Systematic Mortality Risk: An Analysis of Guaranteed Lifetime Withdrawal Benefits in Variable Annuities

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Abstract

Guaranteed lifetime withdrawal benefits (GLWB) embedded in variable annuities provide a type of life annuity which addresses systematic mortality risk while indirectly protecting the policyholders from the downside risk of fund investment. Using tractable equity and stochastic mortality model, we evaluate GLWB in a continuous time framework. The paper provides an extensive study of how different sets of financial and demographic parameters affect the fair guarantee fee being charged, as well as their effects on the profit and loss distribution if they are different to the true parameters of the underlying dynamics. We show that parameter risk is significant since the guarantee is of a very long term nature. We quantify systematic mortality risk component underlying the guarantee and study how different levels of equity exposure chosen by the insured affect the exposure of systematic mortality risk for the guarantee providers. Effectiveness of static hedge of systematic mortality risk is examined with respect to different levels of equity exposure.

Key words: Variable annuity, guaranteed lifetime withdrawal benefits (GLWB), systematic mortality risk, parameter risk, model risk, static hedging

1 Introduction

Variable annuities (VA), introduced in the 1970s in the US, have proven to be popular among investors and retirees especially in North America and Japan. VA are insurance contracts which allow policyholders to invest their retirement savings in the mutual funds. They are attractive because policyholders' retirement savings can have certain exposure to the equity market whereas benefits are based on the performance of the underlying funds, enhanced by certain tax advantages. In addition to the VA, policyholders can elect different types of guarantee to protect their retirement savings by paying various amounts of guarantee fees. Since 1990s, two kinds of embedded guarantees have been offered in such policies. These include the Guaranteed Minimum Death Benefits (GMDB) and the Guaranteed Minimum Living Benefits (GMLB), see Hardy (2003) and Ledlie et al. (2008). There are four types of guarantee that fall into the category of GMLB:

- Guaranteed Minimum Accumulation Benefits (GMAB)
- Guaranteed Minimum Income Benefits (GMIB)
- Guaranteed Minimum Withdrawal Benefits (GMWB)
- Guaranteed Lifetime Withdrawal Benefits (GLWB)

According to LIMRA [1], the election rates of GLWB, GMIB, GMAB and GMWB in the fourth quarter of 2011 in the US corresponded to 59%, 26%, 3% and 2%, respectively. The popularity of GLWB can be attributed to its flexibility in a sense that the policyholder can decide how much he/she wants to withdraw every year, subject to a limited nominal yearly amount, for the lifetime of the insured and regardless of the performance of the investment. After the death of the insured, any savings remaining in the account will be returned to the insured's beneficiary. GLWB is essentially a life annuity with flexible features compared to the traditional life annuity contracts.

Being a type of a life annuity with benefit linked to the equity market, insurers who offer GLWB embedded in variable annuities are subject to several types of risk, namely, equity risk, interest rate risk, withdrawal risk and, in particular, systematic mortality risk. As opposed to unsystematic mortality risk, which is diversifiable, systematic mortality risk, a.k.a. longevity risk, cannot be eliminated through diversification. Systematic mortality risk arises due to stochastic, or unpredictable nature of life expectancy. Insurance providers offering GLWB are subject to systematic mortality risk as the guarantee promises to the insured a stream of income for life, even if the investment account is depleted. Systematic mortality risk can be addressed by means of stochastic mortality modeling, which can be approached via the so-called intensity-based framework, see e.g. Biffis (2005).

1.1 *Motivation and Contribution*

It has been widely documented that starting with the global financial crisis in 2007-2008, VA providers have been suffering from unexpected large losses arising due to the

[1] *http://www.limra.com/default.aspx*

fact that the benefits secured by different types of guarantee were too generous. High volatility of equity markets during this period has also exacerbated the difficulty in managing risks underlying the guarantees because of the unexpectedly high hedging costs.[2] Guarantees turned out to be inherently difficult to price and manage because of their long term nature. In particular, guarantees offering lifetime income incorporate systematic mortality risk due to the fact that people live longer than expected. A prime example is the GLWB.

To our knowledge the existing literature does not study in details the systematic mortality risk underlying the GLWB. Several authors concentrate either on the volatility risk (Kling et al. (2011)) or briefly discuss mortality risk (Piscopo and Haberman (2011)) as part of a wider study on guarantees/annuities (Bacinello et al. (2011)and Ngai and Sherris (2011)). A more detailed analysis of the impact of the systematic mortality risk on valuation and hedging, as well as its interaction with other risks underlying the GLWB is required. These issues are addressed in the present paper.

As discussed in Bacinello et al. (2011), there are three steps involved in a reliable risk management process. First or all, it is risk identification. When looking at the benefit and the premium component of a guarantee, different sources of risk, whether hedgeable or not, can be determined.[3] The second step involves risk assessment. Due to the long term nature and complicated payout of the guarantee, there are likely to be a range of risks involved. Assessing their interactions is required to provide certain insights prior to the set up of an adequate risk management program. Risk assessment can also be useful when determining which sources of risk have larger impact on pricing and risk management of the guarantee. The third step is the risk management action. Risk management of the guarantee generally consists of capital reserving and hedging (Hardy (2003) and Ledlie et al. (2008)). Selecting the appropriate hedge instruments and determining hedging frequency (in particular, static or dynamics hedging) are important issues that should be addressed, while taking into account constraints such as transaction costs and liquidity of the hedge instruments. A reliable risk management process is of a great importance in case of GLWB, as it is a long term contract which lasts, on average, for several decades. Also due to its popularity, VA providers are likely to have a large VA portfolio built up when GLWB is an underlying guarantee.

In this paper we apply the risk management process described above to analyze equity and systematic mortality risks underlying the GLWB, as well as their interactions. Our approach relies on risk measures and the profit and loss (P&L) distribution in order to identify, assess and partially mitigate financial and demographic risks, which occur because of the specific design of the guarantee.

One of the main challenges of the present paper is to asses how different sets of financial and demographic parameters will affect the guarantee fees being charged. In particular, we are interested in their effects on the P&L distributions, if they are set to be different from the “true” parameters of the underlying dynamics. Parameter

[2] See for instance “A challenging environment”, June 2008 and “A balanced structure?”, February 2011 in the Risk Magazine.

[3] For instance, uncertainty stemming from the behavior of policyholders is an unhedgeable risk, which requires insurance companies to assess policyholders general past behavior.

risk turns out to be significant as GLWB is of a very long term nature, lasting for the lifetime of the insured in a portfolio. The paper also contributes to the study of model risk where guarantee providers assume mortality to be deterministic whereas life expectancy of the insured is stochastic in its nature.

Furthermore, this research paper deals with quantification of the systematic mortality risk underlying the guarantee using P&L analysis, which turns out to be particularly useful for determining an appropriate capital reserve required due to unavailability of hedging instruments to hedge the longevity risk. In particular, we show that different levels of equity exposure chosen by the insured have (in presence of systematic mortality risk) a great impact on the GLWB from the guarantee providers' point of view. Finally, the effectiveness of static hedge of systematic mortality risk is assessed with respect to different levels of equity exposure.

1.2 Literature Review

An underlying approach chosen in this paper in order address systematic mortality risk is to model the mortality intensity as a stochastic process. Affine processes have been proposed to model the mortality intensity as they allow closed form expressions for important quantities such as survival probabilities. While Dahl and Moller (2006) suggests a time inhomogeneous square root process, Schrager (2006) and Biffis (2005) propose multi-factor affine processes which allow to capture the evolution of mortality intensity for all ages simultaneously. A concise survey of mortality modeling and the development of longevity market can be found in Cairns et al. (2008).

One of the earliest modeling frameworks that focuses on GLWB is discussed in Holz et al. (2007), who take into account policyholder's behavior and different product features. However, the authors address longevity risk by considering different mortality tables without assuming that the underlying mortality intensity is stochastic. Piscopo and Haberman (2011) decomposes the value of GLWB into the death and living benefit and provide sensitive analysis. Authors address mortality risk by taking into account mortality shocks in an adjusted mortality table. Kling et al. (2011) offer a detailed pricing and hedging analysis of GLWB focusing on the management of volatility risk, and ignore systematic mortality risk. Ngai and Sherris (2011) study a range of life annuity contracts, including GLWB, which are subject to systematic mortality risk with different static hedging strategies for longevity risk management. The current paper provides further detailed analysis that have been conducted in the above mentioned studies, focusing on GLWB.

Papers that investigate the impact of systematic mortality risk on guarantees other than GLWB include Ballotta and Haberman (2006) who value guaranteed annuity options (GAOs) and provide sensitivity analysis taking into account the interest rate risk and the systematic mortality risk. Kling et al. (2012) offer an analysis similar to Ballotta and Haberman (2006), but considering both, GAOs and guaranteed minimal income benefits (GMIB), and taking into account different hedging strategies. Ziveyi et al. (2012) price European options on deferred insurance contracts including pure endowment and deferred life annuity (i.e. GAO) by solving analytically the pricing partial differential equation in a presence of systematic mortality risk.

1.3 Outline

The remainder of the paper is organized as follows. Section 2 describes general features of GLWB including identification of risks and additional product features, and addressing the benefit and the premium component of the guarantee. Section 3 specifies continuous time models to capture the underlying equity and systematic mortality risk, where tractability of the models is considered as paramount since efficient pricing and risk management is of practical importance given the complicated structure of the guarantee. Section 4 evaluates GLWB using two approaches, which are shown to be equivalent. It generalizes the result from pricing of GMWB in Kolkiewicz and Liu (2012) to the case of GLWB. Sensitivity analysis is conducted in Section 5 showing quantitatively which sources of risk have larger impact on pricing of GLWB. Section 6 analyzes the impact of different types of risk on the P&L distributions of the guarantee without incorporating any hedge strategies. Different levels of equity exposure, parameter risk and model risk are considered. Section 7 studies the effectiveness of static hedge of systematic mortality risk via S-forward with respect to different levels of equity exposure. Section 8 summarizes the results and provides further concluding remarks.

2 Features of GLWB

A variable annuity (VA) is an insurance product where a policyholder invests his or her retirement savings in a mutual fund, which forms an investment account managed by an insurance company. VAs are attractive because the investment enjoys certain tax benefits (Cite). While the investment is subject to equity risk, a VA with GLWB rider guarantees that a capped amount of income can be withdrawn from the account every year for the lifetime of the policyholder, even if the investment account is depleted before the policyholder dies. Furthermore, any remaining value in the account will be returned to the policyholder's beneficiary. In return, the policyholder is required to pay guarantee fees every year, which is proportional to the investment account value.

The simplest type of GLWB, which in the following will be referred to as a plain GLWB attached to a VA, is described in what follows in a continuous time framework. Let $A(0)$ be the retirement savings invested in a variable annuity at time $t = 0$. For a plain GLWB, the guaranteed withdrawal amount, that is, the amount which is allowed to be withdrawn from the account per unit of time, is determined as $g \cdot A(0)$ where g is the so called guaranteed withdrawal rate. A typical value of g is 5% for a policyholder aged 65 at $t = 0$. The policyholder has the freedom to decide how much to withdraw per unit of time. However, a penalty may arise if the withdrawal amount is higher than the guaranteed withdrawal amount allowed by the insurer. The guarantee fee paid by the policyholder per unit of time is determined as $\alpha_g \cdot A(t)$ where α_g is the so called guaranteed fee rate and $A(t)$ is the account value at time t. From the insurer's point of view, the liability and the fee structure of a plain GLWB can be described concisely by two scenarios which are captured in Fig. 1 and 2.

For both scenarios we assume a static withdrawal strategy, where the policyholder

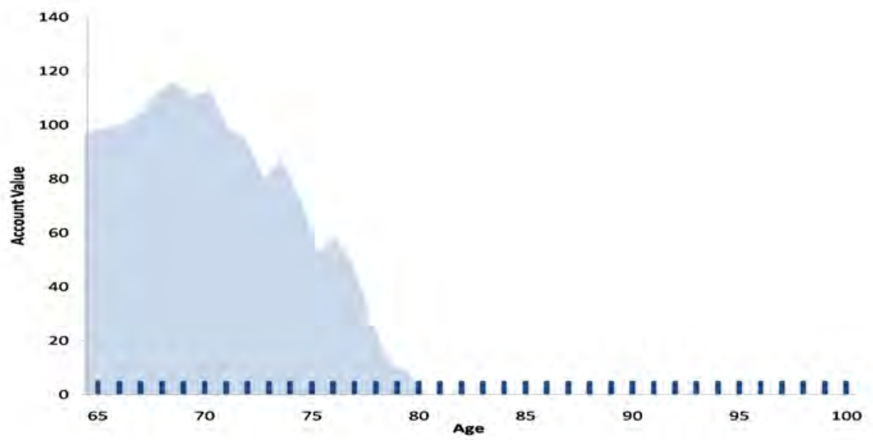

Fig. 1. Scenario 1 (liability for the insurer): policyholder dies at the age of 100 years, 20 years after the account value is depleted.

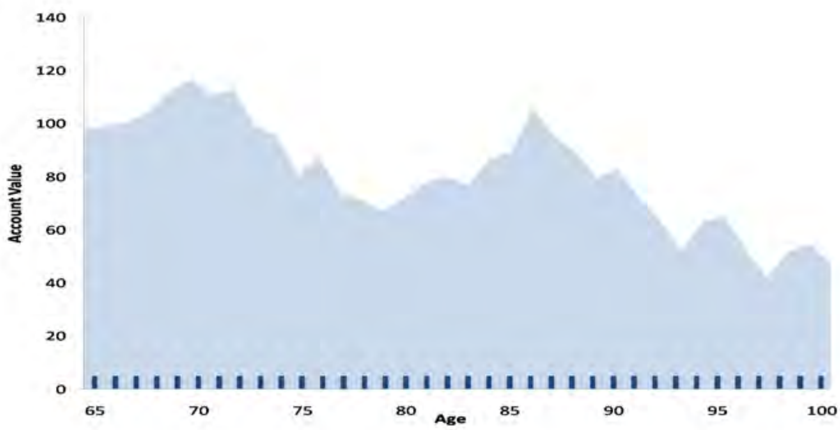

Fig. 2. Scenario 2 (no liability for the insurer): policyholder dies at the age of 100 years, when the account value is still positive.

receives exactly the guaranteed withdrawal amount per unit of time. An alternative strategy, proposed in Milevsky and Salisbury (2006), is a dynamic withdrawal, which leads to an optimal stopping problem and aims to maximize the value of the guarantee. In Fig. 1, the policyholder dies at the age of 100 years and the account value drops to zero when the policyholder is 80 years old. The insurance company receives guarantee fees during the period when the policyholder is aging from 65 to 80. As the account is depleted and the policyholder is still alive at the age of 80, the insurance company has to continue paying the guaranteed withdrawal amount during the period when the policyholder is aging from 80 to 100. In this scenario there is a liability for the insurer. Another scenario is captured in Fig. 2. Again, the policyholder dies at the age of 100, however, there is still some positive amount left in the account at the time of death of the policyholder. In this scenario the insurance company receives the guarantee fees during the lifetime of the policyholder and there is no liability for the insurer.

A plain GLWB can be enriched by adding other features such as a roll-up feature and a step-up (or ratchet) feature. The roll-up feature allows guaranteed withdrawal amount to increase every year by a fixed amount, given the policyholder has not yet started to withdraw money. The contract which contains a set-up option, allows the guaranteed withdrawal amount to increase at specified points in time (set-up dates), if at that point in time the percentage withdrawal rate exceeds the contractually guaranteed amount. In case of deferred version of the contract, as opposed to immediate withdrawals, the policyholder cannot withdraw money during the deferred period. The details on different featured embedded in the guarantee can be found in e.g. Piscopo and Haberman (2011).

From Fig. 1 and Fig. 2 and the discussion above, one may note that GLWB providers are subject to three types of risk: financial risk, demographic risk and behavioral risk. Financial risk includes interest rate risk and equity risk. Demographic risk consists of unsystematic and systematic mortality risk. Different withdrawal behaviors of the policyholder, which might or might not be rational, constitute the behavioral risk.

In the following sections dealing with the model set-up, valuation, hedging and sensitivity analysis, we assume constant interest rate and constant volatility. The withdrawal rate is assumed to be static and there is not lapse. Finally, payments are immediate (not deferred), that is, the withdrawals start immediately and there is no defer period for investment growth. We do not consider possible roll-up or set-up option, but note that the derivations presented below could be easily generalized when additional features are incorporated in the contract. Furthermore, we focus on systematic mortality risk and believe that our results are robust with respect to other features which could be possibly incorporated in the model.

3 Model Specification

Tractable continuous-time models are specified to capture equity and systematic mortality risk underling GLWB in variable annuities. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a filtered probability space where $\mathbb{P}$ is the real world probability measure. The filtration $\mathcal{F}_t$ is constructed as

$$\begin{aligned} \mathcal{G}_t &= \sigma\{(W_1(s), W_2(s)) : 0 \leq s \leq t\} \\ \mathcal{H}_t &= \sigma\{1_{\{\hat{\tau} \leq s\}} : 0 \leq s \leq t\} \\ \mathcal{F}_t &= \mathcal{G}_t \vee \mathcal{H}_t \end{aligned}$$

where $\mathcal{G}_t$ is generated by two independent standard Brownian motions, W_1 and W_2, which are associated with the uncertainties related to equity and mortality intensity, respectively. The subfiltration $\mathcal{H}_t$ represents information set that would indicate whether the death of a policyholder has occurred before time t. The stopping time $\hat{\tau}$ is interpreted as the remaining lifetime of an insured. For a detailed exposition of modeling mortality under the intensity-based framework refer to Biffis (2005).

3.1 Investment Account Dynamics

In our setup the policyholder can invest his or her retirement savings in an investment fund that has both, equity and fixed income exposure. Under the real world probability measure $\mathbb{P}$, we assume that the equity component follows the geometric Brownian motion

$$dS(t) = \mu S(t)dt + \sigma S(t)dW_1(t), \tag{3.1}$$

and the fixed income investment is represented by the money market account $B(t)$ with dynamics $dB(t) = r\,B(t)dt$ where the interest rate r is constant. Let $\delta_1(t)$ and $\delta_2(t)$ denote the number of units invested in $S(\cdot)$ and $B(\cdot)$, respectively. The self-financing investment fund $V(\cdot)$ has the following dynamics

$$\begin{aligned} dV(t) &= \delta_1(t)dS(t) + \delta_2(t)dB(t) \\ &= (\mu\delta_1(t)S(t) + r\delta_2(t)B(t))dt + \sigma\delta_1(t)S(t)dW_1(t) \\ &= (\mu\pi(t) + r(1-\pi(t)))V(t)dt + \sigma\pi(t)V(t)dW_1(t) \end{aligned} \tag{3.2}$$

where the fraction $\pi(\cdot)$ is defined as $\pi(t) = \frac{\delta_1(t)S(t)}{V(t)}$, and is interpreted as the proportion of the retirement savings being invested in the equity component. Since short-selling is not allowed, we set $0 \leq \pi(\cdot) \leq 1$.

Because of the election of GLWB, the investment account $A(\cdot)$ held by the policyholder is being charged continuously with guarantee fees by the insurer. Typically, the guarantee fee charged is proportional to the investment account $A(\cdot)$. Recall that we denote the guarantee fee rate by α_g. We consider the case that a policyholder withdraws a constant amount continuously until death and the withdrawal amount is proportional to the guarantee base, which is assumed to be the initial investment $A(0)$. We denote the (static) guarantee withdrawal rate by g, which is so-named because the withdrawals are guaranteed by the insurer regardless of the investment performance, and define $G = g \cdot A(0)$ to be the withdrawal amount per unit of time. The investment account of the policyholder satisfies the following dynamics:

$$dA(t) = (\mu\pi(t) + r(1-\pi(t)) - \alpha_g)A(t)dt - Gdt + \sigma\pi(t)A(t)dW_1(t) \tag{3.3}$$

with $A(\cdot) \geq 0$, since the investment account cannot be negative. In the following, we assume that $\pi(\cdot)$ is constant, that is, the policyholder invests in a fixed proportion of his/her retirement savings in the equity and fixed income markets throughout the investment period. In particular, we consider five cases where $\pi \in \{0, 0.3, 0.5, 0.7, 1\}$. Each value corresponds to a specific risk-preference of the retiree where a larger value of π indicates that the insured prefers to have a larger equity exposure in his/her savings, which would result in a higher potential growth but will be subject to higher volatility as well. By allowing different levels of equity exposure, we will be able to study the interaction between equity and systematic mortality risk due to the specific design of the product.

3.2 Mortality Model

In order to select an appropriate model for our problem setting, we rely on the following three criteria. First of all, the model should be qualitatively reasonable with mortality intensity being strictly positive. Moreover, as argued in Cairns et al. (2006) and Cairns et al. (2008), it is unreasonable to assume that mortality intensity is mean reverting, even if the mean level is time dependent. [4] The second criterion is tractability. Given the complicated payout structure, a tractable stochastic mortality model is required for an efficient pricing and risk management of GLWB. In particular, the model should allow analytical expressions for important quantities, such as survival probabilities. Further to that, it should be computationally not very burdensome, in a sense that one should be able to simulate fast and accurate the derived solutions paths of underlying dynamics. Finally, since the paper focuses on the impact of systematic mortality risk, the ability of the model to be reduced to a simple deterministic mortality model would provide an accommodating ground to investigate the effect of, and the difference between, systematic and unsystematic mortality risk underlying the guarantee.

Given these criteria, we propose a one-factor, non mean-reverting and time homogeneous affine process for modeling the mortality intensity process $\mu_{x+t}(t)$ of a person aged x at time $t = 0$, as follows:

$$d\mu_{x+t}(t) = (a + b\,\mu_{x+t}(t))dt + \sigma_\mu \sqrt{\mu_{x+t}(t)}\, dW_2(t), \quad \mu_x(0) > 0. \tag{3.4}$$

Here, $a \neq 0$, $b > 0$ and σ_μ represents the volatility of mortality intensity.

In the special case when $\sigma_\mu = a = 0$, the model reduces to the well-known constant Gompertz specification. [5] Hence, we will be able to conveniently compare the impact of stochastic mortality with that of deterministic mortality in the following sections. The values of the parameters a, b and σ_μ are obtained by calibrating the survival curve implied by the mortality model to the survival curve obtained from real data as documented in the Australian Life Tables 2005-2007 [6], for a person aged 65. The result is reported in Table 1.

Table 1
Calibrated parameters for the mortality model.

a	b	σ_μ	$\mu_{65}(0)$
0.001	0.087	0.021	0.01147

The estimated values of the parameters a and σ_μ indicate that the proposed mortality intensity process Eq.(3.4) is strictly positive [7] and is non mean-reverting. Modeling

[4] Mean reverting behavior of mortality intensity would indicate that if mortality improvements have been faster than anticipated in the past then the potential for further mortality improvements will be significantly lower in the future.

[5] From the Gompertz model $\mu_x(0) = ye^{bx}$, we have $\mu_{x+t}(t) = \mu_x(0)e^{bt}$ which satisfies Eq.(3.4) with $\sigma_\mu = a = 0$.

[6] *http://www.aga.gov.au/publications/#life_tables*

[7] It can be shown that if $a \geq \sigma_\mu^2/2$ then the mortality intensity process Eq.(3.4) is strictly positive, see Filipovic (2009).

mortality intensity using a one-factor[8] and time homogeneous affine process can reduce significantly computational time, since the dynamics allow to exactly simulate the model and to derive analytical expressions for survival probabilities. Thus, the proposed model Eq.(3.4) satisfies all criteria stated above.

Since the mortality model proposed is affine and time homogeneous, we have an analytical expression for the survival probability ${}_sP_{x+t}$ of the remaining lifetime $\hat{\tau}$ of an individual aged x at time 0. Assuming that the individual is still alive at $t > 0$ and that $\sigma_\mu > 0$, we obtain

$$ {}_sP_{x+t} = E_t^{\mathbb{Q}}\left(e^{-\int_t^{t+s}\mu_{x+v}(v)dv}\right) = C_1(s)e^{-C_2(s)\mu_{x+t}(t)}, \tag{3.5}$$

where

$$C_1(s) = \left(\frac{2\gamma e^{\frac{1}{2}(\gamma-b)s}}{(\gamma-b)(e^{\gamma s}-1)+2\gamma}\right)^{\frac{2a}{\sigma_\mu^2}}, \quad C_2(s) = \frac{2(e^{\gamma s}-1)}{(\gamma-b)(e^{\gamma s}-1)+2\gamma} \tag{3.6}$$

and $\gamma = \sqrt{b^2 + 2\sigma_\mu^2}$. For details refer to Dahl and Moller (2006). The density function of the remaining lifetime can be calculated as

$$f_{x+t}(s) = -\frac{d}{ds}{}_sP_{x+t} = -\frac{dC_1(s)}{ds}e^{-C_2(s)\mu_{x+t}(t)} + \frac{dC_2(s)}{ds}{}_sP_{x+t}\,\mu_{x+t}(t) \tag{3.7}$$

where

$$\frac{dC_1(s)}{ds} = C_1(s)\frac{2a(\gamma-b)}{\sigma_\mu^2}\left(\frac{1}{2} - \frac{\gamma e^{\gamma s}}{(\gamma-b)(e^{\gamma s}-1)+2\gamma}\right) \tag{3.8}$$

and

$$\frac{dC_2(s)}{ds} = \frac{2\gamma e^{\gamma s}}{(\gamma-b)(e^{\gamma s}-1)+2\gamma}\left(1 - \frac{1}{2}(\gamma-b)C_2(s)\right). \tag{3.9}$$

For the case when $\sigma_\mu = 0$, the survival probability and the density functions can be directly calculated to obtain

$$ {}_sP_{x+t} = e^{\frac{a}{b}s + \frac{1}{b}\left(1-e^{bs}\right)\left(\mu_{x+t}(t)+\frac{a}{b}\right)} \tag{3.10}$$

and

$$f_{x+t}(s) = {}_sP_{x+t}\left(\frac{a}{b}\left(e^{bs}-1\right) + e^{bs}\mu_{x+t}(t)\right), \tag{3.11}$$

respectively. Furthermore, by setting $a = 0$ in Eq.(3.10) and Eq.(3.11) we obtain the survival probability and the density function of the Gompertz model.

Remark 1 *Luciano and Vigna (2008) study a similar model with $a = 0$ and $\sigma_\mu > 0$. However, such specification could be problematic since, as it can be shown from Eq.(3.10), it holds*

$$\lim_{s\to\infty} {}_sP_{x+t} = e^{-\frac{2}{\gamma-b}\mu_{x+t}(t)},$$

[8] One may argue that the past mortality data experiences differences between the evolution of mortality rates for different ages (Cite). However, applying a multi-factor mortality model to the pricing of GLWB requires substantial computational resources (refer to Section 4) and hence, we restrict ourselves to a one-factor model instead.

that is, the survival probability converges to $e^{-\frac{2}{\gamma-b}\mu_{x+t}(t)}$ which only approaches to zero when $\sigma_\mu = 0$. It turns out that if we set $a \neq 0$ this problem is not prsent. Furthermore, the mortality intensity $\mu_{x+t}(t)$ under such a specification has a non zero probability of reaching zero when $\sigma_\mu > 0$ (regardless of the value of b). Therefore, assuming $a \neq 0$, while $\sigma_\mu > 0$, the dynamics specified in Eq.(3.4) produce a more satisfactory stochastic model for the mortality intensity process.

3.3 Risk-Adjusted Measure

For the purpose of no-arbitrage valuation and hedging, we require the dynamics of the account process $A(t)$ and the mortality intensity $\mu_{x+t}(t)$ to be written under the risk-adjusted measure $\mathbb{Q}$. We define $W_1^{\mathbb{Q}}(t)$ and $W_2^{\mathbb{Q}}(t)$ as

$$\begin{aligned} dW_1^{\mathbb{Q}}(t) &= \frac{\mu - r}{\sigma} dt + dW_1(t) \\ dW_2^{\mathbb{Q}}(t) &= \lambda \sqrt{\mu_{x+t}(t)}\, dt + dW_2(t). \end{aligned} \tag{3.12}$$

These are by the Girsarov Theorem, see e.g. Bjork (2009), standard Brownian motions under the $\mathbb{Q}$ measure with $\frac{\mu-r}{\sigma}$ and $\lambda\sqrt{\mu_{x+t}(t)}$ representing the market price of equity risk and systematic mortality risk, respectively. Hence, we can write the investment account process and the mortality intensity under $\mathbb{Q}$ as follows:

$$dA(t) = (r - \alpha_g)A(t)dt - G\, dt + \pi\, \sigma\, A(t)dW_1^{\mathbb{Q}}(t) \tag{3.13}$$

$$d\mu_{x+t}(t) = (a + (b - \lambda\sigma_\mu))\mu_{x+t}(t)dt + \sigma_\mu \sqrt{\mu_{x+t}(t)} dW_2^{\mathbb{Q}}(t). \tag{3.14}$$

By imposing the market price of systematic mortality risk to be $\lambda\sqrt{\mu_{x+t}(t)}$, the mortality intensity process is still a time homogeneous affine process under $\mathbb{Q}$, as can be seen from Eq.(3.14).[9] Thus, this specification preserves tractability of the model when pricing the guarantee under $\mathbb{Q}$. From Eq.(3.13) we observe that the fraction π invested in the equity market affects merely the volatility of the investment account process under $\mathbb{Q}$.

Using parameters estimated for the stochastic model in Table 1 and allowing volatility of mortality σ_μ to vary, we plot survival probabilities and density functions of the remaining lifetime of an individual aged 65 in the left and the right panel of Fig.3, respectively. The estimated parameters produce reasonable survival probabilities and density functions of the remaining lifetime. In particular, an increase in σ_μ leads to an improvement in survival probability under the risk-adjusted measure $\mathbb{Q}$. Hence, higher volatility of mortality leads not only to higher uncertainty about the timing of death of an individual, but also to an increase in the survival probability. Similar figures are obtained when the market price of systematic mortality risk coefficient λ is varying while all other parameters are fixed to the values specified in Table 1, see Fig. 4. Negative values of λ indicate a decline in survival probability, while positive

[9] The mortality intensity process might become mean reverting when $b < \lambda\sigma_\mu$ which is not a favorable, as discussed above.

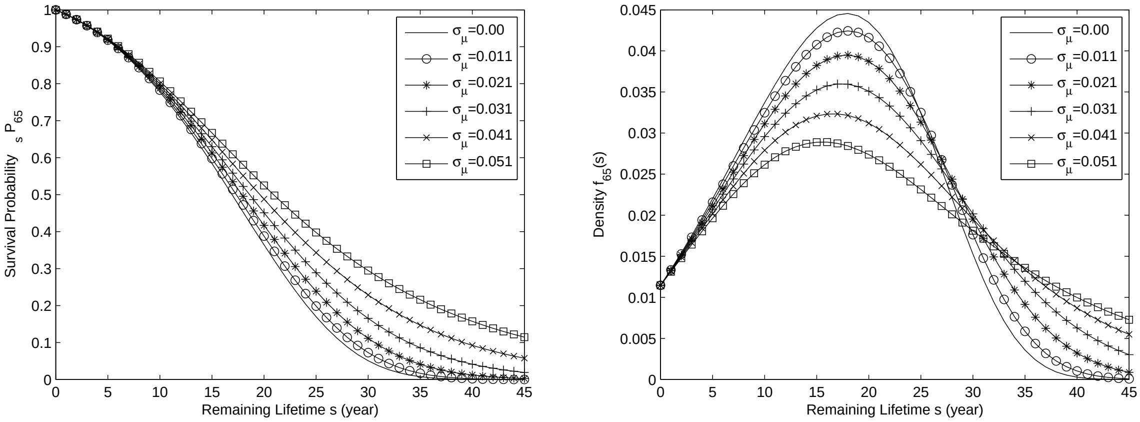

Fig. 3. Survival probabilities and densities of the remaining lifetime of an individual aged 65 with different values of σ_μ. Other parameters are as specified in Table 1 with $\lambda = 0.4$.

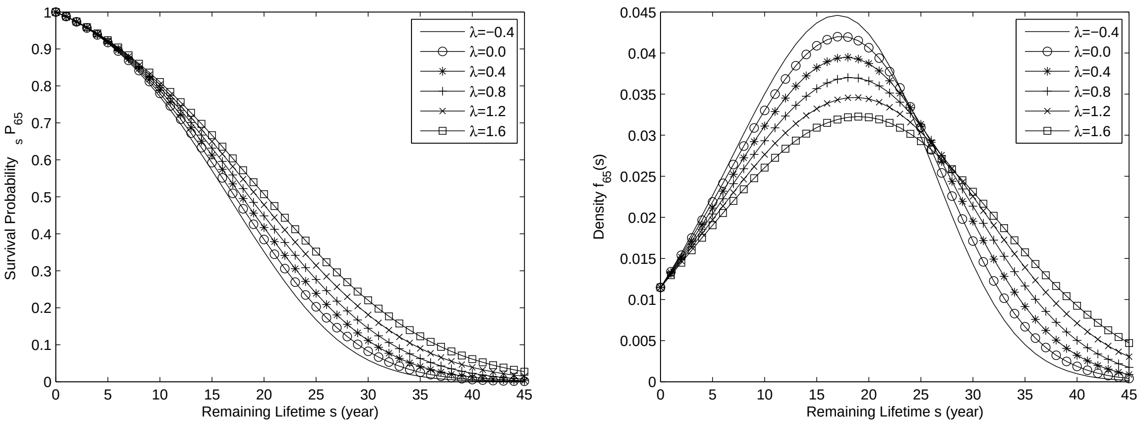

Fig. 4. Survival probabilities and densities of the remaining lifetime of an individual aged 65 with different values of λ. Other parameters are as specified in Table 1. The case $\lambda = 0$ corresponds to the survival curve under $\mathbb{P}$.

values lead to an improvement in survival probability under the risk-adjusted measure $\mathbb{Q}$. Thus, under the proposed model specification, a negative value of λ is suitable for setting the risk premium for a life insurance portfolio while a positive value of λ is adequate for a life annuity business. [10] Corresponding expected remaining lifetime of an individual as a function of σ_μ and λ, given $\mathcal{G}_0$, are plotted in the left and right panel of Fig. 5 respectively. One observes that a linear increase in σ_μ or λ results in an exponential increase in the expected remaining lifetime under the stochastic mortality model.

[10] If the longevity market is matured, then λ can be instead obtained via calibration to market prices of longevity instruments following the no-arbitrage principle, see for instance Russo et al. (2011).

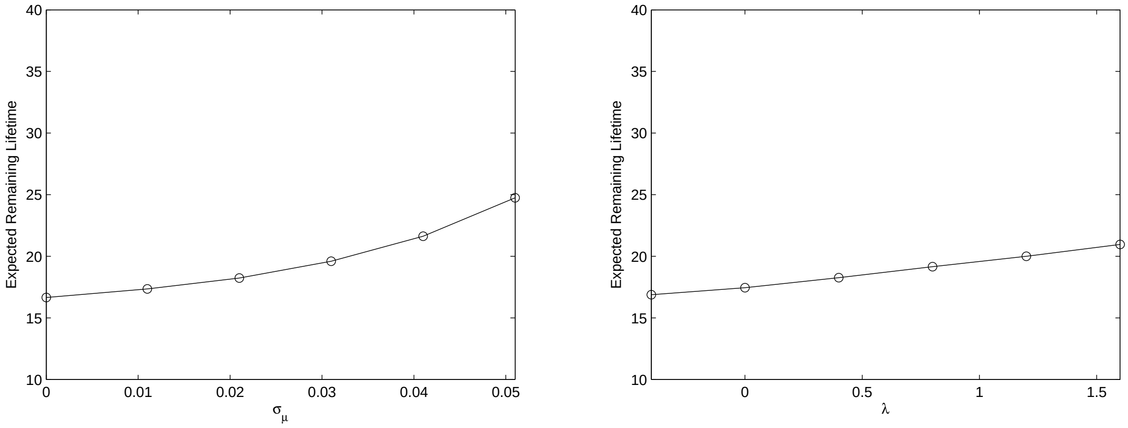

Fig. 5. Expected remaining lifetime of an individual aged 65 as a function of σ_μ (left panel) and λ (right panel), given information $\mathcal{G}_0$.

4 Valuation of the GLWB

We generalize the pricing formulation of guaranteed minimum withdraw benefits (GMWB) in Kolkiewicz and Liu (2012) to the case of GLWB. There are two approaches which suggest to decompose the cashflows involved in the benefit and the premium structure of GMWB, which are shown to be equivalent in Kolkiewicz and Liu (2012). The first approach was suggested by Milevsky and Salisbury (2006) while the second approach was introduced in Aase and Perrson (1994) and Persson (1994) under the "principle of equivalence under $\mathbb{Q}$". In the following, we describe both valuation approaches for the case of GLWB and demonstrate that these are indeed equivalent at any time t. We focus on plain GLWB with assumptions that policyholders exhibit static withdrawal behavior and withdrawals start immediately without any defer period.

4.1 First Approach: Policyholder's Perspective

We recall terminology above, and denote $A(0)$ to be the initial investment, g the guaranteed withdrawal rate, $\hat{\tau}$ the remaining lifetime of an individual aged x at time 0, ω the maximum age allowed in the model and ${}_sP_{x+t}$ the survival probability of an individual aged $x+t$ at time t. From a policyholder's perspective, income from static withdrawals can be regarded as an immediate life annuity. The no-arbitrage value at time t, denoted by $V_1^P(t)$, of an immediate life annuity can be expressed as

$$V_1^P(t) = 1_{\{\hat{\tau}>t\}}\, gA(0) \int_0^{\omega-x-t} {}_sP_{x+t}\, e^{-rs}\, ds, \tag{4.1}$$

where $0 \le t \le \omega - x$ and $1_{\{\hat{\tau}>t\}}$ denotes the indicator function taking value of one if an individual is still alive at time t, and zero otherwise.

Let $\tilde{A}(\cdot)$ be the solution to the SDE (3.3) without the condition that $\tilde{A}(\cdot)$ is absorbed at zero. Since the account value cannot be negative and any remaining amount in the account at the time of death of the policyholder is returned to the policyholder's beneficiary, this cash inflow can be regarded as a call option with payoff $(\tilde{A}(\hat{\tau}))^+$. By the assumption that equity risk and systematic mortality risk are independent, we can write the no-arbitrage value of this call option payoff at time t as

$$V_2^P(t) = 1_{\{\hat{\tau}>t\}} \int_0^{\omega-x-t} f_{x+t}(s) E_t^{\mathbb{Q}} \left(e^{-rs} \, (\tilde{A}(t+s))^+ \right) ds. \tag{4.2}$$

where $f_{x+t}(s) = -\frac{d}{ds}({}_sP_{x+t})$ is the density function of the remaining lifetime of an individual aged $x+t$ at time t.

Both, $V_1^P(t)$ and $V_2^P(t)$ are cash inflows while the amount in the investment account $A(t)$ is viewed as a cash outflow to the VA provider. Under the first approach the value of GLWB, denoted by $V^P(t)$, is defined as

$$V^P(t) = V_1^P(t) + V_2^P(t) - 1_{\{\hat{\tau}>t\}} A(t), \tag{4.3}$$

which can be rewritten as

$$V^P(t) = 1_{\{\hat{\tau}>t\}} \left(\int_0^{\omega-x-t} f_{x+t}(s) \left(\frac{gA(0)}{r} \left(1 - e^{-rs}\right) + E_t^{\mathbb{Q}} \left(e^{-rs} (\tilde{A}(t+s))^+ \right) \right) ds - A(t) \right), \tag{4.4}$$

where integration by part have been applied in order to express Eq.(4.1) in terms of the remaining lifetime density $f_{x+t}(s)$. To make the guarantee fair to both, the policyholder and the insurer, at time $t=0$ we must have $V^P(0)=0$, that is,

$$V_1^P(0) + V_2^P(0) = A(0). \tag{4.5}$$

The fair guarantee fee rate, denoted by α_g^*, is defined as the guarantee fee rate that solves Eq.(4.5).

4.2 Second Approach: Insurer's Perspective

Under the second approach the value of GLWB is defined as the expected discounted benefits minus the expected discounted premiums. Let $\hat{u}$ be random variable defined by

$$\inf\{u \geq 0 \,|\, A(u) = 0\},$$

that is, $\hat{u}$ is the time when account value $A(\cdot)$ is depleted. We can express the expected discounted benefits $V_1^I(t)$ as

$$\begin{aligned} V_1^I(t) &= 1_{\{\hat{\tau}>t\}} \int_0^{\omega-x-t} f_{x+t}(s) \, E_t^{\mathbb{Q}} \left(\int_{t+\hat{u}}^{t+s} gA(0) e^{-r(v-t)} 1_{\{s>\hat{u}\}} \, dv \right) ds \\ &= 1_{\{\hat{\tau}>t\}} \int_0^{\omega-x-t} f_{x+t}(s) \left(\frac{gA(0)}{r} \right) E_t^{\mathbb{Q}} \left(\left(e^{-r\hat{u}} - e^{-rs} \right)^+ \right) ds \end{aligned} \tag{4.6}$$

and the expected discounted premiums $V_2^I(t)$ as

$$V_2^I(t) = 1_{\{\hat{\tau}>t\}} \int_0^{\omega-x-t} f_{x+t}(s) \, E_t^{\mathbb{Q}} \left(\int_t^{t+(\hat{u}\wedge s)} e^{-r(v-t)} \alpha_g \, A(v) \, dv \right) ds, \tag{4.7}$$

where $x_1 \wedge x_2 = \min\{x_1, x_2\}$. Under the second approach, the value of GLWB at time t is defined as

$$V^I(t) = V_1^I(t) - V_2^I(t), \tag{4.8}$$

and the fair guarantee fee rate can be calculated by solving

$$V^I(0) = 0. \tag{4.9}$$

The second approach follows the principle of equivalence under a risk-adjusted measure $\mathbb{Q}$, see Aase and Perrson (1994) and Persson (1994).

4.3 Equivalence of the Two Valuation Approaches

To show that both approaches are equivalent, we consider the following investment strategy. Suppose an individual invests an amount $A(0)$ in a mutual fund-type account held by an insurance company at time $t = 0$. In the future, the individual receives cash flow from the account consisting of, firstly, an amount $K = k \cdot A(0)$ per unit of time and secondly, an amount $\beta \cdot A(t)$ per unit of time, which is proportional to the account value. Both amounts are withdrawn continuously until the account is depleted or individual dies. Finally, any remaining amount in the account will be returned to the individual's beneficiary. In order to value the entire cash flow including the initial investment, we notice that the insurer provides no financial obligation or guarantee to the above arrangement, that is, the role of the insurer is redundant. Thus, the no-arbitrage value of the above cash flow must be zero at time $t = 0$. That is, we have

$$E_0^{\mathbb{Q}}\left(-A(0) + \int_0^{\hat{u}\wedge\hat{\tau}} e^{-rv}(K + \beta A(v))dv + e^{-r\hat{\tau}}(\tilde{A}(\hat{\tau}))^+\right) = 0 \tag{4.10}$$

where $k \geq 0$, $\beta \geq 0$ and $A(\hat{\tau})$ is written as $(\tilde{A}(\hat{\tau}))^+$.

Using the fact that $\int_0^{\hat{u}\wedge\hat{\tau}} = \int_0^{\hat{\tau}} - \int_{\hat{u}}^{\hat{\tau}} \mathbf{1}_{\{\hat{\tau}>\hat{u}\}}$ and rearranging the terms in Eq.(4.10), we obtain

$$E_0^{\mathbb{Q}}\left(-A(0) + \int_0^{\hat{\tau}} K\, e^{-rv}dv + e^{-r\hat{\tau}}(\tilde{A}(\hat{\tau}))^+\right) = E_0^{\mathbb{Q}}\left(\int_{\hat{u}}^{\hat{\tau}} e^{-rv}K\mathbf{1}_{\{\hat{\tau}>\hat{u}\}}dv - \int_0^{\hat{u}\wedge\hat{\tau}} e^{-rv}\beta A(v)dv\right) \tag{4.11}$$

The L.H.S. and R.H.S. of Eq.(4.11) are $V^P(0)$ and $V^I(0)$, respectively. Note that the parameter k can be interpreted as the GLWB guaranteed withdrawal rate g, whereas β is the guarantee fee rate corresponding to α_g. By imposing the principle of equivalence under $\mathbb{Q}$ we note that the R.H.S. of Eq.(4.11) equals zero, which implies that the L.H.S. of Eq.(4.11) must equal to zero as well. Since the argument does not rely on whether the remaining lifetime $\hat{\tau}$ is stochastic or deterministic, it is evident that the equivalence holds for the valuation of GMWB where $\hat{\tau} = T = 1/k$ is deterministic, assuming static withdrawal behavior. Table 2 reports in the last two columns the fair guarantee fee rate α_g^* computed using two equivalent approaches summarized above. One observes that for different values of parameters both approaches lead to the same α_g^*, subject to simulation error with finite sample size.

Table 2
Fair guarantee fees obtained using two valuation approaches. Other parameters are as specified in Table 3.

Case	r	σ	g	σ_μ	λ	Fee (1st App.)	Fee (2nd App.)
1.	4%	20%	5.0%	0.04	0.2	0.49%	0.48%
2.	3%	25%	4.5%	0.02	0.0	0.46%	0.45%
3.	5%	15%	5.5%	0.05	0.8	0.50%	0.49%
4.	4%	20%	6.0%	0.00	0.0	0.73%	0.71%

One can easily generalize Eq.(4.10) to any time $t \geq 0$. Both approaches defined through (Eq.(4.5) and (4.9)), can be applied to valuate GLWB for any time $t \geq 0$. While the first approach is computationally more efficient, the second approach supports theoretical argument that the market reserve of a payment process is defined as the expected discounted benefit minus the expected discounted premium under a risk-adjusted measure, see Dahl and Moller (2006).

Remark 2 *For some special cases the equivalence, formulated in Eq.*(4.11)*, can be verified analytically. Assume that $A(t)$ follows the dynamics in SDE* (3.3) *and*

(1) let $k = 0$ and $\beta \geq 0$. Eq.(4.11) *becomes*

$$A(0) = E_0^{\mathbb{Q}} \left(e^{-r\tau} A(\tau) + \int_0^{\tau} e^{-rs} \beta A(s) ds \right),$$

where the guarantee fee rate β can be interpreted as a continuous dividend yield, see e.g., Bjork (2009). In particular, when $\beta = 0$, it states that the discounted asset is a $\mathbb{Q}$-martingale.

(2) let $k > 0$, $\beta = 0$ and $\sigma = 0$. In this case we have

$$dA(t) = (rA(t) - kA(0))dt$$

which has the solution

$$A(t) = A(0)e^{rt} + \frac{kA(0)}{r} \left(1 - e^{rt}\right).$$

The account $A(t)$ can be interpreted as a money market account where a constant amount is withdrawn continuously. Solving $A(u) = 0$ we have $u = \frac{1}{r} \ln \frac{k}{k-r}$. We can verify Eq.(4.11) *by noticing that*

$$\begin{aligned} E_0^{\mathbb{Q}} \left(\int_0^{u \wedge \hat{\tau}} e^{-rv} K dv + e^{-r\hat{\tau}} (\tilde{A}(\hat{\tau}))^+ \right) &= \int_0^{\omega - x} f_x(s) \left(\int_0^{u \wedge s} e^{-rv} K dv + e^{-rs} (\tilde{A}(s))^+ \right) ds \\ &= A(0) \end{aligned}$$

where the second equality is obtained by considering two cases $s \geq u$ and $s < u$ separately.

5 Sensitivity Analysis

As discussed in Section 1, risk assessment is important for analyzing the underlying risks of an insurance product. Sensitivity analysis is one of such assessments where one can quantify the impact of parameter risk on pricing performance of a product. Given that the underlying guarantee is a long term contract, the effect of parameter risk on pricing can be significant. In this section we study the effect on pricing and the relationship between the fair guarantee fee rate α_g^* and several financial and demographic factors, which include the age of the insured, the equity exposure π, the interest rate level r, the volatility of mortality σ_μ and the market price of systematic mortality risk coefficient λ. Table 3 summarizes parameters for the base case, which will be used in the following when studying the effect of different risks on the fair guarantee fee rate α_g^*.

Table 3
Parameters for the base case.

r	$A(0)$	σ	π	g	a	b	σ_μ	λ	ω
4%	100	25%	0.7	5%	0.001	0.087	0.021	0.4	110

5.1 *Sensitivity to Mortality Risk*

Fig. 6 shows the impact of volatility of mortality σ_μ on the fair guarantee fee rate α_g^* for a policyholder aged 65 and 75 (in the left and the right panel, respectively), with different values of guaranteed withdrawal rate g. One observes that α_g^* is approximately an exponential function of σ_μ, which is consistent with the fact that the expected remaining lifetime increases exponentially with respect to σ_μ, see the left panel of Fig. 5. Values of guaranteed withdrawal rate g have a significant effect on the fair guaranteed fee rate. When g increases from 5% to 5.5% (a relative increase of 10%), α_g^* increases from 0.5% to 0.8% (a relative increase of 60%) for the case when $\sigma_\mu = 0.021$. Thus, it seems to be reasonable for insurance providers to reduce benefits of the guarantee by focusing on g rather than α_g^* (since changing the guaranteed withdrawal rate *appears* to have smaller impact on the benefits than changing the fair guaranteed fee rate from a policyholder's perspective). Furthermore, fair guaranteed fee rates are significantly lower for a 75-years-old compared to a 65-year-old, assuming that they have the same amount of retirement savings. This can be explained by the fact that a 75-year-old has lower probability of exhausting the retirement savings in the account before he or she dies under the condition that the withdrawal amount per year is limited. Even if the account is depleted before policyholder dies, the period of the income guaranteed by the provider for a 75-year-old is likely to be shorter than in case of a 65-year-old.

Sensitivity of the fair guarantee fee rate to the market price of systematic mortality risk coefficient λ is similar to the case of volatility of mortality, see Fig. 7. Explanation for the result can be carried over from the case of volatility of mortality as discussed above. Thus, Fig. 5, 6 and 7 suggest that the effect of varying parameters related to systematic mortality risk on pricing can be indicated by the expected remaining lifetime as a function of the underlying parameters.

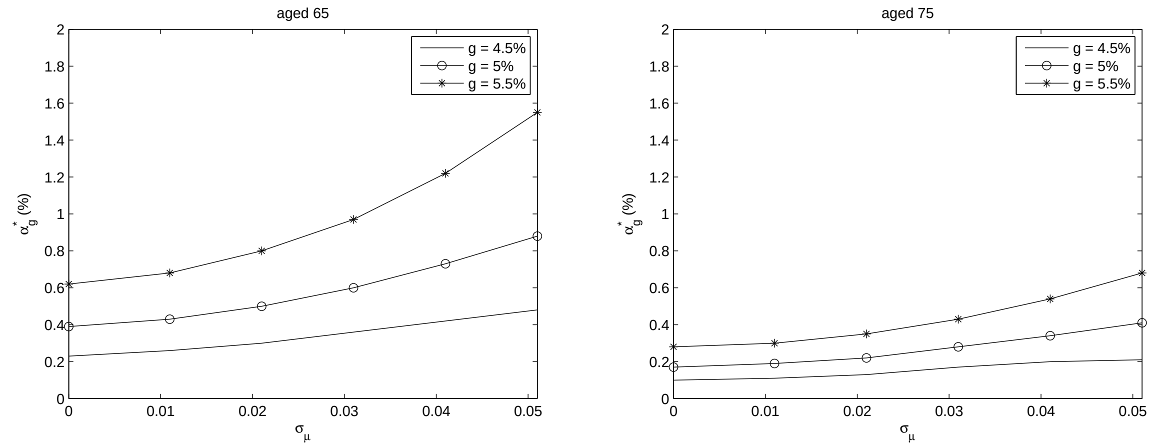

Fig. 6. Sensitivity of the fair guarantee fee rate α_g^* with respect to volatility of mortality represented by σ_μ for policyholders aged 65 and 75.

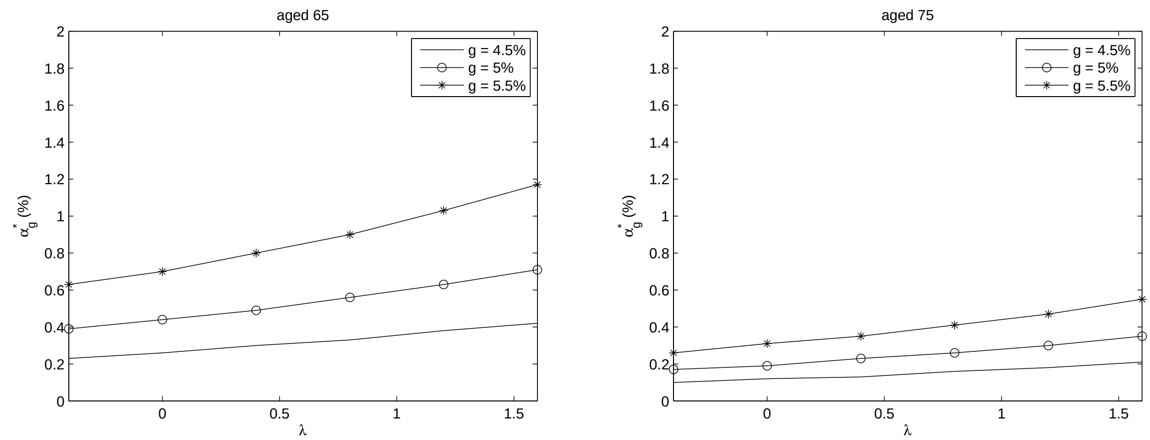

Fig. 7. Sensitivity of the fair guarantee fee rate α_g^* with respect to market price of systematic mortality coefficient λ for policyholders aged 65 and 75.

5.2 Sensitivity to Financial Risk

Sensitivity of the fair guarantee fee rate with respect to the equity exposure π, represented by the volatility of the investment account $\pi \cdot \sigma$, is shown in Fig. 8 where σ is set to be 25% and $\pi \in \{0, 0.3, 0.5, 0.7, 1\}$. The fair guarantee fee rate appears to be very sensitive to the equity exposure as α_g^* increases rapidly when π (and hence, $\pi \cdot \sigma$) increases. Positive relationship between α_g^* and $\pi \cdot \sigma$ is consistent with a financial theory which suggest that options are more expensive when volatility of underlying is high. When π is close to zero, α_g^* approaches zero as well for $g \leq 5\%$. The intuition behind this observation is that in this case there is essentially no liability for the guarantee provider since the guaranteed withdrawal rate is too low for the case when $\pi \cdot \sigma = 0$. Numerical experiment indicates that it takes more than 35 years for the account to drop to zero when $\pi \cdot \sigma = 0$ with $g = 5\%$, and very few policyholders are still alive by that time. This implies that the income guaranteed by the insurer when the account is depleted will never be realized for most situations. Similarly to the case of mortality risk, the fair guarantee fee rate is sensitive to g, as well as to the age of the policyholder.

In contrast to the case of equity exposure and parameters related to mortality risk, the relationship between the interest rate level r and α_g^* is negative, see Fig. 9. It can be explained by considering Eq.(4.8) and Eq.(4.9) where the fair guarantee fee rate is defined as the fee rate that makes the expected discounted benefit equals to the expected discounted premium. As r increases, the present value of the benefit and the premium decreases. However, since benefit is further away in the future than premium, the present value of the benefit will drop faster than the present value of the premium when r is high. As a result, α_g^* will take lower value in the expected discounted premium so that the expected discounted benefit and the expected discounted premium become equal.

Another interesting result arising from Fig. 9 is that the fair guarantee fee rate is very sensitive to interest rates. For instance, when r drops from 2% to 1% for a 65-year-old policyholder while g is set equal 5.5%, α_g^* increases from approx. 3% to 7.5%. This result indicates that it could be a challenging situation for GLWB providers (who issue the guarantee when interest rate was high, and subsequently interest rate declines to a very low level where it remains for a prolonged period of time) as fair guarantee fee rates might be substantially undervalued in this circumstances. [11]

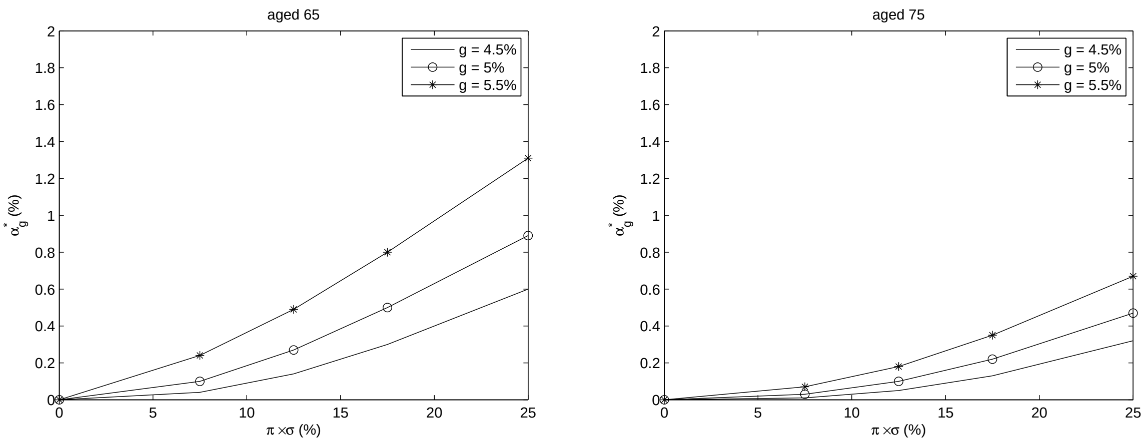

Fig. 8. Sensitivity of the fair guarantee fee rate α_g^* with respect to equity exposure π, represented by volatility of the investment account $\pi \cdot \sigma$, for policyholders aged 65 and 75.

5.3 Sensitivity to Parameter Risk

One of the most important issues related to the long term nature of GLWB, is that the misspecification of parameters at the inception of the contract would have a significant impact on pricing, which in turn would affect the realized profit and loss (P&L) for the guarantee providers. Even if the proposed modeling framework captures effectively qualitative features of the underlying financial and demographic variables, one would

[11] The impact of market variables volatility, including the low interest rate environment in the US starting with the global financial crisis in 2007-2008, is discussed in "American VA providers de-risk to combat market volatility", January 2012, Risk Magazine.

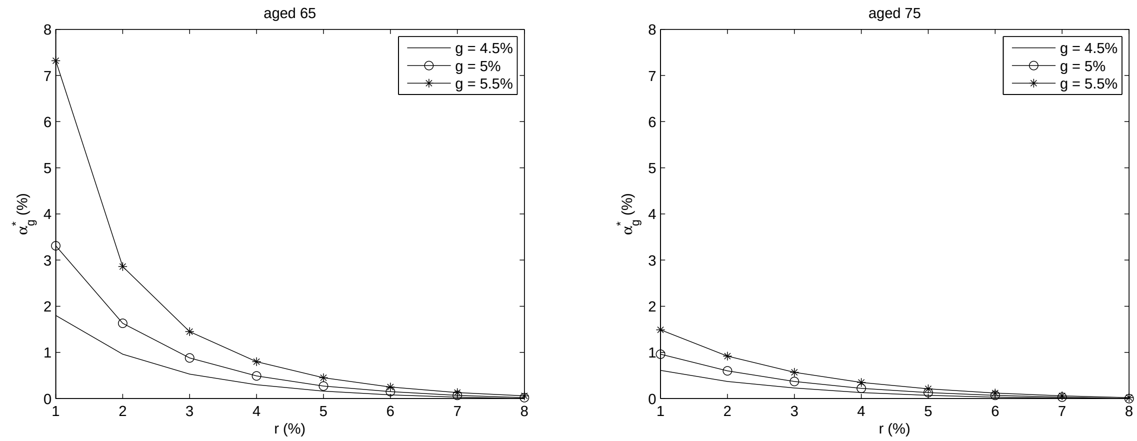

Fig. 9. Sensitivity of the fair guarantee fee rate α_g^* with respect to interest rate level r for policyholders aged 65 and 75.

still face a challenge related to the estimation of model parameters. The challenge arises partly due to the fact that markets for the long term contracts are relatively illiquid and hence, no reliable market prices required for parameter calibration are available. However, one could rely on historical data, perhaps combined with expert judgements, when estimating parameters and pricing a long term contract. Furthermore, given the long term nature of the guarantee, perfect hedging is economically unviable when taking transaction costs and liquidity into account. This suggests that the insurance provider must accept certain level of risk as the uncertainties underlying a guarantee cannot be completely eliminated. Therefore, providing sensitivity analysis for the relative impact of risk inherent in parameter specification (that is, the risk that parameters estimated at the inception of the contract may not be adequate for capturing future dynamics of underlying variables described by the model) on pricing of the guarantee. This is an important step towards understanding the risks undertaken by guarantee providers who make the decision to issue the guarantee.

In the following we study the impact of different model parameters on pricing of GLWB. As shown in Fig. 9, the fair guarantee fee rate is extremely sensitive to low interest rates. This suggests to give first priority to the level of interest when pricing GLWB, especially in a prolonged period of low interest rate environment. Fig.6 and Fig.8 suggest that equity exposure π, and hence, volatility of the investment account $\pi \cdot \sigma$, may have a larger impact on pricing of GLWB than the volatility of mortality σ_μ. However, the parameter risk of σ_μ is non-neglectable, which is illustrated by considering the following scenario. Suppose that the guarantee provider estimates that the fund's volatility corresponds to $\sigma = 20\%$ at the time when guarantee is issued. However, it turns out in the future that the "true" parameter value corresponds to $\sigma = 25\%$. In this scenario the guarantee provider has mis-specified σ by 5%, with $g = 5\%$. This leads to an underestimation of the fee rate α_g^* by approx. 0.2%, see Fig. 8. On the other hand, such a magnitude of mis-pricing of α_g^* corresponds to underestimation of the volatility of mortality σ_μ by approx. $0.04 - 0.021 = 0.019$ (if the original estimation of σ_μ is 0.021 at the inception of the guarantee while the "true" parameter value turns out to be $\sigma_\mu = 0.04$ in the future), see Fig. 3. This indicates that, on average, the remaining lifetime increases unexpectedly by approximately three years, see Fig. 5 and Section 5.1. An underestimation of three years of expected remaining lifetime for a period of 30 to 40 years[12] is possible and the risk of mis-specifying σ_μ cannot be ignored.

Although sensitivity analysis provides us with a quantitative measure for the degree of impact of parameter risk on pricing, it does not summarize the effect of these mis-specifications on the P&L distribution, which is discussed in the next section.

6 Profit and Loss Analysis

To better quantify the underlying risks involved in the guarantee, we simulate profit and loss (P&L) distributions for different situations where no hedging is performed and the liabilities are funded solely by the guarantee fee charged by the insurer.

We assume that a VA portfolio consists of 1000 policyholders all aged 65 who have elected GLWB as the only guarantee to protect their investment. Each individual invests \$100 and chooses the same equity exposure π for their retirement savings investment. A fair guarantee fee is charged according to the valuation result in Section 4. We compute 1000 independent death times by simulating the mortality intensity process μ_{65+t}.[13] The P&L for each individual is obtained by simulating the account process $A(\cdot)$. If a policyholder dies before the account is depleted, the profit for the insurer is determined by the received fee, which grows with an interest rate r. On the other hand, if a policyholder dies after the account value drops to zero, the fee charged by the insurer will be used to fund the liability incurred. The surplus/shortfall is aggregated across all individuals, and discounted to obtain the discounted P&L of the portfolio.[14] The described procedure constitutes one sample. To obtain summary statistics for the discounted P&L distribution, 100,000 simulations are performed. The results reported in Table 4 include the fair guarantee fee rate α_g^*; the mean P&L for the insurer; its standard deviations (Std.dev.); the coefficient of variation (C.V.) defined as the mean P&L per unit of standard deviation; the Value-at-Risk (VaR) defined as the q%-quantile of the P&L distribution; and the Expected Shortfall (ES) determined as the expected loss of portfolio value given that a loss is occurring at or below the q%-quantile. The confidence level $(1-q)$% for the VaR and the ES corresponds to 99.5%. Parameter values used in the simulations are as specified in Table 3 except for the guaranteed withdrawal rate g which is set equal to 6%.[15]

[12] This corresponds approximately to the duration of a VA portfolio with GLWB, assuming retirees are aged 65 years old when the contract starts.

[13] Note that the mortality intensity process is simulated only once. The intensity provides us with a probability distribution, which can be use to draw 1000 independent death times from it.

[14] We find it convenient to use the discounted P&L rather than P&L, since time of death of a policyholder is random and there is no fixed maturity for the contract. Therefore, using discounted P&L allows for more convenient comparison and interpretation of the results at time $t = 0$.

[15] The reason for increasing g from 5% to 6% for the purpose of P&L simulations is as follows. When equity exposure $\pi = 0$, the guarantee is too generous for the insurer. This leads to VaR and ES being positive even for a relatively high confidence level 99.5%. For the purpose of convenience of interpretation of the results, we set $g = 6\%$, which leads to

We first consider interaction between systematic mortality risk and equity exposure π, and its effect on the P&L. Table 4 reports summary statistics for the discounted P&L distributions with systematic mortality risk (with s.m.r) or without systematic mortality risk (no s.m.r.). We first consider the situation when $\pi = 0$, that is, the retirement savings is entirely invested in the money market account with deterministic growth and since the portfolio is large enough, the unsystematic mortality risk is diversified away. In the case when $\pi = 0$ and no s.m.r., the fee charged is able to almost exactly (on average) offset the liability, since the fair fee is obtained by the principle of equivalence under $\mathbb{P}$. Once systematic mortality risk is introduced, the average return, as well as the risk (determined by the Std.dev.) of selling the guarantee, increases, which is consistent with the standard risk-return tradeoff argument. Since systematic mortality risk coefficient $\lambda = 0.4$, there is a positive risk premium associated with the systematic mortality risk, see Fig. 4. C.V. takes its lowest (highest) value of 0.504 (1.375) in a presence of (without) systematic mortality risk and no equity exposure. Generally, distributions with $C.V. < 1$ ($C.V. > 1$) are considered to be low- (high-) variance. For the case of no systematic mortality risk we observe that C.V. decreases with increasing equity exposure. This indicates that for the high values of π the mean return increases faster than the Std.dev., compared to the situation when π is low. Clearly, VaR and ES increase (in absolute terms) when we move from no s.m.r. to s.m.r., and when equity exposure level π increases. In addition, one observes that the effect of systematic mortality risk becomes less pronounced when π increases. This result suggests that if policyholders choose low exposure to equity risk, the guarantee provider should be more careful about systematic morality risk, as it becomes a larger contributor to the overall risk underlying the guarantee.

Table 4
The effect of the presence of systematic mortality risk with respect to different levels of equity exposure π. Summary statistics are computed using discounted P&L distribution per dollar received.

Case	α_g^*	Mean	Std.dev.	C.V.	$\text{VaR}_{0.995}$	$\text{ES}_{0.995}$
			$\pi = 0$			
no s.m.r.	0.09%	0.0011	0.0008	1.375	-0.0010	-0.0013
with s.m.r.	0.23%	0.0062	0.0123	0.504	-0.0470	-0.0556
			$\pi = 0.3$			
no s.m.r.	0.32%	0.0267	0.0243	1.099	-0.0807	-0.0995
with s.m.r.	0.51%	0.0427	0.0363	1.176	-0.1147	-0.1444
			$\pi = 0.5$			
no s.m.r.	0.62%	0.0590	0.0624	0.946	-0.1595	-0.1863
with s.m.r.	0.85%	0.0807	0.0838	0.963	-0.1988	-0.2369
			$\pi = 0.7$			
no s.m.r.	0.97%	0.1049	0.1318	0.796	-0.2333	-0.2619
with s.m.r.	1.25%	0.1362	0.1717	0.793	-0.2718	-0.3148
			$\pi = 1$			
no s.m.r.	1.60%	0.2136	0.3579	0.597	-0.3236	-0.3532
with s.m.r.	1.88%	0.2520	0.4599	0.548	-0.3628	-0.4047

negative VaR and ES across all equity exposures.

6.2 Parameter Risk

Parameter risk constitutes a significant component for the guarantee provider given that the guarantee is generally of a very long term nature. As specified in Section 5.3, parameter risk refers to the situation when parameter(s) from the model are not specified/estimated correctly by the guarantee provider. For example, if volatility of systematic mortality risk σ_μ is set to be 0.021 at the inception of the contract and it turns out to be larger in the future, the resulting mis-specification leads to an inadequate amount of guarantee fee being charged. This is an unfavorable situation for the guarantee provider, even if some hedging program has been implemented.

Table 5 shows distributional statistics for the discounted P&L when different parameters are assumed to be misspecified. These include financial risk parameters (interest rate r and fund volatility σ) as well as demographic risk parameters (volatility of mortality σ_μ and systematic mortality risk coefficient λ). The first column of Table 5 reports parameters r, σ, σ_μ and λ (mis-)specified by the guarantee provider at the initiation of the contract, whereas the "true" parameters correspond to $r = 4\%$, $\sigma = 25\%$, $\sigma_\mu = 0.021$ and $\lambda = 0.4$, and are indicated in bold letters. The second column represents the fair guarantee fee rate calculated using the misspecified parameters, with other parameters set to be "true". For instance, if σ is estimated to be 20%, the fair guarantee fee rate is underestimated by $1.25\% - 0.97\% = 0.28\%$.

As one would expect, the mean P&L depends on the guarantee fee rate. Overcharging or undercharging the guarantee fee leads to a substantial increase or decrease in the mean P&L, respectively. In contrast to the mean P&L, risk measures such as VaR and ES are relatively insensitive to the fair guarantee fee rate; and the standard deviation becomes larger when guarantee fee rate increases. This is due to the fact that the P&L distribution is affected by two factors: the probability of the occurrence of different scenarios and the cash flow involved in each scenario. A miscalculation of guarantee fee rate has no impact on the probability of occurrence of different scenarios, but merely affects the cash flow. As tail risk measures, VaR and ES mostly concern with scenarios represented in Fig. 1. Due to the fact that the guarantee fee rate has larger impact on premiums than on liabilities [16] , and for the worst 0.05% scenarios liabilities are much larger than premiums, overcharging or undercharging the guarantee fee has only small impact on the left tail of the P&L distribution. As a result, VaR and ES are relatively robust to misspecification of guarantee fee rate as they represent the left tail of the P&L distribution.

Contrary to the VaR and the ES, the standard deviation is a measure of dispersion of the data around its mean. Scenarios represented in Fig. 2 are typical when considering standard derivation of the P&L distribution, since most samples around the mean have larger premiums than liabilities. In order to explain why standard deviation is larger when guarantee fee rate increases, we consider two scenarios assuming no

[16] Premiums and liabilities are from an insurer's prospective. Clearly, premiums (or guarantee fees) depend on the guarantee fee rate directly while liabilities implicitly depend on the fee rate as the investment account will decrease (increase) slightly faster when the fee rate is higher (lower).

liabilities. The difference of profits of these two scenarios is calculated as

$$\alpha_g \int_0^\tau (A_1(t) - A_2(t))\, dt. \tag{6.1}$$

Note that samples in the P&L distribution come from a portfolio with 1000 policyholders and hence, the death time τ in Eq.(6.1) is representative only. Note also that the difference in profits indicates how "apart" these two samples are when guarantee fee rate is α_g. Now suppose that the guarantee fee rate is $\hat{\alpha}_g$ instead of α_g. For the same two scenarios the difference of profits become $\hat{\alpha}_g \int_0^\tau (\hat{A}_1(t) - \hat{A}_2(t))\, dt$. Since the integrand is the difference of two account values and the value of guarantee fee rate has only a small effect on the dynamics of $A(t)$, it is reasonable to expect that

$$\int_0^\tau (A_1(t) - A_2(t))\, dt \approx \int_0^\tau (\hat{A}_1(t) - \hat{A}_2(t))\, dt. \tag{6.2}$$

As a result, the samples are more "apart" when $\hat{\alpha}_g > \alpha_g$.

Table 5
Parameter risk: distributional statistics for the discounted P&L per dollar received. The "true" parameters corresponding to $r = 4\%$, $\sigma = 25\%$, $\sigma_\mu = 0.021$ and $\lambda = 0.4$, and are indicated in bold letters.

Case	α_g^*	Mean	Std.dev.	$\mathrm{VaR}_{0.995}$	$\mathrm{ES}_{0.995}$
			r		
2%	5.70%	0.3619	0.3461	-0.2210	-0.2636
3%	2.40%	0.2344	0.2511	-0.2578	-0.3012
4%	**1.25%**	**0.1362**	**0.1717**	**-0.2718**	**-0.3148**
5%	0.70%	0.0731	0.1214	-0.2855	-0.3264
6%	0.39%	0.0326	0.0901	-0.2872	-0.3280
7%	0.20%	0.0055	0.0718	-0.2882	-0.3299
			σ		
10%	0.48%	0.0446	0.0995	-0.2882	-0.3292
15%	0.70%	0.0731	0.1214	-0.2855	-0.3264
20%	0.97%	0.1058	0.1475	-0.2764	-0.3175
25%	**1.25%**	**0.1362**	**0.1717**	**-0.2718**	**-0.3148**
30%	1.50%	0.1607	0.1905	-0.2718	-0.3160
35%	1.90%	0.1962	0.2198	-0.2633	-0.3100
			σ_μ		
0.005	1.00%	0.1079	0.1487	-0.2816	-0.3231
0.010	1.08%	0.1175	0.1564	-0.2733	-0.3152
0.015	1.15%	0.1250	0.1633	-0.2751	-0.3212
0.021	**1.25%**	**0.1362**	**0.1717**	**-0.2718**	**-0.3148**
0.025	1.36%	0.1472	0.1790	-0.2731	-0.3153
0.030	1.54%	0.1645	0.1939	-0.2699	-0.3131
			λ		
0.1	1.14%	0.1244	0.1619	-0.2743	-0.3186
0.2	1.16%	0.1263	0.1637	-0.2772	-0.3162
0.3	1.22%	0.1321	0.1684	-0.2759	-0.3188
0.4	**1.25%**	**0.1362**	**0.1717**	**-0.2718**	**-0.3148**
0.5	1.30%	0.1412	0.1765	-0.2744	-0.3155
0.6	1.35%	0.1458	0.1785	-0.2686	-0.3129

Model risk relates to the situation when the guarantee provider assumes deterministic mortality model, that is, ignoring systematic mortality risk. For the implementation of the deterministic mortality model we impose the conditions $a = 0$ and $\sigma_\mu = 0$ in Eq.(3.4) while parameter b is estimated as discussed in Section 3.2, where we have $b = 0.106$. Table 6 reports the results for the P&L statistics in presence of model risk, comparing stochastic and deterministic mortality models. One observes that when equity exposure is small ($\pi \leq 0.5$), the assumption regarding deterministic mortality leads to an underestimation of the fair guarantee fee rate. As a result, the average return drops while the VaR and the ES worsen. However, for large equity exposures ($\pi > 0.5$) one observes essentially no difference between deterministic or stochastic mortality model specification. This can be explained by the fact that for large π, equity risk dominates systematic mortality risk, which is reflected in the Std.dev., VaR and ES in Table 6.

Table 6
Model risk assuming stochastic vs. deterministic mortality model: distributional statistics for the discounted P&L per dollar received.

Model	α_g^*	Mean	Std.dev.	$\text{VaR}_{0.995}$	$\text{ES}_{0.995}$
			$\pi = 0$		
stochastic	0.23%	0.0062	0.0123	-0.0470	-0.0556
deterministic	0.14%	0.0001	0.0114	-0.0501	-0.0581
			$\pi = 0.3$		
stochastic	0.51%	0.0427	0.0363	-0.1147	-0.1444
deterministic	0.45%	0.0371	0.0345	-0.1166	-0.1507
			$\pi = 0.5$		
stochastic	0.85%	0.0807	0.0838	-0.1988	-0.2369
deterministic	0.82%	0.0786	0.0822	-0.2001	-0.2383
			$\pi = 0.7$		
stochastic	1.25%	0.1362	0.1717	-0.2718	-0.3148
deterministic	1.25%	0.1362	0.1717	-0.2718	-0.3148
			$\pi = 1$		
stochastic	1.88%	0.2520	0.4599	-0.3628	-0.4047
deterministic	1.90%	0.2534	0.4478	-0.3684	-0.4117

7 Static Hedging of Systematic Mortality Risk: A Numerical Example

Several longevity-linked securities proposed in the literature can be applied as hedging instruments for the longevity risk. Here, we consider the so-called S-forward, or ‘survivor’ forward, which has been developed by LLMA (2010). It is a cash settled contract linked to survival rates of a given population cohort which are ultimately derived from mortality rates. The S-forward is the basic building-block for the longevity (survivor) swaps, see Dowd (2003), that have already been used extensively by pension funds and insurance companies to hedge the longevity risk. These longevity swaps can be regarded as a stream of S-forwards with different maturity dates. The advantage of using S-forward as our choice of hedging strategy is that there is no initial capital requirement at the inception of the contract and cash flows only occur at maturity. Moreover, since the underlying of S-forward is the survival probability of a chosen population, it is convenient for the problem at hand given that we have an analytical expression for survival probability. Finally, the lack of a matured longevity market also makes static hedging more realistic compared to the dynamics hedging which requires liquid longevity-linked instruments. S-forward is an agreement between two counterparties to exchange at maturity an amount equal to the realized survival rate of a given population cohort, in return for a fixed survival rate agreed at the inception of the contract, whereas survival rate is calculated from mortality rates. Thus, S-forward is in fact a swap with only one payment at maturity T where the fixed leg pays:

$$N \cdot E_0^{\mathbb{Q}} \left(e^{-\int_0^T \mu_{x+s}(s)ds} \right) \tag{7.1}$$

while the floating leg pays:

$$N \cdot e^{-\int_0^T \mu_{x+s}(s)ds}, \tag{7.2}$$

with N denoting the notional amount of the contract. The fixed leg is determined under the risk-adjusted measure $\mathbb{Q}$ and as discussed in Section 3.3. Since there is a positive risk premium for systematic mortality risk, the systematic mortality risk hedger who pays fixed leg and receives floating leg, bears implicit cost for entering an S-forward.

Since the maturity of GLWB is not fixed, we assume in our numerical example that $T = 20$, which approximately corresponds to the expected remaining lifetime of a 65-years old (Fig. 5). To determine an appropriate notional amount N for the contract, we notice that N should depend on the total investment I (which corresponds to \$100 investment for each of 1000 policyholders: $I = 1000 \times 100 = 100,000$) of a VA portfolio. Assuming N to be a linear function of I ($N = \theta \cdot I$), we found that $\theta \approx 0.35$ provides the most effective hedge in a sense of VaR and ES reduction (in absolute terms) through our numerical experiment.

Table 7
Static hedging of systematic mortality risk via S-forward. Distributions of discounted P&L per dollar received.

Case	Mean	Std.dev.	$\text{VaR}_{0.995}$	$\text{ES}_{0.995}$
		$\pi = 0$		
No hedge	0.0062	0.0123	-0.0470	-0.0556
Static hedge	0.0016	0.0111	-0.0283	-0.0313
		$\pi = 0.3$		
No hedge	0.0427	0.0363	-0.1147	-0.1444
Static hedge	0.0382	0.0396	-0.0995	-0.1251
		$\pi = 0.5$		
No hedge	0.0807	0.0838	-0.1988	-0.2369
Static hedge	0.0761	0.0861	-0.1870	-0.2198
		$\pi = 0.7$		
No hedge	0.1362	0.1717	-0.2718	-0.3148
Static hedge	0.1316	0.1737	-0.2591	-0.2968
		$\pi = 1$		
No hedge	0.2520	0.4599	-0.3628	-0.4047
Static hedge	0.2476	0.4615	-0.3491	-0.3866

Table 7 reports the results for the discounted P&L statistics in presence of static hedge and different levels of equity exposure. We observe that using the simplest form of static hedge be means of S-forward allows us to reduce VaR and ES (in absolute terms) considerably. In particular, the improvement (in percentage terms) of the VaR and the ES is larger when equity exposure is small. Although the average return becomes lower when the hedging strategy is in place, the risk (determined by the standard deviation) remains nearly unchanged, which suggests that the volatility of the fund investment has a significant impact on the outcome of the P&L distribution. Therefore, a guarantee provider should address equity risk carefully in a first place, prior to considering systematic mortality risk, especially when the equity exposure of the VA portfolio is large.

8 Conclusion

This paper presents comprehensive analysis for the guaranteed lifetime withdrawal benefits (GLWB) embedded in variable annuities, which is the most elected guarantee in the U.S. in 2011. GLWB provides a type of life annuity which addresses systematic mortality risk while protecting the policyholders from the downside risk of fund investment. It allows a policyholder to decide how much to withdraw every year for the lifetime, regardless of the performance of the investment. The GLWB guarantee promises to the insured a stream of income for life, even if the investment account is depleted. After the death of the insured, any savings remaining in the account will be returned to the insured's beneficiary. The paper provides a detailed description of GLWB, including product features and identification of risks.

The paper considers pricing of the GLWB using two equivalent approaches, assuming tractable equity and stochastic mortality models. We demonstrate the equivalence of the two approaches using a numerical example. We further study the effect of various financial and demographic variables (such as the interest rate r, volatility of a fund investment σ, volatility of mortality σ_μ and the market price of systematic mortality coefficient λ) on the fair guarantee fee rate α_g^* charged by the insurer, as well as on the profit and loss characteristics from the point of view of insurance provider.

The results indicate that the fair guarantee fee increases exponentially with increasing volatility of mortality σ_μ, or the market price of systematic mortality coefficient λ, which is due to the fact that the expected remaining lifetime increases with increasing σ_μ. Furthermore, the fair guarantee fee appears to be positively related, and highly sensitive to the volatility of the investment account, which is consistent with a financial theory suggesting that, generally, options are more expensive when volatility is high. The relationship between α_g^* and r is negative and α_g^* appears to be highly sensitive to low interest rates.

The results from studying the P&L distribution from the guarantee provider's point of view (and assuming no hedge in place) show that the risk determined via Value-at-Risk (VaR) and expected shortfall (ES) increases with increasing equity exposure. The VaR and the ES are higher (in absolute terms) when systematic mortality risk is present in the model, compared to the situation with no systematic mortality risk. We further show that parameter risk and model risk might result in significant under- or overestimation of the fair guarantee fee rate α_g^*. Finally, a static hedge procedure implemented using S-forward, allows to reduce VaR and ES, but leads to a decrease in the average return for the insurer (especially when the equity exposure is high), suggesting that the equity risk has to be taken care of along with a careful consideration of a systematic mortality risk.

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