

# **System Identification**

## **A Frequency Domain Approach**

**Second Edition**

**Rik Pintelon**  
**Johan Schoukens**

**MATLAB<sup>®</sup>**  
*examples*



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