SYNTOMIC REGULATORS AND p-ADIC INTEGRATION I: RIGID SYNTOMIC REGULATORS

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1. Introduction

The syntomic cohomology, more precisely the cohomology of the sheaves $s(n)$ on the syntomic site of a scheme, where introduced in [FM87] in order to prove comparison isomorphisms between crystalline and p-adic étale cohomology. It can be seen as an analogue of the Deligne-Beilinson cohomology in the p-adic world (for an excellent discussion see [Nek98]). In particular, when X is a smooth scheme over the ring of integers $\mathcal{V}$ of a finite extension K of $\mathbb{Q}_p$ there should exist higher Chern classes from algebraic K-theory into the syntomic cohomology of X. Such classes have been constructed, sometimes under certain additional assumptions, by Gros [Gro90] and by Nizioł [Niz97].

Syntomic cohomology comes in different flavors (much like Deligne-Beilinson cohomology). The versions discussed above are well behaved only for proper schemes. In particular, they do not have the homotopy property for affine spaces. This makes computations difficult because most constructions in K-theory go through non proper schemes.

In [Gro94], Gros introduced, using the rigid cohomology of Berthelot [Ber96, Ber97], *rigid syntomic cohomology* for a scheme X which is smooth over an unramified base. When the scheme X is affine he constructs rigid syntomic regulators,

$$c_{i,j} : K_j(X) \to H^{2i-j}(X, s(i)_{X/K,\mathrm{rig}}) \quad ,$$

from K-theory into his rigid syntomic cohomology. Using these regulators Gros is able to show that the value of the syntomic regulator on certain cyclotomic elements in the higher K-theory of number fields is, when properly normalized, given by the values of p-adic polylogarithms at roots of unity.

It should be mentioned here that there is another method of "controlling" syntomic cohomology, due to Somekawa [Som92]. In this method one assumes X has a compactification where the complement is a relative normal crossings divisor. Somekawa is able to prove the result of Gros for all cyclotomic elements. We should note however that loc. sit. is not yet published to the best of our knowledge.

The following philosophy exists:

Philosophy 1. *There should be a p-adic Beilinson conjecture that relates special values of p-adic L-functions to syntomic regulators.*

Special cases of this are the results of [Gro90] and [KNQD98]. One should be able to derive some general conjecture from [PR95]. For results about CM elliptic curves see the discussion below.

The main result of this work is an extension of the constructions of Gros to an arbitrary smooth $\mathcal{V}$-scheme X. For such a scheme X we define in section 3 syntomic cohomology $H^i_{\mathrm{syn}}(X,n)$ and in section 4 we construct Chern classes from K theory to it. Our definition takes into account more growth conditions than that of Gros: we also consider log singularities. The result is that H_{syn} is always finite dimensional (proposition 3.5). Our cohomology maps when possible to the version of Gros (proposition 6.4) and to the version of Nizioł (proposition 6.7).

Another objective of this work is to begin to develop tools for computations in syntomic cohomology. Our main result here is the construction of a *modified syntomic cohomology*, denoted $H^*_{\mathrm{ms}}(X,*)$, in section 5. This cohomology is related to syntomic cohomology by a natural map (proposition 5.6.2) which is an isomorphism in most cases of interest (proposition 5.6.3). It is significantly easier to compute when the base $\mathcal{V}$ is ramified. We have also found that the original rigid syntomic cohomology of Gros (without log singularities), extended to the case of ramified base, is also useful in some computations (see for example [BdJ98]). It again can come with an original or modified flavor, the latter being most useful.

Let us discuss a bit of applications. In a sequel to this paper [Bes98] we compute the syntomic regulator $K_2(X) \to H^2_{\mathrm{syn}}(X,2)$ when X is smooth and proper of relative dimension 1 over $\mathcal{V}$. We show that there is a precise relation between this regulator and the p-adic regulator constructed by Coleman and de Shalit [CdS88]. In particular, for elliptic curves with complex multiplication their results in conjunction with ours relate the syntomic regulator with special values of a p-adic L-function of E, in line with the philosophy 1.

In [Bes97] we will build on the results of this paper and embed syntomic cohomology in some other "cohomology theory" which has Poincaré duality. This is very useful for computations involving cycles. We will show how to relate p-adic Abel-Jacobi maps to a generalization of Coleman's p-adic integration theory [Col85].

Finally, in [BdJ98] we intend to show how to compute syntomic regulators on the wedge complexes introduced in [DJ95] using p-adic polylogarithms. This is a generalization of the results of Gros on cyclotomic elements described above.

Throughout this work $\mathcal{V}$ is a complete valuation ring with maximal ideal $\mathfrak{p}$, quotient field K and residue field κ of characteristic p. When κ is perfect we let $\mathcal{V}_0 \subset \mathcal{V}$ be the Witt ring of κ and K_0 its quotient field. All schemes will be of finite type over $\mathcal{V}$.

We would like to thank Gros, Berthelot, de Jeu and Scholl for helpful conversations.

2. Rigid and de Rham complexes

In this section we do the preparation to the construction of syntomic cohomology in the next section by constructing certain complexes computing rigid and (filtered parts of) de Rham cohomology. For the purpose of constructing Chern classes, it is very useful to lift cohomology to the level of the derived category, and even to the level of complexes. In the constructions below we will habitually write $\mathbb{R}\Gamma$ but we will actually mean a particular complex representing this object in the derived category and maps between these objects will be represented by maps of complexes commuting "on the nose". We will explain below (proposition 2.17 how such a choice of complexes and maps can be achieved.

The first step is to construct the complexes computing rigid cohomology. Here we assume that $\mathcal{V}$ is a discrete valuation ring. We will consider schemes X which are of finite type over κ.

Definition 2.1. A rigid datum for X over $\mathcal{V}$ consists of an open immersion $j : X \hookrightarrow \overline{X}$ together with a closed immersion $\overline{X} \hookrightarrow \mathcal{P}$ into a formal $\mathcal{V}$-scheme $\mathcal{P}$ which is smooth in a neighborhood of X. We will write this datum as $(\overline{X}, j, \mathcal{P})$. A morphism between data $(\overline{X}, j, \mathcal{P})$ and $(\overline{X}', j', \mathcal{P}')$ is a commutative diagram

$$\begin{tikzcd} X \arrow[r, "j"] \arrow[d, "="'] & \overline{X} \arrow[r] \arrow[d, "\alpha"'] & \mathcal{P} \arrow[d, "u"'] \\ X \arrow[r, "j'"] & \overline{X}' \arrow[r] & \mathcal{P}', \end{tikzcd}$$

where α is proper and u is smooth in a neighborhood of X. The collection of all rigid data becomes a category under this notion of morphisms. We denote this category by $\mathcal{RD}(X, \mathcal{V})$.

Remark 2.2. Without further mention we will always assume that rigid data for X exist. This is certainly the case if X is quasi-projective. If this condition is not satisfied, one can carry out all the constructions using simplicial formal schemes, but we will not discuss this in this work.

Lemma 2.3. *The category $\mathcal{RD}(X, \mathcal{V})$ is filtered.*

Proof. Given $\mathcal{D}_r = (\overline{X}_r, j_r, \mathcal{P}_r)$, $r = 0, 1, 2$ in $\mathcal{RD}(X, \mathcal{V})$, with (resp. without) maps $\mathcal{D}_r \to \mathcal{D}_0$ for $r = 1, 2$, we can consider $\mathcal{D}_3 = (\overline{X}_3, j_3, \mathcal{P}_3) \in \mathcal{RD}(X, \mathcal{V})$ where $\mathcal{P}_3 = \mathcal{P}_1 \times_{\mathcal{P}_0} \mathcal{P}_2$ (resp. $\mathcal{P}_3 = \mathcal{P}_1 \times_{\mathcal{V}} \mathcal{P}_2$), $\overline{X}_3$ is the closure in $\overline{X}_1 \times_{\overline{X}_0} \overline{X}_2$ (resp. $\overline{X}_1 \times \overline{X}_2$) of the image of X and j_3 is the obvious map. Then there clearly exists a commutative diamond

(2.1)
$$\begin{tikzcd} & \mathcal{D}_3 \arrow[dl] \arrow[dr] & \\ \mathcal{D}_1 \arrow[dr] & & \mathcal{D}_2 \arrow[dl] \\ & \mathcal{D}_0, & \end{tikzcd}$$

(resp. without the maps to $\mathcal{D}_0$). $\square$

For a p-adic formal $\mathcal{V}$-scheme $\mathcal{P}$ there is an associated rigid analytic K-space, the *generic fiber* of $\mathcal{P}$, denoted $\mathcal{P}_K$. There is a canonical specialization map $\mathrm{sp} : \mathcal{P}_K \to \mathcal{P}$, which is continuous when $\mathcal{P}_K$ is given its strong Grothendieck topology and $\mathcal{P}$ its Zariski topology. Berthelot introduces the notion of a tube. If Y is a locally closed subset of the special fiber of a formal $\mathcal{V}$-scheme $\mathcal{P}$, the *tube* of Y in $\mathcal{P}$, denoted $]Y[_{\mathcal{P}}$, is a rigid analytic K-subspace of $\mathcal{P}_K$ whose underlying set is the set $\mathrm{sp}^{-1}(Y)$ of points whose specialization is in Y. Now let $(\overline{X}, j, \mathcal{P}) \in \mathcal{RD}(X, \mathcal{V})$ and let $Z = \overline{X} - X$. Berthelot introduces the notion of a *strict neighborhood* of $]X[_{\mathcal{P}}$ inside $]\overline{X}[_{\mathcal{P}}$. By definition this is a subset $U \subset]\overline{X}[_{\mathcal{P}}$, open in the strong Grothendieck topology, such that $\{U,]Z[_{\mathcal{P}}\}$ is a covering of $]\overline{X}[_{\mathcal{P}}$ in the same topology. Berthelot defines a functor $j^\dagger$ from the category of sheaves on $]\overline{X}[_{\mathcal{P}}$ to itself by

$$j^\dagger(F) = \varinjlim_U j_{U*} F,$$

where the direct limit is over all U which are strict neighborhoods of $]X[_{\mathcal{P}}$ in $]\overline{X}[_{\mathcal{P}}$ and j_U is the canonical embedding.

Definition 2.4 (Berthelot)**.** Let $\mathcal{D} = (\overline{X}, j, \mathcal{P}) \in \mathcal{RD}(X, \mathcal{V})$. The *rigid complex* $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}}$ is defined by

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}} := \mathbb{R}\Gamma(]\overline{X}[_{\mathcal{P}}, j^{\dagger}\Omega^{\bullet}_{]\overline{X}[_{\mathcal{P}}}).$$

As remarked above, we will take this as defining an actual complex rather than merely an object of a derived category. It is easy to see that the association $\mathcal{D} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}}$ is a contravariant functor from $\mathcal{RD}(X, \mathcal{V})$ to the category of bounded below complexes of K-vector spaces. A fundamental theorem of Berthelot ([Ber97] Theorem 1.4 and Corrolaire 1.7) asserts that maps of rigid data induce quasi-isomorphisms of rigid complexes. This motivates the following definition.

Definition 2.5. The *rigid complex* of X over K is defines as

$$\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) := \varinjlim_{\mathcal{D} \in \mathcal{RD}(X,\mathcal{V})} \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}}.$$

The complex $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$ is quasi-isomorphic to each of the $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}}$.

We next discuss functoriality. Let X and Y be κ-schemes as above and let $f : X \to Y$ be a κ-morphism. We define a rigid datum for f as a tuple consisting of rigid data for X and Y, $(\overline{X}, j_X, \mathcal{P}_X)$ and $(\overline{Y}, j_Y, \mathcal{P}_Y)$ respectively, and maps $\bar{f} : \overline{X} \to \overline{Y}$, $\hat{f} : \mathcal{P}_X \to \mathcal{P}_Y$ compatible with f in the obvious sense. We denote by $\mathcal{RD}(f, \mathcal{V})$ the category of all rigid data for f. As before, it is easily seen that the category $\mathcal{RD}(f, \mathcal{V})$ is filtered. Let P_1 and P_2 be the two projection functors from $\mathcal{RD}(f, \mathcal{V})$ to $\mathcal{RD}(X, \mathcal{V})$ and $\mathcal{RD}(Y, \mathcal{V})$ respectively.

Lemma 2.6. *The functor P_2 is surjective.*

Proof. Following [Ber97] bottom of page 338 and page 340, Let $(\overline{X}, j_X, \mathcal{P}_X)$ and $(\overline{Y}, j_Y, \mathcal{P}_Y)$ be rigid data for X and Y respectively. We than constructs a new rigid datum for X in the following way. We take $\mathcal{P} = \mathcal{P}_X \times \mathcal{P}_Y$, $\overline{X}'$ to be the closure of the graph of f in $\overline{X} \times \overline{Y}$ and $j : X \to \overline{X}'$ to be the obvious map. Then $(\overline{X}', j, \mathcal{P})$ together with $(\overline{Y}, j_Y, \mathcal{P}_Y)$ and the projections on $\overline{Y}$ and $\mathcal{P}_Y$ define an object of $\mathcal{RD}(f, \mathcal{V})$ mapping to $(\overline{Y}, j_Y, \mathcal{P}_Y)$. $\square$

Corollary 2.7. *The association $X \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$ extends to a contravariant functor from κ-schemes to complexes of K-vector spaces.*

Proof. It is easy to see that to any $\mathcal{D} \in \mathcal{RD}(f, \mathcal{V})$ corresponds a map $\mathbb{R}\Gamma_{\mathrm{rig}}(Y/K)_{P_2(\mathcal{D})} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{P_1(\mathcal{D})}$ and that this map is natural in $\mathcal{D}$. We therefore obtain a diagram

$$\varinjlim_{\mathcal{D}' \in \mathcal{RD}(Y,\mathcal{V})} \mathbb{R}\Gamma_{\mathrm{rig}}(Y/K)_{\mathcal{D}'} \leftarrow \varinjlim_{\mathcal{D} \in \mathcal{RD}(f,\mathcal{V})} \mathbb{R}\Gamma_{\mathrm{rig}}(Y/K)_{P_2(\mathcal{D})} \to$$
$$\varinjlim_{\mathcal{D} \in \mathcal{RD}(f,\mathcal{V})} \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{P_1(\mathcal{D})} \to \varinjlim_{\mathcal{D}'' \in \mathcal{RD}(X,\mathcal{V})} \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}''}.$$

The left pointing arrow is an isomorphism (not just a quasi-isomorphism) by lemma 2.6. This gives the map associated to f. It is easy to check that we get a functor. $\square$

Functoriality allows us to extend the definition of the rigid complex to simplicial schemes in the standard fashion.

Definition 2.8. Let $X_\bullet$ be a simplicial κ-scheme. Applying the functor $\mathbb{R}\Gamma_{\mathrm{rig}}(?/K)$ we obtain a cosimplicial object in the category of complexes of K-vector spaces. We define $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\bullet/K)$ to be the total complex of the associated double complex.

This construction is functorial on the category of simplicial κ-schemes. We have the usual spectral sequence:

Proposition 2.9. *Let $X_\bullet = (X_n)_{n\in\mathbb{Z}_{\geq 0}}$ be a simplicial κ-scheme. Then there exists a spectral sequence*

$$E_2^{i,j} = H^i_{\mathrm{rig}}(X_j/K) \Rightarrow H^{i+j}_{\mathrm{rig}}(X_\bullet/K).$$

Proposition 2.10. *Let X be a κ-scheme and let $\mathcal{U}_\bullet \to X$ be the covering associated to a finite Čech covering of X (we view X as a simplicial scheme which is X in each degree). Then the canonical map $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{U}_\bullet/K)$ is a quasi-isomorphism.*

Proof. Let the Čech covering be $\{U_1, \ldots, U_n\}$. Then

$$\mathcal{U}_n = \coprod_{|I|=n+1} U_I, \qquad U_I := \bigcap_{i\in I} U_i.$$

We choose a compactification $j : X \to \overline{X}$ and an embedding $\overline{X} \hookrightarrow \mathcal{P}$. This then defines a compactification $U_I \to X \to \overline{X}$, denoted j_I, for each U_I and we thus get a rigid datum $\mathcal{D}_I = (\overline{X}, j_I, \mathcal{P})$ for each U_I. The identity maps on $\overline{X}$ and $\mathcal{P}$ define rigid data for all the morphisms between the U_I that appear in the definition of $\mathcal{U}_\bullet$. It follows that $\mathbb{R}\Gamma_{\mathrm{rig}}(\mathcal{U}_\bullet/K)$ is quasi-isomorphic to the total complex of the double complex

$$\bigoplus_{|I|=n+1} \mathbb{R}\Gamma(]\overline{X}[_{\mathcal{P}}, j_I^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}}).$$

It follows from [Ber96, Prop. 2.1.8] or [Ber97, 1.2.ii] that this last complex is quasi-isomorphic to $\mathbb{R}\Gamma(]\overline{X}[_{\mathcal{P}}, j^\dagger \Omega^\bullet_{]\overline{X}[_{\mathcal{P}}})$ and hence to $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$. □

We state 2.11, 2.12 and 2.13 below for schemes but they immediately extend to simplicial schemes as well.

Proposition 2.11. *Let $\mathcal{V} \to \mathcal{V}'$ be a finite map of discrete valuation rings where $\mathcal{V}'$ has residue field κ' and fraction field K' and let X be a κ-scheme. Then there is a canonical base change map*

$$K' \otimes_K \mathbb{R}\Gamma_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X \otimes \kappa'/K'),$$

which is a quasi-isomorphism. The base change map is functorial in the obvious sense with respect to diagrams $\mathcal{V} \to \mathcal{V}' \to \mathcal{V}''$ and commutes with the maps induced by morphisms of κ-schemes.

Proof. Let $\mathcal{D} = (\overline{X}, j, \mathcal{P})$ be a rigid datum for X over $\mathcal{V}$. One obtains a rigid datum $\mathcal{D}' = (\overline{X} \otimes \kappa', j \otimes \kappa', \mathcal{P} \otimes_{\mathcal{V}} \mathcal{V}')$ for $X \otimes \kappa'$ over $\mathcal{V}'$. In this situation the proof of [Ber97, Proposition 1.8] shows the existence of a map $K' \otimes_K \mathbb{R}\Gamma_{\mathrm{rig}}(X/K)_{\mathcal{D}} \to \mathbb{R}\Gamma_{\mathrm{rig}}(X \otimes \kappa'/K')_{\mathcal{D}'}$. Taking direct limits give the required map and the functoriality statements are straightforward. □

Corollary 2.12. *Suppose κ is perfect and recall that $\mathcal{V}_0$ is the Witt ring of κ. Let $\sigma : \mathcal{V}_0 \to \mathcal{V}_0$ be the map induced by the p-power map on κ. Then there exists a canonical and natural σ-semilinear map $\phi : \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0)$.*

Proof. Let π be the projection $X \to \kappa$ and let $X \xrightarrow{\mathrm{Fr}_X \times \pi} X \otimes_{\kappa,\mathrm{Fr}_\kappa} \kappa$ be the relative Frobenius map. Here the map $\kappa \to \kappa$ in the last tensor product is the Frobenius map of κ, i.e., the p-power map. The map ϕ is obtained as the composition

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0) &\xrightarrow{1\otimes \mathrm{id}} K_0 \otimes_\sigma \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0)\\ &\xrightarrow{\text{base change}} \mathbb{R}\Gamma_{\mathrm{rig}}(X \otimes_{\kappa,\mathrm{Fr}_\kappa} \kappa/K_0) \xrightarrow{(\mathrm{Fr}_X \times \pi)^*} \mathbb{R}\Gamma_{\mathrm{rig}}(X/K_0),\end{aligned}$$

where the base change map is with respect to the map σ. Naturality is easily verified. □

The following lemma will be needed for the comparison between syntomic cohomology and modified syntomic cohomology. Its truth is obtained by a careful application of the functoriality properties of the base change.

Lemma 2.13. *Suppose, under the assumptions of corollary* 2.12 *that κ is a finite field with $q = p^r$ elements, which implies that $\mathrm{Fr}^r : X \to X$ is κ-linear. Then $\phi^r = (\mathrm{Fr}^r)^*$ as endomorphisms of $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$.*

It is convenient to use a different model for the rigid complex which takes into account more data.

Definition 2.14. An extended rigid datum for X over $\mathcal{V}$ consists of a rigid datum $\mathcal{D} = (\overline{X}, j, \mathcal{P}) \in \mathcal{RD}(X, \mathcal{V})$ together with a strict neighborhood U of $]X[_\mathcal{P}$ in $]\overline{X}[_\mathcal{P}$. A map from $(\mathcal{D}, U)$ to $(\mathcal{D}', U')$ is a map of rigid data $\mathcal{D} \to \mathcal{D}'$ such that the induced map on tubes takes U into U'. The category of extended rigid data is denoted $\mathcal{ER}(X, \mathcal{V})$. Given a morphism $f : X \to Y$ over κ, an extended rigid datum for f consists of rigid data, $(\mathcal{D}_X, U_X)$ and $(\mathcal{D}_Y, U_Y)$, for X and Y over $\mathcal{V}$ respectively, a map $\bar{f} : \overline{X} \to \overline{Y}$ extending f and a rigid map $U_X \to U_Y$ commuting with the specialization maps to $\overline{X}$ and $\overline{Y}$. The collection of extended rigid data for f forms a category denoted $\mathcal{ER}(X, \mathcal{V})$.

The categories $\mathcal{ER}(X, \mathcal{V})$ and $\mathcal{ER}(f, \mathcal{V})$ are again filtered: In the situation of the proof of lemma 2.3, suppose we were given in addition corresponding strict neighborhoods U_i for $i = 0, 1, 2$. Then one can take $U_3 = (U_1 \times_{U_0} U_2) \cap]\overline{X}_3[_{\mathcal{P}_3}$.

There are obvious functors $\mathcal{RD}(?, ?) \to \mathcal{ER}(?, ?)$ obtained by taking $U =]\overline{X}[_\mathcal{P}$.

Definition 2.15. Let $\mathcal{D} = (\overline{X}, j, \mathcal{P}, U) \in \mathcal{ER}(X, \mathcal{V})$. The rigid complex $\mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)_\mathcal{D}$ is defined by

$$\mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)_\mathcal{D} := \mathbb{R}\Gamma(U, j^\dagger \Omega^\bullet_U)$$

The rigid complex $\mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)$ is given by

$$\mathbb{R}\Gamma'_{\mathrm{rig}}(X/K) := \varinjlim_{\mathcal{D} \in \mathcal{ER}(X,\mathcal{V})} \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)_\mathcal{D}.$$

The complex $\mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)$ is clearly functorial in X by the same argument that proved the functoriality of $\mathbb{R}\Gamma_{\mathrm{rig}}(X/K)$. It follows from [Ber97, 1.2.iv] that for an extended rigid datum $(\mathcal{D}, U)$, with $\mathcal{D} = (\overline{X}, j, \mathcal{P})$, the map induced on rigid complexes by the map $(\mathcal{D}, U) \to (\mathcal{D},]\overline{X}[_\mathcal{P})$ is a quasi-isomorphism. This implies that all maps of extended rigid data induce quasi-isomorphisms on the associated rigid complexes, hence that $\mathbb{R}\Gamma'_{\mathrm{rig}}(X/K) \cong \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)_\mathcal{D}$ for any datum $\mathcal{D}$. There is an obvious natural transformation $\mathbb{R}\Gamma_{\mathrm{rig}} \to \mathbb{R}\Gamma'_{\mathrm{rig}}$ which is a quasi-isomorphism by the result above.

The next step is to define a de Rham complex. This was already done by Huber [Hub95, Chapter 7] so we do not go into all the details. We need to know not only a complex computing de Rham cohomology, but also complexes computing all the filtered parts. Here K can be any field of characteristic 0. Let X be a smooth K-scheme. A de Rham datum for X is an injection $i : X \hookrightarrow Y$ where Y is a smooth and proper K-scheme and $D := Y - X$ is a divisor with normal crossings.

Definition 2.16. To a de Rham datum $\mathcal{D} = (Y, i)$ and to every $k \in \mathbb{Z}_{\geq 0}$ we associate a complex, called the k-th filtered part of the *de Rham complex* of X with respect to the datum $\mathcal{D}$, defined by

$$\operatorname{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_{\mathcal{D}} := \mathbb{R}\Gamma(Y, \Omega_Y^{\geq k}\langle \log D\rangle).$$

The k-th filtered part of the *de Rham complex* of X is defined by

$$\operatorname{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) := \varinjlim_{\mathcal{D}} \operatorname{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)_{\mathcal{D}},$$

where the limit is over all de Rham data $\mathcal{D}$.

We will write $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ for $\operatorname{Fil}^0 \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$. Note that the Fil^k, in spite of their name, are not subcomplexes of $\mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$ but there are natural maps $\operatorname{Fil}^k \mathbb{R}\Gamma_{\mathrm{dR}}(X/K) \to \mathbb{R}\Gamma_{\mathrm{dR}}(X/K)$.

The final ingredient needed for the construction of syntomic cohomology is a comparison between de Rham and rigid cohomology. Let X be a smooth $\mathcal{V}$-scheme with generic fiber X_K and closed fiber X_κ. We will define a functorial map $\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$. We stress that this map is not a quasi-isomorphism in general. The datum required for the definition is a compactification $X \xrightarrow{j} \overline{X}$ together with a de Rham datum for X_K, $i : X_K \hookrightarrow Y$. The compactification j gives rise to an extended rigid datum $(\overline{X}_\kappa, j_\kappa, \hat{\overline{X}}, X_K^{\mathrm{an}})$, where $\hat{\overline{X}}$ is the p-adic completion of $\overline{X}$ and X_K^{an} is the rigid analytic K-space associated with X_K [Ber96, 0.3.3]. It is not so hard to see that indeed X_K^{an} is a strict neighborhood of $]X_\kappa[_{\hat{\overline{X}}}$ inside $]\overline{X}_\kappa[_{\hat{\overline{X}}} = \overline{X}_K^{\mathrm{an}}$. We obtain a map

$$\begin{aligned} \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)_{(i,Y)} &= \mathbb{R}\Gamma(Y, \Omega_Y^\bullet\langle \log(Y - X_K)\rangle) \to \mathbb{R}\Gamma(Y, i_*\Omega_{X_K}^\bullet) \\ &\to \mathbb{R}\Gamma(X_K, \Omega_{X_K}^\bullet) \to \mathbb{R}\Gamma(X_K^{\mathrm{an}}, \Omega_{X_K^{\mathrm{an}}}^\bullet) \\ &\to \mathbb{R}\Gamma(X_K^{\mathrm{an}} j_\kappa^\dagger \Omega_{X_K^{\mathrm{an}}}^\bullet) = \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)_{(\overline{X}_\kappa, j_\kappa, \hat{\overline{X}}, X_K^{\mathrm{an}})}. \end{aligned} \tag{2.2}$$

By taking the limit over all $\overline{X}$ and Y we obtain the required map. Functoriality is evident.

Now it remains to explain why all the constructions can be made on the level of complexes

Proposition 2.17. *There exists a way to choose the complexes below, representing their name sakes in the derived categories of complexes of K or K_0-vector spaces, in such a way that all morphisms in the derived categories sense between them we have used above are in fact represented by maps between these complexes. The complexes are: $\mathbb{R}\Gamma(U, j^\dagger\Omega_U^\bullet)$ and $\mathbb{R}\Gamma(U, \Omega_U^\bullet)$ when U is a strict neighborhood of a tube (the latter required for the case $U = X_K^{\mathrm{an}}$ in (2.2)), $\mathbb{R}\Gamma(X, \Omega_X^\bullet)$ when X is a smooth K-scheme, and $\mathbb{R}\Gamma(Y, \Omega_Y^{\geq k}\langle \log(Y - X)\rangle)$ and $\mathbb{R}\Gamma(Y, i_*\Omega_X^\bullet)$ when $i : X \hookrightarrow Y$ is a de Rham datum for X.*

Proof. The method for constructing these complexes is standard. It uses a construction in [SD72] and was used by Beilinson to construct Zariski sheaves computing Deligne cohomology in [Bei85, 1.6.5]. We only explain how to construct the complexes $\mathbb{R}\Gamma(U, j^\dagger\Omega_U^\bullet)$ in a functorial way with respect to morphisms of extended rigid data and leave the other cases as an exercise.

Consider the category $\mathcal{A}$ whose object are 4-tuples $\mathcal{D} = (X, j : X \to \overline{X}, \mathcal{P}, U)$ where X and $\overline{X}$ are κ-schemes and $(\overline{X}, j, \mathcal{P}, U)$ is an extended rigid datum for X. Maps are the obvious commuting diagrams. Let $\mathcal{B}$ be the category whose objects are pairs $(\mathcal{D}, F)$ where $\mathcal{D} \in \mathcal{A}$ and F is a sheaf on $U = U_{\mathcal{D}}$ in the Grothendieck topology of U. A map $(\mathcal{D}, F) \to (\mathcal{D}', F')$ consists of a map $f : \mathcal{D} \to \mathcal{D}'$ in $\mathcal{A}$ together with a map of sheaves $F' \to f_{U*}F$, where f_U is the map $U_{\mathcal{D}} \to U_{\mathcal{D}'}$ which is part of f. Then $\mathcal{B}$ is, in the terminology of [SD72, 1.2.2], *bifiltered by toposes over* $\mathcal{A}$ (loc. sit., 4.1.0). We can consider the section category $\underline{\Gamma}(\mathcal{B})$ of loc. sit., 1.2.8. Explicitly, an object of $F \in \underline{\Gamma}(\mathcal{B})$ is given by a collection of sheaves $F_{\mathcal{D}}$ on $U_{\mathcal{D}}$, for every $\mathcal{D} \in \mathcal{A}$, together with morphisms of sheaves $f^* : F_{\mathcal{D}'} \to f_{U*}F_{\mathcal{D}}$ for every morphism $f : \mathcal{D} \to \mathcal{D}'$ in $\mathcal{A}$ such that one has

$$(f \circ g)^* = g^* \circ f^*, \quad \mathrm{id}^* = \mathrm{id} \quad . \tag{2.3}$$

By loc. sit., 1.2.12, $\underline{\Gamma}(\mathcal{B})$ is a topos. By loc. sit., 1.3.10 there is a collection $I_{\mathcal{B}}$ of abelian objects of $\underline{\Gamma}(\mathcal{B})$ such that the following two properties hold:

- Any abelian $F \in \underline{\Gamma}(\mathcal{B})$ injects into $I \in I_{\mathcal{B}}$.
- For $I \in I_{\mathcal{B}}$ and for any $\mathcal{D} \in \mathcal{A}$, the sheaf $I_{\mathcal{D}}$ on $U_{\mathcal{D}}$ is flasque.

The association $\mathcal{D} \mapsto j_{\mathcal{D}}^\dagger \Omega_{U_{\mathcal{D}}}^\bullet$, together with the natural maps $j_{\mathcal{D}'}^\dagger \Omega_{U_{\mathcal{D}'}}^\bullet \to f_{U*} j_{\mathcal{D}}^\dagger \Omega_{U_{\mathcal{D}}}^\bullet$ defines a complex of abelian objects of $\underline{\Gamma}(\mathcal{B})$. We choose a resolution of this complex by a complex of objects of $I_{\mathcal{B}}$. This gives us for each $\mathcal{D} \in \mathcal{A}$ a complex of flasque sheaves $I_{\mathcal{D}}^\bullet$ on $U_{\mathcal{D}}$ together with a morphisms of complexes of sheaves $f^* : I_{\mathcal{D}'}^\bullet \to f_{U*} I_{\mathcal{D}}^\bullet$ for every morphism $f : \mathcal{D} \to \mathcal{D}'$ in $\mathcal{A}$ satisfying (2.3). Taking global sections on $U_{\mathcal{D}}$ now gives a functor $\mathcal{D} \to \Gamma(U_{\mathcal{D}}, I_{\mathcal{D}}^\bullet)$ into complexes of vector spaces. This is enough to construct functorially $\mathbb{R}\Gamma'_{\mathrm{rig}}$. □

3. Syntomic cohomology and product structures

In this section we will define syntomic cohomology and state some of its fundamental properties, including the product structure. We begin with a bit of homological algebra.

Suppose we are given complexes $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ with maps $f : X^\bullet \to Z^\bullet$ and $g : Y^\bullet \to Z^\bullet$. Then one can form the naive fibered product $X^\bullet \times_{Z^\bullet} Y^\bullet$ whose n-th component is $X^n \times_{Z^n} Y^n$. It is of course equal to the kernel of $f-g : X^\bullet \oplus Y^\bullet \to Z^\bullet$. Therefore, one should prefer to use instead the slightly different construction, called the *quasi-fibered product*, $X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet := \mathrm{Cone}(f-g)[-1]$. We have the well known

Lemma 3.1. *In the situation above, if the map $f-g$ is surjective, then the two construction are quasi-isomorphic via the map*

$$(x, y) \to (x \oplus y, 0). \tag{3.1}$$

It will be convenient to use both construction in what follows.

Notice that we have canonical maps $Z^\bullet[-1] \xrightarrow{i} X^\bullet \tilde{\times}_{Z^\bullet} Y^\bullet \xrightarrow{p} X^\bullet \oplus Y^\bullet$ coming from the cone construction. Let us write p_A and p_B for the composition of p with the first and second projection respectively. The following construction of the cup product is a variant of one of Nizioł [Niz93], which is itself a variant of a construction of Beilinson. Alternatively, it is a special case of the construction of [Bei86, 1.11]:

Lemma 3.2. *Suppose We are given complexes $X_i^\bullet$, $Y_i^\bullet$, $Z_i^\bullet$ and maps f_i, g_i as above for $i = 1, 2, 3$, and that we are given maps of complexes $\cup : X_1^\bullet \otimes X_2^\bullet \to X_3^\bullet$, and similarly for Y and Z, which are (strictly) compatible with the maps f_i and g_i in the obvious sense. Then,*

1. *There exist a map (bottom horizontal), making the following diagram commute, where the top horizontal map is induced by the maps $\cup$.*
$$\begin{array}{ccc} (X_1^\bullet \times_{Z_1^\bullet} Y_1^\bullet) \otimes (X_2^\bullet \times_{Z_2^\bullet} Y_2^\bullet) & \xrightarrow{\cup} & X_3^\bullet \times_{Z_3^\bullet} Y_3^\bullet \\ \downarrow & & \downarrow \\ (X_1^\bullet \tilde{\times}_{Z_1^\bullet} Y_1^\bullet) \otimes (X_2^\bullet \tilde{\times}_{Z_2^\bullet} Y_2^\bullet) & \xrightarrow{\cup} & X_3^\bullet \tilde{\times}_{Z_3^\bullet} Y_3^\bullet . \end{array}$$
2. *On homology one has the following projection formula for $z \in H^*(Z_1^\bullet)$ and $w \in H^*(X_2^\bullet \tilde{\times}_{Z_2^\bullet} Y_2^\bullet)$:*
$$((i_1)_*(z)) \cup w = (i_3)_* \left[x \cup (g_2)_*(p_{B_2})_* w \right] .$$

Proof. (Compare with [Niz93, Prop. 3.1] or [Bei86, Lemma 1.11]) One chooses a parameter γ and defines the cup product by the formula

$$\begin{aligned} (x_1, y_1, z_1) \cup (x_2, y_2, z_2) = \big(& x_1 \cup x_2, y_1 \cup y_2, \\ & z_1 \cup (\gamma f_2(x_2) + (1-\gamma) g_2(y_2)) \\ & + (-1)^{\deg x_1} ((1-\gamma) f_1(x_1) + \gamma g_1(y_1)) \cup z_2 \big) \quad . \end{aligned} \tag{3.2}$$

All of these products are known to be homotopic for different values of γ. Checking the required properties is straightforward from this formula. For the second part one specifies $\gamma = 0$. ∎

We are now ready to define syntomic cohomology. Let X be a smooth $\mathcal{V}$-scheme. By the constructions of section 2 we have, for any $n \in \mathbb{Z}_{\geq 0}$, the following diagram of complexes and maps between them

$$\begin{aligned} \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa / K) \\ \leftarrow \mathbb{R}\Gamma_{\mathrm{dR}}(X_K / K) \leftarrow \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K / K). \end{aligned} \tag{3.3}$$

We also have a σ-linear map $\phi : \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K_0)$ and both diagram and map are functorial in X.

Definition 3.3. The *syntomic complex* of X twisted by n is defined to be

$$\mathbb{R}\Gamma_{\mathrm{syn}}(X, n) := \mathrm{Cone}\left(1 - \frac{\phi}{p^n}\right)[-1] \tilde{\times}_{\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa / K)} \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K / K),$$

where the two maps defining the fibered product are

$$\begin{aligned} \mathrm{Cone}\left(1 - \frac{\phi}{p^n} : \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K_0)\right)[-1] \\ \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K_0) \to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa / K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa / K) \end{aligned}$$

and the map induced by the left pointing arrows of (3.3). The i-th homology of $\mathbb{R}\Gamma_{\mathrm{syn}}(X, n)$ will be denoted $H^i_{\mathrm{syn}}(X, n)$.

The above construction is evidently functorial in X. We can therefore define $\mathbb{R}\Gamma_{\mathrm{syn}}$ for simplicial schemes as in definition 2.8. We have the analogue of proposition 2.10:

Proposition 3.4. *Let X be a smooth $\mathcal{V}$-scheme and let $\mathcal{U}_\bullet \to X$ be the covering associated to a finite Čech covering of X. Then the canonical map $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{syn}}(\mathcal{U}_\bullet, n)$ is a quasi-isomorphism for any $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Because $\mathbb{R}\Gamma_{\mathrm{syn}}$ is defined as an iterated cone, it is enough to check the statement of the proposition on each of the components of the cone. But for the de Rham components it is well known and for the rigid components it was proved in proposition 2.10. $\square$

We proceed to show some of the fundamental properties of syntomic cohomology.

Proposition 3.5. *There is a long exact sequence,*

(3.4)

$$\begin{aligned}\cdots \to H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) &\xrightarrow{①} H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \oplus H^{i-1}_{\mathrm{rig}}(X_K/K)\\ \to H^i_{\mathrm{syn}}(X,n)&\\ \to H^i_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) &\xrightarrow{②} H^i_{\mathrm{rig}}(X_\kappa/K_0) \oplus H^i_{\mathrm{rig}}(X_K/K) \to \cdots,\end{aligned}$$

where the maps ① and ② are given in the appropriate degrees by

$$(x,y) \mapsto \left(\left(1-\frac{\phi}{p^n}\right)x, x-y\right). \tag{3.5}$$

Here, for the second component we have identified both x and y with their images in $H^i_{\mathrm{rig}}(X_K/K)$. In particular, if K is finite over K_0, then $H^i_{\mathrm{syn}}(X,n)$ is a finite dimensional K_0-vector space for every i and n.

Proof. By writing explicitly the quasi-fibered product in term of cones, one finds

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \cong \mathrm{Cone}(&\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\\ &\to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K))[-1],\end{aligned}$$

where the map defining the cone is given by (3.5) (the reader should compare at this point the construction of [Niz97, 2.1]). This immediately gives the result. $\square$

Remark 3.6. Let us consider the special case where X is a smooth K-scheme considered as a $\mathcal{V}$-scheme. In this case we have $X_\kappa = \emptyset$, so $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/?) = 0$ with $? = K$ or K_0 and the same is true with $\mathbb{R}\Gamma'_{\mathrm{rig}}$. The long exact sequence (3.4) shows that

$$H^i_{\mathrm{syn}}(X,n) \cong \mathrm{Fil}^n H^i_{\mathrm{dR}}(X/K).$$

This is perhaps to be expected since this is the "absolute" cohomology for varieties over a field.

Definition 3.7. The cup product map on syntomic cohomology,

$$\cup : H^i_{\mathrm{syn}}(X,n) \times H^j_{\mathrm{syn}}(X,m) \to H^{i+j}_{\mathrm{syn}}(X,n+m),$$

is constructed as follows: By lemma 3.2 it is enough to construct a product $\mathrm{Cone}(1-\phi_n)\times\mathrm{Cone}(1-\phi_m)\to\mathrm{Cone}(1-\phi_{n+m})$, with $\phi_n=\phi/p^n$. This is achieved by the formula, similar to (3.2),

$$\begin{aligned}(x_1,z_1)\cup(x_2,z_2)=\big(&x_1\cup x_2,\\ &z_1\cup(\gamma x_2+(1-\gamma)\phi_m(x_2))\\ &+(-1)^{\deg x_1}((1-\gamma)x_1+\gamma\phi_n(x_1))\cup z_2\big)\quad.\end{aligned}\tag{3.6}$$

This definition is compatible with the definitions given by Nizioł, Kato, Gros and many others.

4. Construction of syntomic regulators

In this section we construct syntomic Chern classes,

$$c_j^p : K_p(X)\to H_{\mathrm{syn}}^{2j-p}(X,j).$$

The method follows mostly Huber [Hub95, Chapter 18] with some input from Gros [Gro90] and Deligne [Del74].

The main step in the construction is to repeat the computation of the de Rham cohomology of $\mathbf{B}_\bullet\mathrm{GL_n}$ by Deligne [Gro90, Chapter II] for rigid cohomology. We briefly recall the setup from loc. sit., but using the notation of Deligne in [Del74, 6].

We will work simultaneously over any of the bases κ, $\mathcal{V}$, $\mathcal{V}_0$ K or K_0, making the needed adjustments. If G is an algebraic group (over any of the bases above) acting on a scheme X we let $[X/G]_\bullet$ be the simplicial scheme such that $[X/G]_n=(G^{\Delta_n}\times X)/G$ where G acts by $g\cdot(g_0,\dots,g_n,x)=(g_0g^{-1},\dots,g_ng^{-1},gx)$ and the face and degeneracy maps are the obvious ones [Del74, 6.1.2]. Note that the quotients are well defined and in fact there is an isomorphism $G^n\times X\xrightarrow{\sim}[X/G]_n$ given by (for example) $(g_1,\dots g_n,x)\mapsto(1,g_1,\dots g_n,x)$.

Lemma 4.1. *Let X be a principal G-bundle over $S=X/G$. Then the map $[X/G]_\bullet\to S$ induces an isomorphism on rigid cohomology.*

Proof. (sketch). If we knew how to write rigid cohomology as a sheaf cohomology this would follow from [Del74, 6.1.2.2]. We need to check that what we know about rigid cohomology is sufficient for a proof. Let $X_\bullet=\mathrm{cosq}(X\to S)$. There is a canonical isomorphism of simplicial schemes over S,

$$X_\bullet\cong[X/G]_\bullet$$

[Del74, 6.1.2.a]. If there is a section $S\to X$, then it extends to a section $s:S\to X_\bullet$ to the canonical map $\pi:X_\bullet\to S$. It is well known that the map $s\circ\pi$ is homotopic to the identity map of $X_\bullet$ and this homotopy induces a homotopy on the rigid complexes showing the result in this case. In the general case we have a finite covering, $U=\coprod U_i$, of S such that the restriction of X to each U_i has a section. Let $\mathcal{U}_\bullet=\mathrm{cosq}(U\to S)$. An application of the spectral sequence 2.9, proposition 2.10, and the special case of a map with a section discussed above now shows that the cohomology of the bisimplicial set

$$(\mathrm{cosq}(X\times_S\mathcal{U}_n\to\mathcal{U}_n))_{?,n}=(\mathrm{cosq}(X_m\times_S U\to X_m))_{m,?}$$

is isomorphic to the cohomology of $\mathcal{U}_\bullet$, hence of S, on the one hand, and to the cohomology of $X_\bullet$ on the other hand. □

Fix $N \geq n$ in $\mathbb{Z}_{\geq 0}$. Let $E = \mathbb{G}_a^{\oplus n}$ and $F = \mathbb{G}_a^{\oplus N}$ be two vector group schemes and let $\underline{\mathrm{Hom}}(E, F)$ be the corresponding scheme of homomorphisms. There is a filtration of $\underline{\mathrm{Hom}}(E, F)$ by open subschemes

$$\underline{\mathrm{Hom}}(E, F) = U_n \supset U_{n-1} \supset \cdots \supset U_0,$$

where U_l is defined by the invertability of at least one $n - l$ minor.

Lemma 4.2. *The scheme $U_l - U_{l-1}$ is a smooth subscheme of $\underline{\mathrm{Hom}}(E, F)$ of codimension $l(l - n + N)$.*

Proof. This is proved in [Gro90, II.2.4] for schemes over $\mathcal{V}_0$ but the proof is the same in any of the other cases. $\square$

The group $G = \mathrm{GL}_n$ acts on $\underline{\mathrm{Hom}}(E, F)$ in the obvious manner, preserving the filtration by the U_i. The scheme U_0 is the so called Stiefel variety of n-frames on F and is denoted by $\underline{\mathrm{Stief}}(E, F)$. We have

$$\underline{\mathrm{Stief}}(E, F)/G \cong \underline{\mathrm{Grass}}_n(F) \quad , \tag{4.1}$$

where $\underline{\mathrm{Grass}}_n(F)$ is the grassmanian of n-dimensional subspaces of F.

Proposition 4.3. *The canonical map*

$$H^*_{\mathrm{rig}}([\underline{\mathrm{Hom}}(E, F)/G]_\bullet/K) \to H^*_{\mathrm{rig}}([\underline{\mathrm{Stief}}(E, F)/G]_\bullet/K)$$

is an isomorphism in degrees $\leq 2(N - n)$.

Proof. (Compare [Gro90, Corollaire II.2.8]). It is enough to show the same for the map induced on rigid cohomology by each of the inclusions $[U_{l-1}/G]_\bullet \to [U_l/G]_\bullet$. By lemma 4.2 we see that on the n-th component, $[U_l/G]_n - [U_{l-1}/G]_n$ is a closed subscheme of $[U_l/G]_n$ of codimension $l(l-n+N) \geq N-n+1$. By purity for rigid cohomology [Ber97, Corollaire 5.7] the map $H^i_{\mathrm{rig}}([U_l/G]_n/K) \to H^i_{\mathrm{rig}}([U_{l-1}/G]_n/K)$ is an isomorphism if $i \leq 2(N - n)$. The result now follows from the spectral sequence 2.9. $\square$

Proposition 4.4. *There are canonical classes $x_i \in \mathrm{Fil}^i H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K)$ such that we have isomorphisms*

$$K[x_1, \dots, x_n] \xrightarrow{\sim} H^*_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \xrightarrow{\sim} H^*_{\mathrm{rig}}(\mathbf{B}_\bullet \mathrm{GL_n}/K). \tag{4.2}$$

If $K = K_0$ and we identify the classes x_i with their images in H_{rig}, then we have $\phi(x_i) = p^i x_i$.

Proof. Let $*$ be the one point space. Then $\mathbf{B}_\bullet \mathrm{GL_n} = [*/G]_\bullet$ [Del74, 6.1.3]. We have a G-equivariant diagram,

$$\begin{tikzcd}
\underline{\mathrm{Stief}}(E, F) \arrow[rr] \arrow[dr] & & \underline{\mathrm{Hom}}(E, F) \arrow[dl, "\pi"'] \\
& * \arrow[ur, dotted, bend right, "0"'] &
\end{tikzcd}$$

where 0 denotes the 0 section to π. It induces a corresponding diagram of cohomologies,

$$\begin{tikzcd}
H^i_{\mathrm{rig}}([\underline{\mathrm{Stief}}(E,F)/G]_\bullet/K) & & H^i_{\mathrm{rig}}([\underline{\mathrm{Hom}}(E,F)/G]_\bullet/K) \arrow[ll] \arrow[dl, dotted, bend left] \\
& H^i_{\mathrm{rig}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \arrow[ul] \arrow[ur, "\pi^*"'] &
\end{tikzcd}$$

An easy diagram chase using proposition 4.3 shows that the left diagonal map is injective for $i \leq 2(N-n)$. A similar argument shows the same for de Rham cohomology. By lemma 4.1 and (4.1) it now follows that the two horizontal maps in the commutative diagram

$$\begin{tikzcd}
H^i_{\mathrm{rig}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \arrow[r] & H^i_{\mathrm{rig}}(\underline{\mathrm{Grass}}_n(F)/K) \\
H^i_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K) \arrow[r, "\alpha"] \arrow[u] & H^i_{\mathrm{dR}}(\underline{\mathrm{Grass}}_n(F)/K) \arrow[u, "\sim"']
\end{tikzcd} \tag{4.3}$$

are injective for $i \leq 2(N-n)$. The map on the right is an isomorphism since $\underline{\mathrm{Grass}}_n(F)$ is proper.

In de Rham cohomology we have a good theory of characteristic classes. Let $x_i \in \mathrm{Fil}^i H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K)$ be the i-th Chern class of the universal bundle. Then $\alpha(x_i)$ are the Chern classes of the universal vector bundle over $\underline{\mathrm{Grass}}_n(F)$ and it is known that these generate the cohomology ring of $\underline{\mathrm{Grass}}_n(F)$. It follows that α is surjective, hence that if $i \leq 2(N-n)$ all maps in diagram (4.3) are isomorphisms. Varying N we find the isomorphisms (4.2). It now follows that the properties of the classes x_i can be tested in the cohomology of $\underline{\mathrm{Grass}}_n(F)$, where they are well known: As $\underline{\mathrm{Grass}}_n(F)$ is proper we have an isomorphism

$$H^{2i}_{\mathrm{dR}}(\underline{\mathrm{Grass}}_n(F)/K_0) \cong H^{2i}_{\mathrm{cr}}(\underline{\mathrm{Grass}}_n(F)/\mathcal{V}_0) \otimes K_0,$$

under which x_i correspond to the crystalline Chern classes of the universal bundle and therefore have the right behavior under Frobenius. $\square$

We can now define Chern classes in syntomic cohomology. From proposition 4.4 it follows that $\mathbf{B}_\bullet \mathrm{GL_n}$ has cohomology only in even dimensions. Using the long exact sequence (3.4) we easily obtain an isomorphism

$$H^{2i}_{\mathrm{syn}}(\mathbf{B}_\bullet \mathrm{GL_n} \otimes \mathcal{V}, i) \cong \{x \in \mathrm{Fil}^i H^{2i}_{\mathrm{dR}}(\mathbf{B}_\bullet \mathrm{GL_n}/K_0), \phi(x) = p^i x\}.$$

In particular, we see that the classes x_i of proposition 4.4 define classes, denoted C^n_i, in $H^{2i}_{\mathrm{syn}}(\mathbf{B}_\bullet \mathrm{GL_n} \otimes \mathcal{V}, i)$. Considering the usual inductive system of $\mathbf{B}_\bullet \mathrm{GL_n}$-s, obtained by the inclusions "in the upper left corner" $\mathrm{GL}_n \to \mathrm{GL}_{n+1}$, we see that the C^n_i are compatible under the induced maps on cohomology because the de Rham universal classes are known to do so. We thus obtained cohomology classes C_i in the cohomology of the ind-scheme $\mathbf{B}_\bullet \mathrm{GL}$ which we call the *universal syntomic Chern classes.*

Theorem 4.5. *Let X be a smooth $\mathcal{V}$-scheme. There exist functorial Chern classes*

$$c^p_j : K_p(X) \to H^{2j-p}_{\mathrm{syn}}(X, j),$$

such that their composition with the map $H^{2j-p}_{\mathrm{syn}}(X, j) \to \mathrm{Fil}^j H^{2j-p}_{\mathrm{dR}}(X_K/K)$ obtained from the sequence(3.4) *gives the usual Chern classes in de Rham cohomology.*

Proof. We follow Huber's treatment in [Hub95, Chapter 18]. By [Hub95, Proposition 18.1.5] (whose proof is also valid in our case) we have an isomorphism

$$\varinjlim_{U_\bullet} \pi_p \operatorname{Tot}(\mathbb{Z} \times \mathbb{Z}_\infty \mathbf{B}_\bullet \mathrm{GL}(U_\bullet)) \to K_p(X),$$

where the direct limit is over all finite affine Čech coverings $U_\bullet$ of X. By [Hub95, Proposition 18.1.7 b] there are induced maps

$$K_p(X) \to \varinjlim_{U_\bullet} \pi_p \operatorname{Tot}(\mathbb{Z}_\infty \mathbf{B}_\bullet \mathrm{GL}(U_\bullet)), \qquad K_0(X) \to \mathbb{Z}. \tag{4.4}$$

For simplicial schemes $U_\bullet$ and $Y_\bullet$, let $B(U_\bullet, Y_\bullet)$ be the simplicial cosimplicial abelian group which is the $\mathbb{Q}$-vector space generated by $\operatorname{Hom}(U_n, Y_m)$ in degree (m, n) and let $A(U_\bullet, Y_\bullet)$ be the associated complex [Hub95, Definition 18.2.1]. Summation of pullback maps give a map of simplicial cosimplicial groups,

$$B(U_\bullet, Y_\bullet) \to [\operatorname{Hom}(\mathbb{R}\Gamma_{\mathrm{syn}}(Y_m, j), \mathbb{R}\Gamma_{\mathrm{syn}}(U_n, j)]_{m,n} \quad ,$$

where Hom here means in the category of complexes (this is why we insisted on defining the syntomic cohomology on the level of complexes). By taking the associated complexes and then the total complexes we obtain a map

$$A(U_\bullet, Y_\bullet) \to \mathbb{R} \operatorname{Hom}(\mathbb{R}\Gamma_{\mathrm{syn}}(Y_\bullet, j), \mathbb{R}\Gamma_{\mathrm{syn}}(U_\bullet, j)) \quad . \tag{4.5}$$

In the special case that $Y_\bullet = \mathbf{B}_\bullet \mathrm{GL}$, we have by [Hub95, Lemma 18.2.4] a map

$$\pi_p \operatorname{Tot}(\mathbb{Z}_\infty \mathbf{B}_\bullet \mathrm{GL}(U_\bullet)) \to H^{-p}(A(U_\bullet, \mathbf{B}_\bullet \mathrm{GL})) \quad .$$

Composing this with the map induced on homology by (4.5) and applying to the universal class $C_j \in H^{2j}(\mathbb{R}\Gamma_{\mathrm{syn}}(\mathbf{B}_\bullet \mathrm{GL}, j))$ we get a map

$$\pi_p \operatorname{Tot}(\mathbb{Z}_\infty \mathbf{B}_\bullet \mathrm{GL}(U_\bullet)) \to H^{2j-p}_{\mathrm{syn}}(U_\bullet, j).$$

If $U_\bullet$ is as in (4.4), then $H^{2j-p}_{\mathrm{syn}}(U_\bullet, j) \cong H^{2j-p}_{\mathrm{syn}}(X, j)$ by proposition 3.4. This completes the construction. The result about the composition with the projection to de Rham cohomology follows from the universal case and functoriality. $\square$

Remark 4.6. As in [Hub95, Definition 18.2.6], the same construction yields Chern classes for split simplicial smooth $\mathcal{V}$-schemes of finite combinatorial dimension. In particular, we obtain Chern classes in relative cohomology using simplicial cones.

Definition 4.7. The Chern character $ch : K_i(X) \to \bigoplus_j H^{2j-i}_{\mathrm{syn}}(X, j)$ is given by

$$ch = \sum_{j \geq 1} -\frac{(-1)^{j-1}}{(j-1)!} c^i_j \quad (+\,\mathrm{Rank} \quad \text{if} \quad i = j = 0).$$

Proposition 4.8. *The Chern character is multiplicative.*

Proof. This reduces as usual to properties of the universal Chern classes. Because the syntomic cohomology is the same as de Rham cohomology for $\mathbf{B}_\bullet \mathrm{GL_n}$, there is nothing to prove. $\square$

5. Modified syntomic cohomology

In this section we define a certain modification of the rigid syntomic cohomology of section 3. The difference is that we replace the semi-linear Frobenius by a linear Frobenius. This makes the theory easier to compute. For the purpose of computing regulators in higher K-theory the modified theory is as good as the original one.

In this section we need the additional assumption that $\kappa \subset \bar{F}_p$. The following notion is due to Coleman.

Definition 5.1. Let X be a κ-scheme. A *Frobenius endomorphism*, $\varphi : X \to X$, of degree $q = p^r$ is any κ-endomorphism of X obtained in the following way: Let X' be an F_q-scheme and let $\alpha : X \xrightarrow{\sim} X' \otimes_{F_q} \kappa$ be a κ-isomorphism. Then $\varphi = \alpha^{-1} \circ (\mathrm{Fr}^r \otimes \mathrm{id}_\kappa) \circ \alpha$.

It is clear that if φ is a Frobenius endomorphism of degree q then φ^k is a Frobenius endomorphism of degree q^k.

Definition 5.2. The category of Frobenius endomorphisms of X is the category whose objects are Frobenius endomorphisms $\varphi : X \to X$. There is a unique morphism between φ and φ^k for any $k \geq 1$.

Lemma 5.3. *The category of Frobenius endomorphisms of X is filtered.*

Proof. It is not hard to see that sufficiently high powers of any two Frobenius endomorphisms become identical. $\square$

Fix an integer n. We associate to each Frobenius endomorphism a certain complex, in such a way that we get a functor on the category of all Frobenius endomorphisms. To φ of degree q we associate the complex

$$\mathrm{Cone}\left(1 - \frac{\varphi^*}{q^n} : \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)\right)[-1].$$

To the morphism $\varphi \to \varphi^m$ we associate the map of cones induced by the commutative diagram

$$\begin{array}{ccc} \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K) & \xrightarrow{1-(\varphi^*/q^n)} & \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K) \\ {\scriptstyle =}\big\downarrow & & \big\downarrow{\scriptstyle \sum_{s=0}^{m-1}(\varphi^*/q^n)^s} \\ \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K) & \xrightarrow{1-(\varphi^*/q^n)^n} & \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K). \end{array} \tag{5.1}$$

Definition 5.4. The *modified syntomic complex* associated with a Frobenius endomorphism φ is the complex

$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi := \mathrm{Cone}\left(1 - \frac{\varphi^*}{q^n}\right)[-1]\tilde{\times}_{\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)} \mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K),$$

where q is the degree of φ and the cone is the one discussed above. The *modified syntomic complex* of X is

$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) = \varinjlim_{\varphi} \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi,$$

where the direct limit is over the category of all Frobenius endomorphisms and the connecting maps are the ones defined above. The homology of the modified syntomic complex is called *modified syntomic cohomology* and denoted by $H^i_{\mathrm{ms}}(X,n)$.

Lemma 5.5. *The modified syntomic complexes, and hence the modified syntomic cohomologies, are functorial.*

Proof. One need only observe that any morphism of varieties over κ is already defined over some finite field, which implies that for any morphism $f : X \to Y$ and for a cofinal collection of Frobenius endomorphisms $\varphi : Y_\kappa \to Y_\kappa$ there is a Frobenius endomorphism $\varphi' : X_\kappa \to X_\kappa$ making the obvious diagram commute. $\square$

Most of the basic properties of the modified syntomic cohomology are concentrated in the following proposition.

Proposition 5.6. 1. *There is a canonical quasi-isomorphism,*

(5.2)
$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \cong \varinjlim_{\varphi} \mathrm{Cone}\left(1 - \frac{\varphi^*}{q^n} : \mathrm{Fil}^n\, \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)\right)[-1],$$

where the limit is over all Frobenius endomorphisms φ, the notation $1-\varphi^*/q^n$ stands for this map composed with the map $\mathbb{R}\Gamma_{\mathrm{dR}} \to \mathbb{R}\Gamma'_{\mathrm{rig}}$ and the transition maps are constructed using a diagram analogous to (5.1). Furthermore, if φ is any fixed Frobenius endomorphism of degree q, then we also have the quasi-isomorphism

(5.3)
$$\mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \cong \varinjlim_{k} \mathrm{Cone}\left(1 - \left(\frac{\varphi^*}{q^n}\right)^k : \mathrm{Fil}^n\, \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X/K)\right)[-1].$$

2. *If κ is a finite field, then there is a canonical and functorial map*
$$\Xi : \mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{ms}}(X,n) \quad .$$
3. *There are canonical and functorial maps*

(5.4)
$$\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)[-1] \to \mathbb{R}\Gamma_{\mathrm{syn}}(X,n), \qquad \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)[-1] \to \mathbb{R}\Gamma_{\mathrm{ms}}(X,n).$$

When κ is a finite field these maps are compatible with the map Ξ. These maps induce isomorphisms,

(5.5)
$$H^i_{\mathrm{ms}}(X,n) \cong H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\,\mathrm{Fil}^n\, H^{i-1}_{\mathrm{dR}}(X_K/K) \quad ,$$

and, if κ is finite,

(5.6)
$$H^i_{\mathrm{syn}}(X,n) \cong H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\,\mathrm{Fil}^n\, H^{i-1}_{\mathrm{dR}}(X_K/K) \quad ,$$

(at least) in the following two cases:
- *X is proper over $\mathcal{V}$ and $2n \neq i, i-1, i-2$,*
- *X is affine and $n \geq i > \mathrm{reldim}\, X$.*

In particular, if in either of these cases κ is a finite field, then Ξ induces an isomorphism on degree i cohomology.

4. *Suppose $\mathcal{V}'$ is a finite extension of $\mathcal{V}$ with field of fractions K' and let $X' = X\otimes_{\mathcal{V}}\mathcal{V}'$. Then there exists a canonical base change quasi-isomorphism $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)\otimes_K K' \to \mathbb{R}\Gamma_{\mathrm{ms}}(X',n)$.*
5. *There are cup products in modified syntomic cohomology compatible with the products in syntomic cohomology under the map Ξ and also compatible with base change.*

Proof. 1. Let φ be a Frobenius endomorphism of X_κ. In the definition 5.4 of $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi$ we may replace the quasi-fibered product $\tilde{\times}$ by an ordinary product because the cone on the left hand side surjects on $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$. The resulting complex is easily seen to be isomorphic to the level φ complex of (5.2). The second part of the assertion follows because for a fixed φ the collection of powers φ^k is cofinal in the category of Frobenius endomorphisms.
2. We first construct a map $\mathrm{Cone}(1-\phi/p^n : \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \circlearrowleft) \to \mathrm{Cone}(1-\varphi^*/q^n : \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \circlearrowleft)$ for some Frobenius endomorphism φ. Suppose κ is a finite field with $q=p^r$ elements. Then $\varphi = \mathrm{Fr}^r$ is a Frobenius endomorphism of X_κ and by lemma 2.13 we have $\phi^r = \varphi^*$ on $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0)$. It follows that we can define the required map by using a diagram similar to (5.1). This map can then be composed with the extension of scalars map and the canonical map between $\mathbb{R}\Gamma_{\mathrm{rig}}$ and $\mathbb{R}\Gamma'_{\mathrm{rig}}$ to give a map

$$\mathrm{Cone}(1-\phi/p^n : \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \circlearrowleft) \to \mathrm{Cone}(1-\varphi^*/q^n : \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \circlearrowleft) \quad .$$

By the construction of the (modified) syntomic complexes we now obtain a map $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma_{\mathrm{ms}}(X,n)_\varphi$ by taking the identity maps on the other components of the quasi-fibered product. This map we may compose with the map to the limit on all Frobenius endomorphisms to complete the construction. For schemes over κ our particular φ commutes with all maps and this easily gives functoriality.
3. The maps (5.4) are evident from the definition of the (modified) syntomic complexes, as is the compatibility with the map Ξ because we have taken the identity map on $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$ when defining it. We show that these maps induce isomorphisms on cohomology in the stated cases for syntomic cohomology, the proof for modified cohomology being essentially the same. We abbreviate Cone for $\mathrm{Cone}(1-\phi/p^n)[-1]$. From the construction of syntomic cohomology as a quasi-fibered product, which is again a cone, we get the following long exact sequence.

$$\begin{aligned}\cdots \to H^{i-1}(\mathrm{Cone}) \oplus \mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \to H^{i-1}_{\mathrm{rig}}(X_\kappa/K) \to H^i_{\mathrm{syn}}(X,n)\\ \to H^i(\mathrm{Cone}) \oplus \mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) \to H^i_{\mathrm{rig}}(X_\kappa/K) \to \cdots .\end{aligned}$$

It follows that the map $H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \to H^i_{\mathrm{syn}}(X,n)$ is an isomorphism if $H^{i-1}(\mathrm{Cone}) = H^i(\mathrm{Cone}) = 0$ and the map $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) \to H^i_{\mathrm{rig}}(X_\kappa/K)$ is an injection. This last requirement holds in the cases considered because in the proper case $H^i_{\mathrm{dR}}(X_K/K) \cong H^i_{\mathrm{rig}}(X_\kappa/K)$ and when $n > \mathrm{reldim}\, X$ we have $\mathrm{Fil}^n H^i_{\mathrm{dR}}(X_K/K) = 0$. The long exact sequence for the cohomology of Cone,

$$\begin{aligned}\cdots \to H^{i-2}_{\mathrm{rig}}(X_\kappa/K_0) &\xrightarrow{1-\phi/p^n} H^{i-2}_{\mathrm{rig}}(X_\kappa/K_0) \to H^{i-1}(\mathrm{Cone})\\ \to H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) &\xrightarrow{1-\phi/p^n} H^{i-1}_{\mathrm{rig}}(X_\kappa/K_0) \to H^{i-1}(\mathrm{Cone})\\ \to H^{i}_{\mathrm{rig}}(X_\kappa/K_0) &\xrightarrow{1-\phi/p^n} H^{i}_{\mathrm{rig}}(X_\kappa/K_0) \to \cdots \quad ,\end{aligned}$$

shows that the i-th and $i-1$-th cohomologies of Cone vanish when $1-\phi/p^n$ is an isomorphism on $H_{\mathrm{rig}}(X_\kappa/K_0)$ in degrees i, $i-1$ and $i-2$. This now follows from the theory of weights. By [ClS98] and [Chi97] the K_0-linear Frobenius, which is a power of ϕ, has weight j when acting on $H^j_{\mathrm{rig}}(X_\kappa/K_0)$ when X is proper and has mixed weights between j and $2j$ in general. In the proper case it follows that if $2n \neq j$ for $j = i-2$, $i-1$ and i, then the operator ϕ/p^n has no fixed vector on $H^j_{\mathrm{rig}}(X_\kappa/K_0)$ because some power of it does not. It follows that $1-\phi/p^n$ is

injective, hence bijective, on the degree $i-2$, $i-1$ and i cohomologies. In the second case this is no longer true a-priori for $j=i$ but $H^i_{\mathrm{rig}}=0$ because X is affine and $i>\operatorname{reldim} X$.
4. Apply base change (proposition 2.11 for rigid cohomology) in each component.
5. The construction of the cup product is almost identical to the one we did for syntomic cohomology. The cup product on $\mathrm{Cone}(1-\varphi^*/q^n : \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \circlearrowleft)$ is given by the formula (3.6) with $\phi_m=\varphi^*/q^m$. One then needs to check that these products are compatible up to homotopy under the transition maps. This can be done by a direct laborious computation. A much more conceptual and general of understanding this is given in [Bes97]. This type of compatibility also implies that the product is compatible with the map Ξ. Compatibility with base change is clear. □

Remark 5.7. 1. We expect the base change isomorphism of proposition 5.6.4 to exist for infinite extensions as well, at least on the level of cohomology.

2. Using the model (5.3) for modified syntomic cohomology it is easy to see that the cup product is given in level k by the formula (3.6) with ϕ_m being $(\varphi^*/q^n)^k$ composed with $\mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\to\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$.
3. Suppose $K=K_0$. One can extend the map ϕ to $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K_0)$. An argument similar to the proof of proposition 5.6.1 shows that

$$\mathbb{R}\Gamma_{\mathrm{syn}}(X,n)\cong \mathrm{Cone}(\mathrm{Fil}^n\,\mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)\xrightarrow{1-\phi/p^n}\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)) \quad .$$

This gives rise to a long exact sequence

$$\begin{aligned}\cdots\to \mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K)&\xrightarrow{1-\phi/p^n} H^{i-1}_{\mathrm{rig}}(X_\kappa/K)\to H^i_{\mathrm{syn}}(X,n)\\ \to \mathrm{Fil}^n H^{i}_{\mathrm{dR}}(X_K/K)&\xrightarrow{1-\phi/p^n} H^{i}_{\mathrm{rig}}(X_\kappa/K)\to\cdots \quad .\end{aligned}$$

In the cases discussed in proposition 5.6.1 this reduces to a short exact sequence

$$0\to \mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K)\xrightarrow{1-\phi/p^n} H^{i-1}_{\mathrm{rig}}(X_\kappa/K)\to H^i_{\mathrm{syn}}(X,n)\to 0.$$

The isomorphism (5.6) is induced by the map sending $x\in H^{i-1}_{\mathrm{rig}}(X_\kappa/K)$ to the image in $H^i_{\mathrm{syn}}(X,n)$ of $(1-\phi/p^n)x$. A similar analysis applies to modified syntomic cohomology. See proposition 5.9.3 for a special case.

When we compose the syntomic Chern classes with the canonical map $\Xi: H_{\mathrm{syn}}\to H_{\mathrm{ms}}$ we obtain modified syntomic Chern classes and Chern characters. Alternatively, one can construct these directly using the same techniques as before and universal Chern classes which are the images of the syntomic ones under the map Ξ in the cohomology of $\mathbf{B}_\bullet\mathrm{GL_n}$. This makes the following lemma evident

Lemma 5.8. *The modified syntomic Chern classes commute with base change, i.e., when X is an $\mathcal{V}$-scheme, $\mathcal{V}'$ is a finite extension of $\mathcal{V}$ and $X'/\mathcal{V}'$ is the scheme obtained by base change to $\mathcal{V}'$, there is a commutative diagram*

$$\begin{array}{ccc} K_p(X) & \xrightarrow{c_j^p} & H^{2j-p}_{\mathrm{ms}}(X,j)\\ \downarrow & & \downarrow\\ K_p(X') & \xrightarrow{c_j^p} & H^{2j-p}_{\mathrm{ms}}(X',j). \end{array}$$

From here until the end of the section we consider the cohomology $H^i_{\mathrm{ms}}(X,i)$ (so the degree equals the twist) of a smooth affine $\mathcal{V}$-scheme $X = \mathrm{Spec}(A)$. Fix a compactification $j : X \hookrightarrow \overline{X}$ over $\mathcal{V}$. Let $\mathcal{P} = \hat{\overline{X}}$. We get a rigid datum $\mathcal{D} = (\overline{X}_\kappa, j_\kappa, \mathcal{P})$. By the proof of proposition 1.10 in [Ber97] we see that the complex $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)_{\mathcal{D}}$ is quasi-isomorphic to the complex

$$\Omega^\bullet_{A^\dagger,K} := \varinjlim_U \Omega^\bullet_U,$$

where the limit is over all strict neighborhoods of $]X_\kappa[_{\overline{X}_K}$. We remark that this complex is in fact the complex of differentials of the dagger algebra $A^\dagger$ used in the Monsky-Washnitzer cohomology [MW68, vdP86], but we will not need this fact here. Now fix a Frobenius endomorphism φ of degree q of X_κ. It follows from lifting theorems for dagger algebras ([Col85, Thm A-1] or [vdP86, Thm 2.4.4.ii]) that there is a lifting ϕ of φ to the dagger algebra $A^\dagger$. This implies that there are strict neighborhoods $U' \subset U''$ and a map $\phi : U' \to U''$ whose reduction is φ, so the collection,

$$(\overline{X}_\kappa, j_\kappa, \mathcal{P}, U'),\ (\overline{X}_\kappa, j_\kappa, \mathcal{P}, U'') \in \mathcal{ER}(X_\kappa, \mathcal{V}), \quad \phi : U' \to U'',$$

belongs to $\mathcal{ER}(\varphi, \mathcal{V})$. We therefore obtain a commutative diagram

$$\begin{array}{ccc} \Omega^\bullet_{A^\dagger,K} & \xrightarrow{\phi^*} & \Omega^\bullet_{A^\dagger,K} \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) & \xrightarrow{\varphi^*} & \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K), \end{array}$$

where the vertical maps are quasi-isomorphisms.

Now choose any de Rham datum (Y,i) for X. By Hodge theory [Del71, Corollaire 3.2.13.ii] we see that the space

$$\Omega^i(X_K)_{\log} := H^0(Y, \Omega^i_Y\langle\log(Y - X_K)\rangle)$$

is independent of the choice of (Y,i) and is isomorphic to $\mathrm{Fil}^i H^i_{\mathrm{dR}}(X_K/K) = H^i(Y, \Omega^{\geq i}_Y\langle\log(Y - X_K)\rangle)$.

Proposition 5.9. *Let X and ϕ be as above.*

1. *There is a canonical isomorphism*

(5.7)

$$H^i_{\mathrm{ms}}(X,i) \cong \varinjlim_k \Big\{(\omega,h) : \quad \omega \in \Omega^i(X_K)_{\log},$$
$$h \in \Omega^{i-1}_{A^\dagger,K}/d\Omega^{i-2}_{A^\dagger,K}, \quad dh = \left(1 - \left(\frac{\phi^*}{q^i}\right)^k\right)\omega\Big\} \quad ,$$

where we abusively identified ω with its image in $\Omega^i_{A^\dagger,K}$. The connecting map between level k and level km is given by

$$(\omega,h) \mapsto \left(\omega, \sum_{s=0}^{m-1} (\phi^*/q^n)^{sk}\, h\right).$$

2. *The cup product $H^i(X,i)\times H^j(X,j)\to H^{i+j}(X,i+j)$ is given in level k by the formula*

$$\begin{aligned}\omega_1,h_1)\cup(\omega_2,h_2) = \Bigg(&\omega_1\wedge\omega_2,\\ &h_1\wedge\left(\gamma+(1-\gamma)\left(\frac{\phi^*}{q^j}\right)^k\right)\omega_2\\ &+(-1)^i\left(\left((1-\gamma)+\gamma\left(\frac{\phi^*}{q^i}\right)^k\right)\omega_1\right)\wedge h_2\Bigg)\quad .\end{aligned} \tag{5.8}$$

3. *When $i>\operatorname{reldim}X$ the isomorphism $H^{i-1}_{\mathrm{rig}}(X_\kappa/K)\to H^i_{\mathrm{ms}}(X,i)$ of (5.5) is given by the formula*

$$u\in H^{i-1}_{\mathrm{rig}}(X_\kappa/K)\subset \Omega^{i-1}_{A^\dagger,K}/d\Omega^{i-2}_{A^\dagger,K}\mapsto \left((0,(1-(\phi^*/q^i)^k)u)\right)_{k>0}.$$

Proof. The first part follows immediately from the discussion above and (5.3). The second part follows easily from remark 5.7.2. The last part is straightforward (compare remark 5.7.3). ☐

Proposition 5.10. *For $X=\operatorname{Spec}(A)$, φ and ϕ as above, The composed map*

$$A^\times\to K_1(X)\xrightarrow{c_1^1} H^1_{\mathrm{ms}}(X,1),$$

is given as follows: Let $f\in A^\times$ and let $\bar f$ be its reduction. As $\bar f$ is defined over some finite field, there is some power of φ, say φ^k, of degree q^k, such that $\bar f\circ\varphi^k=\bar f^{q^k}$. It follows that $f\circ\phi^k\equiv f^{q^k} \pmod{\mathfrak p}$ and therefore that the rigid function

$$f_0 := \frac{f^{q^k}}{f\circ\phi^k}$$

satisfies $\log f_0\in\Omega^0_{A^\dagger,K}$. With all that, under the isomorphism (5.7) the cohomology class $c_1^1(f)$ is given in degree k by

$$\left(\operatorname{dlog} f,\frac{\log f_0}{q^k}\right)\quad .$$

Proof. By replacing ϕ by ϕ^k we may always assume $k=1$. We will abuse the notation to write $c_1^1(f)$ for the first component of the regulator, which is defined under our assumption. We start with the case $X=\mathbb{G}_m$ (so $A=\mathcal{V}[T,T^{-1}]$), $f=T$ and ϕ is defined by $\phi^*(T)=T^q$. Since the modified syntomic Chern class lifts the de Rham Chern class, which for T is just $\operatorname{dlog}T$, We see that

$$c_1^1(T)=(h,\operatorname{dlog}T),\quad \text{where}\quad dh=\operatorname{dlog}T-\frac{1}{q}\operatorname{dlog}\phi^*(T)=0.$$

The proposition in this case amounts to the statement that $h=0$. To see this we use the involution $\tau: A\to A$ defined by $\tau(T)=T^{-1}$. As the Chern class is functorial, and as τ commutes with ϕ, we see that

$$c_1^1(T^{-1})=\tau^*(h,\operatorname{dlog}T)=(\tau^*h,\tau^*\operatorname{dlog}T)=(h,-\operatorname{dlog}T).$$

As c_1^1 is a group homomorphism, we have $(0,0)=c_1^1(T^{-1}\cdot T)=(2h,0)$, which proves what we wanted in this case.

The next step is to show that the formula of the proposition is consistent with changing the map ϕ. For this we will need a lemma.

Lemma 5.11. *Let $Z = \mathbb{G}_m/\mathcal{V} \hookrightarrow \overline{Z} = \mathbb{P}^1$ and $\mathcal{P} = \hat{\overline{Z}}$. Let U be a strict neighborhood of $]Z_\kappa[_{\mathcal{P}}$ in $\mathcal{P}_K$. Let Δ_{Z_κ}, resp. $\Delta_{\overline{Z}_\kappa}$ be the diagonals in $Z_\kappa \times Z_\kappa$, resp. $\overline{Z}_\kappa \times \overline{Z}_\kappa$, and let $\Delta = (U \times U) \cap]\Delta_{\overline{Z}_\kappa}[_{\mathcal{P}\times\mathcal{P}}$. Finally, let z be the standard parameter on Z_K and let x and y be its two pullbacks to Δ. Then, the image of $\log(x/y) \in A(\Delta)$ in the 0-th component of $\mathbb{R}\Gamma'_{\mathrm{rig}}(Z_\kappa/K)$ is 0.*

Proof. First of all we notice that indeed $\log(x/y)$ is a rigid function on Δ. This is because $x \equiv y \pmod{\mathfrak{p}}$. Let γ be the image of $\log(x/y)$ in the 0-th component. Then $d\gamma$ is the image of $\operatorname{dlog}(x/y) = \operatorname{dlog} x - \operatorname{dlog} y$. But both $\operatorname{dlog} x$ and $\operatorname{dlog} y$ are pullbacks of $\operatorname{dlog} z$ from U, so by the construction of $\mathbb{R}\Gamma'_{\mathrm{rig}}(Z_\kappa/K)$ as a filtered direct limit we have $d\gamma = 0$. It follows that γ defines a class $[\gamma] \in H^0_{\mathrm{rig}}(Z_\kappa/K) \xrightarrow{\sim} K$. To show that the image is 0, we now use the diagonal map $\delta : U \to \Delta$. This map, together with all the associated data, defines an object of $\mathcal{ER}(\mathrm{id}, \mathcal{V})$. It follows that the image of $[\gamma]$ in K is the same as that of $\delta^* \log(x/y) = 0$. $\square$

Remark 5.12. Notice that in the above proof the map δ does not define a morphism of extended rigid data. It is therefore essential to verify first that $d\gamma = 0$.

Corollary 5.13. *Let $X = \operatorname{Spec}(A)$ as above, let $f \in A$ and let ϕ, ϕ' be two morphisms $U' \to U''$ whose reduction is φ. Then the functions $\log(f^q/\phi^*(f))$, $\log[f^q/((\phi')^*(f))] \in A(U')$ have the same image in $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa, \mathcal{V})$.*

Proof. We need to show that $\log[\phi^*(f)/((\phi')^*(f))]$ has image 0 in $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa, \mathcal{V})$. This follows because this function is the pullback from Δ (in the notation of the lemma) of $\log(x/y)$ by the map $U' \to \Delta$ given by $(f \circ \phi, f \circ \phi')$. $\square$

We can now complete the proof of proposition 5.10. Corollary 5.13 tells us that when the result is proved for one endomorphism ϕ, it is true for all of them. In particular, the result is now proved in complete generality for $\mathbb{G}_m$. Suppose now that we have another affine scheme $Y = \operatorname{Spec}(B)$, a map $F : X \to Y$ and $g \in B^\times$. If we can find lifts ϕ and ϕ' of the respective Frobenius endomorphisms making the diagram

$$\begin{tikzcd} B^\dagger \arrow[r, "F^\dagger"] \arrow[d, "\phi'"'] & A^\dagger \arrow[d, "\phi"'] \\ B^\dagger \arrow[r, "F^\dagger"] & A^\dagger \end{tikzcd}$$

commute, and if the result is true for g, then it will also be true for $F^*(g)$.

Consider now $f \in A$. It is the pullback from $T \in \mathcal{V}[T, T^{-1}] = \mathcal{O}(\mathbb{G}_m)^\times$ via the composed map $X \xrightarrow{(\mathrm{id}, f)} X \times \mathbb{G}_m \xrightarrow{p_2} \mathbb{G}_m$. We are left with showing that lifts of Frobenius endomorphisms as above exist for the two maps, $F^\dagger : A^\dagger \hat{\otimes} \mathcal{V}[T, T^{-1}]^\dagger \to A^\dagger$ given by $a \hat{\otimes} T \mapsto a \cdot f$, and $G^\dagger : \mathcal{V}[T, T^{-1}]^\dagger \to A^\dagger \hat{\otimes} \mathcal{V}[T, T^{-1}]^\dagger$ given by $T \mapsto 1 \hat{\otimes} T$. The existence of lifts, and hence proposition 5.10, now follow from application of [Col85, Thm A-1] to the following two diagrams:

$$\begin{tikzcd} A^\dagger & A^\dagger \hat{\otimes} \mathcal{V}[T, T^{-1}]^\dagger \arrow[l, "\phi \circ F^\dagger"'] \arrow[dl, dashed, "\phi'"'] \\ A^\dagger \hat{\otimes} \mathcal{V}[T, T^{-1}]^\dagger \arrow[u, two heads, "F^\dagger"] & \mathcal{V} \arrow[l] \arrow[u] \end{tikzcd}$$

and

$$\begin{tikzcd}
\{0\} & A^\dagger \hat{\otimes} \mathcal{V}[T,T^{-1}]^\dagger \arrow[l] \arrow[dl, dashed, "\phi'"'] \\
A^\dagger \hat{\otimes} \mathcal{V}[T,T^{-1}]^\dagger \arrow[u, two heads] & \mathcal{V}[T,T^{-1}]^\dagger \arrow[l, "G^\dagger \circ \phi"'] \arrow[u, "G^\dagger"']
\end{tikzcd} \quad ,$$

where in each case the theorem guarantees the existence of a diagonal map making the diagram commute and whose reduction is the obvious Frobenius endomorphism making the reduction commute. □

6. Comparison with other cohomology theories

In this section we compare our constructions with some other versions of syntomic cohomology and regulators and then also with étale cohomology and regulators.

As mentioned in the introduction, in [Gro94], Gros defines, for X smooth over $\mathcal{V}$ and $K = K_0$, rigid syntomic cohomology $H^i(X, s(n)_{X/K,\mathrm{rig}})$. When X is affine he further defines higher Chern classes into this cohomology. The main difference with our construction is that no attention to log singularities is given. We generalize his construction to the case $K \neq K_0$ as follows.

We define a filtration on $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$ following [Gro94, I.3.2]. Let $j : X \hookrightarrow \overline{X}$ be a compactification of X. Now complete $(\overline{X}_\kappa, j_\kappa)$ to an extended rigid datum $\mathcal{D} = (\overline{X}_\kappa, j_\kappa, \mathcal{P}, U) \in \mathcal{ER}(X_\kappa, \mathcal{V})$. We have $\overline{X}_K^{\mathrm{an}} \subset]\overline{X}_\kappa[_{\mathcal{P}}$.

Definition 6.1. The n-th filtered part of $\mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)_{\mathcal{D}}$ relative to $\overline{X}$ is defines as

$$\mathrm{Fil}^n_{\overline{X}} \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)_{\mathcal{D}} := \mathbb{R}\Gamma(U, j_\kappa^\dagger \mathrm{Fil}^n_{\overline{X}} \Omega_U^\bullet) \quad ,$$

where

$$\mathrm{Fil}^n_{\overline{X}} \Omega_U^\bullet := I^n \to I^{n-1}\Omega_U^1 \to I^{n-2}\Omega_U^2 \to \cdots \quad ,$$

with I the ideal defining $\overline{X}_K^{\mathrm{an}}$ in $]\overline{X}_\kappa[_{\mathcal{P}}$ and letting $I^k = (1)$ for $k \leq 0$.

By results of Berthelot [Gro94, I.3.3, I.3.5] changing the data induces quasi-isomorphisms between the above defined complexes. We may set

$$\mathrm{Fil}^n_X \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) := \varinjlim_{(\overline{X}, j, \mathcal{P}, U)} \mathrm{Fil}^n_{\overline{X}} \Omega_U^\bullet, \tag{6.1}$$

which is a functorial complex in X with a natural map $\mathrm{Fil}^n_X \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$.

Definition 6.2. We define complexes $\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n)$ and $\widetilde{\mathbb{R}\Gamma}_{\mathrm{ms}}(X,n)$ by taking the definition of $\mathbb{R}\Gamma_{\mathrm{syn}}(X,n)$ and $\mathbb{R}\Gamma_{\mathrm{ms}}(X,n)$ and replacing the map $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$ with the map $\mathrm{Fil}^n_X \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \to \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$. The associated cohomologies will be denoted, as always, by $\tilde{H}^i_{\mathrm{syn}}(X,n)$ and $\tilde{H}^i_{\mathrm{ms}}(X,n)$.

Lemma 6.3. *When $K = K_0$ there is a canonical isomorphism $\tilde{H}^i_{\mathrm{syn}}(X,n) \cong H^i(X, s(n)_{X/K,\mathrm{rig}})$, where the latter cohomology is the one defined by Gros.*

Proof. As in remark 5.7.3, when $K = K_0$ the construction of $\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n)$ simplifies to $\mathrm{Cone}(\mathrm{Fil}^n_X \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \xrightarrow{1-\phi/p^n} \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K))$. By choosing particular data this is easily seen to be quasi-isomorphic to the construction of Gros. □

Proposition 6.4. *There is a functorial commutative square of maps*

$$\begin{array}{ccc} \mathbb{R}\Gamma_{\mathrm{syn}}(X,n) & \longrightarrow & \widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n) \\ \downarrow & & \downarrow \\ \mathbb{R}\Gamma_{\mathrm{ms}}(X,n) & \longrightarrow & \widetilde{\mathbb{R}\Gamma}_{\mathrm{ms}}(X,n). \end{array}$$

In particular, we obtain a functorial map of cohomology theories $H^*_{\mathrm{syn}}(X,n) \to H^*(X, s(n)_{X/K,\mathrm{rig}})$.

Proof. The left vertical map has already been defined and the right vertical map is defined in exactly the same way. To construct the horizontal maps one only has to define maps $\mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K) \to \mathrm{Fil}^n_X \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)$. To that end, let $j : X \hookrightarrow \overline{X}$ be a compactification of X and let $i : X_K \hookrightarrow Y$ be a de Rham datum for X_K. Consider $\mathcal{D} = (\overline{X}_\kappa, j_\kappa, \mathcal{P}, U)$ with $\mathcal{P} = \hat{\overline{X}}$ and $U = X_K^{\mathrm{an}}$. Then we have $\mathrm{Fil}^n_{\overline{X}} \Omega^\bullet_U = \Omega^{\geq n}_U$. We can therefore obtain a map, in a similar manner to (2.2),

$$\begin{aligned} \mathrm{Fil}^n \mathbb{R}\Gamma_{\mathrm{dR}}(X_K/K)_{(i,Y)} &= \mathbb{R}\Gamma(Y, \Omega^{\geq n}_Y \langle \log(Y - X_K)\rangle) \to \mathbb{R}\Gamma(Y, i_*\Omega^{\geq n}_{X_K}) \\ &\to \mathbb{R}\Gamma(X_K, \Omega^{\geq n}_{X_K}) \to \mathbb{R}\Gamma(U, \Omega^{\geq n}_U) \\ &\to \mathbb{R}\Gamma(U, j_\kappa^\dagger \Omega^{\geq n}_U) = \mathrm{Fil}^n_{\overline{X}} \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K)_{\mathcal{D}}. \end{aligned}$$

Taking limits gives the required map. □

In [Niz95] and [Niz97] Nizioł defines another version of syntomic cohomology, this time based on the convergent cohomology of Ogus. This amounts to ignoring both logarithmic singularities and overconvergent singularities. We will need one of the versions of this definition.

Definition 6.5 (Nizioł [Niz97, Proof of Lemma 2.1])**.** Let X be a smooth quasi-projective $\mathcal{V}$-scheme. The f-cohomology of X with values in the sheaf $\mathcal{K}(n)$, $H^*_f(X, \mathcal{K}(n))$, is defined as the homology of the complex

$$\begin{aligned} &\mathbb{R}\Gamma(X, \mathcal{S}^\bullet(n)) = \\ &\qquad \mathrm{Cone}(H((X_\kappa/\mathcal{V}_0)_{\mathrm{conv}}, \mathcal{K}_{X_\kappa/\mathcal{V}_0}) \oplus H((X_\kappa/\mathcal{V})_{\mathrm{conv}}, F^n_X) \\ &\qquad\quad \to H((X_\kappa/\mathcal{V}_0)_{\mathrm{conv}}, \mathcal{K}_{X_\kappa/\mathcal{V}_0}) \oplus H((X_\kappa/\mathcal{V})_{\mathrm{conv}}, \mathcal{K}_{X_\kappa/\mathcal{V}}))[-1]. \end{aligned}$$

Here, H denotes the derived functor of the global section functor, $\mathcal{K}_{X_\kappa/\mathcal{V}}$ is the canonical sheaf on the convergent topos and F^n_X is its standard filtration. The map defining the cone is given by 3.5.

Proposition 6.6. *There is a canonical map*

$$\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n) \to \mathbb{R}\Gamma(X, \mathcal{S}^\bullet(n)),$$

which is an isomorphism if X is proper.

Proof. It follows from [Ogu90, Theorem 0.6.6] that there are functorial maps $\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/L) \to H((X_\kappa/\mathcal{O}_L)_{\mathrm{conv}}, \mathcal{K}_{X_\kappa/\mathcal{O}_L})$ for $L = K$ or K_0, which are quasi-isomorphisms if X is proper. One can check that these maps further induce maps $\mathrm{Fil}^n \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \to$

$H((X_\kappa/\mathcal{V})_{\mathrm{conv}}, F_X^n)$. As in the proof of proposition 3.5 we see that $\widetilde{\mathbb{R}\Gamma}_{\mathrm{syn}}(X,n)$ can be written as

$$\begin{aligned}\mathbb{R}\Gamma_{\mathrm{syn}}(X,n) \cong \mathrm{Cone}(&\mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathrm{Fil}^n \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K) \\ &\to \mathbb{R}\Gamma_{\mathrm{rig}}(X_\kappa/K_0) \oplus \mathbb{R}\Gamma'_{\mathrm{rig}}(X_\kappa/K))[-1] \quad ,\end{aligned}$$

which makes the existence of the required map obvious. When X is proper the map is a quasi-isomorphism because each of its components is. ☐

Proposition 6.7. *For X smooth and quasi-projective there is a functorial map of cohomology theories $H^*_{\mathrm{syn}}(X,n) \to H^*_f(X,\mathcal{K}(n))$ which is an isomorphism when X is proper and which commutes with Chern classes.*

Proof. To construct the map we simply compose $H_{\mathrm{syn}} \to \tilde{H}_{\mathrm{syn}} \to H_f$. We have shown both maps to be isomorphisms when X is proper. To show compatibility with Chern classes it is enough to check that the universal Chern classes in the cohomology of $\mathbf{B}_\bullet\mathrm{GL_n}$ are the same. But we know that $\mathbf{B}_\bullet\mathrm{GL_n}$ only has de Rham, rigid and convergent cohomologies in even degrees. This implies that the universal Chern classes coincide if their projection on de Rham cohomology do. But these projections are simply the corresponding universal de Rham Chern classes. Indeed, this is true by construction for H_{syn} and for H_f it follows from [Niz97, Lemma 2.2]. ☐

Finally, the comparison with Nizioł's cohomology allows us to connect our version of syntomic cohomology with étale cohomology. By [Niz95] and [Niz97, Cor. 3.1] there is a functorial map of cohomology theories $H^*_f(X,\mathcal{K}(n)) \to H^*_{\text{ét}}(X_K,\mathbb{Q}_p(n))$ which is compatible with Chern classes. Here ét denotes continuous étale cohomology as defined by Jannsen [Jan88].

Corollary 6.8. *For X smooth and quasi-projective there is a functorial map of cohomology theories $H^*_{\mathrm{syn}}(X,n) \to H^*_{\text{ét}}(X_K,\mathbb{Q}_p(n))$ which is compatible with Chern classes.*

Remark 6.9. Let X be smooth and projective. For all versions of syntomic cohomology (which are all the same in this case) the following is to be expected: The composed map

$$\begin{aligned}H^{i-1}_{\mathrm{dR}}(X_K/K)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) &\cong H^{i-1}_{\mathrm{rig}}(X_\kappa/K)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \\ &\xrightarrow{5.6.3} H^i_{\mathrm{syn}}(X,n) \to H^i_{\text{ét}}(X_K,\mathbb{Q}_p(n)) \to H^i_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_p(n)),\end{aligned}$$

is 0, where the last map comes from the Hochschild-Serre spectral sequence. This and the spectral sequence in turn give a map

$$H^{i-1}_{\mathrm{dR}}(X_K/K)/\mathrm{Fil}^n H^{i-1}_{\mathrm{dR}}(X_K/K) \to H^1(\mathrm{Gal}(\bar{K}/K), H^{i-1}_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_p(n))).$$

This map is nothing other than the Bloch-Kato exponential map associated with the $\mathrm{Gal}(\bar{K}/K)$ representation $H^{i-1}_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_p(n))$. This can be proved using [Niz95, Cor. 5.1] if $H^0(\mathrm{Gal}(\bar{K}/K), H^i_{\text{ét}}(X_{\bar{K}},\mathbb{Q}_p(n))) = 0$. The author could not understand, however, why this last condition is required, and therefore believes that the result can be proved in general. Private communications with the author of [Niz95] confirms that this may well be true and is likely to be sorted out by the time a final version of [Niz95] is available. An analogous result is proved in complete generality by Nekovář in [Nek98] and one need only assume properness instead of projecitveness. We were not able to compare his version of syntomic cohomology with ours.

Note finally that the condition above is verified whenever the cohomology is the target of Chern classes from higher K-theory by weight considerations.

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