System Reliability and Free Riding

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System reliability often depends on the effort of many individuals, making reliability a public good. It is well-known that purely voluntary provision of public goods may result in a free rider problem: individuals may tend to shirk, resulting in an inefficient level of the public good.

How much effort each individual exerts will depend on his own benefits and costs, the efforts exerted by the other individuals, and the technology that relates individual effort to outcomes. In the context of system reliability, we can distinguish three prototype cases.

Total effort. Reliability depends on the sum of the efforts exerted by the individuals.

Weakest link. Reliability depends on the minimum effort.

Best shot. Reliability depends on the maximum effort.

Each of these is a reasonable technology in different circumstances. Suppose that there is one wall defending a city and the probability of successful defense depends on the strength of the wall, which in turn depends on the sum of the efforts of the builders. Alternatively, think of the wall as having varying height, with the probability of success depending on the height at its lowest point. Or, finally, think of a there being several walls, where only the highest one matters. Of course, many systems involve a mixture of these cases.

1 Literature

Hirshleifer [1983] examined how public good provision varied with the three technologies described above. His main results were:

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1. With the weakest-link technology, there will be a range of Nash equilibria with equal contributions varying from 0 to some maximum, determined by the tastes of one of the agents.

2. The amount of underprovision of the public good rises as the number of contributors increases in the total effort case, but the efficient amount of the public good and the Nash equilibrium amount will be relatively constant as the number of contributors increases.

3. Efficient provision in the best-effort technology generally involves only the agents with the lowest cost of contributing making any contributions at all.

Cornes [1993] builds on Hirshleifer's analysis. In particular he examines the impact of changes in income distribution on the equilibrium allocation. Sandler and Hartley [2001] provide a comprehensive survey of the work on alliances, starting with the seminal contribution of Olson and Zeckhauser [1966]. Their motivating concern is international defense with NATO as a recurring example. In this context, it is natural to emphasize income effects since countries with different incomes may share a greater or lesser degree of the burden of an alliance.

The motivating example for the research reported here is computer system reliability and security where teams of programmers and system administrators create systems whose reliability depends on the effort they expend. In this sort of case, considerations of costs, benefits, and probability of failure become paramount, with income effects being a secondary concern. This difference in focus gives a different flavor to the analysis, although it still retains points of contact with the earlier work summarized in Sandler and Hartley [2001] and the other works cited above.

2 Notation

Let x_i be the effort exerted by agent $i = 1, 2$, and let $P(F(x_1, x_2))$ be the probability of successful operation of the system. Agent i receives value v_i from the successful operation of the system and effort x_i costs the agent $c_i x_i$.

The expected payoff to agent i is taken to be

$$P(F(x_1, x_2))v_i - c_i x_i$$

and the social payoff is

$$P(F(x_1, x_2))[v_1 + v_2] - c_1 x_1 - c_2 x_2.$$

We assume that the function $P(F)$ is differentiable, increasing in F, and is concave, at least in the relevant region.

We examine three specifications for F, motivated by the taxonomy given earlier.

Total effort. $F(x_1, x_2) = x_1 + x_2$.

Weakest link. $F(x_1, x_2) = \min(x_1, x_2)$.

Best shot. $F(x_1, x_2) = \max(x_1, x_2)$.

3 Nash equilibria

We first examine the outcomes where each individual chooses effort unilaterally, and then compare these outcomes to what would happen if the efforts were coordinated so as to maximize social benefits minus costs.

3.1 Total effort

Agent 1 chooses x_1 to solve

$$\max_{x_1} v_1 P(x_1 + x_2) - c_1 x_1,$$

which has first-order conditions

$$v_1 P'(x_1 + x_2) = c_1.$$

Letting G be the inverse of the derivative of P', we have

$$x_1 + x_2 = G(c_1/v_1).$$

Defining $\bar{x}_1 = G(c_1/v_1)$ we have the reaction function of agent 1 to agent 2's choice

$$f_1(x_2) = \bar{x}_1 - x_2.$$

Similarly

$$f_2(x_1) = \bar{x}_2 - x_1.$$

These reaction functions are plotted in Figure 1. It can easily be seen that the unique equilibrium involves only one agent contributing effort, with the other free riding, except in the degenerate case where each agent has the same benefit/cost ratio: $v_2/c_2 = v_1/c_1$.

Let us suppose that $v_2/c_2 > v_1/c_1$. Then, $\bar{x}_2 > \bar{x}_1$, so agent 2 contributes everything and agent 1 free rides.

Fact 1 *In the case of total effort, system reliability is determined by the agent with the highest benefit-cost ratio. All other agents free ride on this agent.*

The fact that we get this extreme form of free riding when utility takes this quasilinear form is well-known; see, for example, Varian [1994] for one exposition.

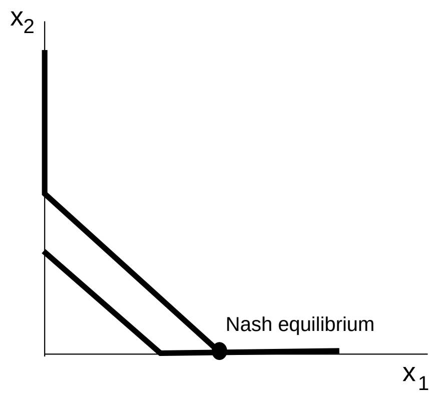

Figure 1: Nash equilibrium in total effort case.

3.2 Weakest link

Agent 1's problem is now

$$\max_{x_1} v_1 P(\min(x_1, x_2)) - c_1 x_1.$$

It is not hard to see that agent 1 will want to match agent 2's effort if $x_2 < \bar{x}_1$, and otherwise set $x_1 = \bar{x}_1$. The two agents' reaction functions are therefore

$$f_1(x_2) = \min(x_2, \bar{x}_1) \quad (1)$$

$$f_2(x_1) = \min(x_1, \bar{x}_2). \quad (2)$$

These reaction functions are plotted in Figure 2. Note that there will be a whole range of Nash equilibria. The largest of these will be at $\min(\bar{x}_1, \bar{x}_2)$. This Nash equilibrium Pareto dominates the others, so it is natural to think of it as the likely outcome.

Fact 2 *In the weakest-link case, system reliability is determined by the agent with the lowest benefit-cost ratio.*

3.3 Best shot

The best shot case is equivalent to the linear case. Obviously only one agent will contribute, the one with the highest benefit-cost ratio.

4 Social optimum

4.1 Total effort

The social problem solves

$$\max_{x_1, x_2} P(x_1 + x_2)[v_1 + v_2] - c_1 x_1 - c_2 x_2.$$

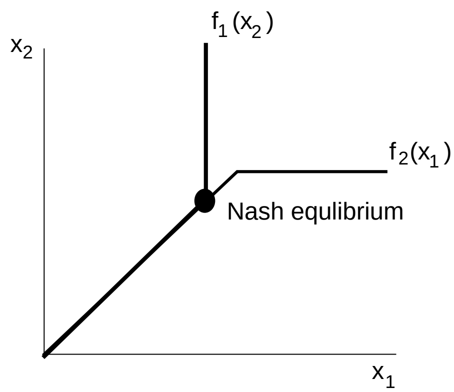

Figure 2: Nash equilibrium in weakest link case.

The first-order conditions

$$P'(x_1 + x_2)[v_1 + v_2] \leq c_1 \quad (3)$$
$$P'(x_1 + x_2)[v_1 + v_2] \leq c_2. \quad (4)$$

At the optimum, the agent with the lowest cost exerts all the effort. Let $c_{min} = \min\{c_1, c_2\}$, so that the optimum is determined by

$$x_1^* + x_2^* = G(c_{min}/(v_1 + v_2)). \quad (5)$$

Summarizing, we have:

Fact 3 *In the total effort case, there is always too little effort exerted in the Nash equilibrium as compared with the optimum. Furthermore, when $v_2/c_2 > v_1/c_1$ but $c_1 < c_2$, the "wrong" agent exerts the effort.*

4.2 Best shot

The social and private outcomes in this case are the same as in the total effort case.

4.3 Weakest link

The social objective is now

$$\max_{x_1, x_2} P(\min(x_1, x_2))[v_1 + v_2] - c_1 x_1 - c_2 x_2.$$

At the social optimum, it is obvious that $x_1 = x_2$ so we can write this problem as

$$\max_x P(x)[v_1 + v_2] - [c_1 + c_2]x,$$

which has first-order conditions

$$P'(x)[v_1 + v_2] = c_1 + c_2,$$

or

$$x_1 = x_2 = x = G((c_1 + c_2)/(v_1 + v_2)). \tag{6}$$

Fact 4 *The probability of success in the socially optimal solution is always lower in the case of weakest link that in the case of total effort.*

This occurs because the weakest link case requires equal effort from all the agents, rather than just effort from any single agent. Hence it is inherently more costly to increase reliability in this case.

5 Identical values, different costs

Let n be the number of agents and, for simplicity, set $v_i = 1$ for all $i = 1, \ldots, n$. In the total-effort case, the social optimum is given by

$$nP'(x) = \min c_i,$$

while the private optimum is determined by

$$P'(x) = \min c_i.$$

In the weakest-link case, the social optimum is determined by

$$nP'(x) = \sum_i c_i,$$

or

$$P'(x) = \bar{c} = \frac{1}{n} \sum_i c_i.$$

while the private optimum is determined by

$$P'(x) = \max c_i.$$

If we think of drawing agents from a distribution, what matters for system reliability are the order statistics—the highest and lowest costs of effort.

Fact 5 *Systems will become increasingly reliable as the number of agents increases in the total efforts case, but increasingly unreliable as the number of agents increases in the weakest link case.*

6 Increasing the number of agents

Let us now suppose that $v_i = c_i = 1$ and that the number of agents is n. In this case, the social optimum in the case of total effort is determined by

$$nP'(\sum_i x_i) = 1,$$

or

$$\sum_i x_i = G1/n).$$

The Nash equilibrium satisfies

$$P'(\sum_i x_i) = 1,$$

or

$$\sum_i x_i = G(1).$$

Fact 6 *In the total efforts case with identical agents, the Nash outcome remains constant as the number of agents is increased, but the socially optimal amount of effort increases.*

In weakest-link case, the social optimum is determined by

$$nP'(x) = n,$$

which means that the socially optimal amount of effort remains constant as n increases. In the Nash equilibrium

$$P'(x) = 1,$$

or

$$x = G(1).$$

Fact 7 *In the weakest-link case with identical agents, the socially optimal reliability and the Nash reliability are identical, regardless of the number of agents.*

7 Fines and liability

7.1 Total effort

Let us return to the two-agent case, for ease of exposition, and consider the optimal fine, that is, the fine that induces the socially optimal levels of effort. Let us start with the total effort case, and suppose that agent 1 has the lowest marginal cost of effort. If we impose a cost of v_2 on agent 1 in the event that the system fails, then agent 1 will want to maximize

$$v_1 P(x_1 + x_2) + v_2[1 - P(x_1 + x_2)] - c_1 x_1.$$

The first order condition is

$$(v_1 + v_2)P'(x_1 + x_2) = c_1,$$

which is precisely the condition for social optimality. This result easily extends to the n-person case, so we have:

Fact 8 *A fine equal to the costs imposed on the other agents should be imposed on the agent who has the lowest cost of reducing the probability of failure.*

Alternatively, we could consider a strict liability rule, in which the amount charged in the case of system failure is paid to the other agent. If the "fine" is paid to agent 2, his optimization problem becomes

$$v_2 P(x_1 + x_2) + [1 - P(x_1 + x_2)]v_2 - c_2 x_2.$$

Simplifying, we have

$$v_2 - c_2 x_2,$$

so agent 2 will want to set $x_2 = 0$. But this is true in the social optimum as well, so there is no distortion. Obviously this result is somewhat delicate; in a more general specification, there would be some distortions from the liability payment since it will, in general, change the behavior of agent 2. If the liability payment is too large, it may induce agent 2 to seek to be injured. This is not merely a theoretical issue, as it seems likely that if liability rules would be imposed, each system failure would give rise to many plaintiffs, each of whom would seek maximal compensation.

The fact that the agents with the least cost of effort to avoid system failure should bear all the liability is a standard result in the economic analysis of tort law, where it is sometimes expressed as the doctrine of the "least-cost avoider." As Shavell [1987], page 17-18, points out, this doctrine is correct only in rather special circumstances of which one is the sum-of-efforts case we are considering.

7.2 Weakest link

How does this analysis work in the weakest-link case? Since an incremental increase in reliability requires effort to be exerted by both parties, each agent must take into account the cost of effort of the other.

One way to do this is to make each agent face the other's marginal cost, in addition to facing a fine in case of system failure. Letting $x = \min\{x_1, x_2\}$, the objective function for agent 1, say, would then be:

$$v_1 P(x) - [1 - P(x)]v_2 - c_1 x_1 - c_2 x_1.$$

Agent 1 would want to choose $x = x_1$ determined by

$$(v_1 + v_2)P'(x) = c_1 + c_2,$$

which is the condition for social optimality. Agent 2 would make exactly the same choice.

Let us now examine a liability rule in which *each* must compensate the other in the case of system failure. The objective functions then take the form

$$\max_{x_1} \quad v_1 P(x) - (1 - P(x))v_2 + (1 - P(x))v_1 - c_1 x_1 \tag{7}$$
$$\max_{x_2} \quad v_2 P(x) - (1 - P(x))v_1 + (1 - P(x))v_2 - c_2 x_2 \tag{8}$$
$$\tag{9}$$

Note that when the system fails, each agent compensates the other for their losses, but is in turn compensated.

Simplifying, we can express the optimization problems as

$$\max_{x_1} \quad v_1 - v_2 + v_2 P(x) - c_1 x_1 \tag{10}$$
$$\max_{x_2} \quad v_2 - v_1 + v_1 P(x) - c_2 x_2 \tag{11}$$

This leads to first order conditions

$$v_2 P'(x) = c_1 \tag{12}$$
$$v_1 P'(x) = c_2 \tag{13}$$

If we are in the symmetric case where $v_1 = v_2$ and $c_1 = c_2$ (or more generally, where $v_1 c_1 = v_2 c_2$), then both of these equations can be satisfied and, somewhat surprisingly, the solution is the social optimum. Of course, if all agents are identical, then there is no reason to impose a liability rule, since individual optimization leads to the social optimum anyway, as shown earlier.

If we are not in the symmetric case, the equilibrium will be determined by $\min\{c_1/v_2, c_2/v_1\}$. In this case, strict liability does not result in the social optimum.

The resolution is to use the negligence rule. Under this doctrine, the court establishes a level of *due care,* $\bar{x}$. In general, this could be different for different parties, but that generality is not necessary for this particular case. If the system fails, there is no liability if the level of care/effort meets or exceeds the due care standard. If the level of care/effort was less than the due care standard, then the party who exerted inadequate care/effort must pay the other the costs of system failure.

Although the traditional analysis of the negligance rule assumes the courts determine the due care standard, an alternative model could involve the insurance companies setting a due care standard. The insurance companies could offer a contract saying that you would be reimbursed for the costs of an accident only if you had exercised an appropriate standard of due care.

Let x^* be the socially optimal effort level; i.e., the level that solves

$$\max_x \; (v_1 + v_2)P(x) - (c_1 + c_2)x.$$

It therefor satisfies the first-order condition

$$(v_1 + v_2)P'(x^*) = c_1 + c_2.$$

We need to show that if the due care standard is set at $\bar{x} = x^*$, then $x_1 = x_2 = \bar{x}$ is a Nash equilibrium.[1]

To prove this, assume that $x_2 = \bar{x}$. We must show that the optimal choice for agent 1 is $x_1 = \bar{x}_1$. Certainly we will never have $x_1 > \bar{x}$ since choosing x_1 larger than $\bar{x}$ has no impact on the probability of system failure and incurs positive cost. Will agent 1 ever want to choose $x_1 < \bar{x}$? Agent 1's objective function is

$$v_1 P(x_1) + (1 - P(x_1))v_2 - c_1 x_1.$$

Computing the derivative, and using the concavity of $P(x)$, we find

$$(v_1 + v_2)P'(x_1) - c_1 > (v_1 + v_2)P'(x^*) - c_1 = c_2.$$

Hence agent 1 will want to increase his level of effort when $x_1 < \bar{x}_1$. Summarizing:

Fact 9 *In the case of weakest link, strict liability is not adequate in general to achieve the socially optimal level of effort, and one must use a negligence rule to induce the optimal effort.*

Again, this is a standard result in liability law, which was first established by Brown [1973]; see Proposition 2.2 in Shavell [1987], page 40. The argument given here is easily modified to show that the negligence rule induces optimal behavior in the sum-of-efforts case as well, or for that matter, for any other form $P(x_1, x_2)$.

8 Sequential moves

8.1 Total effort

Let us now assume that the agents move sequentially, where the agent who moves second can observe the choice of the agent who moves first. The following discussion is based on Varian [1994].

We assume that agent 1 moves first. The utility of agent 1 as a function of his effort is given by,

$$U_1(x_1) = v_1 P(x_1 + f_2(x_1)) - c_1 x_1.$$

which can be written as

$$U_1(x_1) = v_1 P(x_1 + \max\{\bar{x}_2 - x_1, 0\}) - c_1 x_1.$$

We can also write this as

$$U_1(x_1) = \begin{cases} v_1 P(\bar{x}_2) - c_1 x_1 & \text{for } x_1 \le \bar{x}_2 \\ v_1 P(x_1) - c_1 x_1 & \text{for } x_1 \ge \bar{x}_2. \end{cases}$$

[1] Of course, there will be many other Nash equilibria as well, due to the weakest-link technology. The legal due-care standard has the advantage of serving as a focal point to choose the most efficient such equilibrium.

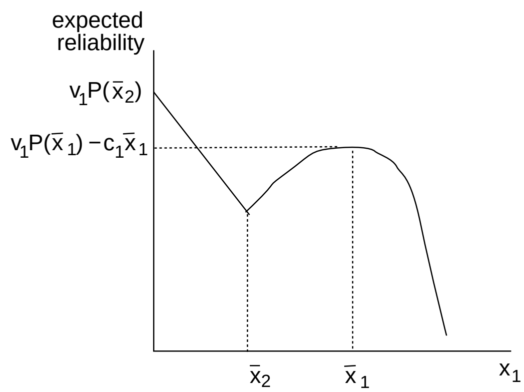

Figure 3: Sequential contribution in total efforts case.

It is clear from Figure 3 that there are two possible optima: either the first agent exerts zero effort and achieves payoff $v_1 P(\bar{x}_2)$ or he contributes $\bar{x}_1$ and achieves utility $v_1 P(\bar{x}_1) - c_1 \bar{x}_1$.

Case 1. *The agent with the lowest value of* v_i/c_i *moves first.* In this case the optimal choice by the first player is to choose zero effort. This is true since

$$v_1 P(\bar{x}_2) > v_1 P(\bar{x}_1) > v_1 P(\bar{x}_1) - c_1 \bar{x}_1.$$

Case 2. *The agent with the highest value of* v_i/c_i *is the first contributor.* In this case, either contributor may free ride. If the agents have tastes that are very similar, then the first contributor will free ride on the second's contribution. However, if the first mover likes the public good *much* more than the second, then the first mover may prefer to contribute the entire amount of the public good himself.

Referring to Figure 3 we see that there are two possible subgame perfect equilibria: one is the Nash equilibrium, in which the agent who has the highest benefit-cost ratio does everything. The other equilibrium is where the agent who has the *lowest* benefit-cost contributes everything. This equilibrium cannot be a Nash equilibrium since the threat to free ride by the agent who likes the public good most is not credible in the simultaneous-move game.

Fact 10 *The equilibrium in the sequential-move, the total-effort game always involves the same or less reliability than the simultaneous-move game.*

Note that it is always advantageous to move first since there are only two possible outcomes and the first mover gets to pick the one he prefers.

Fact 11 *If you want to ensure the highest level of security in the sequential-move game, then you should make sure that the agent with the lower benefit-cost ratio moves first.*

8.2 Best-effort and weakest-link

The best-effort case is the same as the total-effort case. The weakest-link case is a bit more interesting. Since each agent realizes that the other agent will, at most, match his effort, there is no point in choosing a higher level of effort than the agent who cares the least about reliability. On the other hand, there is no need to settle for one of the inefficient Nash equilibria either.

Fact 12 *The unique equilibrium in the sequential-move game will be the Nash equilibrium in the simultaneous-move game that has the highest level of security, namely* $\min(\bar{x}_1, \bar{x}_2)$.

Hirshleifer [1983] recognizes this and uses it as an argument for selecting the Nash equilibrium with the highest amount of the public good as the "reasonable" outcome.

9 Adversaries

Let us now consider what happens if there is an adversary who is trying to increase the probability of system failure. First we consider the case of just two players, then we move to looking at what happens with a team on each side.

We let x be the effort of the defender, and y the effort of the attacker. Effort costs the defender c and the attacker d. The defender gets utility v if the system works, and the attacker gets utility w if the system fails. We suppose that the probability of failure depends on "net effort," $x-y$, and that there is a maximal effort $\hat{x}$ and $\hat{y}$ for each player.

The optimization problems for the attacker and defender can be written as

$$\max_x \quad vP(x-y) - cx \tag{14}$$
$$\max_y \quad w[1 - P(x-y)] - dy. \tag{15}$$

The first-order conditions are

$$vP'(x-y) = c \tag{16}$$
$$wP'(x-y) = d. \tag{17}$$

Let $G(\cdot)$ be the inverse function of $P'(x-y)$. By the second-order condition this has to be locally decreasing, and we will assume it is globally decreasing. We can then apply the inverse function to write the two reaction functions:

$$x - y = G(c/v) \tag{18}$$
$$x - y = G(d/w). \tag{19}$$

Of course, these are only the reaction functions for *interior* optima. Adding in the boundary conditions gives us:

$$x = \min\{\max\{G(c/v) + y, 0\}, \hat{x}\} \tag{20}$$
$$y = \min\{\max\{G(d/w) - x, 0\}, \hat{y}\}. \tag{21}$$

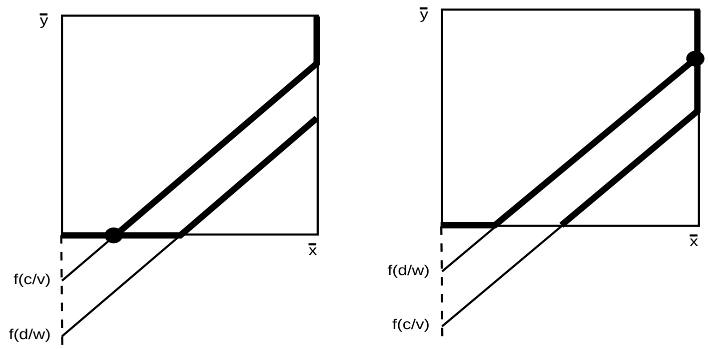

Figure 4: Reaction functions in adversarial case.

We plot these reaction functions in Figure 4. Note that there are two possible equilibrium configurations. If $c/v < d/w$, we have $x^* = G(c/v)$ and $y^* = 0$, while if $c/v > d/w$ we have $x^* = \hat{x}$ and $y^* = \hat{x} - G(d/w)$.

Intuitively, if the cost-benefit ratio of the defender is smaller than that of the attacker, the attacker gives up, and the defender does just enough to keep him at bay. If the ratio is reversed, the defender has to go all out, and the attacker pushes to keep him there.

10 Sum of efforts and weakest link

(Notes: yet to finish)

The reaction functions are:

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{m} y_i = G(c_j/v_j) \quad (22)$$

$$\sum_{i=1}^{n} x_i - \sum_{i=1}^{m} y_i = G(d_j/w_j). \quad (23)$$

Looks like only the one with the lowest cost/benefit ratio exerts effort, everyone else free rides. This becomes a "battle between the champions."

Weakest link case:

$$\min\{x_1, \ldots, x_n\} - \min\{y_1, \ldots, y_m\} = G(c_j/v_j) \quad (24)$$

$$\min\{x_1, \ldots, x_n\} - \min\{y_1, \ldots, y_m\} = G(d_j/w_j). \quad (25)$$

This is a "battle between the losers."

Note that when technology is total effort, large teams have an advantage, whereas weakest link technology confers an advantage to small teams.

11 Future work

There are several avenues worth exploring:

- To what extent to these results generalize to the more general framework of Cornes [1993] and Sandler and Hartley [2001]. The possibility of Pareto improving transfers is particularly interesting. Though Cornes [1993] examined this in the context of income transfers, knowledge transfers would be particularly interesting in our context.

- One case where transfers are important are when agents can subsidize other agents' actions, as in Varian [1994]. The subgame perfect equilibrium of "announce subsidies then choose actions" is Pareto efficient in the case we examine.

- One could look at capacity constraints on the part of the agents. For example, each agent could put in only one unit of effort. Similarly, one could look at increasing marginal cost of effort.

- Imperfect information adds additional phenomena. For example, Hermalin [1998] shows that in a model with uncertainty about payoffs, an agent may choose to move first in order to demonstrate to the other agent that a particular choice is worthwhile. Hence "leadership" plays a role of signaling to the other agents.

- M. and Sandler [2001] examines how results change when a contribution game's structure moves in the direction of best shot or weakest link. This sort of partial comparative statics exercise could be of interest in our context as well.

- One could examine situations where there were communication costs among the cooperating agents, a la team theory. If, for example, there is imperfect information about what others are doing, it might lead to less free riding.

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