chebyshev series expansion for this shape for an appropriate number of coefficients. Then we convert the series expansion to a ratio of Chebyshev polynomials each of degree N (see Appendix II for the transformation equations). We note that this procedure consists of two approximations in cascade. Nevertheless, a visibly good initial approximation to the desired shape is obtained as shown in Fig. 1.

Another procedure which yields an excellent initial guess of the p and q parameters involves first setting up a best polynomial approximation (as described in [1]) instead of the Chebyshev truncated series. The polynomial is simply rewritten in terms of the Chebyshev polynomials and then the ratio of Chebyshev polynomials is obtained of corresponding order.

Upon obtaining the initial approximation corresponding to Fig. 1, we supplied the p and q parameters to the least squares fitting routine [16]. Convergence was achieved and the results are displayed in Fig. 2.

As a final example, we show in Fig. 3 a best approximation to the desired shape indicated using the routine explained in [10].

It is of interest to note that convergence was realized with nominal computation (viz., a few seconds of IBM 360 CPU time) even though our examples involve a 12th-order rational function of z^{-1}. However, in our experience of applying three different Remez routines [7], [8], [10] which seek the best rational function approximation, we have never felt confident of convergence. Each routine has failed to converge from initializing sets of parameters we thought reasonable. The least-squares iterative routine was just as unreliable in achieving convergence.

In contrast to the Remez and least-squares routines, we have found that the differential correction algorithm has had no convergence problems. In Figs. 4 and 5 we show examples of filter design obtained using this routine. Experience has shown that any initial guess will lead to the same solution in about the same amount of time. In the spectral shape shown in Fig. 5, a "don't care" transition

[4] $T_n(x)$ is the Chebyshev polynomial of nth order. See Appendix II for more detail.

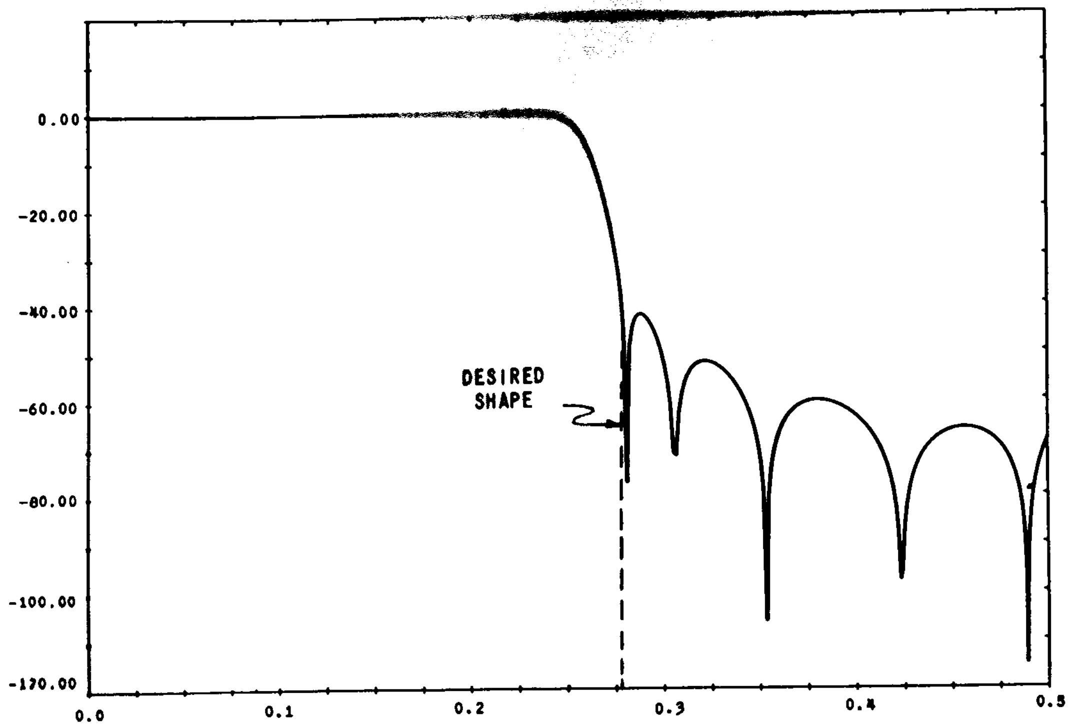

Fig. 2. 12th-order recursive filter design with rational least squares approximation.

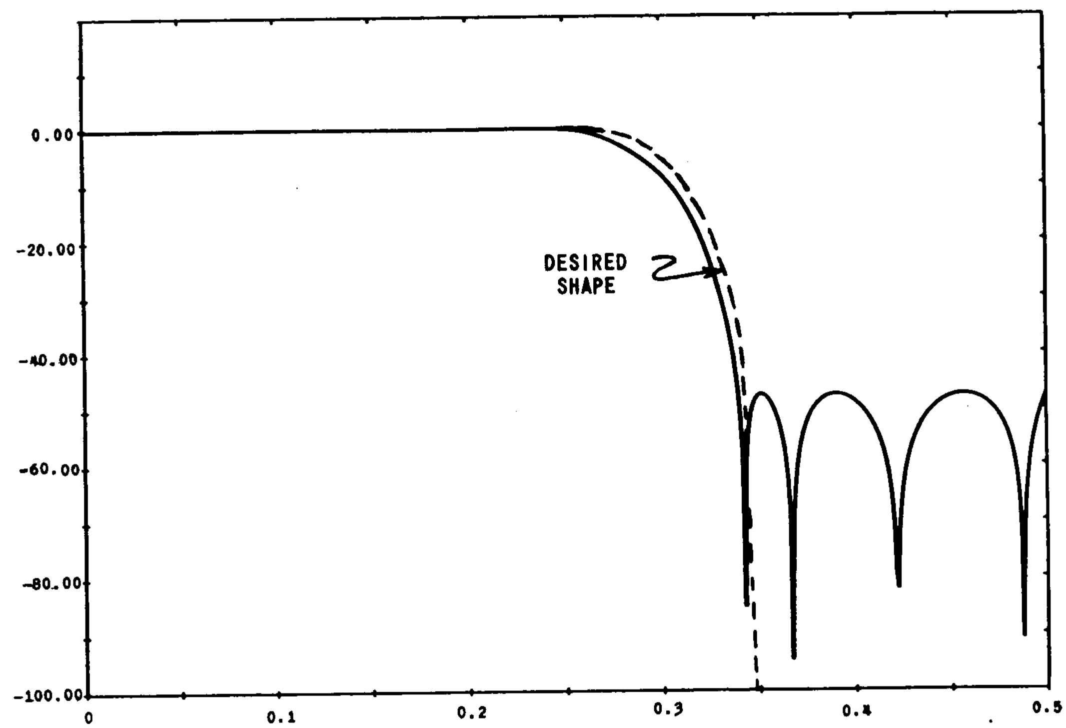

Fig. 3. 12th-order recursive filter design with rational best approximation.

region was included in the design. These advantages make it clear that the differential correction algorithm is far superior for magnitude only design of recursive digital filters.

By way of comparison to known techniques for synthesizing recursive digital filters (magnitude only approximation), the techniques described in this paper are either simpler or algorithmic convergence is faster. Using the least squares criterion, Steiglitz [18] described a synthesis procedure which uses a Fletcher–Powell minimization subroutine. Disadvantages encountered were that no fast initial guess procedure is apparent, and higher order (beyond fourth degree) filters required excessive computation time. Two algorithms using Chebyshev approximation

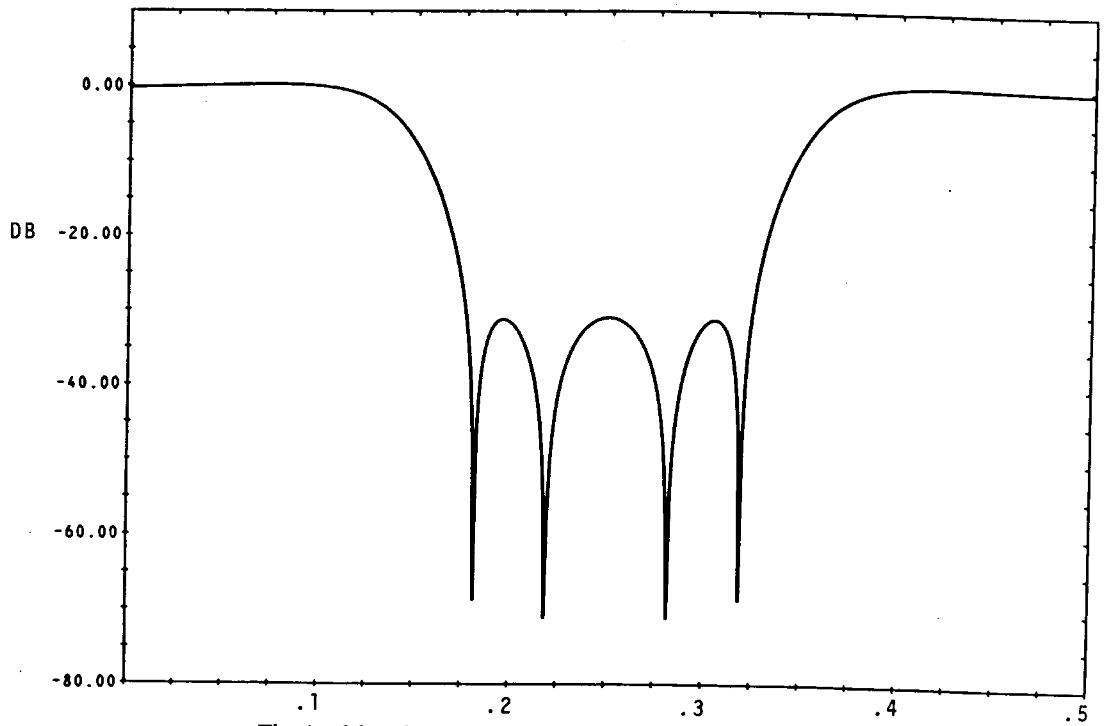

Fig. 4. 8th-order recursive filter design by ODC algorithm.

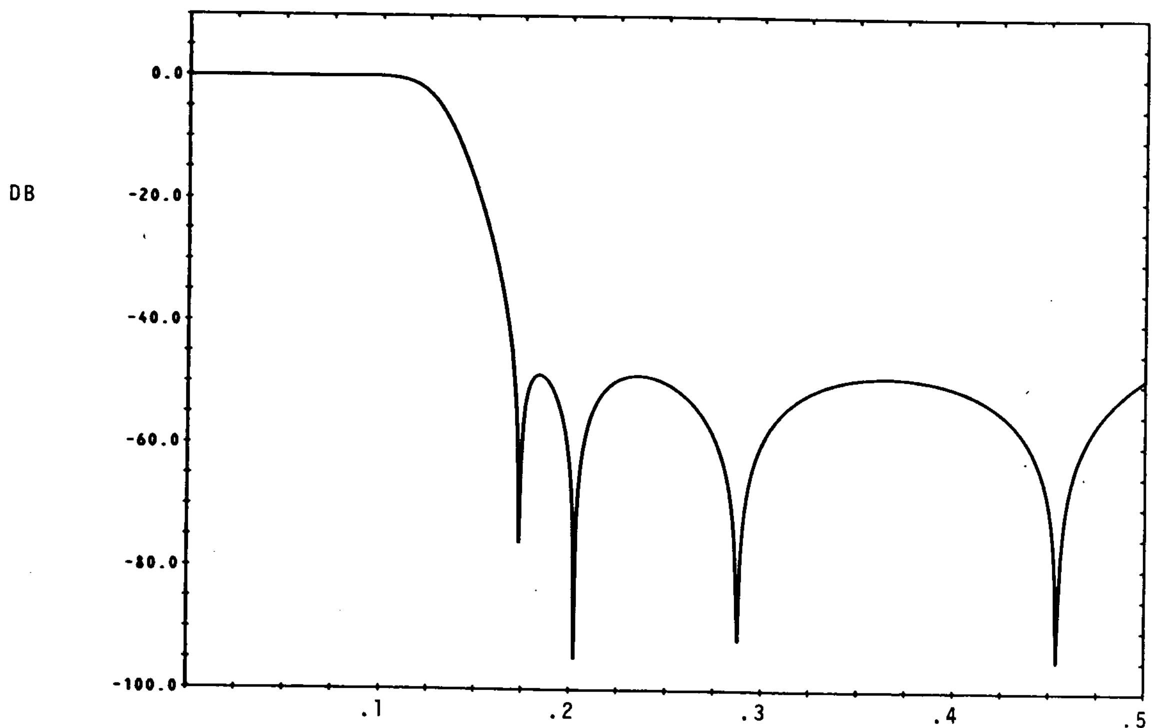

Fig. 5. 8th-order recursive filter design by ODC algorithm.

have appeared recently [19], [20]. The first algorithm employs Remez approximation routines. However, a nonlinear function of the magnitude (logarithm) is the approximating function and this results in complicating both the synthesis procedure and the initialization. In addition, in our experience Remez algorithms are plagued with convergence problems when rational functions are the approximants. The second algorithm is based on a linear programming approach to solving the approximation problem. The speed of convergence is at least linear for such an approach while it is known that the speed is at least quadratic for Remez algorithms. The contribution made by Barrodale *et al.* [12] is that an approximation procedure using linear programming was found which exhibited at least quadratic convergence. As we have reported in this paper, we have applied this faster procedure to the synthesis problem in digital filters and have found it to be a superior design technique.

APPENDIX I
PADÉ RATIONAL FUNCTION APPROXIMATION

The power series expansion[5] of the rational function

$$R(x) = \frac{a_0 + a_1x + a_2x^2 + \cdots + a_Nx^N}{1 + b_1x + b_2x^2 + \cdots + b_Nx^N} \qquad |x| \le 1$$

has for its coefficients

$$\begin{aligned} c_0 &= a_0 \\ c_1 &= -b_1c_0 + a_1 \\ c_2 &= -b_1c_1 + c_0 + a_2 \\ &\vdots \\ c_N &= -\sum_{k=1}^{N} b_kc_{N-k} + a_N \\ &\vdots \\ c_{N+n} &= -\sum_{k=1}^{N} b_kc_{N+n-k} \\ R(x) &= \sum_{k=0}^{\infty} c_kx^k \qquad |x| \le 1. \end{aligned}$$

Hence it is easily seen that the first $2N + 1$ coefficients c_k, $0 \le k \le 2N$ completely determine the rational function. The Padé approximant of order (N,N) to a given power series $G(x) = \sum_{k=0}^{\infty} d_kx^k, |x| \le 1$ is that rational function whose coefficients c_k agree with d_k $0 \le k \le 2N + 1$, i.e., $c_k = d_k$, $k = 0,1,2,\cdots, 2N + 1$.

APPENDIX II
CHEBYSHEV POLYNOMIALS TO RATIONAL FUNCTION TRANSFORMATION

Much like the Padé approximant technique, Maehly's method [17] converts a series expansion to a rational function form. However, the algorithm now involves a Chebyshev series and a rational function composed of a ratio of Chebyshev polynomial expansions.

Specifically, we convert

$$\sum_{k=0}^{\infty} c_kT_k(x) \qquad |x| \le 1$$

to a rational function approximation

$$R(x) = \frac{\sum_{k=0}^{N} a_kT_k(x)}{1 + \sum_{k=1}^{N} b_kT_k(x)} \qquad |x| \le 1$$

where $T_k(x)$ is kth Chebyshev polynomial on $|x| \le 1$. Akin to the Padé method we obtain the recursive relationships (using the identity $T_n(x)T_m(x) = \frac{1}{2}(T_{|n-m|}(x) + T_{n+m}(x))$,

$$\sum_{k=0}^{N} (c_{N+n-k} + c_{N+n+k})b_k = 0, \qquad n = 1,2,\cdots,N$$

where $b_0 \triangleq 1$. The a's are determined from

$$a_0 = c_0 + \frac{1}{2}\sum_{k=1}^{N} c_kb_k$$

$$a_n = c_n + \frac{1}{2}c_0b_n + \frac{1}{2}\sum_{k=1}^{N}(c_{|n-k|} + c_{n+k})b_k, \qquad n = 1,2,\cdots,N.$$

APPENDIX III
REMEZ ALGORITHMS FOR RATIONAL FUNCTION APPROXIMATIONS

The Remez algorithm has been found to be a stable method for finding the best polynomial approximation (in Chebyshev sense) to a continuous function on a finite closed interval(s) [3]. The extension of the Remez technique to rational function approximation does not come without problems. To appreciate the difficulties, we outline the general procedure. Let

$$R(x) = \frac{\sum_{i=0}^{N} a_ix^i}{1 + \sum_{i=1}^{N} b_ix^i} \qquad |x| \le 1$$

where

- x_i^k location of extrema at kth iteration;
- $D(x_i^k)$ value of desired function at x_i^k; and
- E^k $\max_i |R(x_i^k) - D(x_i^k)|$.

By the alternation theorem for the best rational function approximation we expect

$$D(x_i^k) - \frac{a_0^k + a_1^kx_i^k + a_2^k(x_i^k)^2 + \cdots + a_N^k(x_i^k)^N}{1 + b_1^kx_i^k + b_2^k(x_i^k)^2 + \cdots + b_N^k(x_i^k)^N} = (-1)^iE^k, \qquad i = 0,1,2,\cdots, 2N + 1.$$

Clearing the fraction and rearranging terms

$$\begin{aligned} D(x_i^k) = {} & (-1)^iE^k + a_0^k + a_1^kx_i^k + a_2^k(x_i^k)^2 + \cdots \\ & + a_N^k(x_i^k)^N - b_1^kx_i^k(D(x_i^k) - (-1)^iE^k) - \cdots \\ & - b_N^k(x_i^k)^N(D(x_i^k) - (-1)^iE^k), \\ & \qquad i = 0,1,2,\cdots, 2N + 1. \end{aligned} \tag{12}$$

We solve this nonlinear set of equations by first iterating to a solution for E^k. First, we take $E^{0,0} \equiv 0$ and solve $(j = 0)$

$$\begin{aligned} D(x_i^k) = {} & (-1)^iE^{k,j+1} + a_0^k + a_1^kx_i^k + \cdots \\ & + a_N^k(x_i^k)^N - b_1x_i^k \cdot (D(x_i^k) - (-1)^iE^{k,j}) - \cdots \\ & - b_N^k(x_i^k)^N(D(x_i^k) - (-1)^iE^{k,j}). \end{aligned}$$

At this point, we observe that a nonlinear relationship exists between E and the unknown coefficients $\{a_j\}_0^N$ for a given set of points of extrema $\{x_i\}_0^{2N+1}$. In their algorithm, Fraser and Hart [8] suggest iterating first towards a value of E at a given set of points $\{x_i\}_0^{2N+1}$. Upon determining E, the a's and b's are then solved. An $\{x_i\}_0^{2N+1}$ is now determined for these coefficients and the cycle is repeated. Werner [10] iteratively solves for estimates of the unknown coefficients and E by setting up the problem in an eigenvector–eigenvalue form.

[5] Conditions for its existence are presumed here.

A problem often encountered in iterative procedures used for solving (12) is that of cycling. A perhaps more serious problem is that of losing the "correct" number of alternations in the error curve. One particular Remez routine [11] we have used remedies these conditions by falling back on an interpolation procedure in order to recover from cycling or retain the minimum number of sign changes in the error curve. The interpolation is performed by assuming the error curve behaves parabolically between its zeros.

References

[1] T. W. Parks and J. H. McClellen, "Chebyshev approximation for nonrecursive digital filters with linear phase," *IEEE Trans. Circuit Theory*, vol. CT-19, pp. 189–194, 1972.

[2] J. Vlach, *Computerized Approximation and Synthesis of Linear Networks*. New York: Wiley, 1969.

[3] E. W. Cheney, *Introduction to Approximation Theory*. New York: McGraw-Hill, 1966.

[4] E. W. Cheney and T. H. Southard, "Survey of methods for rational approximation with particular reference to a new method based on a formula of Darboux," *SIAM Rev.*, vol. 5, p. 219, 1963.

[5] H. L. Loeb, "Algorithms for Chebychev approximations using the ratio of linear forms," *J. SIAM*, vol. 8, pp. 458–465, 1960.

[6] P. Fox, A. A. Goldstein, and G. Lastman, *Rational Approximations on Finite Point Sets, in Approximation of Functions*, Garabedian, Ed. New York: Elsevier, 1965.

[7] A. Ralston, "On economization of rational functions," *J. Ass. Comput. Mach.*, vol. 10, pp. 278–282, 1963.

[8] W. Fraser and J. F. Hart, "On the computation of rational approximations to continuous functions," *Commun. Ass. Comput. Mach.*, vol. 5, p. 401, 1962.

[9] A. Shenitzer, "Chebyshev approximation of a continuous function by a class of functions," *J. Ass. Comput. Mach.*, vol. 4, pp. 30–35, 1957.

[10] H. Werner, "Rationale Tschebyscheff-approximation, eigenwert theorie und differenzenrechnung," *Arch. Ration. Mech. Anal.*, vol. 13, pp. 330–347, 1963.

[11] R. H. Hardin, "MINT, a program for minimax fitting of tabular data by a rational function," in Bell Lab. Int. Memo., March 6, 1970.

[12] I. Barrodale, M. J. D. Powell, and F. D. K. Roberts, "The differential correction algorithm for rational L_∞-approximation," *SIAM J. Numer. Math.*, vol. 9, pp. 493–504, 1972.

[13] E. N. Novodvorskii and I. S. Pinsker (A. Shenitzer, tr.), "On a process of equalization of maxima," *Usp. Mat Nauk*, vol. 6, p. 46, 1951.

[14] L. Veidinger, "On the numerical determination of the best approximations in the Tchebychev sense," *Numer. Math.*, vol. 2, pp. 99–105, 1960.

[15] E. W. Cheney and H. L. Loeb, "Two new algorithms for rational approximation," *Numer. Math.*, vol. 3, pp. 72–75, 1961.

[16] *IBM System/360 Programming Manual*, Scientific Subroutine Package (360 A-CM-03X), Fortran Program DARAT, p. 265.

[17] M. A. Snyder, *Chebyshev Methods in Numerical Approximation*. Englewood Cliffs, N.J.: Prentice-Hall, 1966.

[18] K. Steiglitz, "Computer-aided design of recursive digital filters," *IEEE Trans. Audio Electroacoust.*, vol. AU-18, pp. 123–129, June 1970.

[19] A. G. Deczky, "Equiripple and minimax (Chebyshev) approximations for recursive digital filters," *IEEE Trans. A.S.S.P.*, vol. ASSP-22, pp. 98–111, Apr. 1974.

[20] N. Y. Graham, H. D. Helms, and L. R. Rabiner, "Linear programming design of IIR digital filters with arbitrary magnitude function," *IEEE Trans. A.S.S.P.*, vol. ASSP-22, pp. 117–123, Apr. 1974.