

SYNTOMIC REGULATORS AND p -ADIC INTEGRATION II: K_2 OF CURVES

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1. INTRODUCTION

Let C/\mathbb{C} be a smooth complete curve, $f, g \in \mathbb{C}(C)$ two rational functions on C . In [Bei80], Beilinson defines the complex regulator of f and g ,

$$r_C(\{f, g\}) \in H^1(C^{an}, \mathbb{R}(1)),$$

such that the following formula is satisfied when $\omega \in H^0(C, \Omega_{C/\mathbb{C}}^1)$ is a holomorphic 1-form on C :

$$(1.1) \quad r_C(\{f, g\}) \cup [\omega] = \frac{1}{2\pi i} \int_{C(\mathbb{C})} \log |g|^2 \overline{d \log f} \wedge \omega.$$

One can show that r_C is antisymmetric and satisfies the Steinberg relation and therefore defines a map

$$r_C : K_2(\mathbb{C}(C)) \rightarrow H^1(C^{an}, \mathbb{R}(1)).$$

Beilinson, following Bloch, shows the following theorem:

Theorem 1 (Bloch, Beilinson). *If E/\mathbb{Q} is an elliptic curve with complex multiplication, then there exist $f, g \in \mathbb{Q}(E)$ such that*

$$(1.2) \quad r_E(\{f, g\}) \cup [\omega] = a_{f,g} \Omega L^*(E, 0),$$

where $\omega \in H^0(E, \Omega_{E/\mathbb{Q}}^1)$, $a_{f,g} \in \mathbb{Q}$ and Ω is a known transcendental period. The L function $L^*(E, s)$ is the usual L function multiplied by the Gamma factor.

In [CdS88], Coleman and de Shalit gave a p -adic analogue of formula (1.1). Suppose that C/\mathbb{C}_p is a smooth complete curve with good reduction (the results of [CdS88] apply in some greater generality). Then they define a p -adic regulator

$$r_{p,C} : K_2(\mathbb{C}_p(C)) \rightarrow \text{Hom}(H^0(C, \Omega_{C/\mathbb{C}_p}^1), \mathbb{C}_p) \quad .$$

The value of $r_{p,C}$ on a symbol $\{f, g\}$ with $f, g \in \mathbb{C}_p(C)$ is defined to be the functional on holomorphic 1 forms given by

$$(1.3) \quad r_{p,C}(\{f, g\})(\omega) := \int_{(f)} \log(g) \cdot \omega.$$

Briefly, the meaning of this formula is as follows: Coleman's p -adic integration theory allows (as will be explained below) to define a function

$$F_{\log(g) \cdot \omega} : C(\mathbb{C}_p) \rightarrow \mathbb{C}_p,$$

unique up to a constant. If the divisor of f is $(f) = \sum n_i(x_i)$, then one defines

$$\int_{(f)} \log(g) \cdot \omega := \sum n_i F_{\log(g) \cdot \omega}(x_i).$$

Coleman and de Shalit used their regulator to derive an analogue of theorem 1.

Theorem 2 (Coleman, de-Shalit). *For the same E , f, g , and ω as in theorem 1, Let p be a prime that splits in the CM field of E . Then*

$$r_{p,E}(\{f, g\})(\omega) = a_{f,g} \Omega_p L_p(E, 0),$$

where $L_p(E, s)$ is the p -adic L -function of E , $a_{f,g}$ is the same as in theorem 1 and Ω_p is a p -adic period (into which we have pushed an Euler factor at p to keep the exposition simple).

We remark that the point 0 is outside the interpolation range, so the p -adic formula can not be recovered from the complex one.

The formula (1.3) is derived in an ad hoc manner from its complex counterpart (1.1). However, the relation with the value of a p -adic L -function seems to indicate that it is in some sense the “correct” p -adic formula. On the other hand, there is a general method of assigning p -adic regulators to elements of K -theory, namely the construction of syntomic regulators. The purpose of this work is to show that on K_2 of curves these constructions are very closely related.

Let L/\mathbb{Q}_p be a finite extension with residue field κ and let Z/\mathcal{O}_L smooth and surjective (i.e., not an L scheme) with generic fiber Z_L and special fiber Z_κ . Then Gros [Gro90] and Nizioł [Niz97] define regulators (= Chern characters) from the K -theory of Z into syntomic cohomology. This last cohomology has several versions by now. They are all essentially the same in the projective case. We will use the version developed by us in [Bes98]. With the notation of loc.sit, the regulator takes the form:

$$ch_{i,j} : K_j(Z) \rightarrow H_{syn}^{2i-j}(Z, i).$$

In this work we will only be interested in the case where $i = j = 2$ and Z is projective and of relative dimension 1. Then it can be shown that

$$H_{syn}^2(Z, 2) \cong H_{dR}^1(Z_L/L).$$

As Z_κ is a smooth and projective curve over a finite field, it follows from [Har77] that $K_i(Z_\kappa)$ is torsion for $i \geq 1$. The localization sequence in K -theory shows that $K_2(Z) \otimes \mathbb{Q} \cong K_2(Z_L) \otimes \mathbb{Q}$. Since the target of the Chern character is an L -vector space, it induces a map $r_{syn} : K_2(Z_L) \otimes \mathbb{Q} \rightarrow H_{dR}^1(Z_L/L)$.

Let $C = Z_L \otimes \mathbb{C}_p$. Then one can write the following diagram:

$$(1.4) \quad \begin{array}{ccc} K_2(Z_L) \otimes \mathbb{Q} & \xrightarrow{r_{syn}} & H_{dR}^1(Z_L/L) \\ \downarrow & & \downarrow \\ K_2(C) \otimes \mathbb{Q} & & H_{dR}^1(C/\mathbb{C}_p) \\ \downarrow & & \downarrow \text{Poincaré duality} \\ K_2(\mathbb{C}_p(C)) \otimes \mathbb{Q} & & \text{Hom}(H_{dR}^1(C/\mathbb{C}_p), \mathbb{C}_p) \\ \downarrow r_{p,C} & & \downarrow \\ \text{Hom}(H^0(C, \Omega_{C/\mathbb{C}_p}^1), \mathbb{C}_p) & \xlongequal{\quad} & \text{Hom}(H^0(C, \Omega_{C/\mathbb{C}_p}^1), \mathbb{C}_p). \end{array}$$

Here, the map given by Poincaré duality is (to fix signs)

$$\alpha \mapsto (\beta \mapsto \text{tr}(\beta \cup \alpha)) \quad .$$

Our main result is then

Theorem 3. *The diagram above commutes.*

Note that what the theorem says is essentially that the regulator of Coleman and de Shalit computes “part” of the syntomic regulator, namely the value of the syntomic regulator, thought of as a functional on H_{dR}^1 via Poincaré duality, on the subspace of holomorphic forms. The theorem of Coleman and de Shalit suggests that, rather than that their regulator gives only part of the information, it is in fact the syntomic regulator that should be modified to land in a smaller subspace. In the corresponding complex situation, one uses the action of complex conjugation to cut down the target space to the right size. Is there a similar procedure in the p -adic situation?

The proof of the main theorem turns out to be fairly simple, once one makes explicit the two sides of diagram (1.4). The new idea involved is a kind of a residue theorem, corollary 4.11, which is similar in spirit to the reciprocity law proved by Coleman in [Col89].

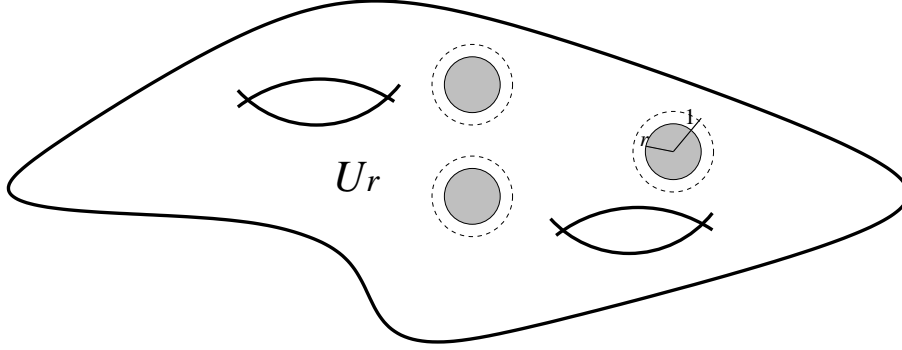
The structure of the paper is as follows: In section 2 we recall the basics of the theory of Coleman integration we need from [CdS88] and in particular explain the formula (1.3) for the p -adic regulator. In section 3 we use the theory developed in [Bes98] to write an explicit one-form representing the syntomic regulator of an element of K_2 on an open part of Z . Using the formula of Serre for the cup product in de Rham cohomology on a curve we reduce the proof of the main theorem to a formula (proposition 3.4) relating residues and Coleman integrals. In section 4 we define *local indices*, which are some kind of a generalization of residues that make sense out of the residue of the log function in some cases. Our main reciprocity law, proposition 4.10, should be considered as an extension of the residue theorem to these generalized residues. The final section verifies proposition 3.4 and thereby completes the proof of the main theorem.

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2. p -ADIC INTEGRATION

Our basic setup throughout this work will be as follows: K is a complete subfield of \mathbb{C}_p with ring of integers \mathcal{O}_K and residue field κ . Let X/\mathcal{O}_K be a smooth projective and surjective scheme of relative dimension 1 with generic fiber X_K and special fiber X_κ . Let $Y \subset X$ be an open affine subscheme, smooth and surjective over \mathcal{O}_K . The special fiber of the complement of Y is a union of a finite number of points: $X_\kappa - Y_\kappa = \{e_1, \dots, e_n\}$.

To the situation above one associates a “basic wide open”, in Coleman’s terminology (see [CdS88, 2.1]). For $r < 1$, consider the rigid space U_r obtained by “removing discs of radius r around e_i ”



The formal definition of U_r is, following Berthelot, as follows: If Y_κ is locally in X_κ given by the equation $\bar{h} = 0$, with \bar{h} the reduction of a function h on X , then U_r is locally given by the inequality $|h| > r$. While the definition depends on the choice of the local lifts h , the inverse limit $U = \varprojlim_{r \rightarrow 1} U_r$ does not, in the sense that any two choices of local lifts give the same U_r for sufficiently large r [Ber96]. The spaces of functions and one-forms, $A(U) := \varinjlim_{r \rightarrow 1} A(U_r)$ and $\Omega^1(U) := \varinjlim_{r \rightarrow 1} \Omega^1(U_r)$, are the spaces of “overconvergent” functions and one-forms. Inside U one has the “underlying affinoid” in Coleman’s terminology, which is the space obtained by throwing away from X_K the corresponding open discs of radius 1.

We will need the notion of Frobenius endomorphisms on U . One first defines these on Y_κ . We have some Y'/F_q , for some $q = p^r$ such that $Y_\kappa \cong Y' \otimes_{F_q} \kappa$. Consider the κ -morphism $\varphi = \text{Fr}_{Y'}^r \otimes \text{id}_\kappa$.

Definition 2.1. A Frobenius endomorphism of Y_κ is any φ obtained in the way described above.

An important remark is that any power of a Frobenius endomorphism is again a Frobenius endomorphism.

Theorem 2.2 (Coleman [CdS88, Theorem 2.2.]). *For any Frobenius endomorphism φ there exists a rigid analytic map $\phi : U \rightarrow U$ reducing to φ (i.e., $\phi : U_r \rightarrow U_s$ with $s < r$ sufficiently near 1). Any such ϕ will be called a Frobenius endomorphism of U .*

The proof in loc.cit is for $K = \mathbb{C}_p$ but works in general. We will call a map ϕ as above a Frobenius endomorphism of U . Clearly one obtains an operator ϕ^* on $A(U)$ and $\Omega^1(U)$.

Example 2.3. For $X = \mathbb{P}_{\mathcal{O}_{\mathbb{C}_p}}^1$ and $Y = \mathbb{G}_m$ we can take $U_r = \{r < |z| < 1/r\}$, $\phi(z) = z^p$, $\phi : U_r \rightarrow U_{rp}$.

We now sketch Coleman’s integration theory on U . For a full account see [CdS88]. For simplicity and compatibility with loc.cit we will assume from now until the end of this section that $K = \mathbb{C}_p$, hence $\kappa = \bar{F}_p$, and set $C = X_K$. The space U decomposes *set theoretically* into a disjoint union of residue discs U_x over $x \in X(\bar{F}_p)$. When $x \in Y(\bar{F}_p)$, U_x is the collection of closed points of C reducing to x and is isomorphic to the open unit disc $\{|z| < 1\}$ (because X is smooth) via some local parameter which we denote z_x . For each $r < 1$ and each e_i , the residue disc of e_i in U_r is the collection of closed points in U_r reducing to e_i and is isomorphic to an open annulus $\{r < |z| < 1\}$ via a local parameter z_{e_i} . Here we assume that z_{e_i} extends

to a local parameter for the residue disc of e_i in C . This fixes an *orientation* for the annulus e_i (see [Col89, II] and the discussion below in section 4). The pro-space U_{e_i} is the inverse limit of these annuli. We have

$$\Omega^1(U_x) = A(U_x)dz_x$$

and

$$A(U_x) = \{f(z) = \sum_{n=0}^{\infty} a_n z^n \text{ converging for } |z| < 1\},$$

when $x \in Y_s(\bar{F}_p)$, or

$$A(U_x) = \{f(z) = \sum_{n=-\infty}^{\infty} a_n z^n \text{ converging for } r < |z| < 1 \text{ for some } r < 1\},$$

when $x = e_i$ for some i . In both cases we have set $z = z_x$.

Definition 2.4. The annuli U_{e_i} , with the orientation discussed above, are called the annuli ends of U and their collection is denoted $End(U)$.

In fact, we will abuse the notation and will usually write e_i for U_{e_i} . To confuse things further, we will sometime use e_i to refer to the full residue disc of e_i in C . The intention should be clear from the context.

Definition 2.5. The residue of the form $\omega = \sum_{n=-\infty}^{\infty} a_n z_{e_i}^n dz_{e_i}$ along the annuli e_i is given by

$$\text{Res}_{e_i} \omega = a_{-1}$$

The residue is independent of the choice of parameter.

Coleman's theory allows one to integrate certain locally analytic one forms. One first needs to make a choice of a branch of the p -adic logarithm.

Definition 2.6. A branch of the p -adic logarithm is any locally analytic homomorphism $\log : \mathbb{C}_p^\times \rightarrow \mathbb{C}_p$ with the usual expansion for \log around 1. Such a function is determined by choosing $\pi \in \mathbb{C}_p$ such that $|\pi| < 1$ and declaring $\log(\pi) = 0$.

Suppose a branch of the p -adic logarithm has been chosen. One defines $A_{\log}(U_x) := A(U_x)$ if $x \in Y_s(\bar{F}_p)$ and to be the polynomial ring in the function $\log(z_{e_i})$ over $A(U_{e_i})$ if $x = e_i$. This ring is independent of the choice of z_{e_i} because it can be shown that the difference of the logs of two local parameters is analytic on U_{e_i} . Set $\Omega_{\log}^1(U_x) := A_{\log}(U_x)dz_x$. Then one defines locally analytic functions and one forms on U by

$$A_{\log}(U) := \prod_x A_{\log}(U_x), \quad \Omega_{\log}^1(U) := \prod_x \Omega_{\log}^1(U_x).$$

There is an obvious differential $d : A_{\log}(U) \rightarrow \Omega_{\log}^1(U)$. One easily checks that this is surjective. The point is that by adding logs we are able to integrate dz/z . The inverse image under d is very big, because one can choose a different constant of integration at every x . Coleman's theory isolates a subclass of locally analytic differential forms that can be integrated uniquely up to a global constant. This is done using the Frobenius endomorphism ϕ . It acts on locally analytic functions and differential forms in a way compatible with the differential d . Coleman's idea is now as follows: One constructs a certain subspace $M(U)$ of $A_{\log}(U)$, which we call the space of *Coleman functions*, and a vector space map (integration), which

we denote by \int or by $\omega \mapsto F_\omega$, from $M(U) \otimes_{A(U)} \Omega^1(U)$ to $M(U)/\mathbb{C}_p$. The map \int is characterized by three properties:

1. It is a primitive for the differential in the sense that $dF_\omega = \omega$.
2. It is Frobenius equivariant in the sense that $\int(\phi^*\omega) = \phi^* \int(\omega)$.
3. If $g \in A(U)$, then $F_{dg} = g + \mathbb{C}_p$.

The construction relies on a simple principle: If \int has already been defined on some space W , and $\omega \in \Omega_{loc}^1(U)$ is such that there is a polynomial $P(t)$ with \mathbb{C}_p coefficients such that $P(\phi^*)\omega = \eta \in W$, then the conditions on the integral force the equality $P(\phi^*)F_\omega = F_\eta + \text{Const}$. When P has no roots of unity as roots this condition fixes F_ω up to a constant. Starting with $W_0 = dA(U)$ one finds a unique way of integration all $\omega \in W_1 = \Omega^1(U)$. One defines recursively $W_{i+1} = (\int(W_i)) \cdot \Omega^1(U)$ and checks that the principle above permits extending \int uniquely to W_{i+1} . Finally one sets $M(U) = \bigcup_i \int W_i$. The entire theory turns out to be independent of the choice of ϕ .

In particular, suppose ω, η are rigid forms on U . Coleman's theory then finds a canonical (up to constant) $F_\eta : U \rightarrow \mathbb{C}_p$ such that $dF_\eta = \eta$, and $F_{F_\eta \cdot \omega}$ such that $dF_{F_\eta \cdot \omega} = F_\eta \cdot \omega$. One can show that if $g \in A(U)$, then $F_{\text{dlog}(g)} = \log(g)$. Now we can continue to define

$$F_{\log(g) \cdot \omega} = F_{F_{\text{dlog}(g)} \cdot \omega}.$$

Coleman and de-Shalit show that for a rational function $g \in \mathbb{C}_p(C)$ there is a canonical extension of $F_{\log(g) \cdot \omega}$ to $C - \{x : \text{ord}_x g \neq 0\}$. The extension is obtained by covering this set by basic wide opens and gluing the resulting integrals.

In some situations, notably when f and g in (1.3) have a common singular point, we will need to extend $F_{\log(g) \cdot \omega}$ to a singular point x_0 of g as well. The extension is not part of a general theory but is done ad hoc in [CdS88, 3.2]. As we will see in proposition 5.5, it is in fact the correct choice.

Definition 2.7. Suppose $\text{ord}_{x_0}(g) \neq 0$ and a choice of $F_{\log(g) \cdot \omega}$ has been made. We define $F_{\log(g) \cdot \omega}(x_0)$ as follows: We choose an integral F_ω such that $F_\omega(x_0) = 0$. Then we choose $\int F_\omega \text{dlog } g$ in such a way that the integration by parts formula is satisfied, i.e.,

$$F_{\log(g) \cdot \omega} + \int F_\omega \text{dlog } g = \log(g) \cdot F_\omega.$$

We now define $F_{\log(g) \cdot \omega}(x_0) = -(\int F_\omega \text{dlog } g)(x_0)$.

The motivation for this formula is that we should expect $F_\omega \log g$ to tend to 0 as x tends to x_0 just as in the complex case. If everything is defined over a discrete valuation field, then Coleman and de Shalit show that definition 2.7 gives the unique continuous extension. Another motivation is given by the following easy lemma

Lemma 2.8. *With the extension of definition 2.7, the function $(g, \omega) \rightarrow F_{\log(g) \cdot \omega}(x_0)$ is bilinear in the following sense: if we are given forms ω and η and we choose $F_{\log(g) \cdot \omega}$, $F_{\log(g) \cdot \eta}$ and $F_{\log(g) \cdot (\omega + \eta)} = F_{\log(g) \cdot \omega} + F_{\log(g) \cdot \eta}$, then*

$$F_{\log(g) \cdot \omega}(x_0) + F_{\log(g) \cdot \eta}(x_0) = F_{\log(g) \cdot (\omega + \eta)}(x_0).$$

A similar formula holds if we fix ω and consider functions g_1, g_2 and $g_3 = g_1 \cdot g_2$. In addition, the extension coincides with the integral if $\text{ord}_x(g) = 0$.

The formula (1.3) for the p -adic regulator is now clear.

3. THE SYNTOMIC REGULATOR

In this section we compute an explicit representative for the syntomic regulator evaluated on an element of K_2 of a curve. We begin by reviewing some of the results of [Bes98] in the present context. Rather than discussing syntomic cohomology here, we will work with the *modified syntomic cohomology* introduced in [Bes98, Section 5].

We keep the notation introduced in the previous section but assume for the moment that the field K is a finite extension of \mathbb{Q}_p . Recall that X is a smooth projective \mathcal{O}_K -scheme, Y is an open subscheme of X , smooth and surjective over \mathcal{O}_K and that to Y one associates a basic wide open U . We fix a Frobenius endomorphism $\phi : U \rightarrow U$. According to [Bes98, 5.6.3] There is a canonical isomorphism, $H_{\text{syn}}^2(X, 2) \xrightarrow{\sim} H_{\text{ms}}^2(X, 2)$, which is compatible with Chern classes by definition. Since we will only work with this degree of cohomology there is no need to introduce syntomic cohomology and we may work with the modified variant throughout. As Y is affine, we can write part of its modified syntomic cohomology quite easily. Namely, by [Bes98, 5.9.1] we have

$$H_{\text{ms}}^i(Y, i) = \varinjlim_k \left\{ \omega \in \Omega^i(Y_K)_{\log}, \quad h \in \Omega^{i-1}(U)/d\Omega^{i-2}(U) : \quad dh = \left(1 - \left(\frac{\phi^*}{q^i} \right)^k \right) \omega \right\}.$$

The notation differs somewhat from loc.sit.: what we denote here by $\Omega^j(U)$ is denoted by $\Omega_{A^\dagger, K}^j$ there, where A is a weak complete closure of Y in the sense of Monsky and Washnitzer. Also, $\Omega^i(Y_K)_{\log}$ is the space of degree i algebraic differential forms on Y_K with logarithmic singularities along $X_K - Y_K$. We have also identified $\omega \in \Omega^i(Y_K)_{\log}$ with its pullback under $U \hookrightarrow Y_K$. The connecting map from the k -th level to the km -th level is given by

$$(\omega, h) \mapsto \left(\omega, \sum_{s=0}^{m-1} (\phi^*/q^n)^{sk} h \right).$$

One can show directly that this definition is independent up to isomorphism of all choices. In particular, it is clear that it is unchanged if we replace ϕ by some power ϕ^k .

We next recall the computation of the modified syntomic regulator for functions, i.e., $ch_{1,1} : \mathcal{O}_Y^\times \rightarrow H_{\text{ms}}^1(Y, 1)$. We have $ch_{1,1} = -c_1^1$ so it is enough to compute the Chern class c_1^1 . Suppose we are given $f \in \mathcal{O}_Y^\times$. By replacing ϕ by ϕ^k for some k if necessary we can assume $\bar{f} \circ \bar{\phi} = \bar{f}^q$. We will find $c_1^1(f)$ in the first term of the directed system, which is given explicitly by

$$\{(\omega, h) : \omega \in \Omega^1(Y_K)_{\log}, h \in A(U), dh = (1 - \phi^*/q)\omega\}.$$

Set $f_0 := f^q/\phi^*f$. Then $f_0 \equiv 1$ and therefore $\log f_0 \in A(U)$ is a well defined rigid function. According to [Bes98, Prop. 5.10]

$$ch_1^1(f) := (d\log f, \frac{1}{q} \log f_0).$$

To define regulators on symbols we need to recall the cup product in modified syntomic cohomology. We again only say what it is in the case we need, i.e.,

$H_{\text{ms}}^1(Y, 1) \times H_{\text{ms}}^1(Y, 1) \rightarrow H_{\text{ms}}^2(Y, 2)$. In this case, according to [Bes98, 5.9.2] it is given on representatives as above by the formula

$$(\omega_1, h_1) \cup (\omega_2, h_2) = (\omega_3, h_3)$$

with

$$\omega_3 = \omega_1 \wedge \omega_2, \quad h_3 = (1/q)h_1\phi^*\omega_2 - h_2\omega_1.$$

The relevant condition, which the reader may easily check for himself, for example using the trick below, is $dh_3 = (1 - \phi^*/q^2)\omega_3$. This is only one of a family of possible homotopic cup products but we make no use of any other possibility.

Remark 3.1. The following trick is quite helpful in computations: consider a field F and two vector spaces V and W over F , and let T and S be operators on V and W respectively. The polynomial ring $F[t, s]$ acts on $V \otimes W$ by letting t acts as $T \otimes 1$ and s as $1 \otimes S$. The operator $T \otimes S$ then corresponds to the action of ts . We will use this in a situation where V and W are spaces of differential forms (or functions) and both S and T correspond to the action of Frobenius. Then $V \otimes W$ maps to a space of differential forms and $T \otimes S$ corresponds to the action of Frobenius on this space. This allows to translate relations in $F[t, s]$ to relations on the Frobenius action. The main example to be used in this paper is the relation

$$1 - \frac{ts}{q^2} = \left(1 - \frac{t}{q}\right) \frac{s}{q} + \left(1 - \frac{s}{q}\right),$$

which implies that for a function f and a one form ω we have

$$(3.1) \quad \left(1 - \frac{\phi^*}{q^2}\right)(f\omega) = \left(1 - \frac{\phi^*}{q}\right)f \cdot \frac{\phi^*}{q}\omega + f \cdot \left(1 - \frac{\phi^*}{q}\right)\omega.$$

All the above was true for Y of arbitrary dimension. In our case, taking into account that Y is of relative dimension 1, we obtain the following formula for the Chern character evaluated on the symbol $\{f, g\}$ with $f, g \in \mathcal{O}_Y^\times$: As the Chern character is multiplicative we have

$$ch_{2,2}(\{f, g\}) = (\text{dlog } f, (1/q) \log f_0) \cup (\text{dlog } g, (1/q) \log g_0) = (0, \eta_0(f, g)),$$

with

$$(3.2) \quad \eta_0(f, g) = \frac{1}{q^2} \log f_0 \text{dlog } \phi^* g - \frac{1}{q} \log g_0 \text{dlog } f \quad .$$

From the description of the modified syntomic cohomology above it is evident that $H_{\text{ms}}^2(Y, 2) \cong H^1(U)$. The naive isomorphism has to be twisted, however, to get the correct identification, a point which becomes clear once one considers the way the naive identification changes with k . The following result is a special case of [Bes98, 5.6.3 and 5.9.3] and is given here only to fix notation.

Proposition 3.2. *There exists a natural isomorphism $H^1(U) \rightarrow H_{\text{ms}}^2(Y, 2)$ which is given on the k -th level by the map*

$$\eta \mapsto (0, (1 - (\phi^*/q^2)^k)\eta).$$

Proof. It is clear that the map above commutes with all the transition maps. It is an isomorphism because the eigenvalues of ϕ^* on $H^1(U)$ are Weil numbers with absolute values $q^{1/2}$ or q by [CdS88, 2.5]. \square

For easy reference we write down the resulting formula for the image of the regulator on the symbol $\{f, g\}$ in de Rham cohomology.

Proposition 3.3. *Let $f, g \in \mathcal{O}_Y^\times$ and suppose we have chosen a Frobenius endomorphism ϕ such that $\bar{f} \circ \bar{\phi} = \bar{f}^q$ and $\bar{g} \circ \bar{\phi} = \bar{g}^q$. Then the image of $ch_{2,2}(\{f, g\})$ in $H^1(U)$ is given by the class of any form $\eta(f, g) \in \Omega^1(U)$ satisfying*

$$(3.3) \quad (1 - \phi^*/q^2)\eta(f, g) = \eta_0(f, g) + d(\).$$

Suppose now that $K = \mathbb{C}_p$. While we expect the theory of modified syntomic cohomology to work in this case as well, a verification of this possibility requires some facts in rigid cohomology which are not known to us. Nevertheless, the explicit computations we have performed in this section do not depend on the assumption that K is finite over \mathbb{Q}_p and we may simply take proposition 3.3 as a definition of a modified syntomic regulator for symbols on Y . In particular, in the formulation of the following key result we are free to assume that f and g are defined over \mathbb{C}_p .

We now explain the main result leading to the proof of theorem 3. Suppose again that $K = \mathbb{C}_p$ and denote, as in section 2, the generic fiber of X by C . For $f, g \in \mathbb{C}_p(C)$ and $x \in C$ let $t_x(f, g)$ be the tame symbol of f and g at x , defined by

$$t_x(f, g) = \left[(-1)^{\text{ord}_x(f) \text{ord}_x(g)} f^{\text{ord}_x(g)} g^{-\text{ord}_x(f)} \right] (x).$$

The tame symbol is known to satisfy the Steinberg relations and therefore to extend to a map $t_x : K_2(\mathbb{C}_p(C)) \rightarrow \mathbb{C}_p^\times$. It is well known that an element in $K_2(\mathbb{C}_p(C))$ extends to $K_2(C)$ if and only if its tame symbols at all points of C are 1.

Proposition 3.4. *Let $f, g \in \mathbb{C}_p(C)$ and let ω be a holomorphic 1-form in $H^0(C, \Omega_{C/\mathbb{C}_p}^1)$. Then*

$$(3.4) \quad \sum_{e \in \text{End}(U)} \text{Res}_e(F_\omega \cdot \eta(f, g)) = \sum_{x \in C} (\log t_x(f, g)) \cdot F_\omega(x) + \int_{(f)} \log g \cdot \omega.$$

The next two sections will be devoted to the proof of this proposition.

We may now turn to the proof of theorem 3. We first prove a lemma

Lemma 3.5. *Let Y be a smooth schemes, surjective and of relative dimension 1 over \mathcal{O}_K , and let Y' be a closed proper subscheme. Then the composition*

$$(3.5) \quad K'_2(Y') \rightarrow K_2(Y) \xrightarrow{ch_{2,2}} H_{ms}^2(Y, 2)$$

is 0.

Proof. If Y' is supported in the closed fiber, then the result follows easily because $K_2(Y'_\kappa)$ is torsion by [Har77]. If it is not, then, using the compatibility of the modified syntomic regulator with base change [Bes98, 5.8], we are reduced to the case where Y' is the image of a section $\text{Spec } \mathcal{O}_K \rightarrow Y$. In particular, Y' is smooth over \mathcal{O}_K . It is well known that under this assumption we may factor the pushforward map in K -theory, $K'_2(Y') \rightarrow K_2(Y)$, as $K'_2(Y') \cong K_2(Y, Y - Y') \rightarrow K_2(Y)$, where $K_2(Y, Y - Y')$ is the K -theory of Y relative to $Y - Y'$. It follows that the image of (3.5) is contained in the image of $H_{ms}^2(Y, Y - Y', 2) \rightarrow H_{ms}^2(Y, 2)$ and it will therefore be enough to show that the relative modified syntomic cohomology group $H_{ms}^2(Y, Y - Y', 2)$ is trivial. The modified syntomic cohomology can be written as the limit of cohomologies of certain cones [Bes98, 5.6.1]. Writing the associated long exact sequence, which are valid also for relative cohomology, we get

$$\cdots \rightarrow H_{\text{rig}}^1(Y_\kappa, Y_\kappa - Y'_\kappa/K) \rightarrow H_{ms}^2(Y, Y - Y', 2) \rightarrow \text{Fil}^2 H_{dR}^2(Y_K, Y_K - Y'_K/K) \rightarrow \cdots$$

The relative cohomologies on either side can be rewritten as cohomologies with support and since Y' is smooth over \mathcal{O}_K we can use purity (known for rigid cohomology by [Ber97, Corollaire 5.7]) to get

$$H_{\text{rig}}^1(Y_\kappa, Y_\kappa - Y'_\kappa/K) \cong H_{\text{rig}, Y'_\kappa}^1(Y_\kappa/K) = 0$$

and

$$\text{Fil}^2 H_{dR}^2(Y_K, Y_K - Y'_K/K) \cong \text{Fil}^2 H_{dR, Y'_K}^2(Y_K/K) \cong \text{Fil}^1 H_{dR}^0(Y'_K/K) = 0 \quad .$$

□

Corollary 3.6. *With Y as in the previous lemma, if $s, s' \in K_2(Y)$ have the same restriction to $K_2(K(Y))$, then $ch_{2,2}(s) = ch_{2,2}(s')$.*

Proof of theorem 3. Let $s \in K_2(Z)$. We can write the restriction of s to the function field $L(Z)$ as $\sum \{f_i, g_i\}$ with f_i, g_i in $L(Z)$. Let π be a uniformizer of L . We may write each $h \in L(Z)$ as $h_1 \pi^k$ where $k \in \mathbb{Z}$ and the divisor of h_1 does not include the special fiber Z_κ . Using that we may rewrite the restriction to $L(Z)$ as

$$s|_{L(Z)} = \sum \{f_i, g_i\} + \{\pi, f^n\}, \quad \text{with} \quad \text{ord}_{Z_\kappa} f_i = \text{ord}_{Z_\kappa} g_i = \text{ord}_{Z_\kappa} f = 0.$$

Furthermore, the tame symbols of $s|_{L(Z)}$ must all be trivial. The tame symbol on Z_κ equals $t_{Z_\kappa}(\pi, f^n) = f^n|_{Z_\kappa}$, so $f^n|_{Z_\kappa} = 1$. It follows that we may write $f^n - 1 = \pi^k h$ where $k \in \mathbb{Z}_{>0}$, $h \in L(Z)$ and $\text{ord}_{Z_\kappa} h = 0$. In $K_2(L(Z))$ we therefore have

$$(3.6) \quad -k\{\pi, f^n\} = \{\pi^{-k}, f^n\} = \{\pi^{-k}(f^n - 1), f^n\} = \{h, f^n\}.$$

Let $Y \subset Z$ be an affine open on which f_i, g_i, f and h become invertible, and let U be the corresponding basic wide open. It follows from corollary 3.6 that

$$ch_{2,2}(s)|_Y = \sum ch_{2,2}(\{f_i, g_i\}) - \frac{1}{k} ch_{2,2}(\{h, f^n\}) \quad \text{in} \quad H_{\text{ms}}^2(Y, 2).$$

Let $[\eta]$ be the image of $s \in K_2(X)$ in $H_{dR}^1(Z_L)$, represented by the one-form η . It follows from the considerations above that

$$(3.7) \quad [\eta]|_U = \sum_i [\eta(f_i, g_i)] - 1/k [\eta(h, f^n)] \in H^1(U) \quad .$$

We choose a branch of the p -adic logarithm such that $\log(\pi) = 0$. This implies that at any point $x \in Z_L$ we have $\log t_x(\pi, f^n) = 0$. Since the tame symbols of $s|_{L(Z)}$ are all 1 this implies that

$$(3.8) \quad \sum_i \log t_x(f_i, g_i) = 0, \quad \text{for all} \quad x \in Z_L.$$

At this point we extend scalars to \mathbb{C}_p and set as usual $C = Z_{\mathbb{C}_p}$. Let $\omega \in \Omega^1(C)$. We need to show that

$$[\omega] \cup [\eta] = \sum_i \int_{(f_i)} \log g_i \cdot \omega.$$

The cup product formula of Serre tells us that the left hand side can be computed as $\sum_{x \in C} \text{Res}_x(F_\omega \cdot \eta)$. Here, F_ω need only be a local integral of ω around x and the result is independent of the choices. If we want to write the cup product as a sum of terms corresponding to the decomposition (3.7), however, these independent choices are not sufficient because the forms $\eta(f_i, g_i)$ have residues along the annuli

e so the result depends on the choice of local integrals. By choosing F_ω to be the Coleman integral of ω , however, this splitting,

$$(3.9) \quad [\omega] \cup [\eta] = \sum_i \sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \cdot \eta(f_i, g_i) - \frac{1}{k} \sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \cdot \eta(h, f^n) \quad ,$$

becomes possible. This is because the Coleman integral is unique up to a constant and the sum of residues of a rigid form over all annuli ends is 0 by [Col89, Prop. 4.3]. We treat the last term first. According to proposition 3.4 we have

$$\sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \cdot \eta(h, f^n) = \sum_{x \in C} (\log t_x(h, f^n)) \cdot F_\omega(x) + \int_{(h)} \log f^n \cdot \omega \quad .$$

While the left hand side is only defined because h and f^n are units in \mathcal{O}_Y , the right hand side is defined for any two rational functions and it factors through K_2 . By (3.6) we see that the above expression equals

$$-k \left(\sum_{x \in C} (\log t_x(\pi, f^n)) \cdot F_\omega(x) + \int_{(\pi)} \log f^n \cdot \omega \right) = 0 \quad .$$

It now follows, again by 3.4, that the right hand side of (3.9) equals

$$\begin{aligned} \sum_i \left(\sum_{x \in C} (\log t_x(f_i, g_i)) \cdot F_\omega(x) + \int_{(f_i)} \log g_i \cdot \omega \right) \\ = \sum_i \int_{(f_i)} \log g_i \cdot \omega + \sum_{x \in C} F_\omega(x) \left(\sum_i \log t_x(f_i, g_i) \right) . \end{aligned}$$

The last summand is 0 by (3.8) and so the theorem is proved. \square

4. RESIDUES AND A RECIPROCITY LAW

In this section we prove a reciprocity law, corollary 4.11, which will be used in the next section to prove proposition 3.4 and hence the main theorem. To state it, we will make a certain extension to the p -adic (or algebraic) notion of residue. We start by briefly recalling the setup, to be found in [Col89, II].

Definition 4.1. An open annulus is a rigid space over \mathbb{C}_p isomorphic to $A(r, s) := \{x \in \mathbb{C}_p : r < |x| < s\}$ with $r, s \in |\mathbb{C}_p^\times|$. A uniformizing parameter for an annulus V is a rigid function giving an isomorphism of V with some $A(r, s)$.

Given an annulus V and a uniformizing parameter z on V we can define residues in the usual way. A rigid form $\omega \in \Omega^1(V)$ can be written as $\omega = \sum_{n=-\infty}^{\infty} a_n z^n dz$. The residue of ω with respect to z is the constant a_{-1} . Clearly the residue of dh is 0 if $h \in A(V)$. It can be shown that the residue is independent of the parameter z up to a (unique) sign. Clearly the sign is reversed if we switch from z to z^{-1} .

Definition 4.2. An orientation of an annulus is a choice of a residue function $\text{Res} : \Omega^1(U) \rightarrow \mathbb{C}_p$, equal to the residue function with respect to some uniformizing parameter. The reverse orientation is given by the function $-\text{Res}$. An annulus together with an orientation is called an oriented annulus.

The residue gives an isomorphism $H^1(V) \rightarrow \mathbb{C}_p$. Suppose $f : V \rightarrow W$ is a rigid map between oriented annuli. Since $f^*dA(W) \subset dA(V)$, the following definition makes sense.

Definition 4.3. The degree of a map f is the unique number $\deg f$ such that $\text{Res}_V f^* \omega = \deg f \cdot \text{Res}_W \omega$.

To extend the definition of residues to somewhat more general “functions” we make use of the following trivial linear algebra lemma.

Lemma 4.4. *Let F be a field of characteristic different from 2. Let B be an F vector space and let $r : B \rightarrow F$ be a non zero linear map. Suppose we are given a bilinear pairing $\langle \cdot, \cdot \rangle : \text{Ker}(r) \times B \rightarrow F$ whose restriction to $\text{Ker}(r) \times \text{Ker}(r)$ is anti-symmetric. Then there is a unique anti-symmetric extension of $\langle \cdot, \cdot \rangle$ to $B \times B$.*

Proof. Suppose we are given such an extension $\ll \cdot, \cdot \gg$. Choose $x \in B$ such that $r(x) \neq 0$. For $y, z \in B$ we have unique $\alpha, \beta \in F$ and $y', z' \in \text{Ker}(r)$ such that $y = y' + \alpha x$ and $z = z' + \beta x$. We then must have

$$(4.1) \quad \begin{aligned} \ll y, z \gg &= \ll y' + \alpha x, z' + \beta x \gg = \ll y', z' \gg + \beta \ll y', x \gg - \alpha \ll z', x \gg \\ &= \langle y', z' \rangle + \beta \langle y', x \rangle - \alpha \langle z', x \rangle. \end{aligned}$$

This shows uniqueness. To show existence one only has to show that the formula (4.1) in fact defines an anti-symmetric extension of $\langle \cdot, \cdot \rangle$. The anti-symmetry is obvious once one substitutes $y' = z'$ and $\alpha = \beta$ in (4.1) to get 0 using the anti-symmetry of $\langle \cdot, \cdot \rangle|_{\text{Ker}(r) \times \text{Ker}(r)}$. If $y = y'$ and $\alpha = 0$ we get in (4.1)

$$\ll y, z \gg = \langle y, z' \rangle + \beta \langle y, x \rangle = \langle y, z \rangle,$$

which shows that $\ll \cdot, \cdot \gg$ indeed extends $\langle \cdot, \cdot \rangle$. \square

We apply this lemma to the following situation: Let V be an oriented annulus. We let $B = \int \Omega^1(V)$, the collection of all Coleman integrals of rigid forms on V . The map r will be given by $r(F) := \text{Res}(dF)$, which is well defined because $dF \in \Omega^1(V)$. Our pairing will be defined by $\langle F, G \rangle := \text{Res } FdG$. This is well defined when $r(F) = 0$ as this is equivalent to $F \in A(V)$. If also $G \in A(V)$, then we have

$$0 = \text{Res } d(FG) = \text{Res}(FdG + GdF) = \langle F, G \rangle + \langle G, F \rangle,$$

showing the antisymmetry. Our lemma therefore gives the following result.

Proposition 4.5. *There is a unique anti-symmetric function,*

$$\text{ind}_V : \int \Omega^1(V) \times \int \Omega^1(V) \rightarrow \mathbb{C}_p,$$

such that $\text{ind}_V(F, G) = \text{Res}_V(FdG)$ whenever $F \in A(V)$.

An immediate consequence of the uniqueness is the following functoriality property.

Lemma 4.6. *If $f : V \rightarrow W$ is a map of oriented annuli, then for any $F, G \in \int \Omega^1(W)$ we have*

$$\text{ind}_V(f^*F, f^*G) = \deg f \cdot \text{ind}_W(F, G).$$

In particular, if W' is the annulus W with the reversed orientation, then $\text{ind}_{W'}(F, G) = -\text{ind}_W(F, G)$.

We now globalize. Suppose U is a basic wide open in a curve C . Given two Coleman functions, F and G , such that their restrictions to all annuli ends $e \in \text{End}(U)$ are of the type described above, we would like to describe the sum $\sum_e \text{ind}_e(F, G)$. The first result along these lines is

Lemma 4.7. *If $F \in A(U)$ and $dG \in \Omega^1(U)$, then $\sum_{e \in \text{End}(U)} \text{ind}_e(F, G) = 0$.*

Proof. As $\text{Res}_e dF = 0$, for all $e \in \text{End}(U)$, we see that $\text{ind}_e(F, G) = \text{Res}_e(FdG)$ for all e . As $FdG \in \Omega^1(U)$, the result follows because by [Col89, Proposition 4.3], for any $\omega \in \Omega^1(U)$, $\sum_{e \in \text{End}(U)} \text{Res}_e \omega = 0$. \square

Proposition 4.8. *There is a canonical split exact sequence*

$$0 \rightarrow H^1(C) \xrightarrow{j^*} H^1(U) \xrightarrow{\text{Res}} \bigoplus_{e \in \text{End}(U)} \mathbb{C}_p.$$

Proof. The short exact sequence is just part of the standard long exact sequence in rigid cohomology

$$\cdots \rightarrow \bigoplus H_{\text{rig}, x}^1(X_{\bar{F}_p}/\mathbb{C}_p) \rightarrow H^1(X_{\bar{F}_p}/\mathbb{C}_p) \rightarrow H^1(Y_{\bar{F}_p}/\mathbb{C}_p) \rightarrow \bigoplus H_{\text{rig}, x}^2(X_{\bar{F}_p}/\mathbb{C}_p) \rightarrow \cdots.$$

Here X and Y are the smooth $\mathcal{O}_{\mathbb{C}_p}$ schemes that give rise to C and U as in section 2 and the direct sums are over all $x \in X_{\bar{F}_p} - Y_{\bar{F}_p}$. The first direct sum is 0 and in the second each summand is \mathbb{C}_p by purity. Explicitly, we can always represent a cohomology class in $H^1(C)$ by a form of the second kind ω with no poles on U . The map j^* is then given simply by restriction to U . The map Res is given by $\eta \mapsto (\text{Res}_e \eta)_e$. Considering the action of Frobenius we see that the eigenvalues on $H^1(C)$ are, by the Weil conjectures for crystalline cohomology [KM74], algebraic integers with absolute value \sqrt{q} . On $\bigoplus \mathbb{C}_p$ Frobenius acts as multiplication by q . This gives the splitting, which is easily seen to be independent of the choice of ϕ . \square

We obtain a canonical projection $\mathbf{p} : H^1(U) \rightarrow H^1(C)$.

Lemma 4.9. *For any $h \in A(U)$, $\mathbf{p}(\text{dlog } h) = 0$.*

Proof. First we notice that if $U' \hookrightarrow U$ is an injection of basic wide open spaces in C , then there is a commutative diagram

$$\begin{array}{ccc} H^1(U) & \longrightarrow & H^1(U') \\ \mathbf{p} \downarrow & & \downarrow \mathbf{p} \\ H^1(C) & \xlongequal{\quad} & H^1(C). \end{array}$$

We may consider an underlying affinoid Z in U and assume that $\|h|_Z\| = 1$. We then remove from U the residue discs of points where the reduction of $h|_Z$ is 0 and obtain a new wide open U' with an underlying affinoid on which $|h| = 1$. The proof of [CdS88, Lemma 2.5.1] shows that for some Frobenius endomorphism ϕ on U' of a large enough degree q we have $(\phi^* - q) \text{dlog } h \in dA(U')$. This gives what we want. \square

We can now state the main result of this section.

Proposition 4.10. *Let F and G be Coleman functions on a basic wide open U such that $dF, dG \in \Omega^1(U)$. Denote by $[dF], [dG]$ the corresponding cohomology classes in $H^1(U)$. Then*

$$\sum_{e \in \text{End}(U)} \text{ind}_e(F, G) = \mathbf{p}[dF] \cup \mathbf{p}[dG].$$

Corollary 4.11. *If $h \in A(U)$ and F is a Coleman function on U with $dF \in \Omega^1(U)$, then*

$$\sum_{e \in \text{End}(U)} \text{ind}_e(F, \log h) = 0.$$

Proof of proposition 4.10. It follows from lemma 4.7 that setting

$$\langle [dF], [dG] \rangle := \sum_i \text{ind}_{e_i}(F, G)$$

gives a well defined pairing $\langle \cdot, \cdot \rangle : H^1(U) \times H^1(U) \rightarrow \mathbb{C}_p$. Let $\phi : U \rightarrow U$ be a Frobenius endomorphism of degree q , preserving all ends. It is easy to see that ϕ has degree q at all ends. It therefore follows from lemma 4.6 that $\langle \phi^*x, \phi^*y \rangle = q \langle x, y \rangle$ for any $x, y \in H^1(U)$. The formula of Serre for the cup product on $H^1(C)$ implies that $\langle j^*(x), j^*(y) \rangle = x \cup y$ if $x, y \in H^1(C)$. All we need therefore to show is that $\langle H^1(U), \text{Ker } \mathbf{p} \rangle = 0$. This is now clear from weight considerations: Suppose $x \in \text{Ker } \mathbf{p}$. Then $\phi^*x = qx$, which implies that for any $y \in H^1(U)$ we have

$$\langle \phi^*y, x \rangle = q^{-1} \langle \phi^*y, \phi^*x \rangle = \langle y, x \rangle.$$

Thus, pairing with x provides a Frobenius invariant functional on $H^1(U)$, which is clearly impossible unless this functional is 0. \square

Remark 4.12. It seems that one can give a simple proof of the main theorem of [Col89] using this proposition. In particular, the projection \mathbf{p} should extend the map in Theorem 4.10 there.

Proposition 4.13. *Let G and H be Coleman functions on U such that $G \in A(U)$, $dH \in \Omega^1(U)$ and $\text{Res}_e dH = 0$ at all $e \in \text{End}(U)$. Let I be a Coleman function such that $dI = HdG$. Let $f \in A(U)$. Then*

$$\sum_{e \in \text{End}(U)} \text{Res}_e GH \, d\log f = \sum_{e \in \text{End}(U)} \text{ind}_e(GH, \log f) = \sum_{e \in \text{End}(U)} \text{ind}_e(I, \log f).$$

Proof. The assumptions on H imply that $H|_e \in A(e)$ for all e , and therefore the same holds true for GH . This implies the first equality. It also shows that $dI|_e \in \Omega^1(e)$ and therefore the expression $\text{ind}_e(I, \log f)$ makes sense. We have $GdH \in \Omega^1(U)$ and we can find a Coleman function J with $dJ = GdH$. As $d(GH) = GdH + HdG = dJ + dI$ we see that $GH = I + J + \text{Const}$ (this is just a complicated way to write integration by parts). Corollary 4.11 shows that $\sum_e \text{ind}_e(J + \text{Const}, \log f) = 0$, giving the result. \square

5. COMPUTATION OF THE REGULATOR

To prove proposition 3.4, and hence the main theorem, we will introduce a new regulator, which we will show equals each side of (3.4) in turn.

Proposition 5.1. *Let U be a basic wide open, $f, g \in A(U)$ and $\omega \in \Omega^1(U)$ with $\text{Res}_e \omega = 0$ for all ends e of U . Let F_ω be a Coleman integral of ω and choose a Coleman integral $\int F_\omega \, d\log g$. Then the expression*

$$\rho(f, g)([\omega]) := \sum_{e \in \text{End}(U)} \text{ind}_e(\log f, \int F_\omega \, d\log g)$$

is well defined and depends only on the cohomology class $[\omega] \in H^1(C) \subset H^1(U)$. It therefore defines a map $\rho : A(U) \otimes A(U) \rightarrow \text{Hom}(H^1(C), \mathbb{C}_p)$.

Proof. The condition on ω implies that $F_\omega|_e \in A(e)$ for all e , showing that the expression $\text{ind}_e(\log f, \int F_\omega \text{dlog } g)$ is well defined. If ω is exact, then $F_\omega \in A(U)$ hence $F_\omega \text{dlog } g \in \Omega^1(U)$ and $\sum_e \text{ind}_e(\log f, \int F_\omega \text{dlog } g) = 0$ by corollary 4.11. \square

Remark 5.2. Because morally, $\rho(f, g)([\omega]) = \sum_i \text{Res}_{e_i} \log f \text{dlog } g F_\omega = [\log f \text{dlog } g] \cup [\omega]$, this regulator should be compared with the complex regulator (1.1).

Proposition 5.3. *Let f, g and ω be as above and let $\eta(f, g)$ be as in proposition 3.3. Then*

$$\rho(f, g)([\omega]) = \sum_{e \in \text{End}(U)} \text{Res}_e F_\omega \eta(f, g).$$

Before giving the actual proof we give a heuristic proof. This proof demonstrates that the proposition would be very easy if we could treat logs as good functions, and in particular take their residues. The actual proof repeats the same considerations using the local indices.

The idea is that by the trick of remark 3.1, and in particular (3.1), we have

$$\begin{aligned} \left(1 - \frac{\phi^*}{q^2}\right) (\log f \text{dlog } g) &= \left(1 - \frac{\phi^*}{q}\right) \log f \cdot \frac{\phi^*}{q} \text{dlog } g + \log f \cdot \left(1 - \frac{\phi^*}{q}\right) \text{dlog } g \\ &= \frac{1}{q^2} \log f_0 \text{dlog } \phi^* g + \frac{1}{q} \log f \text{dlog } g_0 \\ &= \left(1 - \frac{\phi^*}{q^2}\right) \eta(f, g) + d(\log f \log g_0) + d(\quad), \end{aligned}$$

where the last equality follows from proposition 3.3. In other words, $\left(1 - \frac{\phi^*}{q^2}\right) (\log f \text{dlog } g)$ and $\left(1 - \frac{\phi^*}{q^2}\right) \eta$ “have the same cohomology class”. Now one should argue that $\left(1 - \frac{\phi^*}{q^2}\right)$ is invertible on cohomology. Let us make this a bit more reasonable. We have

$$\phi^* \omega \cup \left(1 - \frac{\phi^*}{q^2}\right) \eta = \phi^* \omega \cup \eta - \frac{1}{q} \omega \cup \eta = \left(\phi^* - \frac{1}{q}\right) \omega \cup \eta.$$

As the same is valid with $\log f \text{dlog } g$ in place of η and as $\left(\phi^* - \frac{1}{q}\right)$ is invertible on $H^1(C)$ we are done. Now for the actual proof

Proof of proposition 5.3. We note first that both side of the equation depend only on the cohomology class of ω . We will evaluate both sides on $(\phi^* - 1/q)\omega$. Put $\langle \epsilon, \theta \rangle = \sum_e \text{Res}_e F_\epsilon \theta$ for shorthand, and also $\eta = \eta(f, g)$, $\eta_0 = \eta_0(f, g)$. As $(1 - \phi^*/q^2)\eta = \eta_0 + d(\quad)$ (3.3) we have

$$(5.1) \quad \langle \eta, (\phi^* - 1/q)\omega \rangle = \langle \eta, \phi^* \omega \rangle - (1/q^2) \langle \phi^* \eta, \phi^* \omega \rangle = \langle \eta_0, \phi^* \omega \rangle.$$

Let $\theta = \phi^* \omega$. Then θ is a one form on U whose residues along the ends of U is 0. we have

$$\begin{aligned} \langle \eta_0, \theta \rangle &= \frac{1}{q^2} \sum_e \text{Res}_e \log f_0 F_\theta \text{dlog } \phi^* g - \frac{1}{q} \sum_e \text{Res}_e F_\theta \log g_0 \text{dlog } f \\ &= \frac{1}{q^2} \sum_e \text{ind}_e(\log f_0, \int F_\theta \text{dlog } \phi^* g) + \frac{1}{q} \sum_e \text{ind}_e(\log f, \int F_\theta \text{dlog } g_0), \end{aligned}$$

where we have used proposition 4.13 to rewrite the second summand. Making sure that the local indices make sense at each stage, we now have

$$\begin{aligned}
\langle \eta_0, \theta \rangle &= \frac{1}{q} \sum_e \text{ind}_e(\log f - \frac{\phi^*}{q} \log f, \int F_\theta \, d\log \phi^* g) \\
&\quad + \sum_e \text{ind}_e(\log f, \int F_\theta (d\log g - \frac{\phi^*}{q} d\log g)) \\
&= \sum_e \text{ind}_e(\log f, \int F_\theta \frac{\phi^*}{q} d\log g) - \sum_e \text{ind}_e(\frac{\phi^*}{q} \log f, \int F_\theta \frac{\phi^*}{q} d\log g) \\
&\quad + \sum_e \text{ind}_e(\log f, \int F_\theta d\log g) - \sum_e \text{ind}_e(\log f, \int F_\theta \frac{\phi^*}{q} d\log g) \quad .
\end{aligned}$$

The first and last terms cancel out, leaving us with

$$= \sum_e \text{ind}_e(\log f, \int F_\theta d\log g) - \sum_e \text{ind}_e(\frac{\phi^*}{q} \log f, \int F_\theta \frac{\phi^*}{q} d\log g) \quad ,$$

and using the fact that $\theta = \phi^* \omega$ and the equivariance of the local index (lemma 4.6)

$$\begin{aligned}
&= \sum_e \text{ind}_e(\log f, \int F_{\phi^* \omega} d\log g) - \frac{1}{q} \sum_e \text{ind}_e(\log f, \int F_\omega d\log g) \\
&= \sum_e \text{ind}_e(\log f, \int F_{(\phi^* - 1/q)\omega} d\log g) = \rho(f, g)((\phi^* - 1/q)[\omega]).
\end{aligned}$$

Together with (5.1) we get the equation

$$\rho(f, g)((\phi^* - 1/q)[\omega]) = \sum_e \text{Res}_e \eta(f, g) F_{(\phi^* - 1/q)\omega} \quad ,$$

proving the proposition for $(\phi^* - 1/q)\omega$ instead of ω , and therefore for any form in the cohomology class $[(\phi^* - 1/q)\omega]$. But the operator $(\phi^* - 1/q)$ is invertible on $H^1(C)$ so the proposition follows. \square

The last step is to compute the local indices on an annulus e occurring in the definition of $\rho(f, g)([\omega])$ in terms of the singular points of f and g inside the corresponding disc. By scaling it is enough to consider the open unit disc $D = \{|z| < 1\}$. We need to recall the following.

Lemma 5.4. *Let $e = A(s, 1) \subset D$ be an open annulus.*

1. *Let ω be a rigid differential form on D which is analytic on e and has at most a finite number of poles otherwise. Then $\text{Res}_e \omega = \sum_{x \in D} \text{Res}_x \omega$.*
2. *Let $f \in A(D)^\times$. Then $\log(f) \in A(D)$.*

Proof. The first part is just [Col89, Prop 2.3]. The disc D is in particular a basic wide open space and e is its unique annulus end. By the first part we have $\text{ord}_e(f) := \text{Res}_e d\log f = 0$. It therefore follows from [Col89, Lemma 4.8] that $|cf - 1|$ is bounded above by $\alpha < 1$ for some $c \in \mathbb{C}_p$. It follows that $\log(cf) = \log c + \log f$ is a convergent power series in $cf - 1$ and is therefore analytic. \square

Proposition 5.5. *Let f and g be meromorphic functions on D with a finite number of poles and zeros. Let $\omega \in \Omega^1(D)$ and choose an integral F_ω . Let $e = A(s, 1)$ be an*

annulus on which both f and g are invertible. Let H be a Coleman function on D such that $dH = \log g \cdot \omega$ and let $G = \log g \cdot F_\omega - H$, so that $dG = F_\omega d\log g$. Then

$$\mathrm{ind}_e(\log f, G) = \sum_{x \in D} (\log t_x(f, g) \cdot F_\omega(x) + \mathrm{ord}_x(f) \cdot H(x)),$$

where we use definition 2.7 to compute $H(x)$ when needed.

Proof. First we claim that the truth of the proposition is independent of the choice of H . Indeed, if we replace H by $H + C$, where C is a constant, then G is replaced by $G - C$. With the convention of definition 2.7 this is true even at the support of the divisor of g . Thus, the claim follows from

$$\mathrm{ind}_e(\log f, C) = -\mathrm{Res}_e C d\log f = -C \sum_{x \in D} \mathrm{ord}_x(f),$$

where the last equality follows from lemma 5.4. Let

$$S = \{x \in D : \mathrm{ord}_x(f) \neq 0 \text{ or } \mathrm{ord}_x(g) \neq 0\}.$$

The proof will be by induction on the size of S . When it is 0 both sides vanish, the right hand side trivially and the left hand side because by lemma 5.4 both $\log f$ and G are analytic on D . If $S \neq \emptyset$, then we may assume that $0 \in S$ by translation. When $S = \{0\}$ the proof will be given below. Suppose then that S has at least 2 elements. Let $r = \max_{x \in S} |x|$ and set $S_r = \{x \in S, |x| = r\}$. There is some $t < r$ such that

$$W := \{z \in D : \forall x \in S, |z - x| > t\}$$

is a basic wide open disjoint from S . It is obtained by removing from \mathbb{P}^1 closed discs D_i , $i = 0, \dots, k$, with $D_0 = \{|z| \geq 1\}$, $D_i = D[a_i, t]$ for some $a_i \in D$ for $i \geq 1$, and such that $S \subset \bigcup_{i=1}^k D_i$. Let e_i , $i = 0, \dots, k$ be the corresponding annuli ends of W . Then e_0 is just e with the reversed orientation. Therefore, corollary 4.11 and lemma 4.6 imply that

$$\mathrm{ind}_e(\log f, G) = \sum_{i=1}^k \mathrm{ind}_{e_i}(\log f, G).$$

This shows that the statement of the proposition is true for D if it is true for the discs $D(a_i, t)$ after translation and scaling to the open unit disc. Since 0 and any $x \in S_r$ are not in the same disc D_i , we are done by induction.

It remain to prove the result when $S = \{0\}$. Both sides of the equation are bilinear in f and g (with respect to multiplication) and linear in F_ω . This follows from our claim at the beginning of the proof and from lemma 2.8. It is therefore sufficient to consider the cases listed below.

If $\mathrm{ord}_0 f \neq 0$ and $\mathrm{ord}_0 g = 0$, then $dG = F_\omega d\log g \in \Omega^1(D)$, hence $G \in A(D)$. It follows that

$$\begin{aligned} \mathrm{ind}_e(\log f, G) &= -\mathrm{Res}_e G d\log f = -\mathrm{ord}_0(f) \cdot G(0) \\ &= \mathrm{ord}_0(f) \cdot (H(0) - \log g(0) \cdot F_\omega(0)) = \log t_0(f, g) \cdot F_\omega(0) + \mathrm{ord}_0(f) \cdot H(0). \end{aligned}$$

If $\mathrm{ord}_0 f = 0$ and $\mathrm{ord}_0 g \neq 0$, then $\log f \in A(D)$, so

$$\begin{aligned} \mathrm{ind}_e(\log f, G) &= \mathrm{Res}_e \log f dG = \mathrm{Res}_e \log f F_\omega d\log g = \mathrm{Res}_0 \log f F_\omega d\log g \\ &= \mathrm{ord}_0(g) \cdot \log f(0) \cdot F_\omega(0) = \log t_0(f, g) \cdot F_\omega(0). \end{aligned}$$

If $f(z) = g(z) = z$ and $F_\omega(0) = 0$, then we still have $dG \in \Omega^1(D)$ so we can compute as in the first case. Our convention exactly says that because $F_\omega(0) = 0$ we have $-G(0) = H(0)$. It follows that

$$\text{ind}_e(\log f, G) = H(0) = \log t_0(z, z) \cdot F_\omega(0) + \text{ord}_0(z) \cdot H(0).$$

Finally, if $f(z) = g(z) = z$ and $F_\omega = C$, where C is a constant function, then $dH = 0$ so H is constant and $G = C \log z - H$. It follows that

$$\begin{aligned} \text{ind}_e(\log f, G) &= \text{ind}_e(\log z, C \log z - H) \\ &= -\text{ind}_e(\log z, H) = \text{Res}_e H \, d\log z = H = H(0). \end{aligned}$$

The proof is complete. \square

Proof of proposition 3.4. Let D_e be the residue disc corresponding to the annulus e . Then

$$\begin{aligned} & \sum_{e \in \text{End}(U)} \text{Res}_e(F_\omega \cdot \eta(f, g)) \\ &= \sum_{e \in \text{End}(U)} \text{ind}_e(\log f, \int F_\omega \, d\log g) \quad \text{Prop. 5.3} \\ &= \sum_{e \in \text{End}(U)} \sum_{x \in D_e} \left((\log t_x(f, g)) \cdot F_\omega(x) + \text{ord}_x(f) \cdot \left(\int \log g \cdot \omega \right)(x) \right) \text{Prop. 5.5} \\ &= \sum_{x \in C} (\log t_x(f, g)) \cdot F_\omega(x) + \int_{(f)} \log g \cdot \omega. \end{aligned}$$

\square

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