SYSTEM IDENTIFICATION PROBLEMS AND THE METHOD OF MOMENTS

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1. Introduction

Let $X(t)$ and $W(t)$ be vectors of dimension $N > 0$. We are concerned with the problem of computing an $N \times N$ matrix A such that

$$X'(t) = AX(t) + W(t), \quad t > 0 \tag{1.1}$$

where $X'(t)$ is the rate of change of $X(t)$ with respect to time t. Such problems frequently arise in the biosciences although it is not always immediately evident that they may be posed in the form of Eq. (1.1). In applications the data for $X(t)$ and $W(t)$ is obtained from experiments and the collection of such data is not always performed over equal time intervals nor is it always the case that data is obtained for the scalar functions $X_i(t)$ and $W_i(t)$, which form the components of $X(t)$ and $W(t)$ respectively, at the same instances of time. Also, certain entries in the matrix A are known a priori or relationships between entries must be satisfied.

The object of this paper is to introduce a method by which the system identification problem, i.e. the problem discussed above, may be analyzed. In particular, we consider the moments

$$\text{mom}_n(X) = \int_0^\infty t^n X(t)\, dt, \quad n = 0, 1, \cdots \tag{1.2}$$

of the data; or in the event that such integrals do not exist, we compute the moments of $X(t)e^{-pt}$ where $p > 0$ is sufficiently large.

The method of moments may be developed from the principle that the matrix A should be determined in such a manner so as to achieve the best fit between the moments of $X(t)$ as computed from the data and the moments of $X(t)$ as computed from the differential equation (1.1). This is the point of view taken in [1].

It is interesting and somewhat surprising that the equations resulting from the above principle form a finite dimensional linear system (because of relations between the moments), and the matrix A, which results from the solution of the consistent system of linear equations, provides an exact fit.

In addition to the general theory we present specific examples which are representative of three classes of problems.

The first example is fluorescence decay in which the object is to compute the solution I(t) of an integral equation of Volterra type as an exponential decay sum. I. Isenberg and co-workers [2] [3] [4] [5] have developed the method of moments for this particular application where is has particular advantage in avoiding non-random errors [4] [6]. Our present discussion enlarges the class of problems to which the method may apply.

Our second example is a compartmental analysis model for liver disorder. Such a model has been developed from the stochastic process point of view [7], but we consider it here as a system of differential equations of the form (1.1). There is a mathematical relationship between these two types of models [8]. The computer algorithm contructed for the stochastic model [7] attempts to calculate the best fit, in the least squares sense, by a combination of methods: Monte Carlo trials, Gauss Least Squares, Direct Pattern Search. There are several difficulties with this approach: ill-conditioning, lack of convergence, and even in the case of a "convergent sequence" it is not clear that the "limit" is indeed the best fit. In fact, the induced nonlinear regression problem may not have a unique solution.

In contrast, we shall see that the moment approach tends to alleviate such difficulties. For the liver model in question, we obtain a linear, 4×4, system of equations.

For our third example we consider a scalar differential equation of the form

$$C_0 Y^{(N)} + C_1 Y^{(N-1)} + \cdots + C_N Y(t) = f(t) \tag{1.3}$$

which arises in cardiology. In this example N = 2 and the coefficients C_0, C_1, C_2 are dependent on two unknown parameters.

These examples were chosen to indicate the versatility of the method to different types of applications.

2. The method of moments. Consider first the case where the integrals defining the moments of X(t) and W(t) are absolutely convergent. In particular, it is necessary that

$$X(t) \to 0, \quad W(t) \to 0 \quad \text{as } t \to \infty. \tag{2.1}$$

Define the vectors

$$G(s+1) = \text{mom}_s (X) / s!, \; s = 0, 1, \cdots \tag{2.2}$$

$$M(s+1) = \text{mom}_s (W) / s!, \quad s = 0, 1, \cdots \tag{2.3}$$

Then from Eq. (1.1)

$$(s!)^{-1} \text{mom}_s (X) = (s!)^{-1} \text{mom}_s (A X) + (s!)^{-1} \text{mom}_s (W) \tag{2.4}$$

If s = 0 the term on the right may be integrated and one obtains, in view of (2.1),

$$X(0) = -AG(1) - M(1) \tag{2.5}$$

If s > 0, we integrate (2.4) by parts to obtain

$$AG(s+1) = -G(s) - M(s+1), \quad s = 1,2,\cdots . \tag{2.6}$$

Without prior knowledge of the entries in A we require N equations of the form (2.6) to determine A. Let $\Phi(s)$ be the N x N matrix whose columns are $G(s + 1), \cdots, G(s + N)$, and let $X(s)$ be the N x N matrix whose columns are $M(s + 1), \cdots, M(s + N)$. Then, from (2.6),

$$A\Phi(s) = -\Phi(s - 1) - X(s), \quad s = 1,2,\cdots \tag{2.7}$$

Thus formally,

$$A = -[\Phi(s-1) + X(s)]\Phi^{-1}(s). \tag{2.8}$$

One may compute A by one of the standard Gaussian elimination techniques.

In practice, as we will show by examples, the number of constants we wish to determine is less than N^2 so if we use a set of N equations form (2.6) to determine the unknown parameters in A we will have redundancies. Thus it is better to work directly with (2.6) with A expressed in terms of its assumed form and unknown parameters. After this is done we see precisely how many linear equations are needed to compute the unknown parameters in A.

In any event one obtains a system of linear equations for determining A. In particular, it becomes obvious when a unique solution may be computed from the data.

Once A is determined we may use (2.5) to compute the initial values of X(t). Even if the initial values could be determined directly from the data such a determination may not be as reliable as that obtained from (2.5) which makes use of all available data and not just the points near t = 0. Having obtained both the matrix A and the initial values one could then compute the solution X(t) of the differential equation by a numerical algorithm. This would serve to check the calculation of A.

We now proceed to the general case where (2.1) need not be satisfied. We assume however that there exists a positive number p such that

$$X(t)e^{-pt} \to 0, \quad W(t)e^{-pt} \to 0 \quad \text{as } t \to \infty \tag{2.9}$$

Such an assumption is almost always satisfied in applications. Let

$$X(t,p) = X(t)e^{-pt}, \; W(t,p) = W(t)e^{-pt} \; . \tag{2.10}$$

Then Eq. (1.1) becomes

$$X'(t,p) = A(p)X(t,p) + W(t,p) \tag{2.11}$$

where

$$A(p) = A - pI \tag{2.12}$$

and I is the identity matrix. One may then apply the above method to compute A(p) and thus A.

3. Fluorescence Decay. In fluorescence decay experiments, discrete data are collected for the fluorescence decay, F(t), and for the lamp flash excitation, E(t). The impulse function I(t) is related to F(t) and E(t) by convolution,

$$F(t) = \int_0^t E(t - s)\, I(s)\, ds \; . \tag{3.1}$$

One assumes a "model" for I(t),

$$I(t) = \sum_{i=1}^{N} \alpha_i \exp(-t/\tau_i), \; t \geq 0 \tag{3.2}$$

where the time constants τ_i are distinct positive numbers and the amplitudes α_i are positive.

According to the model (3.2), I(t) is a solution of an Nth order scalar differential equation

$$I^{(N)} + C_1 I^{(N-1)} + \cdots + C_N I(t) = 0 \tag{3.3}$$

whose characteristic polynomial

$$P(\lambda) = \lambda^N + C_1 \lambda^{N-1} + \cdots + C_N \tag{3.4}$$

has negative distinct roots $-\tau_i^{-1}$. In order to pose the problem in the form (1.1) we define the vector X(t) whose components $X_i(t)$ satisfy

$$X_i(t) = I^{(i)}(t), \quad i = 0, 1, \cdots, N-1 \tag{3.5}$$

Then X(t) is a solution of

$$X'(t) = AX(t) \tag{3.6}$$

where the matrix A has the form

$$A = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -C_N & -C_{N-1} & -C_{N-2} & \cdots & -C_1 \end{pmatrix} \tag{3.7}$$

We, have from (2.6) with M = 0,

$$AG(s+1) = -G(s), \quad s = 1, 2, \cdots \tag{3.8}$$

where G(s) is the vector defined in (2.2). Observe that X(t) as defined in

(3.5) has, in view of (3.2), the appropriate behavior at $t = \infty$ so that a transformation of the type (2.9) is not needed in this example. When one writes out the first N - 1 equations given by the matrix equation one observes the identity between the components of $G(s)$, $\{G_i(s)\}_{i=1}^{N}$,

$$G_{i+1}(s) = (-1)^i G_1(s-1), \quad s \geq i+1, \; i = 1,2,\cdots \tag{3.9}$$

which may also be obtained by integration by parts in the definition of $G(s)$. The last equation of the N equations presented by (3.8) may be expressed, with the aid of the identities (3.9), in the form

$$G_1(s) = C_1 G_1(s+1) - C_2 G_1(s+2) + \cdots + (-1)^{N+1} C_N G_1(s+N), \; s \geq 1 . \tag{3.10}$$

Equation (3.10) is known as the "backward moment formula" [1]. By considering N different values of s one obtains a system of linear equations whose solutions are the coefficients C_i. Of course, the constants $G_1(s)$ must be determined first. Isenberg et al. [3] presents a linear recursive formula for calculating the $G_1(s)$ in terms of the moments of F(t) and E(t) whose discrete values are obtained from the fluorescence experiment. Having calculated $-\tau_i^{-1}$ as the roots of the polynomial (3.4), the next step is to determine the amplitudes α_i. Now it is not difficult to see, by direct integration, that $G_1(s)$ may be expressed alternatively as

$$G_1(s) = \sum_{i=1}^{N} \alpha_i \tau_i^s \tag{3.11}$$

Thus N linear equations of the form (3.11) may be used to determine the α_i.

For further details concerning the application of the method of moments to the problem of fluorescence decay and the general problem of exponential separation (see [1] [2] [3] [4] [5]).

4. Liver disorders: four compartment model. Distinguishing hepatitis from biliary atresia is a difficult task especially in infants. One method of diagnosis is to follow the path of radioactive Rose Bengal tracer as is transported through the biliary system. Tracer data is typically collected from blood, urine, and feces samples. From this information one may also assess the portion of Rose Bengal tracer in the liver compartment. A four compartment model based on Markov processes has been introduced in [7] and studied further in [8]. A five compartmental model has been introduced in [9].

Let the compartments be labeled: 1 for blood, 2 for liver, 3 for urine, 4 for feces. Let $X_i(t)$ denote the portion of Rose Bengal tracer in compartment i at time t and let Γ_{ij} denote the rate constant from compartment j to compartment i. Then it turns out that the differential equations are

$$X_1'(t) = -(\Gamma_{31} + \Gamma_{21})\, X_1(t) + \Gamma_{12}\, X_2(t) \tag{4.1}$$

$$X_2'(t) = \Gamma_{21}\, X_1(t) - (\Gamma_{42} + \Gamma_{12})\, X_2(t) \tag{4.2}$$

$$X_3'(t) = \Gamma_{31}\, X_1(t) \tag{4.3}$$

$$X_4'(t) = \Gamma_{42}\, X_2(t) \tag{4.4}$$

Observe that the first two equations (4.1) and (4.2) form a closed system

$$X'(t) = AX(t) \tag{4.5}$$

where

$$A = \begin{pmatrix} -(\Gamma_{31} + \Gamma_{21}) & \Gamma_{12} \\ \Gamma_{21} & -(\Gamma_{42} + \Gamma_{12}) \end{pmatrix} \tag{4.6}$$

It is easy to show that the eigenvalues of A are distinct and negative and the two vector X(t) has the form

$$X(t) = \alpha_1 \exp(-t/\tau_1) + \alpha_2 \exp(-t/\tau_2) \tag{4.7}$$

where τ_1 and τ_2 are distinct positive numbers. The form (4.7) should be expected since Rose Bengal tracer is eventually depleted from the blood and liver and terminates in the urine and feces compartments. In particular the assumptions needed for applying the method of moments are satisfied. Applying the method in the form of (2.7) for a particular value of s(here $\chi \equiv 0$) we determine $a_{11} = -(\Gamma_{31} + \Gamma_{21})$, $a_{12} = \Gamma_{12}$, $a_{21} = \Gamma_{21}$ and $a_{22} = -(\Gamma_{42} + \Gamma_{12})$. These equations may be inverted to give $\Gamma_{12} = a_{12}$, $\Gamma_{21} = a_{21}$, $\Gamma_{31} = a_{21} - a_{11}$, and $\Gamma_{42} = a_{12} - a_{22}$.

Thus one may identify the system (4.1) - (4.4) from the moments in the blood and liver compartments. However such data may not always be available. The most reliable data are collected from the blood and urine compartments. From these two compartments alone one may determine the rate coefficients as follows. We use a transformation of the type (2.9). Choose a positive number p such that $X_1(t)e^{-pt}$ and $X_3(t)e^{-pt} \to 0$ as $t \to \infty$. Define

$$X_3(t,p) = X_3(t)e^{-pt}, \tag{4.8}$$

$$X_1(t,p) = X_1(t)e^{-pt}.$$

Then Eq. (4.3) becomes

$$X_3'(t,p) = \Gamma_{31} X_1(t,p) - p X_3(t,p). \tag{4.9}$$

We apply the method of moments to (4.9) to obtain the relationship

$$\Gamma_{31} G_1(s+1) - p G_3(s+1, P) = -G_3(s,p), \; s \geq 1 \tag{4.10}$$

where $G_1(s,p)$ and $G_3(s,p)$ are the moments of $X_1(t,p)$ and $X_3(t,p)$. These moments are easily obtained from the moments of $X_1(t)$. Using (4.10) Γ_{31} may then be computed.

The method of moments affords a different approach than that used in [7] and [8] where all the available data is used to compute all the parameters. The moment approach allows us to determine a subset of the parameters from a subset of the data. However one requires sufficient data to estimate the moments.

5. Stiffness of the ventricle. The following simple model for describing muscle mechanics is presented in [10]:

$$\frac{1}{K} P''(t) + \frac{1}{\eta} P'(t) + \frac{1}{M} P(t) = \frac{1}{\alpha} V''(t), \; t \geq 0 \tag{5.1}$$

The pressure perturbation P(t), and the volume V(t) are assumed to be sinusoidal functions. The equivalent mass m and the constant α, relating linear displacement to volume change, are also known. We wish to determine the elastic stiffness K and the viscous damping η (we assume the mass m is known).

To apply the method of moments we introduce the vectors X(t,p), W(t,p)

$$X_1(t,p) = P(t)e^{-pt}, \; X_2(t,p) = P'(t)e^{-pt} \tag{5.2}$$

$$W_1(t,p) = 0, \; W_2(t,p) = (1/\alpha)V''(t)e^{-pt}. \tag{5.3}$$

Then the model (5.1) takes the form

$$X'(t,p) = A(p)X(t,p) + K\,W(t,p) \tag{5.4}$$

where

$$A(p) = \begin{pmatrix} -p & 1 \\ -K/m & -((K/\eta)+p) \end{pmatrix} \tag{5.5}$$

The moment relationship (2.5) for this model is

$$-p\,G_1(s+1,\,p) + G_2(s+1,\,p) = -G_1(s,\,p) \tag{5.6}$$

$$-(K/m)\,G_1(s+1,\,p) - (K/\eta + p)\,G_2(s+1,\,p) = -G_2(s,p) + K\,M_2(s+1,p) \tag{5.7}$$

Solve for $G_2(s,\,p)$ and $G_2(s+1,\,p)$ in terms of $G_1(s-1,\,p)$, $G_1(s,\,p)$ and $G_1(s+1,\,p)$ from Eq. (5.6). Substitute into Eq. (5.7) and divided by K to obtain

$$K^{-1}\,H(s,\,p) + \eta^{-1}\,Q(s,\,p) = R(s,\,p) \tag{5.8}$$

where

$$H(s,p) = 2p\,G_1(s,p) - G_1(s-1,\,p) - p^2\,G_1(s+1,\,p), \tag{5.9}$$

$$Q(s,\,p) = G_1(s,\,p) - p\,G_1(s+1,\,p), \tag{5.10}$$

$$R(s,\,p) = M_2(s+1,\,p) + G_1(s+1,\,p)/m. \tag{5.11}$$

By choosing two distinct s values (≥ 2) in (5.8) one can solve for K^{-1} and η^{-1} (and hence K and η) since $H(s,\,p)$, $Q(s,\,p)$ and $R(s,\,p)$ are known.

6. Numerical Examples. To illustrate the performance of the method of moments, we present in this section two numerical examples of three component analysis, using synthetic data which were generated by the computer-simulated fluorescence decay experiments. For comparison, we also present the results obtained by analysis of the data by the method of nonlinear estimation in the least squares sense. All computer programs used in this study were written or modified by the authors in FORTRAN-10. These programs have been run on the DEC system-10 in time-sharing conversational fashion.

In the computer-simulated fluorescence decay experiments, synthetic data were generated as if they were collected by a photon-counting technique. The exciting light was specified to have a flash characteristic of

$$E(t) = \beta \sin \pi t/T \qquad 0 \leq t \leq T$$
$$= 0 \qquad\qquad T < t$$

where β is the maximum number of counts for a single channel, T is the flash time duration, and $E(t)$ is the number of counts at a time t, measured in nanoseconds. For instance, we may choose $\beta = 10^5$ and $T = 30$ in accordance with actual experiments. After choosing a set of appropriate parameters α_i, τ_i to model $I(t)$, equation (3.2), the convolution equation (3.1) is calculated to simulate the fluorescence decay $F_p(t)$. When synthetic noise is added to $F_p(t)$, we then have

$$F(t) = F_p(t) + \varepsilon_t \sqrt{F_p(t)} \qquad -1 \leq \varepsilon_t \leq 1 \tag{6.1}$$

where ε_t is obtained from the random number generator [11]. The noise is distributed in a Gaussian fashion in accordance with actual experiments. Signal-to-noise ratio at each channel is $\varepsilon_t\sqrt{F_p(t)}$; thus the larger the $F(t)$, the better the signal-to-noise ratio. However, $F(t)$ is proportional to β, and so is $E(t)$. Thus $E(t)$ measures the noise level in the system.

Taking 500 channels, $\beta = 10^5$, $T = 30$ and two sets of three component fluorescence decay parameters: $\alpha_1 = 0.1$, $\tau_1 = 5$; $\alpha_2 = 0.05$, $\tau_2 = 15$; $\alpha_3 = 0.024$, $\tau_3 = 30$; and $\alpha_1 = 0.1$, $\tau_1 = 5$; $\alpha_2 = 0.017$, $\tau_2 = 30$; $\alpha_3 = 0.0084$, $\tau_3 = 60$, two sets of three component data of 500 channels were obtained. Table I gives a comparison of the actual decay parameters and the calculated results obtained by analysis of the data by the method of moments. The total number of fluorescence counts is also presented. It should be pointed out that exponential depression [3], moment index displacement [6] [12] and scaling the calculated moments were employed in the computation. For comparison of the accuracy of the estimated parameters with the actual values, the relative errors (in percent) of the estimated parameters are presented in Table II, along with the results obtained by analysis of the data by the method of weighted nonlinear least squares using three different algorithms: Marquardt's, Hartley's, and the gradient method [13] [14] [15] [16]. The method of nonlinear least squares using Marquardt's algorithm in fluorescence decay analysis has been proposed in [17] [18]. An important point to note is that, because the nonlinear least squares algorithms require an initial estimate of the decay parameters, the computation was set up in

such a way that the estimated values obtained by the method of moments served as initial estimates for the least squares algorithms. Good initial values will often lead to a faster convergence, poor initial values may result in convergence to an unwanted stationary point of the error sum of squares surface (i.e. a local minimum) [16]. Study of table II reveals that even if the error sum of squares is decreased, there is no significant improvement in relative error by nonlinear least squares analysis. In other words, in these cases the method of nonlinear least squares does not give better results than the method of moments. It also should be pointed out that the CPU time (central processing unit time, in seconds) of nonlinear least squares is much more than that of the method of moments. Another comment we can also make from these and other experiments we have carried out is that, for data channels up to 500, the nonlinear least squares analysis often leads to finding a local minimum of the error sum of squares, instead of an absolute minimum. As a matter of fact, we are aware that without some additional and independent knowledge of the system, nonlinear least squares analysis has a certain degree of difficulty; and without some additional modification and revision, the nonlinear least squares analysis has less value than the method of moments in fluorescence decay analysis. Moreover, the method of moments can be used to solve various identification problems in addition to the fluorescence decay problems effectively and efficiently.

		α_1	τ_1	α_2	τ_2	α_3	τ_3	TOTAL COUNTS
I	ACTUAL VALUES	0.10	5.00	0.050	15.00	0.024	30.00	
	CALCULATED VALUES	0.11	5.20	0.038	15.49	0.026	30.29	3.76×10^6
II	ACTUAL VALUES	0.10	5.00	0.017	30.00	0.0084	60.00	
	CALCULATED VALUES	0.10	5.10	0.013	31.37	0.0097	60.41	2.89×10^6

TABLE I
Comparison of the Estimated Parameters with the Actual Values by the Method of Moments

		$\Delta\alpha_1$	$\Delta\tau_1$	$\Delta\alpha_2$	$\Delta\tau_2$	$\Delta\alpha_3$	$\Delta\tau_3$	CPU	FINAL WEIGHTED ERROR SUM OF SQUARES
I	METHOD OF MOMENTS	15.96	3.96	24.49	3.28	10.34	0.98	3.43	6.13×10^7
	MARQUARDT	1.66	4.31	9.99	1.86	6.35	1.54	1:27.17	5.23×10^6
	HARTLEY	16.00	4.01	24.00	4.25	8.33	1.79	1:25.20	5.57×10^7
	GRADIENT	16.02	3.96	24.05	3.28	9.35	0.98	1:59.06	5.88×10^7
II	METHOD OF MOMENTS	5.82	2.17	23.75	4.58	15.96	2.36	6.08	2.49×10^8
	MARQUARDT	1.05	1.76	32.80	10.58	45.36	5.50	1:57.16	7.72×10^7
	HARTLEY	0.00	2.02	29.42	7.23	15.48	5.93	2:13.40	2.32×10^8
	GRADIENT	0.03	2.00	30.12	4.57	30.36	2.36	2:5.32	1.21×10^8

TABLE II
Comparison of the Accuracy of the Parameters as Estimated by Different Methods
(Relative error in the estimated parameters in percentage)

7. Discussion. We have shown how the method of moments can be used to solve various identification problems. This method of course requires the calculating of moments of functions generated by discrete values of the independent variable. Consequently the accuracy of the calculated unknown parameters depends on the type of data that is available.

In many problems, for example, the data is collected for finite time T. The error in computing the moments due to this lack of data for $t > T$ is called the cut-off error. This error, in the case of fluorescence decay, is rectified by a method of successive approximations [3]. Further results on the cut-off errors for general systems have been recently developed and will appear in [19].

In addition to the cut-off error there are errors in the data collected caused for example by instrumentation limitations. Thus an error analysis is necessary in order to ascertain the precision of the parameters. This problem we address ourselves elsewhere.

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