# SYSTEMATIC ORBIFOLD CONSTRUCTIONS OF SCHELLEKENS' VERTEX OPERATOR ALGEBRAS FROM NIEMEIER LATTICES 

GERALD HÖHN ${ }^{\text {a }}$ AND SVEN MÖLLER ${ }^{\text {b,c }}$


#### Abstract

We present a systematic, rigorous construction of all 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with non-zero weight-one space $V_{1}$ as cyclic orbifold constructions associated with the 24 Niemeier lattice vertex operator algebras $V_{N}$ and certain 226 short automorphisms in $\operatorname{Aut}\left(V_{N}\right)$.

We show that up to algebraic conjugacy these automorphisms are exactly the generalised deep holes, as introduced in MS19, of the Niemeier lattice vertex operator algebras with the additional property that their orders are equal to those of the corresponding outer automorphisms.

Together with the constructions in Höh17] and MS19] this gives three different uniform constructions of these vertex operator algebras, which are related through 11 algebraic conjugacy classes in $\mathrm{Co}_{0}$.

Finally, by considering the inverse orbifold constructions associated with the 226 short automorphisms, we give the first systematic proof of the result that each strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with non-zero weight-one space $V_{1}$ is uniquely determined by the Lie algebra structure of $V_{1}$.


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## 1. Introduction

The programme to classify the strongly rational, holomorphic vertex operator algebras of central charge 24 was initiated by Schellekens in 1993. He showed that the weight-one subspace $V_{1}$ of such a vertex operator algebra $V$ is one of 71 reductive Lie algebras called Schellekens' list [Sch93 (see also DM04a DM06 EMS20a]). He conjectured that all potential Lie algebras are realised and that the $V_{1}$-structure fixes the vertex operator algebra $V$ up to isomorphism.

By contributions of many authors over the last three decades the following classification result is now proved:

[^0]Theorem. Up to isomorphism there are exactly 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$. Such a vertex operator algebra is uniquely determined by its $V_{1}$-structure.

The original proof relies mainly on cyclic orbifold constructions but is based on a case-by-case analysis with a variety of different approaches.

In Höh17, a uniform proof of the existence part of the theorem was given, depending on a conjecture on orbifolds of lattice vertex operator algebras proved in Lam20. Each vertex operator algebra $V$ is realised as a simple-current extension of a certain dual pair in $V$.

Another systematic proof of the existence part of the theorem was recently given in MS19] (see also [CLM22]) by considering orbifold constructions $V_{\Lambda}^{\text {orb }(g)}$ associated with generalised deep holes $g$ of the Leech lattice vertex operator algebra $V_{\Lambda}$. The corresponding inverse orbifold constructions are described in ELMS21 and are used to give a simpler proof of Schellekens' list of 71 Lie algebras.

In this work we shall describe a third systematic construction of the strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$, namely as orbifold constructions starting from the vertex operator algebras $V_{N}$ associated with the 24 Niemeier lattices $N$, the positive-definite, even, unimodular lattices of rank 24 , which include the Leech lattice $\Lambda$.

At the centre of these three constructions, as was first observed in Höh17, are 11 algebraic conjugacy classes (i.e. conjugacy classes of cyclic subgroups, see $\left.\left[\mathrm{CCN}^{+} 85\right]\right)$ in the isometry group $\mathrm{O}(\Lambda) \cong \mathrm{Co}_{0}$ of the Leech lattice $\Lambda$, namely those uniquely specified by the Frame shapes $1^{24}, 1^{8} 2^{8}, 1^{6} 3^{6}, 2^{12}, 1^{4} 2^{2} 4^{4}, 1^{4} 5^{4}, 1^{2} 2^{2} 3^{2} 6^{2}$, $1^{3} 7^{3}, 1^{2} 2^{1} 4^{1} 8^{2}, 2^{3} 6^{3}$ and $2^{2} 10^{2}$ (see Table 1). Evidently, these Frame shapes have only non-negative exponents, but they are not characterised in $\mathrm{Co}_{0}$ by this property (see Table 3 in the appendix).

Given a Niemeier lattice $N$, the outer automorphism group $\operatorname{Aut}\left(V_{N}\right) / K$ of the corresponding lattice vertex operator algebra $V_{N}$ is isomorphic to $H=\mathrm{O}(N) / W$ where $W$ is the Weyl group of $N$, and $H$ can be embedded into $\mathrm{O}(N)$.

Let $g$ be an automorphism of finite order $n$ of a Niemeier lattice vertex operator algebra $V_{N}$. Up to conjugation $g$ is of the form $\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ with $\nu \in H$ and $h \in \pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right)$ where $\pi_{\nu}$ is the projection onto the elements that are fixed by $\nu$ (see Theorem 2.13). Then $g$ is called short (see Definition 5.1) if
(1) $g$ has type 0 (so that the cyclic orbifold construction $V_{N}^{\text {orb }(g)}$ exists),
(2) $\nu$ (i.e. the projection of $g$ to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ ) has order $n$ and
(3) $h \bmod \left(N^{\nu}\right)^{\prime}$ has order $n$.

As the main result of the present text we establish the existence part of the classification theorem in a systematic way:
Theorem Theorem 5.4. Corollary 5.5. The cyclic orbifold constructions $V_{N}^{\mathrm{orb}(g)}$, where $N$ runs through the 24 Niemeier lattices and $g$ through the short automorphisms of the corresponding lattice vertex operator algebras $V_{N}$, realise all 70 nonzero Lie algebras $\mathfrak{g}$ on Schellekens' list as weight-one spaces $\left(V_{N}^{\operatorname{orb}(g)}\right)_{1} \cong \mathfrak{g}$.

We classify the short automorphisms:
Theorem Proposition 5.2. There are exactly 226 algebraic conjugacy classes of short automorphisms of the Niemeier lattice vertex operator algebras $V_{N}$, listed in Table 2. The Frame shapes of their projections to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ are given by the 11 Frame shapes in Table 1.

We then show that the short automorphisms $g$ of the Niemeier lattice vertex operator algebras $V_{N}$ are all generalised deep holes as introduced in MS19], i.e.
(1) $g$ has type 0 ,
(2) the dimension of $\left(V_{N}^{\operatorname{orb}(g)}\right)_{1}$ attains the upper bound provided by the dimension formula in MS19 (see Theorem 3.1), and
(3) the weight-one Lie algebras satisfy the orbifold rank condition $\operatorname{rk}\left(\left(V_{N}^{g}\right)_{1}\right)=$ $\operatorname{rk}\left(\left(V_{N}^{\operatorname{orb}(g)}\right)_{1}\right)$,
and additionally
(4) the order of $g$ equals the order of the projection of $g$ to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$.

Conditions (2) to (4) should be understood as extremality requirements.
The last condition (4) entails that the orders of the automorphisms of $V_{N}$ that we consider are relatively small, namely equal to the orders of the corresponding 11 Frame shapes listed above. We contrast this to the uniform construction in [MS19], which uses generalised deep holes of only the Leech lattice vertex operator algebra $V_{\Lambda}$ projecting to the same 11 Frame shapes in $\mathrm{O}(\Lambda)$ but with orders up to 46 .

We then prove that the 226 short automorphisms are already characterised by these four properties:
Theorem Theorem 5.10. The short automorphisms of the Niemeier lattice vertex operator algebras $V_{N}$ are exactly the automorphisms of $V_{N}$ satisfying (1) to (4) (and with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)>0$ in the case of the Leech lattice $\left.\Lambda\right)$.

Finally, we give the first uniform proof of the uniqueness statement in the classification theorem at the beginning of the introduction:
Theorem Theorem 6.5. Let $\mathfrak{g}$ be a non-zero Lie algebra on Schellekens' list. Then there is a Niemeier lattice $N$ and a short automorphism $g \in \operatorname{Aut}\left(V_{N}\right)$ such that any strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \cong \mathfrak{g}$ satisfies $V \cong V_{N}^{\mathrm{orb}(g)}$. In particular, the vertex operator algebra structure of $V$ is uniquely determined by the Lie algebra structure of $V_{1}$.

The proof follows the strategy laid out in LS19 and uses the inverse orbifold constructions associated with certain 157 of the 226 short automorphisms. Moreover, the uniqueness of the decomposition of $V$ into $\left\langle V_{1}\right\rangle$-modules proved in Sch93] is used.

We also describe how the constructions in Höh17, MS19] and in this text are related (see end of Section 4.2 and Proposition 5.7).

Outline. The paper is organised as follows: In Section 2 we review lattice vertex operator algebras. Then we prove some results about automorphism groups of vertex operator algebras and apply them to lattice vertex operator algebras.

In Section 3 we recall the cyclic orbifold theory for holomorphic vertex operator algebras developed in EMS20a, Möl16] and state a dimension formula from [EMS20b, MS19] for central charge 24. We also prove an orbifold rank criterion.

In Section 4 we summarise what is known about the classification of strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 and provide the context, mainly from Höh17, MS19, for the uniform description in this text.

In Section 5 we define the notion of a short automorphism of a unimodular lattice vertex operator algebra and prove that all 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$ can be obtained as orbifold constructions from the 24 Niemeier lattice vertex operator algebras $V_{N}$ associated with such automorphisms. We also study some properties of these automorphisms, classify them and show that they can be characterised as certain generalised deep holes in the sense of MS19].

In Section 6 we give a uniform proof of the uniqueness of a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with any given non-zero weight-one Lie algebra $V_{1}$.

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Computer Calculations. We remark that some of the computations in Section 5 and Section 6 involving the isometry groups $\mathrm{O}(N)$ of the Niemeier lattices $N$ were performed on the computer using Magma BCP97.

All vertex operator algebras are assumed to be complex. Lie algebras are complex and finite-dimensional.

## 2. Lattice Vertex Operator Algebras and Automorphism Groups

In this section we review lattice vertex operator algebras Bor86, FLM88, Don93] (for details see, e.g., Kac98, LL04). Then we describe automorphism groups of vertex operator algebras, in particular the conjugacy classes, and apply these results to lattice vertex operator algebras.

For an introduction to vertex operator algebras and their representation theory we refer the reader to [FLM88, FHL93, LL04]. A vertex operator algebra $V$ is called strongly rational if it is rational (as defined in, e.g., DLM97), $C_{2}$-cofinite (or lisse), self-contragredient (or self-dual) and of CFT-type. This also implies that $V$ is simple. A vertex operator algebra $V$ is called holomorphic (or self-dual or meromorphic) if it is rational and the only irreducible $V$-module is $V$ itself (implying that $V$ is simple and self-contragredient). The definition of twisted modules follows the sign convention in, e.g., DLM00] as opposed to some older texts. A vertex operator subalgebra is called full if it has the same Virasoro vector as the containing vertex operator algebra.

In a vertex operator algebra $V=\bigoplus_{n=0}^{\infty} V_{n}$ of CFT-type the weight-one space $V_{1}$ carries the structure of a (complex, finite-dimensional) Lie algebra via $[u, v]:=u_{0} v$ for all $u, v \in V_{1}$. If $V$ is strongly rational, then this Lie algebra is reductive, i.e. a direct sum of an abelian and a semisimple Lie algebra DM04b.

If a vertex operator algebra $V$ is self-contragredient and of CFT-type, then there exists a non-degenerate, invariant bilinear form $\langle\cdot, \cdot\rangle$ on $V$, which is unique up to a non-zero scalar and symmetric [FHL93, Li94]. This bilinear form restricts to a non-degenerate, invariant bilinear form on the Lie algebra $V_{1}$. It is common to choose the normalisation $\langle\mathbf{1}, \mathbf{1}\rangle=-1$ where $\mathbf{1}$ denotes the vacuum vector.
2.1. Lattice Vertex Operator Algebras. Let $L$ be a positive-definite, even lattice, i.e. a free abelian group $L$ of finite $\operatorname{rank} \operatorname{rk}(L)$ equipped with a positive-definite, symmetric bilinear form $\langle\cdot, \cdot\rangle: L \times L \rightarrow \mathbb{Z}$ such that $\langle\alpha, \alpha\rangle \in 2 \mathbb{Z}$ for all $\alpha \in L$. By $\mathfrak{h}:=L \otimes_{\mathbb{Z}} \mathbb{C}$ we denote the complexified lattice. The discriminant form $L^{\prime} / L$ with dual lattice $L^{\prime}=\left\{\alpha \in L \otimes_{\mathbb{Z}} \mathbb{Q} \mid\langle\alpha, \beta\rangle \in \mathbb{Z}\right.$ for all $\left.\beta \in L\right\}$ naturally carries the structure of a non-degenerate finite quadratic space. The lattice $L$ is called unimodular if $L^{\prime}=L$, i.e. if the discriminant form is trivial.

The lattice vertex operator algebra $V_{L}=M_{\hat{\mathfrak{h}}}(1) \otimes \mathbb{C}_{\varepsilon}[L]$ associated with $L$ is strongly rational and of central charge $c=\operatorname{rk}(L)$. The definition of $V_{L}$ involves a choice of group 2-cocycle $\varepsilon: L \times L \rightarrow\{ \pm 1\}$ satisfying $\varepsilon(\alpha, \beta) / \varepsilon(\beta, \alpha)=(-1)^{\langle\alpha, \beta\rangle}$ for all $\alpha, \beta \in L$.

The irreducible $V_{L}$-modules $V_{\alpha+L}, \alpha+L \in L^{\prime} / L$, are indexed by the elements of the discriminant form $L^{\prime} / L$. In particular, if $L$ is unimodular, then $V_{L}$ is holomorphic.

We now describe the weight-one Lie algebra $\left(V_{L}\right)_{1}$ of a lattice vertex operator algebra $V_{L}$. Let $L$ be an even, positive-definite lattice. Then the set of normtwo vectors $\Phi:=\{\alpha \in L \mid\langle\alpha, \alpha\rangle=2\}$ forms a simply-laced root system. Let $R:=\operatorname{span}_{\mathbb{Z}}(\Phi) \subseteq L$ denote the root sublattice generated by $\Phi$.

The weight-one Lie algebra of $V_{L}$ is given by

$$
\left(V_{L}\right)_{1}=\mathcal{H} \oplus \operatorname{span}_{\mathbb{C}}\left(\left\{1 \otimes \mathfrak{e}_{\alpha} \mid \alpha \in \Phi\right\}\right)
$$

where $\mathcal{H}:=\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}\right\} \cong \mathfrak{h}$ is a choice of a Cartan subalgebra of $\left(V_{L}\right)_{1}$. Note that the restriction to $\mathcal{H}$ of the invariant bilinear form $\langle\cdot, \cdot\rangle$ on $V_{L}$ normalised such that $\langle\mathbf{1}, \mathbf{1}\rangle=-1$ is precisely the bilinear form $\langle\cdot, \cdot\rangle$ on $L$ bilinearly extended to the complexification $\mathfrak{h}=L \otimes_{\mathbb{Z}} \mathbb{C}$ under the identification of $\mathcal{H}$ with $\mathfrak{h}$.

It is easy to verify (see, e.g., Section 7.8 in Kac90]) that $\left(V_{L}\right)_{1}$ is a reductive Lie algebra of $\operatorname{rank} \operatorname{rk}(L)$ with a semisimple part of $\operatorname{rank} \operatorname{rk}(R)$ and an abelian part of rank $\operatorname{rk}(L)-\operatorname{rk}(R)$ and that the root system of the semisimple part of $\left(V_{L}\right)_{1}$ is exactly $\Phi$ (viewed in $\mathcal{H}^{*} \cong \mathfrak{h}^{*}$ via $\langle\cdot, \cdot\rangle$ ).
2.2. Automorphism Groups. For any vertex operator algebra $V$ of CFT-type $K:=\left\langle\left\{\mathrm{e}^{v_{0}} \mid v \in V_{1}\right\}\right\rangle$ defines a normal subgroup of $\operatorname{Aut}(V)$, called the inner automorphism group (see [DN99, Section 2.3). We call Aut $(V) / K$ the outer automorphism group of $V$.

Since vertex operator algebra automorphisms are grading-preserving, there is a restriction homomorphism $r: \operatorname{Aut}(V) \rightarrow \operatorname{Aut}\left(V_{1}\right)$, i.e. automorphisms of the vertex operator algebra $V$ restrict to Lie algebra automorphisms of $V_{1}$. This homomorphism $r$ is in general not surjective. However, it follows from the definition of $K$ and the relation $r\left(\mathrm{e}^{v_{0}}\right)=\mathrm{e}^{\mathrm{ad}_{v}}$ that $r(K)=\operatorname{Inn}\left(V_{1}\right)$, i.e. every inner automorphism of $V_{1}$ can be extended to an inner automorphism of $V$.

Since $r(K) \subseteq \operatorname{Inn}\left(V_{1}\right), r$ induces a homomorphism $\tilde{r}: \operatorname{Aut}(V) / K \rightarrow \operatorname{Out}\left(V_{1}\right)$. Moreover, $\tilde{r}$ is injective if and only if

$$
\begin{equation*}
\operatorname{ker}(r) \subseteq K \tag{A}
\end{equation*}
$$

In the following we assume that (A) holds, or equivalently that $r^{-1}\left(\operatorname{Inn}\left(V_{1}\right)\right)=K$. This excludes, in particular, the case of a vertex operator algebra with $V_{1}=\{0\}$ and $\operatorname{Aut}(V) \neq\{\mathrm{id}\}$, like, for example, the Moonshine module $V^{\natural}$. Then we have the following commutative diagram with exact columns:


Since $\tilde{r}$ is injective, the outer automorphism $\operatorname{group} \operatorname{Aut}(V) / K$ is finite whenever $\operatorname{Out}\left(V_{1}\right)$ is, for example, if $V_{1}$ is semisimple. We shall see that $\operatorname{Aut}(V) / K$ is finite if $V$ is a lattice vertex operator algebra, even when $V_{1}$ is not semisimple. One might speculate that $\operatorname{Aut}(V) / K$ is finite for any sufficiently regular (e.g., strongly rational) vertex operator algebra $V$.

Note that $\operatorname{ker}(r)$, the subgroup of $\operatorname{Aut}(V)$ acting trivially on $V_{1}$, was introduced as inertia group $I(V)$ in LS20a and studied for a few examples of strongly rational, holomorphic vertex operator algebras of central charge 24 (see also Remark 2.3).

First, we describe inner automorphisms of vertex operator algebras up to conjugacy. The following result is immediate:
Proposition 2.1. Let $V$ be a vertex operator algebra of CFT-type and $g \in \operatorname{Aut}(V)$, and assume that holds. Then the following are equivalent:
(1) The automorphism $g$ is inner, i.e. $g \in K$.
(2) The restriction $r(g)$ is inner, i.e. $r(g) \in \operatorname{Inn}\left(V_{1}\right)$.

From now on, let us assume that $V_{1}$ is reductive, i.e. a direct sum of an abelian and a semisimple Lie algebra. If necessary, we fix a choice $\mathcal{H}$ of Cartan subalgebra of $V_{1}$.

We then consider the subgroup $T:=\left\langle\left\{\mathrm{e}^{v_{0}} \mid v \in \mathcal{H}\right\}\right\rangle=\left\{\mathrm{e}^{v_{0}} \mid v \in \mathcal{H}\right\}$ of $K$, which is abelian since $\mathcal{H}$ is and because $[u, v]=0$ implies that $\left[u_{0}, v_{0}\right]=0$ for $u, v \in V_{1}$ by Borcherds' identity. We shall assume in the following, strengthening the statement of (A), that

$$
\begin{equation*}
\operatorname{ker}(r) \subseteq T \tag{B}
\end{equation*}
$$

for the choice $\mathcal{H}$ of Cartan subalgebra of $V_{1}$. But then $(\mathrm{B})$ already holds for all Cartan subalgebras of $V_{1}$. Conditions $(A)$ and $(\bar{B})$ are satisfied, for example, for any lattice vertex operator algebra by Lemma 2.5 in [DN99.

Proposition 2.2. In the situation of Proposition 2.1, assume that $V_{1}$ is reductive and that also (B) holds. Then the items in Proposition 2.1 are equivalent to:
(3) The automorphism $g$ is conjugate in $K$ to $\mathrm{e}^{v_{0}} \in T$ for some $v \in \mathcal{H}$.

Proof. That (3) implies (1) is clear.
To see that (2) implies (3) suppose $r(g) \in \operatorname{Inn}\left(V_{1}\right)$. Then by (the extension to reductive Lie algebras of) Proposition 8.1 in Kac90 $r(g)$ is conjugate to $\mathrm{e}^{\mathrm{ad}_{v}}$ for some $v \in \mathcal{H}$. In fact, since any two Cartan subalgebras are conjugate by an inner automorphism, it follows from the proof of this result that $r(g)$ and $\mathrm{e}^{\mathrm{ad}_{v}}$ are conjugate under some element in $\operatorname{Inn}\left(V_{1}\right)$, which lifts to some $k \in K$. Then $r(g)=r\left(k e^{v_{0}} k^{-1}\right)$ or $r\left(\mathrm{e}^{v_{0}} k^{-1} g^{-1} k\right)=$ id. By condition (B) this means that $\mathrm{e}^{v_{0}} k^{-1} g^{-1} k=\mathrm{e}^{w_{0}}$ for some $w \in \mathcal{H}$, which we can rewrite as $g=k \mathrm{e}^{(v-w)_{0}} k^{-1}$, using the commutativity of $v_{0}$ and $w_{0}$.

Remark 2.3. Evidently, if $V_{1}$ is semisimple, then

$$
T \cap \operatorname{ker}(r)=\left\{\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \mid v \in \mathcal{H}, \mathrm{e}^{(2 \pi \mathrm{i}) \mathrm{ad}_{v}}=\operatorname{id}_{V_{1}}\right\}=\left\{\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \mid v \in P^{\vee}\right\}
$$

where $P^{\vee} \subseteq \mathcal{H}$ denotes the coweight lattice of $V_{1}$. Then condition $(\mathrm{B})$ is equivalent to $\operatorname{ker}(r)=\left\{\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \mid v \in P^{\vee}\right\}$. The latter is shown to hold for some examples of strongly rational, holomorphic vertex operator algebras of central charge 24 in LS20a, Remark 6.6. In fact, Ching Hung Lam informed us that he can prove the statement for all strongly rational, holomorphic vertex operator algebras of central charge 24 with non-zero (and semisimple) weight-one Lie algebra. The proof uses the result from Höh17] that these vertex operator algebras are simple-current extensions of a certain dual pair (see Theorem 4.2.

We remark that condition $\sqrt{A}$ implies condition $(\bar{B})$ if one assumes that $K$ is a complex Lie group.

We also note:
Proposition 2.4. In the situation of Proposition 2.1, assume that $V_{1}$ is reductive and that $g \in \operatorname{Aut}(V)$ has finite order. Then also $V_{1}^{g}$ is reductive and the items in Proposition 2.1 are equivalent to:
(4) $\operatorname{rk}\left(V_{1}^{g}\right)=\operatorname{rk}\left(V_{1}\right)$.

Proof. The proof follows from Proposition 8.6 in Kac90.

In the following we characterise (not necessarily inner) automorphisms of $V$ up to conjugacy. For ease of presentation we shall assume that $V_{1}$ is semisimple. It is however not difficult to extend the results to abelian or reductive Lie algebras.

Note that there is a non-degenerate, symmetric, invariant bilinear form $(\cdot, \cdot)$ on $V_{1}$, which restricts to a non-zero multiple of the Killing form on each simple ideal of $V_{1}$. Upon fixing a choice of this form we may identify the Cartan subalgebra $\mathcal{H}$ with its dual $\mathcal{H}^{*}$. In particular, we may view the roots of $V_{1}$ as elements of $\mathcal{H}$. Usually, $(\cdot, \cdot)$ is normalised such that the long roots have norm 2. If $V$ is self-contragredient, then $(\cdot, \cdot)$ and the invariant bilinear form $\langle\cdot, \cdot\rangle$ on $V$ agree up to multiplication by a non-zero scalar on each simple ideal of $V_{1}$.

In addition to fixing a Cartan subalgebra $\mathcal{H}$ of $V_{1}$, for the following results let us also fix a choice of simple roots $\Delta$. Then, given an outer (or diagram) automorphism $\mu_{0} \in \operatorname{Out}\left(V_{1}\right)=\operatorname{Aut}\left(V_{1}\right) / \operatorname{Inn}\left(V_{1}\right)$, there is a standard lift $\mu \in \operatorname{Aut}\left(V_{1}\right)$ of $\mu_{0}$ (defined in (7.9.2) and (7.10.1) in Kac90 and called diagram automorphism there) such that $\mu_{0}=\mu \operatorname{Inn}\left(V_{1}\right)$ and $\mu$ fixes the Cartan subalgebra $\mathcal{H}$ and the simple roots $\Delta$ setwise.

Moreover, for all outer automorphisms $\mu_{0} \in \tilde{r}(\operatorname{Aut}(V) / K) \subseteq \operatorname{Out}\left(V_{1}\right)$ and their standard lifts $\mu \in \operatorname{Aut}\left(V_{1}\right)$ we also fix choices of extensions of $\mu$ to $\tilde{\mu} \in \operatorname{Aut}(V)$. By the above commutative diagram, such an extension from $\operatorname{Aut}\left(V_{1}\right)$ to $\operatorname{Aut}(V)$ exists precisely for those automorphisms in $\operatorname{Aut}\left(V_{1}\right)$ that project to $\tilde{r}(\operatorname{Aut}(V) / K) \subseteq$ $\operatorname{Out}\left(V_{1}\right)$.

The next lemma shows that the automorphisms in $\operatorname{Aut}(V)$ can be conjugated by elements in $K$ into $\operatorname{Aut}(V)_{\{\mathcal{H}\}}=\{g \in \operatorname{Aut}(V) \mid g(\mathcal{H}) \subseteq \mathcal{H}\}$, the setwise stabiliser of $\mathcal{H}$ in $\operatorname{Aut}(V)$. (Note that $\operatorname{Aut}(V)_{\{\mathcal{H}\}}=r^{-1}\left(\operatorname{Aut}\left(V_{1}\right)_{\{\mathcal{H}\}}\right)$.) In fact, they can be conjugated into $\operatorname{Aut}(V)_{\{\Delta\}}$, the automorphisms fixing the simple roots $\Delta$ setwise.

Lemma 2.5. Let $V$ be a vertex operator algebra of CFT-type such that $V_{1}$ is semisimple and $(\bar{B})$ holds. Let $g \in \operatorname{Aut}(V)$ be of finite order. Then $g$ is conjugate under an automorphism in $K$ to an automorphism of the form

$$
\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}
$$

for some $v \in \mathcal{H}^{\mu}$ where $\mu \in \operatorname{Aut}\left(V_{1}\right)$ is the standard lift of an outer automorphism in $\tilde{r}(\operatorname{Aut}(V) / K) \subseteq \operatorname{Out}\left(V_{1}\right)$ and $\tilde{\mu}$ the choice of extension to $\operatorname{Aut}(V)$.

Note that $\tilde{\mu}$ and $\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ commute since $v \in \mathcal{H}^{\mu}$, i.e. $\tilde{\mu} v=v$. More generally, $g \mathrm{e}^{v_{0}}=\mathrm{e}^{(g v)_{0}} g$ for any $g \in \operatorname{Aut}(V)$ and $v \in V_{1}$.

Proof. Consider $r(g) \in \operatorname{Aut}\left(V_{1}\right)$. Again, by (the proof of) Proposition 8.1 in Kac90] $r(g)$ is conjugate under an inner automorphism, say $r(k)$ for some $k \in K$, to $\mu \mathrm{e}^{(2 \pi \mathrm{i}) \mathrm{ad}_{v}}$ for some $v \in \mathcal{H}$ where $\mu$ is the standard lift of an outer automorphism $\mu_{0}$. This outer automorphism $\mu_{0}$ is in $\tilde{r}(\operatorname{Aut}(V) / K)$ since $r(g)$ projects to an element in $\tilde{r}(\operatorname{Aut}(V) / K) \subseteq \operatorname{Out}\left(V_{1}\right)$. Hence $r(g)=r(k) \mu \mathrm{e}^{(2 \pi \mathrm{i}) \operatorname{ad}_{v}} r(k)^{-1}=r\left(k \tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} k^{-1}\right)$ where $\tilde{\mu}$ is the choice of extension of $\mu$ to $\operatorname{Aut}(V)$. Then $r\left(k^{-1} g^{-1} k \tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}\right)=\mathrm{id}$, i.e. $k^{-1} g^{-1} k \tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}=\mathrm{e}^{(2 \pi \mathrm{i}) w_{0}}$ for some $w \in \mathcal{H}$ by condition (B). This proves that $g$ is conjugate under $k$ to $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i})(v-w)_{0}}$. Finally, by Lemma 8.3 in EMS20b we may assume that $v-w \in \mathcal{H}^{\mu}$.

Lemma 2.6. In Lemma 2.5 it suffices, up to conjugacy in $\operatorname{Aut}(V)$, to let $\mu$ be from a fixed set of (standard lifts of) representatives of the conjugacy classes of $\tilde{r}(\operatorname{Aut}(V) / K) \cong \operatorname{Aut}(V) / K$.

Proof. Consider the automorphism $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ from the above result. It is clear that $\tilde{r}\left(\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} K\right)=\mu \operatorname{Inn}\left(V_{1}\right)$. Now suppose $\mu \operatorname{Inn}\left(V_{1}\right)$ is conjugate to some $\nu \operatorname{Inn}\left(V_{1}\right)$ in $\tilde{r}(\operatorname{Aut}(V) / K) \subseteq \operatorname{Out}\left(V_{1}\right)$ where $\nu \in \operatorname{Aut}\left(V_{1}\right)$ is again the standard lift of the outer automorphism $\nu \operatorname{Inn}\left(V_{1}\right)$. Then there is a $\tau \operatorname{Inn}\left(V_{1}\right) \in \tilde{r}(\operatorname{Aut}(V) / K) \subseteq \operatorname{Out}\left(V_{1}\right)$ such
that $\tau \mu \tau^{-1} \operatorname{Inn}\left(V_{1}\right)=\nu \operatorname{Inn}\left(V_{1}\right)$. Consider the choices of extensions $\tilde{\nu}, \tilde{\tau} \in \operatorname{Aut}(V)$ with $r(\tilde{\nu})=\nu$ and $r(\tilde{\tau})=\tau$. Then

$$
\begin{aligned}
\tilde{r}\left(\tilde{\tau} \tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \tilde{\tau}^{-1} K\right) & =\tilde{r}(\tilde{\tau} K) \tilde{r}\left(\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} K\right) \tilde{r}\left(\tilde{\tau}^{-1} K\right)=\tau \mu \tau^{-1} \operatorname{Inn}\left(V_{1}\right) \\
& =\nu \operatorname{Inn}\left(V_{1}\right)=\tilde{r}(\tilde{\nu} K)
\end{aligned}
$$

and hence $\tilde{\tau} \tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \tilde{\tau}^{-1} K=\tilde{\nu} K$. This implies that $\tilde{\tau} \tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \tilde{\tau}^{-1}=\tilde{\nu} k$, i.e. $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ is conjugate to $\tilde{\nu} k$ for some $k \in K$. By the above lemma, this has to be conjugate under an automorphism in $K$ to some $\tilde{\sigma} \mathrm{e}^{(2 \pi \mathrm{i}) w_{0}}$. But because it is conjugate under an automorphism in $K$, we know that $\tilde{\nu} K=\tilde{\nu} k K=\tilde{\sigma} \mathrm{e}^{(2 \pi \mathrm{i}) w_{0}} K=\tilde{\sigma} K$, which implies that $\nu \operatorname{Inn}\left(V_{1}\right)=\tilde{r}(\tilde{\nu} K)=\tilde{r}(\tilde{\sigma} K)=\sigma \operatorname{Inn}\left(V_{1}\right)$. But then $\nu=\sigma$ (and $\tilde{\nu}=\tilde{\sigma})$ because these were some fixed standard lifts (and extensions).

In total, we have shown that $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ is conjugate to $\tilde{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) w_{0}}$ for some $w \in \mathcal{H}^{\nu}$ if $\mu \operatorname{Inn}\left(V_{1}\right)$ is conjugate to $\nu \operatorname{Inn}\left(V_{1}\right)$. This proves the claim.

For $g \in \operatorname{Aut}(V)_{\{\mathcal{H}\}}$ (and assuming that $r(g)$ has order $n$ ) we define the projection $\operatorname{map} \pi_{g}: \mathcal{H} \rightarrow \mathcal{H}^{g}$ by $v \mapsto \frac{1}{n} \sum_{i=0}^{n-1} g^{i} v$ for $v \in \mathcal{H}$. Then $\mathcal{H}^{g}=\pi_{g}(\mathcal{H})$. We also define the subgroup

$$
P:=\left\{v \in \mathcal{H} \mid \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}=\mathrm{id}_{V}\right\}
$$

of the Cartan subalgebra $\mathcal{H}$ acting trivially on $V$ via $\mathrm{e}^{(2 \pi \mathrm{i})(\cdot)}$.
Lemma 2.7. In Lemma 2.5 it suffices, up to conjugacy in $\operatorname{Aut}(V)$, to select $v$ from a fixed set of representatives for $\mathcal{H}^{\mu} / \pi_{\mu}(P)$.

Proof. Let $w \in P$. Then $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}=\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \mathrm{e}^{(2 \pi \mathrm{i}) w_{0}}=\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i})(v+w)_{0}}$ is conjugate to $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i})\left(v+\pi_{\mu}(w)\right)_{0}}$ by Lemma 8.3 in EMS20b.

In summary we find:
Proposition 2.8 (Conjugacy Classes). Let $V$ be a vertex operator algebra of CFTtype such that $V_{1}$ is semisimple and ( $\bar{B}$ ) holds. Let $g \in \operatorname{Aut}(V)$ be of finite order. Then $g$ is conjugate to an automorphism of the form

$$
\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}
$$

where $\mu \in \operatorname{Aut}\left(V_{1}\right)$ is from a fixed set of (standard lifts of) representatives of the conjugacy classes of $\tilde{r}(\operatorname{Aut}(V) / K) \subseteq \operatorname{Out}\left(V_{1}\right)$ and $\tilde{\mu}$ the choice of extension to $\operatorname{Aut}(V)$, and $v$ is from a fixed set of representatives of $\mathcal{H}^{\mu} / \pi_{\mu}(P)$.

Two conjugate automorphisms in $\operatorname{Aut}(V)$ must project to the same conjugacy class in $\operatorname{Aut}(V) / K$. However, two automorphisms $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ and $\tilde{\mu} \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}^{\prime}}$ where $v$ and $v^{\prime}$ represent different classes in $\mathcal{H}^{\mu} / \pi_{\mu}(P)$ may still be conjugate.

In the lattice case we shall refine this result by further considering orbits under the action of $\operatorname{Aut}(V)_{\{\mathcal{H}\}} / \operatorname{Aut}(V)_{\mathcal{H}}$, the quotient of the setwise stabiliser by the pointwise stabiliser, on $\mathcal{H}$.
2.3. Automorphism Groups of Lattice Vertex Operator Algebras. In the following we specialise the previous discussion to a lattice vertex operator algebra $V_{L}=M_{\hat{\mathfrak{h}}}(1) \otimes \mathbb{C}_{\varepsilon}[L]$ for an even, positive-definite lattice $L$.

The group of isometries (or automorphisms) of $L$ is denoted by $\mathrm{O}(L)$. An automorphism $\nu \in \mathrm{O}(L)$ and a function $\eta: L \rightarrow\{ \pm 1\}$ satisfying $\eta(\alpha) \eta(\beta) / \eta(\alpha+\beta)=$ $\varepsilon(\alpha, \beta) / \varepsilon(\nu \alpha, \nu \beta)$ for all $\alpha, \beta \in L$ define a lift $\phi_{\eta}(\nu)$ of $\nu$, acting on $\mathbb{C}_{\varepsilon}[L]=$ $\bigoplus_{\alpha \in L} \mathbb{C}_{\alpha}$ as $\phi_{\eta}(\nu)\left(\mathfrak{e}_{\alpha}\right)=\eta(\alpha) \mathfrak{e}_{\nu \alpha}$ for $\alpha \in L$ and on $M_{\hat{\mathfrak{h}}}(1)$ as $\nu$ in the obvious way, and the automorphisms obtained in this way form the subgroup $\mathrm{O}(\hat{L}) \subseteq \operatorname{Aut}\left(V_{L}\right)$ (see, e.g., [FLM88, Bor92]). The following sequence is exact:

$$
1 \longrightarrow \operatorname{Hom}(L,\{ \pm 1\}) \longrightarrow \mathrm{O}(\hat{L}) \xrightarrow{-} \mathrm{O}(L) \longrightarrow 1
$$

The injection is given by $\lambda \mapsto \phi_{\lambda}(\mathrm{id})$ and the surjection by $\phi_{\eta}(\nu) \mapsto \overline{\phi_{\eta}(\nu)}=\nu$.
A lift $\phi_{\eta}(\nu) \in \mathrm{O}(\hat{L})$ is called standard lift if the restriction of $\eta$ to the fixedpoint sublattice $L^{\nu} \subseteq L$ is trivial. Standard lifts always exist Lep85, and all standard lifts of a given lattice automorphism $\nu$ are conjugate in Aut $\left(V_{L}\right)$ EMS20a. Standard lifts appear in the definition of twisted modules of lattice vertex operator algebras DL96, BK04. For convenience, we shall fix a choice $\hat{\nu}$ of standard lift for all $\nu \in \mathrm{O}(L)$.

If $\nu \in \mathrm{O}(L)$ has order $m$ and $\hat{\nu}$ is a standard lift of $\nu$, then $\hat{\nu}$ has order $m$ if $m$ is odd or if $m$ is even and $\left\langle\alpha, \nu^{m / 2} \alpha\right\rangle \in 2 \mathbb{Z}$ for all $\alpha \in L$, and $\hat{\nu}$ has order $2 m$ otherwise. In the latter case we say that $\nu$ exhibits order doubling.

When expressing powers of automorphisms $\hat{\nu}$ in $\operatorname{Aut}\left(V_{L}\right)$ a small complication arises due to the fact that if $\hat{\nu}$ is a standard lift of $\nu \in \mathrm{O}(L), \hat{\nu}^{i}$ is not necessarily a standard lift of $\nu^{i}$. However, there is a vector $s_{i}$ in $(1 / 2)\left(L^{\prime}\right)^{\nu^{i}}=(1 / 2)\left(\pi_{\nu^{i}}(L)\right)^{\prime}$ such that

$$
\hat{\nu}^{i}=\widehat{\left(\nu^{i}\right)} \mathrm{e}^{-(2 \pi \mathrm{i}) s_{i}(0)}
$$

where $\widehat{\left(\nu^{i}\right)}$ is a standard lift of $\nu^{i}$. The minus sign is a convention related to the sign convention in the definition of twisted modules.

Note that $s_{m} \in(1 / 2) L^{\prime}$ (with $\nu$ of order $m$ ) can be taken to be zero if and only if $\nu$ does not exhibit order doubling.

We review some well-known facts about the isometry group $\mathrm{O}(L)$. Recall that $\Phi$ denotes the simply-laced root system comprised of the norm-two vectors in $L$. The Weyl group $W \subseteq \mathrm{O}(L)$ is defined as the normal subgroup generated by the reflections about the hyperplanes orthogonal to the roots in $\Phi$.

The automorphism group $\mathrm{O}(L)$ of $L$ is a split extension $W$ : $H$, i.e. $H:=\mathrm{O}(L) / W$ and the short exact sequence

$$
1 \longrightarrow W \longrightarrow \mathrm{O}(L) \longrightarrow H \longrightarrow 1
$$

is right split (see, e.g., Section 1 of Bor87]). In other words, $H$ is isomorphic to a subgroup of $\mathrm{O}(L)$ and, in fact, given a choice of simple roots $\Delta \subseteq \Phi$, the setwise stabiliser of $\Delta$ in $\mathrm{O}(L)$

$$
H_{\Delta}:=\mathrm{O}(L)_{\{\Delta\}}
$$

is isomorphic to $H$.
Recall that for any vertex operator algebra $V$ of CFT-type the normal subgroup $K=\left\langle\left\{\mathrm{e}^{v_{0}} \mid v \in V_{1}\right\}\right\rangle$ of $\operatorname{Aut}(V)$ is the inner automorphism group of $V$. For a lattice vertex operator algebra $V_{L}$ it was shown in DN99, Theorem 2.1, that

$$
\operatorname{Aut}\left(V_{L}\right)=\mathrm{O}(\hat{L}) K
$$

and that the outer automorphism group $\operatorname{Aut}\left(V_{L}\right) / K$ is isomorphic to a quotient group of $\mathrm{O}(L)$. More precisely, $\operatorname{Aut}\left(V_{L}\right) / K$ is isomorphic to $\mathrm{O}(\hat{L}) /(K \cap \mathrm{O}(\hat{L}))$, which, since both $\mathrm{O}(\hat{L})$ and $K$ contain the subgroup $\operatorname{Hom}(L,\{ \pm 1\})$, is in turn isomorphic to $\mathrm{O}(L) / \overline{K \cap \mathrm{O}(\hat{L})}$, i.e.

$$
\operatorname{Aut}\left(V_{L}\right) / K \cong \mathrm{O}(\hat{L}) /(K \cap \mathrm{O}(\hat{L})) \cong \mathrm{O}(L) / \overline{K \cap \mathrm{O}(\hat{L})}
$$

where $\overline{K \cap \mathrm{O}(\hat{L})}$ denotes the image of $K \cap \mathrm{O}(\hat{L})$ under the projection from $\mathrm{O}(\hat{L})$ to $\mathrm{O}(L)$. In particular, $\operatorname{Aut}\left(V_{L}\right) / K$ is finite since $\mathrm{O}(L)$ is finite for a positive-definite lattice $L$. We shall further simplify $\operatorname{Aut}\left(V_{L}\right) / K$ below.

In the following, we choose the Cartan subalgebra $\mathcal{H}=\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}\right\} \cong \mathfrak{h}$ of the reductive (simply-laced) Lie algebra $\left(V_{L}\right)_{1}$ (see Section 2.1) if necessary. With this choice, $\Phi$ is the root system of $\left(V_{L}\right)_{1}$ and $W$ the Weyl group of $\left(V_{L}\right)_{1}$.

Recall that the automorphisms $T=\left\{\mathrm{e}^{v_{0}} \mid v \in \mathcal{H}\right\}$ define an abelian subgroup of $K$. It is easy to describe the action of $T \subseteq \operatorname{Aut}\left(V_{L}\right)$ on a lattice vertex operator algebra $V_{L}=\bigoplus_{\alpha \in L} M_{\hat{\mathfrak{h}}}(1) \otimes \mathfrak{e}_{\alpha}$, which is naturally graded by $L$. Indeed, for $v=h(-1) \otimes \mathfrak{e}_{0} \in \mathcal{H}, h \in \mathfrak{h}$, the automorphism $\mathrm{e}^{v_{0}}=\mathrm{e}^{h(0)}$ acts by multiplication with $\mathrm{e}^{\langle h, \alpha\rangle}$ on the graded component $M_{\hat{\mathfrak{h}}}(1) \otimes \mathfrak{e}_{\alpha}, \alpha \in L$.

Hence, the group $P=\left\{v \in \mathcal{H} \mid \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}=\mathrm{id}_{V}\right\}$ defined above is given by

$$
P=\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in L^{\prime}\right\} \cong L^{\prime}
$$

with the dual lattice $L^{\prime}$ of $L$. Also note that

$$
\operatorname{ker}\left(^{-}\right)=\mathrm{O}(\hat{L}) \cap T=\left\{\mathrm{e}^{(2 \pi \mathrm{i}) h(0)} \mid h \in L^{\prime} / 2\right\} \cong \operatorname{Hom}(L,\{ \pm 1\})
$$

where ${ }^{-}$denotes the projection map $\mathrm{O}(\hat{L}) \rightarrow \mathrm{O}(L)$.
Remark 2.9. While the projection $\operatorname{Aut}\left(V_{L}\right) \rightarrow \operatorname{Aut}\left(V_{L}\right) / K \cong \mathrm{O}(L) / \overline{K \cap \mathrm{O}(\hat{L})}$ is independent of the choice of a Cartan subalgebra of $\left(V_{L}\right)_{1}$, a projection to $\mathrm{O}(L)$ can in general only be defined involving such a choice. Indeed, choosing the Cartan subalgebra $\mathcal{H}=\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}\right\}$, the setwise stabiliser of $\mathcal{H}$ is given by

$$
\operatorname{Aut}\left(V_{L}\right)_{\{\mathcal{H}\}}=\mathrm{O}(\hat{L}) T
$$

while the pointwise stabiliser of $\mathcal{H}$ is

$$
\operatorname{Aut}\left(V_{L}\right)_{\mathcal{H}}=T
$$

Then the quotient $\operatorname{Aut}\left(V_{L}\right)_{\{\mathcal{H}\}} / \operatorname{Aut}\left(V_{L}\right)_{\mathcal{H}}$, which acts (faithfully) on $\mathcal{H}$, becomes

$$
\operatorname{Aut}\left(V_{L}\right)_{\{\mathcal{H}\}} / \operatorname{Aut}\left(V_{L}\right)_{\mathcal{H}}=\mathrm{O}(\hat{L}) T / T \cong \mathrm{O}(\hat{L}) /(\mathrm{O}(\hat{L}) \cap T) \cong \mathrm{O}(L)
$$

since $\mathrm{O}(\hat{L}) \cap T \cong \operatorname{Hom}(L,\{ \pm 1\})$ are exactly the lifts of id $\in \mathrm{O}(L)$. Explicitly, $\phi_{\eta}(\nu) \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ projects to $\nu$ for any lift $\phi_{\eta}(\nu)$ of $\nu$ and any $\mathrm{e}^{(2 \pi \mathrm{i}) h(0)} \in T$.

We return to the quotient $\operatorname{Aut}\left(V_{L}\right) / K$ :
Proposition 2.10 (Outer Automorphisms). Let L be an even, positive-definite lattice and $V_{L}$ the associated lattice vertex operator algebra. Then $\overline{K \cap O(\hat{L})}=W$ so that the outer automorphism group satisfies

$$
\operatorname{Aut}\left(V_{L}\right) / K \cong \mathrm{O}(L) / W=H
$$

where $W$ is the Weyl group of the root system $\Phi$, i.e. the group generated by the reflections about the norm-two vectors in L, which is also the Weyl group of the reductive (and simply-laced) Lie algebra $\left(V_{L}\right)_{1}$.

Proof. Like the Weyl group $W$, the inner automorphisms in $\operatorname{Inn}\left(\left(V_{L}\right)_{1}\right)_{\{\mathcal{H}\}}$ act by definition on the Cartan subalgebra $\mathcal{H}$. In fact, it is well known that for a simple Lie algebra $\mathfrak{g}$ the image of the restriction map $\operatorname{Inn}(\mathfrak{g})_{\{\mathcal{H}\}} \rightarrow \operatorname{Aut}(\mathcal{H})$ is the Weyl group $W$, i.e. the automorphisms of the Cartan subalgebra that are in the Weyl group are exactly those that can be extended to inner automorphisms of the whole Lie algebra $\mathfrak{g}$. This result remains true if $\mathfrak{g}$ is reductive. Equivalently, since $\operatorname{Inn}(\mathfrak{g})_{\mathcal{H}}$ is the kernel of this restriction map, $W \cong \operatorname{Inn}(\mathfrak{g})_{\{\mathcal{H}\}} / \operatorname{Inn}(\mathfrak{g})_{\mathcal{H}}$.

Given the vertex operator algebra $V_{L}$ with reductive Lie algebra $\left(V_{L}\right)_{1}$, since inner automorphisms of $\left(V_{L}\right)_{1}$ are exactly the restrictions of the inner automorphisms of $V_{L}$, it follows that

$$
W \cong \operatorname{Inn}\left(\left(V_{L}\right)_{1}\right)_{\{\mathcal{H}\}} / \operatorname{Inn}\left(\left(V_{L}\right)_{1}\right)_{\mathcal{H}} \cong K_{\{\mathcal{H}\}} / K_{\mathcal{H}} .
$$

This implies that $W \subseteq \overline{K \cap \mathrm{O}(\hat{L})}$. On the other hand, by intersecting $K$ with the expressions in Remark 2.9 we obtain

$$
\begin{aligned}
W & \cong K_{\{\mathcal{H}\}} / K_{\mathcal{H}}=(K \cap \mathrm{O}(\hat{L}) T) / T=T(K \cap \mathrm{O}(\hat{L})) / T \\
& \cong(K \cap \mathrm{O}(\hat{L})) /(T \cap \mathrm{O}(\hat{L}))=\overline{K \cap \mathrm{O}(\hat{L})}
\end{aligned}
$$

Hence, $W=\overline{K \cap \mathrm{O}(\hat{L})}$.
Recall that it is shown in Lemma 2.5 of DN99 that $\operatorname{ker}(r) \subseteq T$ for any lattice vertex operator algebra $V_{L}$, i.e. conditions (A) and (B) are satisfied. Then $\operatorname{ker}(r)$ is given by the expression in Remark 2.3.

Remark 2.11. Let $L$ be an even, positive-definite lattice and $V_{L}$ the associated lattice vertex operator algebra. Assume for simplicity that $\left(V_{L}\right)_{1}$ is semisimple. Then

$$
\begin{aligned}
\operatorname{ker}(r) & =\left\{\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \mid v=h(-1) \otimes \mathfrak{e}_{0} \text { for } h \in R^{\prime}\right\}=\left\{\mathrm{e}^{(2 \pi \mathrm{i}) h(0)} \mid h \in R^{\prime}\right\} \\
& \cong R^{\prime} / L^{\prime}
\end{aligned}
$$

where $R^{\prime}$ is the dual lattice of the lattice $R$ generated by the root system $\Phi$.
Since condition $(B)$ is satisfied, we can give the following characterisation of inner automorphisms:

Proposition 2.12. Let $L$ be an even, positive-definite lattice, $V_{L}$ the associated lattice vertex operator algebra and $g \in \operatorname{Aut}\left(V_{L}\right)$ of finite order. Then the following are equivalent:
(1) The automorphism $g$ is inner, i.e. $g \in K$.
(2) The restriction $r(g)$ is an inner automorphism of the Lie algebra $\left(V_{L}\right)_{1}$.
(3) The rank of the fixed-point Lie subalgebra $\left(V_{L}^{g}\right)_{1}$ equals $\operatorname{rk}\left(\left(V_{L}\right)_{1}\right)=\operatorname{rk}(L)$.
(4) The automorphism $g$ is conjugate in $\operatorname{Aut}\left(V_{L}\right)$ (even in $K$ ) to $\mathrm{e}^{v_{0}}$ for some $v \in \mathcal{H}$. More precisely, since $g$ has finite order, $v=(2 \pi \mathrm{i}) h(-1) \otimes \mathfrak{e}_{0}$ for some $h \in L \otimes_{\mathbb{Z}} \mathbb{Q}$.
(5) The fixed-point vertex operator subalgebra $V_{L}^{g}$ is isomorphic to a lattice vertex operator algebra $V_{K}$ for some sublattice $K$ of $L$ of full rank, i.e., $V_{K}$ is a full vertex operator subalgebra of $V_{L}$.
Proof. The equivalence of (1) to (4) is immediate with Proposition 2.1, Proposition 2.2 and Proposition 2.4 noting that $\operatorname{rk}\left(\left(V_{L}\right)_{1}\right)=\operatorname{rk}(L)$.

For (5) implies (3): The fixed-point vertex operator subalgebra $V_{L}^{g}$ and $V_{L}$ have the same central charge $c$, which equals $\operatorname{rk}(L)$ but also $\operatorname{rk}(K)$ if $V_{L}^{g} \cong V_{K}$. As explained above, the Lie rank of $\left(V_{K}\right)_{1} \cong\left(V_{L}^{g}\right)_{1}$ equals $\operatorname{rk}(K)$, which gives $\operatorname{rk}\left(\left(V_{L}^{g}\right)_{1}\right)=\operatorname{rk}\left(\left(V_{K}\right)_{1}\right)=\operatorname{rk}(K)=c=\operatorname{rk}(L)=\operatorname{rk}\left(\left(V_{L}\right)_{1}\right)$.

For (4) implies (5): The fixed-point vertex operator subalgebra under $\mathrm{e}^{v_{0}}=$ $\mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ for some $h \in L \otimes_{\mathbb{Z}} \mathbb{Q}$ is given by the lattice vertex operator algebra $V_{K}$ with $K=\{\alpha \in L \mid\langle\alpha, h\rangle \in \mathbb{Z}\}$. Since $g$ and $\mathrm{e}^{v_{0}}$ are conjugate, they have isomorphic fixed-point vertex operator subalgebras, which implies $V_{L}^{g} \cong V_{K}$.

Finally, we state the main result of this section, a description of the automorphisms of $V_{L}$ up to conjugacy. Given an even, positive-definite lattice $L$ and an automorphism $\nu \in \mathrm{O}(L)$ (of order $m$ ) we may consider the projection $\pi_{\nu}: \mathfrak{h} \rightarrow \mathfrak{h}^{\nu}$, $\pi_{\nu}=\frac{1}{m} \sum_{i=0}^{m-1} \nu^{i}$, of $\mathfrak{h}=L \otimes_{\mathbb{Z}} \mathbb{C}$ onto $\mathfrak{h}^{\nu}$, the elements of $\mathfrak{h}$ fixed by $\nu$. Also recall that $\hat{\nu} \in \mathrm{O}(\hat{L})$ denotes a fixed choice of standard lift for every $\nu \in \mathrm{O}(L)$.

With the above preparations, it is easy to prove the following specialisation of Proposition 2.8 We may drop the assumption that the weight-one Lie algebra $\left(V_{L}\right)_{1}$ is semisimple, which was merely made for convenience.

Theorem 2.13 (Conjugacy Classes). Let $L$ be an even, positive-definite lattice and $V_{L}$ the corresponding vertex operator algebra. Let $g \in \operatorname{Aut}\left(V_{L}\right)$ of finite order. Then $g$ is conjugate to an automorphism of the form

$$
\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}
$$

where $\nu$ is from a fixed set of representatives of the conjugacy classes of $H_{\Delta} \subseteq \mathrm{O}(L)$ and $\hat{\nu}$ the choice of standard lift in $\mathrm{O}(\hat{L}) \subseteq \operatorname{Aut}\left(V_{L}\right)$, and $h$ is from a fixed set of orbit representatives of the action of the centraliser $C_{\mathrm{O}(L)}(\nu)$ on $\mathfrak{h}^{\nu} / \pi_{\nu}\left(L^{\prime}\right)$.

Note that $\pi_{\nu}\left(L^{\prime}\right)=\left(L^{\nu}\right)^{\prime}$. The automorphisms $\hat{\nu}$ and $\mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ commute since $h \in \mathfrak{h}^{\nu}$. Moreover, since $g$ has finite order, $h$ is actually in $L^{\nu} \otimes_{\mathbb{Z}} \mathbb{Q}=\pi_{\nu}\left(L \otimes_{\mathbb{Z}} \mathbb{Q}\right)$ rather than $\mathfrak{h}^{\nu}=L^{\nu} \otimes \mathbb{C}$.

Proof. If $\left(V_{L}\right)_{1}$ is abelian (i.e. if $L$ has no vectors of norm 2), then the Weyl group is trivial and the assertion was already proved in MS19. To be precise, it is only stated there for the Leech lattice $\Lambda$, but the proof carries over almost verbatim to any positive-definite, even lattice $L$ without norm- 2 vectors, the only difference being that $L$ in contrast to $\Lambda$ does not have to be unimodular.

In the following, let us assume that $\left(V_{L}\right)_{1}$ is semisimple. The general case when $\left(V_{L}\right)_{1}$ is reductive, i.e. a direct sum of an abelian and a semisimple Lie algebra, follows from the abelian and semisimple cases.

We apply Proposition 2.8 Let $\mu_{0}$ be a representative of a conjugacy class of $H \cong \tilde{r}\left(\operatorname{Aut}\left(V_{L}\right) / K\right) \subseteq \operatorname{Out}\left(\left(V_{L}\right)_{1}\right)$. Then the standard lift $\mu \in \operatorname{Aut}\left(\left(V_{L}\right)_{1}\right)$ of $\mu_{0}$ as described in Kac90, preserving the choice of Cartan subalgebra $\mathcal{H}$ and the choice of simple roots $\Delta$ is exactly the restriction to $\left(V_{L}\right)_{1}$ of $\hat{\nu} \in \mathrm{O}(\hat{L})$ times some element in $T$ where $\nu$ is the element in $H_{\Delta} \subseteq \mathrm{O}(L)$ corresponding to $\mu_{0}$. Hence, modulo $T$, we may take $\tilde{\mu}=\hat{\nu}$.

Indeed, both $\tilde{\mu}$ and $\hat{\nu}$ fix the Cartan subalgebra $\mathcal{H}$ setwise, i.e. they are in $\operatorname{Aut}\left(V_{L}\right)_{\{\mathcal{H}\}}=\mathrm{O}(\hat{L}) T$ (see Remark 2.9). Both their projections modulo $T$ to $\mathrm{O}(L)$ lie in the subgroup $H_{\Delta} \cong H$, which is clear for $\hat{\nu}$ by definition and follows for $\tilde{\mu}$ since it fixes $\Delta$ setwise. But then these projections coincide and $\tilde{\mu} T=\hat{\nu} T$.

Recall that $P \cong L^{\prime}$. Then, by Proposition 2.8 $g$ is conjugate to $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ where $\nu$ is from a set of representatives of the conjugacy classes of $H_{\Delta} \subseteq \mathrm{O}(L)$ and $h$ is from a set of representatives of $\mathfrak{h}^{\nu} / \pi_{\nu}\left(L^{\prime}\right)$.

It remains to show that it suffices to let $h$ be from a set of orbit representatives of the action of $C_{\mathrm{O}(L)}(\nu)$ on $\mathfrak{h}^{\nu} / \pi_{\nu}\left(L^{\prime}\right)$. To this end, let $h$ and $h^{\prime}$ in $\mathfrak{h}^{\nu}$ such that $h^{\prime}+\pi_{\nu}\left(L^{\prime}\right)=\tau h+\pi_{\nu}\left(L^{\prime}\right)$ for some $\tau \in C_{\mathrm{O}(L)}(\nu)$. We want to show that $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ and $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h^{\prime}(0)}$ are conjugate.

The automorphism $\left(\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h^{\prime}(0)}\right)^{-1} \hat{\tau} \hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)} \hat{\tau}^{-1}$ is in $\operatorname{Aut}\left(V_{L}\right)_{\mathcal{H}}=T$, i.e. equal to $\mathrm{e}^{(2 \pi \mathrm{i}) f(0)}$ for some $f \in \mathfrak{h}$. Hence, $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ is conjugate to $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i})\left(h^{\prime}+f\right)(0)}$, which is conjugate to $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i})\left(h^{\prime}+\pi_{\nu}(f)\right)(0)}$.

On the other hand, $\left(\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h^{\prime}(0)}\right)^{-1} \hat{\tau} \hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)} \hat{\tau}^{-1}$ acts on $\mathfrak{e}_{\alpha} \in \mathbb{C}_{\varepsilon}[L]$ by multiplication with $\mathrm{e}^{(2 \pi \mathrm{i})\left\langle\tau h-h^{\prime}, \alpha\right\rangle} \eta_{\tau}\left(\nu \tau^{-1} \alpha\right) \eta_{\nu}\left(\tau^{-1} \alpha\right) / \eta_{\tau}\left(\tau^{-1} \alpha\right) / \eta_{\nu}(\alpha)$ for all $\alpha \in L$. This defines a homomorphism $L \rightarrow\{ \pm 1\}$. Suppose that $\alpha \in L^{\nu}$. Then, since $\hat{\nu}$ is a standard lift, the homomorphism becomes 1 . This shows that $\langle f, \alpha\rangle \in \mathbb{Z}$ for all $\alpha \in L^{\nu}$ or equivalently that $\pi_{\nu}(f) \in\left(L^{\nu}\right)^{\prime}=\pi_{\nu}\left(L^{\prime}\right)$.

The result shows that all the automorphisms in $\operatorname{Aut}\left(V_{L}\right)$ can be conjugated into $\operatorname{Aut}\left(V_{L}\right)_{\{\mathcal{H}\}}=\mathrm{O}(\hat{L}) \cdot T$ or more precisely into $\operatorname{Aut}(V)_{\{\Delta\}}=\left\{g \in \mathrm{O}(\hat{L}) \mid \bar{g} \in H_{\Delta}\right\} \cdot T$ (cf. comment before Lemma 2.5).

Two conjugate automorphisms in $\operatorname{Aut}\left(V_{L}\right)$ must project to the same conjugacy class in $H_{\Delta}$, but we cannot exclude the possibility that $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ and $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h^{\prime}(0)}$ for representatives $h$ and $h^{\prime}$ from different orbits are still conjugate.

The special case of Theorem 2.13 for the Leech lattice vertex operator algebra $V_{\Lambda}$ was already proved in MS19.

Remark 2.14. It is not difficult to formulate Theorem 2.13for algebraic conjugacy classes, i.e. conjugacy classes of cyclic subgroups. In that case, not surprisingly, it suffices to let $\nu$ be from a fixed set of representatives of the algebraic conjugacy classes of $H_{\Delta} \subseteq \mathrm{O}(L)$. Moreover, we may replace the action of the centraliser $C_{\mathrm{O}(L)}(\nu)$ by the action of the normaliser $N_{\mathrm{O}(L)}(\langle\nu\rangle)$.

More precisely, suppose $\nu$ has order $m$ and let $\tau \in N_{\mathrm{O}(L)}(\langle\nu\rangle)$. Then $\tau \nu \tau^{-1}=\nu^{i}$ for some $i \in \mathbb{Z}_{m}$ with $(i, m)=1$ and $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i}) h(0)}$ is algebraically conjugate to $\hat{\nu} \mathrm{e}^{(2 \pi \mathrm{i})\left(i^{-1} \tau h\right)(0)}$ where $i^{-1}$ denotes the inverse of $i$ modulo $m$.

We conclude this section by describing the orders of the automorphisms in $\operatorname{Aut}\left(V_{L}\right)$. Let $\nu \in \mathrm{O}(L)$ of order $m$. Then the order of an automorphism $g=$ $\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)} \in \operatorname{Aut}\left(V_{L}\right)$, where $h \in \mathfrak{h}^{\nu}$ so that $\hat{\nu}$ and $\mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ commute, must be a multiple of $m$. Of particular relevance in this text will be automorphisms $g$ having the same order as $\nu$. There are two cases to consider, depending on whether $\nu$ exhibits order doubling or not.

First, suppose that $\nu$ does not have order doubling. In this case $\hat{\nu}$ is of order $m$ and any element $h$ in $(1 / m) L^{\prime} \cap \mathfrak{h}^{\nu}=(1 / m)\left(L^{\prime}\right)^{\nu}=(1 / m)\left(\pi_{\nu}(L)\right)^{\prime}$ will yield an inner automorphism $\mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ of order dividing $m$ so that $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ has order $m$.

On the other hand, if $\nu$ exhibits order doubling, then $\hat{\nu}$ has order $2 m$ with $\hat{\nu}^{m} \mathfrak{e}_{\alpha}=(-1)^{m\left\langle\pi_{\nu}(\alpha), \pi_{\nu}(\alpha)\right\rangle} \mathfrak{e}_{\alpha}=(-1)^{\left\langle\alpha, \nu^{m / 2} \alpha\right\rangle} \mathfrak{e}_{\alpha}$ for all $\alpha \in L$. However, there exists a vector $s$ in $(1 /(2 m))\left(L^{\prime}\right)^{\nu}=(1 /(2 m))\left(\pi_{\nu}(L)\right)^{\prime}$ defining an inner automorphism $\mathrm{e}^{-(2 \pi \mathrm{i}) s(0)}$ of order $2 m$ such that $\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) s(0)}$ has order $m$.

Again, we may multiply by $\mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ for any $h \in(1 / m)\left(L^{\prime}\right)^{\nu}=(1 / m)\left(\pi_{\nu}(L)\right)^{\prime}$ and obtain an automorphism of order $m$. In total, we have proved:

Proposition 2.15. Let $L$ be a positive-definite, even lattice and $\hat{\nu}$ a standard lift of $\nu \in \mathrm{O}(L)$. Then an automorphism $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ of $V_{L}$ with $h \in \mathfrak{h}^{\nu}$ has order $m=|\nu|$ if and only if

$$
h \in \begin{cases}(1 / m)\left(L^{\prime}\right)^{\nu} & \text { if }|\hat{\nu}|=m \\ s+(1 / m)\left(L^{\prime}\right)^{\nu} & \text { if }|\hat{\nu}|=2 m\end{cases}
$$

with $s \in(1 /(2 m))\left(L^{\prime}\right)^{\nu}$ as defined above.

## 3. Orbifold Construction, Dimension and Rank Formulae

In this section we recall the cyclic orbifold construction and present tools to compute the dimension and rank of the corresponding weight-one space, in particular for lattice vertex operator algebras.
3.1. Cyclic Orbifold Construction. We summarise the cyclic orbifold theory for holomorphic vertex operator algebras developed in EMS20a, Möl16. It is a tool used to construct new vertex operator algebras from known ones. Special cases of this orbifold construction have been studied earlier, for example, the construction of the Moonshine module $V^{\natural}$ from the Leech lattice vertex operator algebra $V_{\Lambda}$ as orbifold construction of order 2 [FLM88].

Let $V$ be a strongly rational, holomorphic vertex operator algebra (necessarily of central charge $c \in 8 \mathbb{Z}_{\geq 0}$ ) and $G=\langle g\rangle$ a finite, cyclic group of automorphisms of $V$ of order $n \in \mathbb{Z}_{>0}$.

By DLM00] there is an up to isomorphism unique irreducible $g^{i}$-twisted $V$ module $V\left(g^{i}\right)$ for each $i \in \mathbb{Z}_{n}$. The uniqueness of $V\left(g^{i}\right)$ implies that there is a representation $\phi_{i}: G \rightarrow \operatorname{Aut}_{\mathbb{C}}\left(V\left(g^{i}\right)\right)$ of $G$ on the vector space $V\left(g^{i}\right)$ such that

$$
\phi_{i}(g) Y_{V\left(g^{i}\right)}(v, x) \phi_{i}(g)^{-1}=Y_{V\left(g^{i}\right)}(g v, x)
$$

for all $v \in V, i \in \mathbb{Z}_{n}$. This representation is unique up to an $n$-th root of unity. Denote the eigenspace of $\phi_{i}(g)$ in $V\left(g^{i}\right)$ corresponding to the eigenvalue $\mathrm{e}^{(2 \pi \mathrm{i}) j / n}$ by $W^{(i, j)}$. On $V\left(g^{0}\right)=V$ we choose $\phi_{0}(g)=g$.

By recent results Miy15, CM16 (and [DM97) the fixed-point vertex operator subalgebra $V^{g}=W^{(0,0)}$ (also called orbifold) is again strongly rational. It has exactly $n^{2}$ irreducible modules, namely the $W^{(i, j)}, i, j \in \mathbb{Z}_{n}$ MT04 DRX17. One can further show that the conformal weight $\rho(V(g))$ of $V(g)$ is in $\left(1 / n^{2}\right) \mathbb{Z}$, and we define the type $t \in \mathbb{Z}_{n}$ of $g$ by $t=n^{2} \rho(V(g))(\bmod n)$.

In the following assume that $g$ has type 0 , i.e. that $\rho(V(g)) \in(1 / n) \mathbb{Z}$. Then it is possible to choose the representations $\phi_{i}$ such that the conformal weights satisfy

$$
\rho\left(W^{(i, j)}\right) \in \frac{i j}{n}+\mathbb{Z}
$$

and $V^{g}$ has fusion rules

$$
W^{(i, j)} \boxtimes W^{(l, k)} \cong W^{(i+l, j+k)}
$$

for all $i, j, k, l \in \mathbb{Z}_{n}$ (see EMS20a, Section 5), i.e. the fusion ring of $V^{g}$ is the group ring $\mathbb{C}\left[\mathbb{Z}_{n} \times \mathbb{Z}_{n}\right]$. In particular, all $V^{g}$-modules are simple currents.

In general, a simple vertex operator algebra $V$ is said to satisfy the positivity condition if the conformal weights satisfy $\rho(W)>0$ for any irreducible $V$-module $W \not \approx V$ and $\rho(V)=0$.

Now, if $V^{g}$ satisfies the positivity condition (it is shown in Möl18 that this condition is almost automatically satisfied), then the direct sum of $V^{g}$-modules

$$
V^{\mathrm{orb}(g)}:=\bigoplus_{i \in \mathbb{Z}_{n}} W^{(i, 0)}
$$

admits an up to isomorphism unique strongly rational, holomorphic vertex operator algebra structure extending the given $V^{g}$-module structure. One calls $V^{\mathrm{orb}(g)}$ the orbifold construction associated with $V$ and $g$ EMS20a. Two algebraically conjugate automorphisms in $\operatorname{Aut}(V)$ yield isomorphic orbifold constructions. Note that $\bigoplus_{j \in \mathbb{Z}_{n}} W^{(0, j)}$ is just the old vertex operator algebra $V$.

We briefly describe the inverse (or reverse) orbifold construction. Suppose that the strongly rational, holomorphic vertex operator algebra $V^{\operatorname{orb}(g)}$ is obtained in an orbifold construction as described above. Then via $\zeta v:=\mathrm{e}^{(2 \pi \mathrm{i}) i / n} v$ for $v \in W^{(i, 0)}$, $i \in \mathbb{Z}_{n}$, we define an automorphism $\zeta$ of $V^{\operatorname{orb}(g)}$ of order $n$ and type 0 , and the unique irreducible $\zeta^{j}$-twisted $V^{\operatorname{orb}(g)}$-module is $V^{\operatorname{orb}(g)}\left(\zeta^{j}\right)=\bigoplus_{i \in \mathbb{Z}_{n}} W^{(i, j)}, j \in \mathbb{Z}_{n}$ (see Möl16, Theorem 4.9.6). Then

$$
\left(V^{\operatorname{orb}(g)}\right)^{\operatorname{orb}(\zeta)}=\bigoplus_{j \in \mathbb{Z}_{n}} W^{(0, j)} \cong V,
$$

i.e. orbifolding with $\zeta$ is inverse to orbifolding with $g$.
3.2. Dimension Formula. Continuing in the setting of the previous subsection, we recall a formula for the dimension of $V_{1}^{\operatorname{orb}(g)}$, the weight-one Lie algebra of the orbifold construction. In contrast to the other results in this section, this dimension formula is particular to central charge 24:

Theorem 3.1 (Dimension Formula, MS19]). Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 and $g$ an automorphism of $V$ of order $n>1$ and type 0 . Assume that $V^{g}$ satisfies the positivity condition. Then

$$
\operatorname{dim}\left(V_{1}^{\operatorname{orb}(g)}\right) \leq 24+\sum_{d \mid n} c_{n}(d) \operatorname{dim}\left(V_{1}^{g^{d}}\right)
$$

where the $c_{n}(d)$ are defined by

$$
c_{n}(d)=\frac{n}{d^{2}} \prod_{p \mid d}(-p) \prod_{p \left\lvert\,\left(d, \frac{n}{d}\right)\right.}\left(1-p^{-1}\right) \prod_{p \left\lvert\, \frac{n}{d}\right.}\left(1+p^{-1}\right)
$$

with $p$ prime, or equivalently by the system of equations $\sum_{d \mid n} c_{n}(d)(t, d)=n / t$ for all $t \mid n$.

A special case of this formula for orders $n$ such that the genus of the modular curve $X_{0}(n)=\overline{\Gamma_{0}(n) \backslash \mathbb{H}}$ is zero was proved in EMS20b. In fact, the orders of the automorphisms appearing in this text happen to satisfy the genus-zero condition.

An automorphism $g \in \operatorname{Aut}(V)$ is called extremal if the upper bound in the dimension formula is attained.

The obstruction to extremality is a certain linear combination of the dimensions of the subspaces of the twisted modules $V\left(g^{i}\right), i \neq 0(\bmod n)$, with weights strictly between 0 and 1 . Hence, the following corollary is immediate:

Corollary 3.2. Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 and $g$ an automorphism of $V$ of order $n>1$ and type 0 . Suppose that the conformal weights of the twisted modules obey $\rho\left(V\left(g^{i}\right)\right) \geq 1$ for all $i \neq 0$ $(\bmod n)$. Then $g$ is extremal and

$$
\operatorname{dim}\left(V_{1}^{\operatorname{orb}(g)}\right)=24+\sum_{d \mid n} c_{n}(d) \operatorname{dim}\left(V_{1}^{g^{d}}\right)
$$

The automorphisms considered in Section 5 will turn out to satisfy $\rho\left(V\left(g^{i}\right)\right) \geq 1$ for all $i \neq 0(\bmod n)$ so that they are all extremal (see Proposition 5.9). But note that in general extremal automorphisms need not satisfy this condition (see the examples in MS19]).
3.3. Rank Criterion. Corollary 3.2 allows us to compute the dimension of the weight-one Lie algebra $V_{1}^{\text {orb }(g)}$ if the central charge is 24 . In this section we shall discuss how to determine the rank of $V_{1}^{\mathrm{orb}(g)}$ (in any central charge). More precisely, we derive a sufficient criterion for the orbifold rank condition, namely that the rank of $V_{1}^{g}$ equals the rank of $V_{1}^{\operatorname{orb}(g)}$.

First, we recall some probably known results about graded Lie algebras. To this end, let $\mathfrak{g}=\bigoplus_{i \in \mathbb{Z}_{n}} \mathfrak{g}_{i}$ be a reductive, $\mathbb{Z}_{n}$-graded Lie algebra for some $n \in \mathbb{Z}_{>0}$ (see Kac90, Section 8.1). Equivalently, let $g$ be an automorphism of $\mathfrak{g}$ with $g^{n}=\mathrm{id}$ and the grading of $\mathfrak{g}$ obtained as eigenvalue decomposition with respect to $g$, i.e. $\mathfrak{g}_{i}=\left\{x \in \mathfrak{g} \mid g(x)=\mathrm{e}^{(2 \pi \mathrm{i}) i / n} x\right\}$ for all $i \in \mathbb{Z}_{n}\left(\right.$ and $\left.\mathfrak{g}_{0}=\mathfrak{g}^{g}\right)$.

It is well known that $\mathfrak{g}_{0}$ is also reductive and that $\operatorname{rk}\left(\mathfrak{g}_{0}\right) \leq \operatorname{rk}(\mathfrak{g})$ with equality if and only if the automorphism $g$ is inner (see, e.g., Proposition 8.6 in Kac90).

We now describe the relationship between Cartan subalgebras of $\mathfrak{g}_{0}$ and $\mathfrak{g}$ :
Lemma 3.3. Let $\mathcal{H}_{0}$ be a Cartan subalgebra of $\mathfrak{g}_{0}$. Then the centraliser $\widetilde{\mathcal{H}}:=$ $C_{\mathfrak{g}}\left(\mathcal{H}_{0}\right)$ is a Cartan subalgebra of $\mathfrak{g}$.
Proof. This is stated in Lemma 8.1 in Kac90] for the case that $\mathfrak{g}$ is simple but it is not difficult to see that the result extends to reductive Lie algebras.

On the other hand:

Lemma 3.4. Let $\mathcal{H}$ be a Cartan subalgebra of $\mathfrak{g}$. Then $\operatorname{dim}\left(\mathcal{H}^{g}\right) \leq \operatorname{rk}\left(\mathfrak{g}_{0}\right)$ with equality if and only if $\mathcal{H}^{g}$ is a Cartan subalgebra of $\mathfrak{g}_{0}=\mathfrak{g}^{g}$.
Proof. To prove the inequality we note that for any Cartan subalgebra $\mathcal{H}$ of $\mathfrak{g}$, the fixed points $\mathcal{H}^{g}$ are contained in a Cartan subalgebra of $\mathfrak{g}_{0}=\mathfrak{g}^{g}$. One implication of the equivalence is trivial. For the other assume that $\operatorname{dim}\left(\mathcal{H}^{g}\right)=\operatorname{rk}\left(\mathfrak{g}_{0}\right)$. Since $\mathcal{H}^{g}$ can be extended to a Cartan subalgebra of $\mathfrak{g}_{0}$, it must already be a Cartan subalgebra.
Remark 3.5. Note that given a finite-order automorphism $g$ of a reductive Lie algebra $\mathfrak{g}$ it is always possible to find a Cartan subalgebra $\mathcal{H}$ of $\mathfrak{g}$ such that $\mathcal{H}^{g}$ is a Cartan subalgebra of $\mathfrak{g}_{0}=\mathfrak{g}^{g}$.

Equivalently, if we fix the Cartan subalgebra $\mathcal{H}$, any finite-order automorphism $g$ is conjugate (under an inner automorphism) to an automorphism $g^{\prime}$ such that $\mathcal{H}^{g^{\prime}}$ is a Cartan subalgebra of $\mathfrak{g}^{g^{\prime}}$.
Proof. A chosen Cartan subalgebra $\mathcal{H}_{0}$ of $\mathfrak{g}_{0}$ can be extended to a Cartan subalgebra $\mathcal{H}$ of $\mathfrak{g}$. Then $\mathcal{H}_{0} \subseteq \mathcal{H}^{g} \subseteq \mathfrak{g}_{0}$ and the result follows.

Specifically, if $g$ is of the form $\mu \mathrm{e}^{(2 \pi \mathrm{i}) \mathrm{ad}_{v}}$ where $\mu \in \operatorname{Aut}(\mathfrak{g})$ is the standard lift of an outer automorphism in $\operatorname{Out}(\mathfrak{g})$ (preserving the choice of Cartan subalgebra and simple roots) and $v$ is in $\mathcal{H}^{\mu}$ (cf. Proposition 8.1 in Kac90]), then equality in the above lemma holds. This is true in particular for the representatives of the conjugacy classes in Proposition 2.8, restricted to the Lie algebra $V_{1}$.

We return to the orbifold setting, i.e. we let $V$ be a strongly rational, holomorphic vertex operator algebra and $g$ an automorphism of $V$ of order $n$ and type 0 such that $V^{g}$ satisfies the positivity condition.

The Lie algebras $\mathfrak{g}:=V_{1}$ and $\tilde{\mathfrak{g}}:=V_{1}^{\text {orb }(g)}$ are both reductive and $\mathbb{Z}_{n}$-graded with the grading on $\mathfrak{g}=\bigoplus_{i \in \mathbb{Z}_{n}} \mathfrak{g}_{i}$ given by the eigenvalue decomposition with respect to $g$ and the one on $\tilde{\mathfrak{g}}=\bigoplus_{i \in \mathbb{Z}_{n}} \tilde{\mathfrak{g}}_{i}$ with respect to the inverse orbifold automorphism of $g$. Then $\mathfrak{g}_{0}=\tilde{\mathfrak{g}}_{0}=V_{1}^{g}$, which is also reductive.

We say the orbifold rank condition is satisfied if $\operatorname{rk}\left(V_{1}^{\operatorname{orb}(g)}\right)=\operatorname{rk}\left(V_{1}^{g}\right)$.
Note that for a subset $S$ of a Lie algebra $\mathfrak{g}$ the centraliser is defined as

$$
\begin{aligned}
C_{\mathfrak{g}}(S) & =\{x \in \mathfrak{g} \mid[x, s]=0 \text { for all } s \in S\} \\
& =\left\{x \in \mathfrak{g} \mid \operatorname{ad}_{s}(x)=0 \text { for all } s \in S\right\} .
\end{aligned}
$$

Now suppose that $M$ is a $\mathfrak{g}$-module. Analogously, we define

$$
C_{M}(S)=\{v \in M \mid s \cdot v=0 \text { for all } s \in S\}
$$

simply replacing the adjoint action of $\mathfrak{g}$ on $\mathfrak{g}$ with the module action of $\mathfrak{g}$ on $M$.
Let $\mathcal{H}_{0}$ be a Cartan subalgebra of $V_{1}^{g}$. Then by Lemma 3.3 we know that the centraliser $\widetilde{\mathcal{H}}:=C_{\tilde{\mathfrak{g}}}\left(\mathcal{H}_{0}\right)$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}$.
Lemma 3.6. The Cartan subalgebra $\widetilde{\mathcal{H}}$ of $\tilde{\mathfrak{g}}$ satisfies

$$
\widetilde{\mathcal{H}}=\mathcal{H}_{0} \oplus \bigoplus_{i \in \mathbb{Z}_{n} \backslash\{0\}} C_{\tilde{\mathfrak{g}}_{i}}\left(\mathcal{H}_{0}\right)
$$

Proof. Consider

$$
\widetilde{\mathcal{H}}=C_{\tilde{\mathfrak{g}}}\left(\mathcal{H}_{0}\right)=C_{\bigoplus_{i \in \mathbb{Z}_{n}} \tilde{\mathfrak{g}}_{i}}\left(\mathcal{H}_{0}\right)
$$

which, because the Lie bracket respects the $\mathbb{Z}_{n}$-grading, equals

$$
\bigoplus_{i \in \mathbb{Z}_{n}} C_{\tilde{\mathfrak{g}}_{i}}\left(\mathcal{H}_{0}\right)=C_{\mathfrak{g}_{0}}\left(\mathcal{H}_{0}\right) \oplus \bigoplus_{i \in \mathbb{Z}_{n} \backslash\{0\}} C_{\tilde{\mathfrak{g}}_{i}}\left(\mathcal{H}_{0}\right)=\mathcal{H}_{0} \oplus \bigoplus_{i \in \mathbb{Z}_{n} \backslash\{0\}} C_{\tilde{\mathfrak{g}}_{i}}\left(\mathcal{H}_{0}\right)
$$

In the last step we used that as Cartan subalgebra $\mathcal{H}_{0}$ is self-centralising.

The lemma allows us to determine a Cartan subalgebra and hence the rank of $\tilde{\mathfrak{g}}$ by only considering the module action of $V^{g}$ on the orbifold construction $V^{\text {orb }(g)}$ rather than the full Lie algebra structure of $\tilde{\mathfrak{g}}$, which would involve intertwining operators between submodules of the twisted $V$-modules $V\left(g^{i}\right)$ for $i \in \mathbb{Z}_{n}$.

In the same way that $[u, v]:=u_{0} v$ for $u, v \in V_{1}$ defines a Lie bracket on $V_{1}$, does $u \cdot v:=u_{0} v$ for $u \in V_{1}$ and $v \in W$ define a Lie algebra module of $V_{1}$ on any (untwisted) $V$-module $W$ and on any graded component of $W$.

Finally, note that the twisted $V$-modules $V\left(g^{i}\right)$ for $i \in \mathbb{Z}_{n}$ are untwisted $V^{g_{-}}$ modules, the graded components $V\left(g^{i}\right)_{1}$ are thus Lie algebra modules of $V_{1}^{g}$, and therefore the expression $C_{V\left(g^{i}\right)_{1}}\left(\mathcal{H}_{0}\right)$ is well-defined.

We state a sufficient criterion for the orbifold rank condition:
Proposition 3.7 (Rank Criterion). Let $\mathcal{H}_{0}$ be a Cartan subalgebra of $V_{1}^{g}$. Furthermore, assume that $C_{V\left(g^{i}\right)_{1}}\left(\mathcal{H}_{0}\right)=\{0\}$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$. Then $\mathcal{H}_{0}$ is a Cartan subalgebra of $\tilde{\mathfrak{g}}:=V_{1}^{\operatorname{orb}(g)}$, i.e. the orbifold rank condition $\operatorname{rk}\left(V_{1}^{\operatorname{orb}(g)}\right)=\operatorname{rk}\left(V_{1}^{g}\right)$ is satisfied.
Proof. Clearly, $C_{\tilde{\mathfrak{g}}_{i}}\left(\mathcal{H}_{0}\right) \subseteq C_{V\left(g^{i}\right)_{1}}\left(\mathcal{H}_{0}\right)$ since $W^{(i, 0)} \subseteq V\left(g^{i}\right)$. Then Lemma 3.6 implies the assertion.
Remark 3.8. Note that $C_{V\left(g^{i}\right)_{1}}\left(\mathcal{H}_{0}\right)=\{0\}$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$ with $(i, n)=1$ is a necessary criterion for the orbifold rank condition to be satisfied.
3.4. Lattice Vertex Operator Algebras. We now specialise to the case where $V=V_{L}$ is the holomorphic vertex operator algebra associated with a positivedefinite, even, unimodular lattice $L$ and let $g$ be an automorphism of $V_{L}$ of order $n$ and type 0 . By Theorem 2.13 we may assume that $g$ is of the form $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ for some $\nu \in \mathrm{O}(L)$ and some $h \in \pi_{\nu}\left(L \otimes_{\mathbb{Z}} \mathbb{Q}\right)$. The minus sign in the exponent is a convention related to the sign convention for twisted modules (see the remark at the beginning of Section 2.

In order to apply the dimension formula in Corollary 3.2 we have to compute the conformal weights of the twisted $V_{L}$-modules, which can be described by combining [DL96, BK04] with Section 5 of Li96]. Since $L$ is unimodular, $V_{L}$ is holomorphic and there is a unique $g$-twisted $V_{L}$-module $V_{L}(g)$. Suppose that $\nu$ has order $m$ (necessarily dividing $n$ ) and Frame shape (or cycle shape) $\prod_{t \mid m} t^{b_{t}}$ with $b_{t} \in \mathbb{Z}$, i.e. the extension of $\nu$ to $\mathfrak{h}$ has characteristic polynomial $\prod_{t \mid m}\left(x^{t}-1\right)^{b_{t}}$. Note that $\sum_{t \mid m} t b_{t}=\operatorname{rk}(L)$. Then the conformal weight of $V_{L}(g)$ is given by

$$
\rho\left(V_{L}(g)\right)=\frac{1}{24} \sum_{t \mid m} b_{t}(t-1 / t)+\min _{\alpha \in \pi_{\nu}(L)+h}\langle\alpha, \alpha\rangle / 2 \geq 0
$$

The number

$$
\rho_{\nu}:=\frac{1}{24} \sum_{t \mid m} b_{t}(t-1 / t)
$$

is called the vacuum anomaly of $V_{L}(g)$. Note that $\rho\left(V_{L}(g)\right)>0$ for $g \neq \mathrm{id}$, i.e. any fixed-point vertex operator subalgebra $V_{L}^{g}$ of a lattice vertex operator algebra $V_{L}$ satisfies the positivity condition.

We shall apply the rank criterion from Section 3.3 to $V_{L}$. A Cartan subalgebra of $\mathfrak{g}=\left(V_{L}\right)_{1}$ is given by $\mathcal{H}=\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}\right\} \cong \mathfrak{h}$. Then Lemma 3.4 becomes:
Lemma 3.9. The inequality $\operatorname{rk}\left(\left(V_{L}^{g}\right)_{1}\right) \geq \operatorname{dim}\left(\mathfrak{h}^{\nu}\right)=\operatorname{rk}\left(L^{\nu}\right)$ holds; with equality if and only if

$$
\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}^{\nu}\right\} \cong \mathfrak{h}^{\nu}
$$

is a Cartan subalgebra of $\mathfrak{g}_{0}=\left(V_{L}^{g}\right)_{1}$.

We note that if $\nu \in H_{\Delta} \subseteq \mathrm{O}(L)$ (see Section 2.3), then equality in the lemma holds. This is in particular true for the representatives of the conjugacy classes in Theorem 2.13. This follows from the corresponding statement after Remark 3.5

To formulate Proposition 3.7 in the special case of lattice vertex operator algebras, recall from Section 2.3 that

$$
\hat{\nu}^{i}=\widehat{\left(\nu^{i}\right)} \mathrm{e}^{-(2 \pi \mathrm{i}) s_{i}(0)}
$$

for some vector $s_{i} \in(1 / 2)\left(L^{\nu^{i}}\right)^{\prime} \subseteq \mathfrak{h}^{\nu^{i}}$ where $\widehat{\left(\nu^{i}\right)}$ is a standard lift of $\nu^{i}$. Consequently,

$$
g^{i}=\widehat{\left(\nu^{i}\right)} \mathrm{e}^{-(2 \pi \mathrm{i})\left(s_{i}+i h\right)(0)}
$$

and the unique irreducible $g^{i}$-twisted $V_{L}$-module [DL96, BK04, Li96] is of the form

$$
V_{L}\left(g^{i}\right)=M_{\hat{\mathfrak{h}}}(1)\left[\nu^{i}\right] \otimes \mathfrak{e}_{s_{i}+i h} \mathbb{C}\left[\pi_{\nu^{i}}(L)\right] \otimes \mathbb{C}^{d\left(\nu^{i}\right)}
$$

with a grading by the lattice coset $s_{i}+i h+\pi_{\nu^{i}}(L)$ and defect $d\left(\nu^{i}\right) \in \mathbb{Z}_{>0}$.
Proposition 3.10 (Rank Criterion). Suppose that $\operatorname{rk}\left(V_{L}^{g}\right)_{1}=\operatorname{rk}\left(L^{\nu}\right)$ (i.e. that $\mathcal{H}_{0}:=\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}^{\nu}\right\} \cong \mathfrak{h}^{\nu}$ is a Cartan subalgebra of $\left.\left(V_{L}^{g}\right)_{1}\right)$. Furthermore assume that $\pi_{\nu}\left(s_{i}\right)+i h \notin \pi_{\nu}(L)$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$. Then the orbifold rank condition $\operatorname{rk}\left(V_{1}^{\text {orb }(g)}\right)=\operatorname{rk}\left(\left(V_{L}^{g}\right)_{1}\right)$ is satisfied.
Proof. Let $v=k(-1) \otimes \mathfrak{e}_{0}$ with $k \in \mathfrak{h}^{\nu}$ be in the Cartan subalgebra $\mathcal{H}_{0}$. Let $0 \neq w \in V_{L}\left(g^{i}\right)$ be in the homogeneous space of degree $s_{i}+i h+\beta$ for the lattice coset grading for some $\beta \in \pi_{\nu^{i}}(L)$. Then

$$
v \cdot w=v_{0} w=k(0) w=\left\langle k, s_{i}+i h+\beta\right\rangle w .
$$

Now suppose that $w \in C_{V_{L}\left(g^{i}\right)}\left(\mathcal{H}_{0}\right)$, i.e. that

$$
0=\left\langle k, s_{i}+i h+\beta\right\rangle=\left\langle k, \pi_{\nu}\left(s_{i}\right)+i h+\pi_{\nu}(\beta)\right\rangle
$$

for all $k \in \mathfrak{h}^{\nu}$. Since $\langle\cdot, \cdot\rangle$ is non-degenerate on $\mathfrak{h}^{\nu}$, this is equivalent to $\pi_{\nu}\left(s_{i}\right)+i h+$ $\pi_{\nu}(\beta)=0$. If $\pi_{\nu}\left(s_{i}\right)+i h \notin \pi_{\nu}(L)$, then this equation cannot be satisfied and hence $C_{V_{L}\left(g^{i}\right)}\left(\mathcal{H}_{0}\right)=\{0\}$. In particular, $C_{V_{L}\left(g^{i}\right)_{1}}\left(\mathcal{H}_{0}\right)=\{0\}$ and by Proposition 3.7 the assertion follows.

In the above proposition we demanded that $C_{V_{L}\left(g^{i}\right)}\left(\mathcal{H}_{0}\right)=\{0\}$. Of course, we can weaken the assumption by only requiring that $C_{V_{L}\left(g^{i}\right)_{1}}\left(\mathcal{H}_{0}\right)=\{0\}$, resulting in the slightly more technical condition that there is no $\alpha \in s_{i}+i h+\pi_{\nu^{i}}(L)$ with $\pi_{\nu}(\alpha)=0$ and $\rho_{\nu^{i}}+\langle\alpha, \alpha\rangle / 2+k(i, m) / m=1$ for some $k \in \mathbb{Z}_{\geq 0}$.

It will turn out that the automorphisms in Section 5 satisfy the assumptions of Proposition 3.10 and hence the orbifold rank condition so that we can easily determine the rank of the weight-one Lie algebra $V_{1}^{\text {orb }(g)}$ in these cases.
Remark 3.11. Reversely, if the orbifold rank condition $\operatorname{rk}\left(\left(V_{L}^{\text {orb }(g)}\right)_{1}\right)=\operatorname{rk}\left(\left(V_{L}^{g}\right)_{1}\right)$ is satisfied, then, by Remark 3.8 $1-\rho_{\nu} \notin(1 / m) \mathbb{Z}_{\geq 0}$ or $s_{i}+i h \notin \pi_{\nu}(\Lambda)$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$ with $(i, n)=1$.
Remark 3.12. Finally, note that if $L$ is of rank $24, \nu \in H_{\Delta} \subseteq \mathrm{O}(L)$ (which, as mentioned above, implies that $\mathcal{H}_{0}$ is a Cartan subalgebra of $\left.\left(V_{L}^{g}\right)_{1}\right)$ and $\nu$ has order $m=n$, then the condition $\pi_{\nu}\left(s_{i}\right)+i h \notin \pi_{\nu}(L)$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$ in Proposition 3.10 does not only imply the orbifold rank condition but also $\rho\left(V_{L}\left(g^{\imath}\right)\right) \geq 1$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$ and hence the the extremality of $g$ by Corollary 3.2 Indeed, since $g$ is of type $0, \rho\left(V_{L}\left(g^{i}\right)\right) \in((i, n) / n) \mathbb{Z}$, and $\rho_{\nu^{i}} \geq 1-(i, m) / m=1-(i, n) / n$ for all $\nu \in H_{\Delta}$ by an explicit calculation of all vacuum anomalies $\rho_{\nu}$. Then, $s_{i}+i h \notin \pi_{\nu^{i}}(L)$ since $\pi_{\nu}\left(s_{i}\right)+i h \notin \pi_{\nu}(L)$ and $\rho\left(V_{L}\left(g^{i}\right)\right)$ must be greater than $1-(i, n) / n$ and consequently at least 1 .

## 4. Schellekens' List

In this section we review the classification of strongly rational, holomorphic vertex operator algebras. By work of Zhu Zhu96 the central charge $c$ of such a vertex operator algebra $V$ is a non-negative multiple of 8 and its weight-one Lie algebra $V_{1}$ is reductive DM04b].

The most interesting case is that of central charge 24, in particular because of its connection to bosonic string theory Bor92]. In 1993 Schellekens proved that the weight-one space $V_{1}$ must be one of 71 semisimple or abelian Lie algebras of rank at most 24, which are listed in Table 1 of Sch93] and called Schellekens' list (see also DM04a, DM06, Höh17, EMS20a).

Examples of such vertex operator algebras are the vertex operator algebras $V_{N}$ associated with the 24 Niemeier lattices $N$, i.e. the positive-definite, even, unimodular lattices of rank 24 (of which the Leech lattice $\Lambda$ is the unique one without roots). These are exactly the cases in which the weight-one Lie algebra has rank 24 DM04b. At the other end of the spectrum is the famous Moonshine module $V^{\natural}$ [FLM88] with $V_{1}^{\natural}=\{0\}$.

If $V_{1}$ is one of the 69 semisimple Lie algebras in central charge 24, then the (simple) affine vertex operator algebra $\left\langle V_{1}\right\rangle \cong L_{\hat{\mathfrak{g}}_{1}}\left(k_{1}, 0\right) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_{s}}\left(k_{s}, 0\right)$ generated by $V_{1}$ (with levels $k_{i} \in \mathbb{Z}_{>0}$ ) is a full vertex operator subalgebra of $V$. Since $L_{\hat{\mathfrak{g}}_{1}}\left(k_{1}, 0\right) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_{s}}\left(k_{s}, 0\right)$ is rational, $V$ decomposes into the direct sum of finitely many irreducible $\left\langle V_{1}\right\rangle$-modules. Schellekens also showed that the isomorphism type of $V_{1}$ uniquely determines the vertex operator algebra $\left\langle V_{1}\right\rangle$ and the decomposition of $V$ into irreducible $\left\langle V_{1}\right\rangle$-modules up to outer automorphisms of $V_{1}$.

In a combined effort by numerous authors (see FLM88, DGM90, Don93, DGM96, Lam11, Miy13, LS12, LS15, SS16, LS16b, EMS20a, LS16a, Möl16, LL20] for the existence and DM04a, LS15, LS19, KLL18, LL20, EMS20b, LS20b, LS20a for the uniqueness) the following classification result was proved:

Theorem 4.1 (Classification). Up to isomorphism there are exactly 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$. Such a vertex operator algebra is uniquely determined by the Lie algebra $V_{1}$.

While the zero Lie algebra is realised by the Moonshine module $V^{\natural}$, as of now it is not known whether $V^{\natural}$ is the unique strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1}=\{0\}$.

Unfortunately, the proof of the above theorem uses a variety of different methods and one might be led to believe that Schellekens' classification is largely sporadic. That this is not the case was shown in two independent papers by the authors of this text [Höh17, MS19] (see also CLM22]) who described different uniform constructions of the 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$.

In both works it was observed that the 70 vertex operator algebras fall into 11 families, listed in Table 1, associated with certain (algebraic) conjugacy classes of automorphisms in $\mathrm{Co}_{0} \cong \mathrm{O}(\Lambda)$, the automorphism group of the Leech lattice $\Lambda$, or equivalently with certain lattice genera (see Section 4.1. We note that the 11 conjugacy classes are uniquely characterised by their Frame shapes. A complete list of the vertex operator algebras in each family can be found in the appendix.

In Section 5 we add another systematic construction of the 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$, namely as orbifold constructions associated with the 24 Niemeier lattice vertex operator algebras $V_{N}$.

Table 1. The 11 Frame shapes and lattice genera associated with the strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$.

| Family | $\mathrm{O}(\Lambda)$ | Ord. Doubl. | Lattice Genus | Class No. | No. of VOAs |
| :---: | :--- | :---: | :--- | ---: | ---: |
| A | $1^{24}$ | no | $I I_{24,0}$ | 24 | 24 |
| B | $1^{8} 2^{8}$ | no | $I I_{16,0}\left(2_{I I}^{+10}\right)$ | 17 | 17 |
| C | $1^{6} 3^{6}$ | no | $I I_{12,0}\left(3^{-8}\right)$ | 6 | 6 |
| D | $2^{12}$ | yes | $I I_{12,0}\left(2_{I I}^{-10} 4_{I I}^{-2}\right)$ | 2 | 9 |
| E | $1^{4} 2^{2} 4^{4}$ | no | $I I_{10,0}\left(2_{2}^{+2} 4_{I I}^{+6}\right)$ | 5 | 5 |
| F | $1^{4} 5^{4}$ | no | $I I_{8,0}\left(5^{+6}\right)$ | 2 | 2 |
| G | $1^{2} 2^{2} 3^{2} 6^{2}$ | no | $I I_{8,0}\left(2_{I I}^{+6} 3^{-6}\right)$ | 2 | 2 |
| H | $1^{3} 7^{3}$ | no | $I I_{6,0}\left(7^{-5}\right)$ | 1 | 2 |
| I | $1^{2} 2^{1} 4^{1} 8^{2}$ | no | $I I_{6,0}\left(2_{5}^{+1} 4_{1}^{+1} 8_{I I}^{+4}\right)$ | 1 | 1 |
| J | $2^{3} 6^{3}$ | yes | $I I_{6,0}\left(2_{I I}^{+4} 4_{I I}^{-2} 3^{+5}\right)$ | 1 | 1 |
| K | $2^{2} 10^{2}$ | yes | $I I_{4,0}\left(2_{I I}^{-2} 4_{I I}^{-2} 5^{+4}\right)$ | 1 | 2 |
|  |  |  |  | 1 |  |

Moreover, we use the corresponding inverse orbifold constructions to systematically show that each such vertex operator algebra $V$ is indeed uniquely determined by the Lie algebra $V_{1}$ (see Section 6).
4.1. Leech-Lattice Approach. To provide some context, we explain the constructions in Höh17] and MS19, beginning with the former. Due to recent advancements we are able to formulate some of the conjectural statements in Höh17] as theorems.

Let $V$ be a strongly rational vertex operator algebra. Then the Lie algebra $V_{1}$ is reductive DM04b. Let $\mathfrak{g}=\mathfrak{g}_{1} \oplus \cdots \oplus \mathfrak{g}_{s}$ be the semisimple part of $V_{1}$. Then the (in general non-full) vertex operator subalgebra $\langle\mathfrak{g}\rangle$ of $V$ generated by $\mathfrak{g}$ is isomorphic to the simple affine vertex operator algebra $\langle\mathfrak{g}\rangle \cong L_{\hat{\mathfrak{g}}_{1}}\left(k_{1}, 0\right) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_{s}}\left(k_{s}, 0\right)$ with positive integer levels $k_{i} \in \mathbb{Z}_{>0}$ [DM06]. This entails that $\langle\cdot, \cdot\rangle=k_{i}(\cdot, \cdot)$ restricted to $\mathfrak{g}_{i}$ where $\langle\cdot, \cdot\rangle$ is the invariant bilinear form on $V$ normalised such that $\langle\mathbf{1}, \mathbf{1}\rangle=-1$ and $(\cdot, \cdot)$ the invariant bilinear form on $\mathfrak{g}_{i}$ normalised such that the long roots have norm 2. Moreover, it is well known that $\langle\mathfrak{g}\rangle$ contains the lattice vertex operator algebra $V_{Q_{\mathfrak{g}}}$ (in general a non-full vertex operator subalgebra of $\langle\mathfrak{g}\rangle$ ) where $Q_{\mathfrak{g}}:=\sqrt{k_{1}} Q_{1}^{l} \oplus \cdots \oplus \sqrt{k_{s}} Q_{s}^{l}$ and $Q_{i}^{l}$ is the lattice spanned by the long roots of $\mathfrak{g}_{i}$ normalised to have squared norm 2 (see, e.g., Corollary 5.7 in (DM06).

Let $\mathcal{H} \cong Q_{\mathfrak{g}} \otimes_{\mathbb{Z}} \mathbb{C}$ be the standard Cartan subalgebra of $\left(V_{Q_{\mathfrak{g}}}\right)_{1} \subseteq \mathfrak{g}$, which is also a Cartan subalgebra of $\mathfrak{g}$. Then the vertex operator subalgebra $H:=\langle\mathcal{H}\rangle$ of $V$ generated by $\mathcal{H}$ is a Heisenberg vertex operator algebra and $H$ is a full vertex operator subalgebra of $V_{Q_{g}}$. We then consider the commutant (or centraliser) $W:=$ $\operatorname{Com}_{V}(H)=\operatorname{Com}_{V}\left(V_{Q_{\mathfrak{g}}}\right)$. The double commutant $\operatorname{Com}_{V}(W)=\operatorname{Com}_{V}\left(\operatorname{Com}_{V}(H)\right)$ must be a lattice vertex operator algebra, extending $V_{Q_{\mathfrak{g}}}$, i.e. $\operatorname{Com}_{V}(W)=V_{L}$ for some lattice extension $L \supseteq Q_{\mathfrak{g}}$, and $W$ is strongly rational by Section 4.3 in [CKLR19. Note that $V_{L}$ and $W$ form a dual (or Howe or commuting) pair in $V$, i.e. they are their mutual commutants, and $V$ is isomorphic to a (full) vertex operator algebra extension of $W \otimes V_{L}$.

Now, additionally suppose that $V$ is holomorphic. Then the central charge $c$ of $V$ is a non-negative multiple of 8 , and the rank $r$ of $V_{1}$ is bounded from above by $c$. Let us also assume that $V_{1}=\mathfrak{g}$ is semisimple, i.e. has no abelian part. Then $\mathcal{H}$ is a Cartan subalgebra of $V_{1}$ and $H \subseteq V_{Q_{\mathfrak{g}}} \subseteq V_{L}$ all have central charge $r$ while $W$
has central charge $c-r$. By construction, $W_{1}=\{0\}$. The main results in CKM22, Lin17, and the fact that $V$ is holomorphic imply that there is a braid-reversed equivalence between the module categories of $W$ and $V_{L}$. More concretely, since the irreducible $V_{L}$-modules, the non-integral parts of their conformal weights and the fusion rules are parametrised by the finite quadratic space $L^{\prime} / L$, the irreducible $W$-modules are also simple currents and along with the non-integral parts of their conformal weights and the fusion rules parametrised by some finite quadratic space $R(W)$, and there is an isometry $\tau: L^{\prime} / L \longrightarrow \overline{R(W)}$ such that

$$
V \cong \bigoplus_{\alpha+L \in L^{\prime} / L} W_{\tau(\alpha+L)} \otimes V_{\alpha+L}
$$

i.e. $V$ is a simple-current extension of $W \otimes V_{L}$. Here, $\overline{R(W)}$ denotes the finite quadratic space obtained from $R(W)$ by multiplying the quadratic form by -1 .

We specialise to central charge $c=24$. In that case $V_{1}$ is one of the 71 Lie algebras in Schellekens' list [Sch93] and hence either zero, abelian of rank $r=24$ or semisimple DM04a. We assume that $V_{1} \neq\{0\}$ or equivalently that $r>0$. If $V_{1}$ is abelian, then $V$ is isomorphic to the Leech lattice vertex operator algebra $V_{\Lambda}$ DM04b and the results of the above paragraph are true with $L=\Lambda$; so we may assume that $V_{1}$ is semisimple. Then $\left\langle V_{1}\right\rangle$ is actually a full vertex operator subalgebra of $V$. It is observed in Höh17 that Schellekens' classification implies that the lattice $L=L_{\mathfrak{g}}$, called orbit lattice, belongs to one of the 11 genera listed in Table 1 and labelled by letters A to K.

Furthermore, is shown in Höh17, Theorem 4.7, that $W$ has to be isomorphic to one of 11 vertex operator algebras of the form $V_{\Lambda_{\nu}}^{\hat{\nu}}$ for the conjugacy classes $\nu \in \mathrm{O}(\Lambda)$ listed in Table 1 and corresponding to the genus of the lattice $L_{\mathfrak{g}}$. Here $\Lambda_{\nu}=\left(\Lambda^{\nu}\right)^{\perp}$ denotes the coinvariant lattice and $\hat{\nu}$ a lift of $\nu$ to $\operatorname{Aut}\left(V_{\Lambda_{\nu}}\right)$. Note that since $\nu$ acts fixed-point freely on $\Lambda_{\nu}$, all its lifts are standard and conjugate. The result in Höh17] depended on the conjecture that $R\left(V_{\Lambda_{\nu}}^{\hat{\hat{}}}\right) \cong R(W) \cong \overline{R\left(V_{L_{\mathfrak{g}}}\right)}$, which was proved in full in Lam20 by computing $R\left(V_{\Lambda_{\nu}}^{\hat{\nu}}\right)$, with partial results in Möl16] and by others.

In summary:
Theorem 4.2 (Leech-Lattice Picture, Höh17). Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 with $V_{1} \neq\{0\}$. Then $V$ is isomorphic to a simple-current extension of

$$
V_{\Lambda_{\nu}}^{\hat{\nu}} \otimes V_{L_{\mathfrak{g}}}
$$

where $L_{\mathfrak{g}} \supseteq Q_{\mathfrak{g}}$ is a certain lattice determined by $\mathfrak{g}$ in one of the 11 genera listed in Table 1 and $\nu$ is from the corresponding conjugacy class in $\mathrm{O}(\Lambda)$. Moreover, $V_{\Lambda_{\nu}}^{\hat{\nu}}$ and $V_{L_{\mathfrak{g}}}$ form a dual pair in $V$.

Since every vertex operator algebra of the form $V_{\Lambda_{\nu}}^{\hat{\nu}} \otimes V_{L}$ where $L$ is in the genus associated with $\nu$ may be extended to a strongly rational, holomorphic vertex operator algebra of central charge 24 , the lattices $L_{\mathfrak{g}}$ where $\mathfrak{g}$ runs through the Lie algebras on Schellekens' list exhaust all isomorphism classes of lattices in the 11 genera.

The existence of a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \cong \mathfrak{g}$ for each of the 70 non-zero weight-one Lie algebras $\mathfrak{g}$ on Schellekens' list Sch93 (see Theorem 4.1) is then proved by realising $V$ as holomorphic extension of $V_{\Lambda_{\nu}}^{\hat{\hat{L}}} \otimes V_{L_{\mathfrak{g}}}$ (see Theorem 4.4 in Höh17).

In all families except D and J the lattices $L_{\mathfrak{g}}$ are non-isomorphic for different Lie algebras $\mathfrak{g}$. For genera D and J the Lie algebra $\mathfrak{g}=V_{1}$ (and thus the
vertex operator algebra $V$ ) additionally depends on the choice of the isometry $\tau: L_{\mathfrak{g}}^{\prime} / L_{\mathfrak{g}} \longrightarrow \overline{R\left(V_{\Lambda_{\nu}}^{\hat{\nu}}\right)}$; see the last two columns in Table 1

In principle, the approach in Höh17 can also be used to prove the uniqueness of a strongly rational, holomorphic vertex operator algebra of central charge 24 with a given Lie algebra $\mathfrak{g}=V_{1} \neq\{0\}$ (see Theorem 4.1). By Theorem 4.2 it suffices to classify the holomorphic extensions of $V_{\Lambda_{\nu}}^{\nu} \otimes V_{L_{\mathfrak{g}}}$ for all orbit lattices in the 11 genera. The general approach is described in Höh17, Theorem 4.1. Partial results are obtained in Section 4.3 of Höh17. Except for genera D and J one has to show that each orbit lattice $L_{\mathfrak{g}}$ admits only one holomorphic extension of $V_{\Lambda_{\nu}}^{\hat{\nu}} \otimes V_{L_{\mathfrak{g}}}$ up to isomorphism, i.e. that the choice of the isometry $\tau$ does not matter.

Finally, note that if $V_{1}=\{0\}$ so that $H=V_{L_{\mathfrak{g}}} \cong \mathbb{C} \mathbf{1}$, then the above decomposition degenerates to $V \cong W \otimes \mathbb{C} \mathbf{1} \cong W$ and becomes ineffective.
4.2. Generalised-Deep-Hole Approach. We first recall the notion of generalised deep hole from MS19 and then explain how it was used in MS19 to systematically realise the Lie algebras on Schellekens' list by cyclic orbifold constructions starting from the Leech lattice vertex operator algebra $V_{\Lambda}$.
Definition 4.3 (Generalised Deep Hole, MS19). Let $V$ be a strongly rational, holomorphic vertex operator algebra of central charge 24 and $g \in \operatorname{Aut}(V)$ of finite order $n$. We further assume that $V^{g}$ satisfies the positivity condition. Then $g$ is called a generalised deep hole of $V$ if $g$
(1) has type 0 ,
(2) is extremal (see Theorem 3.1) and
(3) satisfies the orbifold rank condition (see Section 3.3).

Schellekens' list can be realised by orbifold constructions from the Leech lattice vertex operator algebra $V_{\Lambda}$, thus also proving the existence part of Theorem 4.1
Theorem 4.4 (Generalised-Deep-Hole Construction, MS19]). Let $\mathfrak{g}$ be one of the 71 Lie algebras on Schellekens' list. Then there exists a generalised deep hole $g \in \operatorname{Aut}\left(V_{\Lambda}\right)$ such that the corresponding orbifold construction has weight-one Lie $\operatorname{algebra}\left(V_{\Lambda}^{\operatorname{orb}(g)}\right)_{1} \cong \mathfrak{g}$.

In addition, it is proved:
Theorem 4.5 (Holy Correspondence, MS19]). The cyclic orbifold construction $g \mapsto V_{\Lambda}^{\operatorname{orb}(g)}$ defines a bijection between the algebraic conjugacy classes of generalised deep holes $g$ in $\operatorname{Aut}\left(V_{\Lambda}\right)$ with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)>0$ and the isomorphism classes of strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$.

Moreover, it is shown that the generalised deep holes of $V_{\Lambda}$ with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)>0$ project to the 11 conjugacy classes in $\mathrm{O}(\Lambda)$ listed in Table 1 under $\operatorname{Aut}\left(V_{\Lambda}\right) \rightarrow$ $\operatorname{Aut}\left(V_{\Lambda}\right) / K \cong \mathrm{O}(\Lambda)$ (in the case of the Leech lattice $\Lambda$ the Weyl group $W$ is trivial since $\Lambda$ has no roots), again grouping the 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$ in the same way as in Höh17 into the 11 families A to K.

The exactly 38 algebraic conjugacy classes of generalised deep holes $g$ in $V_{\Lambda}$ with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)=0$ all yield the Moonshine module $V^{\natural}$ with $V_{1}^{\natural}=\{0\}$ in the orbifold construction Car18.

The two pictures from Höh17, MS19 are related as follows. First, note that for any automorphism $\nu \in \mathrm{O}(\Lambda)$, the fixed-point (or invariant) sublattice $\Lambda^{\nu}$ and the coinvariant sublattice $\Lambda_{\nu}=\left(\Lambda^{\nu}\right)^{\perp}$ are their mutual orthogonal complements in the Leech lattice $\Lambda$, and $\Lambda$ is a finite-index extension of $\Lambda^{\nu} \oplus \Lambda_{\nu}$. Hence, $V_{\Lambda^{\nu}} \otimes V_{\Lambda_{\nu}}$ is a dual pair in $V_{\Lambda}$, i.e. $V_{\Lambda^{\nu}}$ and $V_{\Lambda_{\nu}}$ are their mutual commutants in $V_{\Lambda}$.

Let $g \in \operatorname{Aut}\left(V_{\Lambda}\right)$ be of finite order and type 0 and let $V \cong V_{\Lambda}^{\text {orb }(g)}$ be the corresponding orbifold construction. Up to conjugacy, $g$ is of the form $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ with $\nu \in \mathrm{O}(\Lambda)$ and $h \in \pi_{\nu}\left(\Lambda \otimes_{\mathbb{Z}} \mathbb{Q}\right)$. As such a $g$ acts on the two tensor factors of $V_{\Lambda^{\nu}} \otimes V_{\Lambda_{\nu}}$ separately, $V_{\Lambda^{\nu, h}} \otimes V_{\Lambda_{\nu}}^{\hat{N}}$ is a full vertex operator subalgebra of $V_{\Lambda}^{g}$ where $\Lambda^{\nu, h}:=\left\{\alpha \in \Lambda^{\nu} \mid\langle\alpha, h\rangle \in \mathbb{Z}\right\}$ and $\hat{\nu}$ is a standard lift of the restriction of $\nu$ to $\Lambda_{\nu}$ (recall that $\nu$ acts fixed-point freely on $\Lambda_{\nu}$ so that all lifts of $\nu$ are standard and conjugate). In fact, $V_{\Lambda^{\nu, h}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}}$ is a dual pair in the fixed-point vertex operator subalgebra $V_{\Lambda}^{g}$.

Now, let $V$ be a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $\mathfrak{g}:=V_{1} \neq\{0\}$ and $g$ the corresponding generalised deep hole of $V_{\Lambda}$ with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)=\operatorname{rk}\left(\Lambda^{\nu}\right)>0$. Then $V_{\Lambda}^{\operatorname{orb}(g)} \cong V$. Moreover, it follows from the classification result in [MS19] that $\nu$ is one of the 11 automorphisms in Table 1 and that the weight-one Lie algebra $\mathfrak{g} \cong\left(V_{\Lambda}^{\operatorname{orb}(g)}\right)_{1}$ is such that the orbit lattice $L_{\mathfrak{g}}$ is in the genus corresponding to $\nu$. Then, by the results in Höh17], $V \cong V_{\Lambda}^{\operatorname{orb}(g)}$ is an extension of the dual pair $V_{L_{\mathfrak{g}}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}}$ (with the same $\nu$ ).

Finally, consider the inverse orbifold construction, which is given by the inner automorphism $\sigma=\mathrm{e}^{-(2 \pi \mathrm{i}) u(0)} \in \operatorname{Aut}(V)$ for a certain $u \in L_{\mathfrak{g}} \otimes_{\mathbb{Z}} \mathbb{Q}$ described in [ELMS21], i.e. $V^{\operatorname{orb}\left(\sigma_{u}\right)} \cong V_{\Lambda}$. Then also $V_{L_{\mathfrak{g}}^{u}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}}$ with $L_{\mathfrak{g}}^{u}:=\left\{\alpha \in L_{\mathfrak{g}} \mid\langle\alpha, u\rangle \in \mathbb{Z}\right\}$ forms a dual pair in the fixed-point vertex operator subalgebra $V_{\Lambda}^{g} \cong V^{\sigma_{u}}$, which entails that $\Lambda^{\nu, h} \cong L_{\mathfrak{g}}^{u}$. Therefore, we obtain the following diagram, in which all inclusion arrows denote full vertex operator subalgebras and the vertical arrows are embeddings of dual pairs:


## 5. Systematic Orbifold Construction

In this section we present a systematic construction of the 70 strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 with $V_{1} \neq\{0\}$ (the existence part of Theorem 4.1), as cyclic orbifold constructions from the 24 Niemeier lattice vertex operator algebras $V_{N}$. We shall also connect this picture to the Leechlattice approach from Höh17. Finally, we provide a characterisation in terms of generalised deep holes as defined in MS19].
5.1. Short Automorphisms. We shall define certain short automorphisms of the Niemeier lattice vertex operator algebras. It will turn out that there are exactly 226 short automorphisms up to algebraic conjugacy and that their corresponding orbifold constructions give all the 70 vertex operator algebras in a systematic way. These automorphisms, too, can be divided into 11 classes corresponding to the 11 Frame shapes in Table 1

Recall that by Theorem 2.13, given a Niemeier lattice $N$, any finite-order automorphism $g \in \operatorname{Aut}\left(V_{N}\right)$ is conjugate to an automorphism of the form $\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ where $\nu$ is from a list of representatives of the conjugacy classes of $H_{\Delta} \subseteq \mathrm{O}(N)$ and
$h \in \pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right)$ projects to a list of orbit representatives of the action of $C_{\mathrm{O}(N)}(\nu)$ on $\pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right) /\left(N^{\nu}\right)^{\prime}$.

Also recall that $H_{\Delta} \cong H$ where $H=\mathrm{O}(N) / W$ with Weyl group $W$ and the isometry group of $N$ is a split extension $\mathrm{O}(N) \cong W: H$. For ease of presentation, we identify $H$ with $H_{\Delta}$ in the following.

Definition 5.1 (Short Automorphism). Let $N$ be a positive-definite, even, unimodular lattice and $V_{N}$ the corresponding holomorphic lattice vertex operator algebra. Let $g$ be an automorphism of $V_{N}$ of finite order $n$ conjugate to $\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ with $\nu \in H$ and $h \in \pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right)$. Then $g$ is called short if
(1) $g$ has type 0 ,
(2) $\nu$ has order $n$ and
(3) $h \bmod \left(N^{\nu}\right)^{\prime}$ has order $n$.

We comment on this definition. Item (1) guarantees that the orbifold construction exists. Explicitly, it reads

$$
\rho\left(V_{N}(g)\right)=\rho_{\nu}+\min _{\alpha \in\left(N^{\nu}\right)^{\prime}+h} \frac{\langle\alpha, \alpha\rangle}{2} \in \frac{1}{n} \mathbb{Z}
$$

Item (2) can be thought of as a minimality condition on the order of $g$. It means that the order of $g$ equals the order of the projection of $g$ to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$. Explicitly,

$$
h \in \begin{cases}\frac{1}{n} N^{\nu}, & \text { if }|\hat{\nu}|=n \\ s+\frac{1}{n} N^{\nu}, & \text { if }|\hat{\nu}|=2 n\end{cases}
$$

with $s \in \frac{1}{2 n} N^{\nu}$ from Proposition 2.15. Item (3) can be rephrased as the restriction of $g$ to $V_{N^{\nu}}$ having the same order as $g$ or equivalently $N^{\nu, h}$ being an index- $n$ sublattice of $N^{\nu}$. The motivation behind this condition will be explained at the end of this section.

We classify all short automorphisms of the Niemeier lattices, including the Leech lattice $\Lambda$, up to algebraic conjugacy.

Proposition 5.2 (Classification of Short Automorphisms). There are exactly 226 algebraic conjugacy classes of short automorphisms of the Niemeier lattice vertex operator algebras $V_{N}$, listed in Table 2. The Frame shapes of their projections to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ are given by the 11 Frame shapes in Table 1.

Proof. By Theorem 2.13, for each Niemeier lattice $N$ and for each conjugacy class in $H$ (see Table 3 in the appendix), represented by $\nu$, we have to determine the orbits under the action of $C_{\mathrm{O}(N)}(\nu)$ on $\pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right) /\left(N^{\nu}\right)^{\prime}$. A computer search using Magma BCP97] reveals that there are exactly 230 orbits (with representatives $h$ ), corresponding to at most 230 non-conjugate automorphisms $g$ in $\operatorname{Aut}\left(V_{N}\right)$. Note that we cannot in principle exclude the possibility that two or more orbit representatives $h$ give conjugate automorphisms $g$ in $\operatorname{Aut}\left(V_{N}\right)$.

By considering the normaliser action (see Remark 2.14), this reduces to 226 orbits, corresponding to at most 226 algebraic conjugacy classes in $\operatorname{Aut}\left(V_{N}\right)$. Finally, by studying some invariants, it is easy to see that these 226 short automorphisms indeed represent 226 distinct algebraic conjugacy classes.

The short automorphisms are listed in Table 2 The first two columns list the 24 Niemeier lattices $N$, labelled as in Höh17 and by their root lattices. The next 11 columns are labelled by the Frame shape of the corresponding outer automorphism in $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ considered as element in $\mathrm{O}(N)$ as listed in Table 1. The entry for a Niemeier lattice $N$ and a Frame shape consists of a comma-separated list providing for each corresponding algebraic conjugacy class in $H$ (see Table 3in the
appendix) the number of short automorphisms in $\operatorname{Aut}\left(V_{N}\right)$. The two last columns and two bottom rows count the total number of algebraic conjugacy classes of short automorphisms and of their projections to $H$.

Table 2. The 226 short automorphisms of the Niemeier lattice vertex operator algebras $V_{N}$.

| $N$ | Outer Automorphism Class |  |  |  |  |  |  |  |  |  |  |  |  | $\begin{gathered} \text { No. in } \\ H \quad \operatorname{Aut}\left(V_{N}\right) \\ \hline \end{gathered}$ |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | A | B | C | D | E | F |  | G | H |  |  | J | K |  |  |
| A1 $D_{24}$ | 1 | . | . | . | . | . |  | . | . |  |  | . |  | 1 | 1 |
| A2 $D_{16} E_{8}$ | 1 | . | . | . | . |  |  | . |  |  |  | . |  | 1 | 1 |
| A3 $E_{8}^{3}$ | 1 | 1 | . | . | . |  |  | . |  |  |  | . |  | 2 | 2 |
| A4 $A_{24}$ | 1 | . | . | 1 | . |  |  | . |  |  |  |  |  | 2 | 2 |
| A5 $D_{12}^{2}$ | 1 | . | . | 1 | . |  |  | . |  |  |  |  |  | 2 | 2 |
| A6 $A_{17} E_{7}$ | 1 | 2 | . | . | . |  |  | . | . |  |  |  |  | 2 | 3 |
| A7 $D_{10} E_{7}^{2}$ | 1 | 2 | . | . | . |  |  | . |  |  |  |  |  | 2 | 3 |
| A8 $A_{15} D_{9}$ | 1 | 4 | . | . | . |  |  | . |  |  |  |  |  | 2 | 5 |
| A9 $D_{8}^{3}$ | 1 | 3 | . | . | . | . |  | . | . |  |  |  |  | 2 | 4 |
| A10 $A_{12}^{2}$ | 1 | . | . | 1 | . | . |  | . | . |  |  |  |  | 2 | 2 |
| A11 $A_{11} D_{7} E_{6}$ | 1 | 5 | . | . | . | . |  | . | . |  |  |  |  | 2 | 6 |
| A12 $E_{6}^{4}$ | 1 | 2, 2 | 2 | . | . |  |  | 1 |  |  |  |  |  | 5 | 8 |
| A13 $A_{9}^{2} D_{6}$ | 1 | 4 | . |  | 2 | . |  | . |  |  |  |  |  | 3 | 7 |
| A14 $D_{6}^{4}$ | 1 | 3 | 2 | 1 | . |  |  | . |  |  |  |  |  | 4 | 7 |
| A15 $A_{8}^{3}$ | 1 | 2 |  | 1,1 | . |  |  | . | . |  |  |  |  | 4 | 5 |
| $\mathrm{A} 16 A_{7}^{2} D_{5}^{2}$ | 1 | 6, 4, 4 | . | . | . | . |  | . | . |  |  |  |  | 4 | 15 |
| $\mathrm{A} 17 A_{6}^{4}$ | 1 | . | 3 | 1 | . | . |  | . |  |  |  | 1 |  | 4 | 6 |
| $\mathrm{A} 18 A_{5}^{4} D_{4}$ | 1 | 4, 5 | 5 | . | 4 |  |  | 4 |  | 1 |  |  |  | 7 | 24 |
| A19 $D_{4}^{6}$ | 1 | 2, 3 | 2, 3 | 1 | 2 | 1 |  | , 3 | . |  |  | . |  | 10 | 20 |
| A20 $A_{4}^{6}$ | 1 | 3 |  | 1,1 | 5 | 2 |  | . |  |  |  |  | 1 | 7 | 14 |
| $\mathrm{A} 21 A_{3}^{8}$ | 1 | 3,6 | 3 | 1 | 7 |  |  | 4 | 1 |  |  | 1 |  | 9 | 27 |
| A22 $A_{2}^{12}$ | 1 | 2 | 4 | 1,1 | 5 | 2 |  | 4 |  | 2 |  | 1,2 | 1 | 12 | 26 |
| $\mathrm{A} 23 A_{1}^{24}$ | 1 | 3 | 2 | 1 | 5 | 2 |  | 7 | 1 | 2 |  |  | 1 | 10 | 25 |
| A24 $\Lambda$ | 1 | 1 | 1 | 1 | 1 | 1 |  | 1 | 1 | 1 |  | 1 | 1 | 11 | 11 |
| No. in $H$ | 24 | 24 | 10 | 15 | 8 | 5 |  | 8 | 3 |  |  | 5 | 4 | 110 |  |
| No. in $\operatorname{Aut}\left(V_{N}\right)$ | 24 | 76 | 27 | 15 | 31 | 8 |  | 26 | 3 | 6 |  | 6 | 4 |  | 226 |

Quite remarkably, only outer automorphisms corresponding to the 11 Frame shapes in Table 1 admit short automorphisms (cf. Table 3).

We also study the powers of the short automorphisms:
Proposition 5.3 (Powers of Short Automorphisms). A power of a short automorphism is short. The non-trivial powers of the 226 short automorphisms are tabulated in Table 15 to Table 19 in the appendix.

With the classification in Proposition 5.2 one may prove this result in a case-bycase computation. However, a more conceptual proof will be given in Section 5.3

We now formulate our main result:
Theorem 5.4 (Main Result). Let $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ be one of the 226 short automorphisms of the Niemeier lattice vertex operator algebras $V_{N}$. Then $\left(V_{N}^{\mathrm{orb}(g)}\right)_{1}$ is isomorphic to the Lie algebra tabulated in Table 4 to Table 14 in the appendix.

We shall prove Theorem 5.4 in Section 5.3 We thus obtain a systematic construction of strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 realising all non-zero Lie algebras $\mathfrak{g}$ on Schellekens' list as weight-one spaces $V_{1} \cong \mathfrak{g}$ (the existence statement in Theorem 4.1):
Corollary 5.5 (Systematic Orbifold Construction). The cyclic orbifold constructions $V_{N}^{\text {orb }(g)}$, where $N$ runs through the 24 Niemeier lattices and $g$ through the short automorphisms of the corresponding lattice vertex operator algebras $V_{N}$, realise all 70 non-zero Lie algebras $\mathfrak{g}$ on Schellekens' list as weight-one spaces $\left(V_{N}^{\operatorname{orb}(g)}\right)_{1} \cong \mathfrak{g}$.

We note by inspecting Table 15 to Table 19 in the appendix:
Remark 5.6. Each of the 226 algebraic conjugacy classes of short automorphisms $g$ is uniquely determined by the algebraic conjugacy class of its projection to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ and the orbifold constructions $V_{N}^{\text {orb }\left(g^{d}\right)}$ for all $d$ dividing the order $n$ of $g$.
5.2. Relation with the Leech-Lattice Picture. We describe the connection of our orbifold construction with the Leech lattice description of the strongly rational, holomorphic vertex operator algebras of central charge 24 in Höh17 reviewed in Section 4.1 .

Given a positive-definite, even lattice $L$, there is an induced action of $\mathrm{O}(L)$ on the discriminant form $L^{\prime} / L$, leading to a short exact sequence

$$
1 \longrightarrow \mathrm{O}_{0}(L) \longrightarrow \mathrm{O}(L) \longrightarrow \overline{\mathrm{O}}(L) \longrightarrow 1
$$

where $\overline{\mathrm{O}}(L)$ is the subgroup of $\mathrm{O}\left(L^{\prime} / L\right)$ induced by $\mathrm{O}(L)$ and $\mathrm{O}_{0}(L)$ are the automorphisms of $L$ that act trivially on $L^{\prime} / L$.

Now, let $N$ be a Niemeier lattice other than the Leech lattice $\Lambda$. As mentioned before, $\mathrm{O}(N) \cong W: H$. The coinvariant lattice $N_{H}=\left(N^{H}\right)^{\perp}$, which is a lattice without norm-two vectors, has an up to conjugation unique $H$-equivariant embedding into $\Lambda$ in such a way that $H \cong \mathrm{O}_{0}\left(N_{H}\right) \cong \mathrm{O}_{0}\left(\Lambda_{H}\right)$ (see Nik80, Remark 1.14.7 and Proposition 1.14.8, and HM16). This embedding allows us to consider an element $\nu \in H$ as an automorphism in $\mathrm{O}(\Lambda)$, which we also denote by $\nu$.

Note that, as in the case of the Leech lattice, $V_{N^{\nu}} \otimes V_{N_{\nu}}$ is a dual pair in $V_{N}$ for any automorphism $\nu \in \mathrm{O}(N)$ where we recall the coinvariant lattice $N_{\nu}=\left(N^{\nu}\right)^{\perp}$.

Let $g \in \operatorname{Aut}\left(V_{N}\right)$ be of finite order and type 0 and let $V \cong V_{N}^{\text {orb }(g)}$ be the corresponding orbifold construction. Up to conjugacy, $g$ is of the form $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ with $\nu \in H$ and $h \in \pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right)$. Then the vertex operator algebra $V_{N^{\nu, h}} \otimes V_{N_{\nu}}^{\hat{\nu}}$ with $N^{\nu, h}:=\left\{\alpha \in N^{\nu} \mid\langle\alpha, h\rangle \in \mathbb{Z}\right\}$ is a full vertex operator subalgebra of the fixed-point vertex operator subalgebra $V_{N}^{g}$ and of $V_{N}^{\operatorname{orb}(g)}$. In fact, $V_{N^{\nu, h}} \otimes V_{N_{\nu}}^{\hat{\nu}}$ is a dual pair in $V_{\Lambda}^{g}$. Note that $N_{\nu} \cong \Lambda_{\nu}$. Since $\nu$ acts fixed-point freely on $N_{\nu}$ and $\Lambda_{\nu}$, all its lifts are standard and conjugate so that $V_{N_{\nu}}^{\hat{\nu}} \cong V_{\Lambda_{\nu}}^{\hat{\nu}}$.

For the short automorphisms, by Proposition 5.2, $\nu$ is from one of the 11 algebraic conjugacy classes of $\mathrm{O}(\Lambda)$ in Table 1] On the other hand, by Höh17, given a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24, $V_{L_{\mathfrak{g}}} \otimes V_{\Lambda_{\mu}}^{\hat{\mu}}$ forms a dual pair in $V$ where $L_{\mathfrak{g}}$ is the orbit lattice associated with $\mathfrak{g}=V_{1}$ and $\mu$ is the one of the 11 algebraic conjugacy classes in Table 1 corresponding to the genus of $L_{\mathfrak{g}}$.

Not surprisingly, all short automorphisms have the property that $\mu=\nu$, i.e. the orbifold Lie algebra $\mathfrak{g}$ we determine in Theorem 5.4 is such that $V_{L_{\mathfrak{g}}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}}$ forms a dual pair in $V_{N}^{\mathrm{orb}(g)}$. As in Section 4.2 this allows us to combine the orbifold picture with the Leech-lattice picture from Höh17. $V_{L_{\mathfrak{g}}}$ must be an extension of $V_{N^{\nu, h}}$
with the same Virasoro element and we get the inclusion $V_{N^{\nu, h}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}} \hookrightarrow V_{L_{\mathfrak{g}}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}}$. In particular, the orbit lattice $L_{\mathfrak{g}}$ is an extension of $N^{\nu, h}$.

In fact, by item (3) of their definition, the short automorphisms satisfy that $L_{\mathfrak{g}} \cong N^{\nu, h}$, implying that $V_{N^{\nu, h}} \otimes V_{\Lambda_{\nu}}^{\hat{\nu}}$ itself is already a dual pair in $V \cong V_{N}^{\text {orb } g}$, in addition to being a dual pair in $V_{N}^{g}$. Indeed, that $N^{\nu, h}$ is an index- $n$ sublattice of $N^{\nu}$ is equivalent to $L_{\mathfrak{g}} \cong N^{\nu, h}$ since

$$
\left|L_{\mathfrak{g}}^{\prime} / L_{\mathfrak{g}}\right|=\left|R\left(V_{\Lambda_{\nu}}^{\hat{\nu}}\right)\right|=n^{2}\left|\left(\Lambda_{\nu}\right)^{\prime} / \Lambda_{\nu}\right|=n^{2}\left|\left(N_{\nu}\right)^{\prime} / N_{\nu}\right|=n^{2}\left|\left(N^{\nu}\right)^{\prime} / N^{\nu}\right|
$$

by the results in Lam20 and because $N^{\nu, h}$ is a sublattice of $L_{\mathfrak{g}}$. This finally motivates item (3) in the definition of short automorphisms (Definition 5.1).

We proved:
Proposition 5.7. The short automorphisms $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ of the Niemeier lattice vertex operator algebras $V_{N}$ and the corresponding orbifold constructions $V_{N}^{\mathrm{orb}(g)} \cong V$ satisfy the following diagram, in which all inclusion arrows represent full vertex operator subalgebras and the vertical arrows are embeddings of dual pairs. In particular, $N^{\nu, h} \cong L_{\mathfrak{g}}$.


Remark 5.8. Using Proposition 5.7 it is clear why all strongly rational, holomorphic vertex operator algebras of central charge 24 belonging to one of the 11 genera in Table 1 can be realised as an orbifold construction of a Niemeier lattice vertex operator algebra by a short automorphism if at least one of the vertex operator algebras for this genus can be obtained in this way: the vertex operator subalgebra $V_{N^{\nu}} \otimes V_{N_{\nu}}$ of $V_{N}$ is obtained from the vertex operator subalgebra $V_{N^{\nu, h}} \otimes V_{N_{\nu}}^{\hat{\nu}}$ of $V$ by extending the lattice $N^{\nu, h}$ by a certain isotropic vector $h^{*}$ in its discriminant form and by extending $V_{N_{\nu}}^{\hat{\nu}} \cong V_{\Lambda_{\nu}}^{\hat{\nu}}$ to $V_{\Lambda_{\nu}}$. When $L$ runs through the different lattices in the genus of $N^{\nu, h}$ and is extended by an isotropic vector in the same $\mathrm{O}\left(\left(N^{\nu, h}\right)^{\prime} / N^{\nu, h}\right)$-orbit as $h^{*}$, then the resulting lattices $K$ are all contained in the genus of $N^{\nu}$ and the resulting extensions of $V_{K} \otimes V_{\Lambda_{\nu}}$ are lattice vertex operator algebras associated with a Niemeier lattice.
5.3. Generalised-Deep-Hole Characterisation. First, we show that the short automorphisms of the Niemeier lattice vertex operator algebras are generalised deep holes in the sense of MS19] (see Definition 4.3). This will then help us in determining the orbifold construction $V_{N}^{\text {orb(g) }}$ and proving Theorem 5.4

Proposition 5.9. Let $g$ be a short automorphism of a Niemeier lattice vertex operator algebra $V_{N}$. Then $g$ is a generalised deep hole.

Proof. Let $n$ denote the order of $g$. We check that all short automorphisms $g$ satisfy $\pi_{\nu}\left(s_{i}\right)+i h \notin \pi_{\nu}(N)$ for all $i \in \mathbb{Z}_{n} \backslash\{0\}$. (For $(i, n)=1$ this is immediate from item (3) in Definition 5.1 but for the other powers of $g$ we need to check this case by case using the classification result in Proposition 5.2) As explained in Remark 3.12, this not only implies the orbifold rank condition by Proposition 3.10 but also the
extremality of $g$ by Corollary 3.2 Since by definition a short automorphism is of type 0 , it follows that $g$ is a generalised deep hole.

By the previous result, for each of the 226 short automorphisms we know the rank and the dimension of $\left(V_{N}^{\text {orb }(g)}\right)_{1}$ by Corollary 3.2 and Proposition 3.10 Using Schellekens' list of possible weight-one Lie algebras $V_{1}$ of strongly rational, holomorphic vertex operator algebras of central charge 24, together with a few additional data, we are then able to determine the Lie algebra structure of $\left(V_{N}^{\operatorname{orb}(g)}\right)_{1}$ in each case by exclusion.

Proof of Theorem 5.4. In addition to the rank and the dimension of $\left(V_{N}^{\operatorname{orb}(g)}\right)_{1}$ (see Proposition 5.9) we know:
(1) By the inverse orbifold construction the Lie algebra $\left(V_{N}^{g}\right)_{1}$ is a fixed-point Lie subalgebra of $\left(V_{N}^{\text {orb }(g)}\right)_{1}$ under an automorphism of order dividing the order $n$ of $g$. On the other hand, the possible fixed-point Lie subalgebras of a Lie algebra on Schellekens' list under finite-order automorphisms can be classified using [Kac90], Chapter 8.
(2) By Proposition 5.9 we know that $\left\{h(-1) \otimes \mathfrak{e}_{0} \mid h \in \mathfrak{h}^{\nu}\right\} \cong \mathfrak{h}^{\nu}$ is a Cartan subalgebra of $\left(V_{N}^{\mathrm{orb}(g)}\right)_{1}$. It is straightforward to compute the action of the Cartan subalgebra on the left- and right-hand side of the inclusion

$$
\left(V_{N}^{g}\right)_{1} \oplus \bigoplus_{\substack{i \in \mathbb{Z}_{n} \\(i, n)=1}}\left(V_{N}\left(g^{i}\right)\right)_{1} \subseteq\left(V_{N}^{\operatorname{orb}(g)}\right)_{1} \subseteq\left(V_{N}^{g}\right)_{1} \oplus \bigoplus_{\substack{i \in \mathbb{Z}_{n} \\ i \neq 0}}\left(V_{N}\left(g^{i}\right)\right)_{1}
$$

The corresponding (potential) root vectors must be compatible with the root system of $\left(V_{N}^{\mathrm{orb}(g)}\right)_{1}$, including their lengths with respect to the unique nondegenerate, invariant bilinear form on $V_{N}^{\mathrm{orb}(g)}$ normalised such that $\langle\mathbf{1}, \mathbf{1}\rangle=-1$.

Note that the long roots of the root systems of the simple components $\mathfrak{g}_{i}$ of the Lie algebras on Schellekens' list have lengths $2 / k_{i}$ with respect to this bilinear form where the $k_{i} \in \mathbb{Z}_{>0}$ are the levels (see Section 4.1).
Both items together with Schellekens' list are sufficient to reduce the possible Lie algebra structures of $\left(V_{N}^{\mathrm{orb}(g)}\right)_{1}$ to just one case for all 226 automorphisms $g$.

We showed in Proposition 5.9 that the short automorphisms $g$ of the Niemeier lattice vertex operator algebras are generalised deep holes. By construction, they satisfy that the order of $g$ equals the order of the projection of $g$ to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$. The short automorphisms are in fact characterised by these properties:

Theorem 5.10. The short automorphisms of the vertex operator algebras $V_{N}$ associated with the 24 Niemeier lattices $N$ are exactly the generalised deep holes $g$ of $V_{N}$ whose orders equal the orders of their respective projections to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ (and with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)>0$ in the case of the Leech lattice $\Lambda$ ).

Proof. In the following we classify the algebraic conjugacy classes of generalised deep holes of the Niemeier lattice vertex operator algebras whose orders are equal to the orders of their respective projections to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ (with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)>0$ in the case of the Leech lattice $\Lambda$ ). We obtain an upper bound of 226 classes, which then must be short because there are 226 classes of short automorphisms by Proposition 5.2 and all of these are generalised deep holes (with the additional properties) by Proposition 5.9

For each Niemeier lattice $N$ we let $\nu$ vary through the algebraic conjugacy classes in $\operatorname{Aut}\left(V_{\Lambda}\right) / K \cong \mathrm{O}(N) / W=H$ (see Table 3 in the appendix). This includes but is not limited to those in Table 2 Then we let $h$ vary through the orbits (computed using Magma BCP97) of the action of $C_{\mathrm{O}(N)}(\nu)$ on $\pi_{\nu}\left(N \otimes_{\mathbb{Z}} \mathbb{Q}\right) /\left(N^{\nu}\right)^{\prime}$
and consider the automorphisms of the form $g=\hat{\nu} \mathrm{e}^{-(2 \pi \mathrm{i}) h(0)}$ of order $n$ and type 0 (see Theorem 2.13 and Remark 2.14).

We then eliminate those entries from this finite list that cannot be extremal since $\rho\left(V_{N}\left(g^{i}\right)\right)<1$ for some $i \in \mathbb{Z}_{n}$ with $(i, n)=1$ (see [MS19]) or cannot satisfy the orbifold rank condition $\operatorname{rk}\left(\left(V_{N}^{\operatorname{orb}(g)}\right)_{1}\right)=\operatorname{rk}\left(\left(V_{N}^{g}\right)_{1}\right)$ by Remark 3.11 and items (1) and (2) in the proof of Theorem 5.4 in combination with Schellekens' list of possible weight-one Lie algebras of strongly rational, holomorphic vertex operator algebras of central charge 24.

This leaves us with 226 automorphisms of the Niemeier lattice vertex operator algebras (all corresponding to the 11 algebraic conjugacy classes in $\mathrm{O}(\Lambda)$ listed in Table 1), representing the 226 distinct algebraic conjugacy classes of short automorphisms.

Note that the 38 algebraic conjugacy classes of generalised deep holes $g$ in $V_{\Lambda}$ with $\operatorname{rk}\left(\left(V_{\Lambda}^{g}\right)_{1}\right)=0$ trivially satisfy the property that the order of the projection of $g$ to $\operatorname{Aut}\left(V_{\Lambda}\right) / K \cong \mathrm{O}(\Lambda)$ is equal to the order of $g$ but they are not short as item (3) in Definition 5.1 is not satisfied.

We conclude this section with the proof of Proposition 5.3
Proof of Proposition 5.3. Let $g$ be a short automorphism of a Niemeier lattice vertex operator algebra $V_{N}$. Then, as stated in the proof of Proposition 5.9 g satisfies the assumptions of Remark 3.12 It is easy to see that any power $g^{i}$ of $g$ still satisfies these assumptions so that $g^{i}$ is a generalised deep hole whose projection to $\operatorname{Aut}\left(V_{N}\right) / K$ has the same order as $g^{i}$. Hence, by Theorem 5.10 $g^{i}$ is short.

## 6. Systematic Uniqueness Proof

In this section we show that, given a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \neq\{0\}$, the Lie algebra structure of $V_{1}$ uniquely determines the vertex operator algebra $V$ (uniqueness statement in Theorem 4.1). Together with the existence result in Corollary 5.5 this implies that there are exactly 70 such vertex operator algebras up to isomorphism. We give the first systematic proof of this statement. We note that we also use the uniqueness of the decomposition of $V$ into $\left\langle V_{1}\right\rangle$-modules proved in [Sch93].

The proof follows the strategy proposed in LS19 to find for all vertex operator algebras $V$ with a certain Lie algebra $V_{1}$ an inner automorphism $\zeta \in \operatorname{Aut}(V)$ of type 0 such that $V^{\operatorname{orb}(\zeta)}$ is isomorphic to some Niemeier lattice vertex operator algebra $V_{N}$ and such that the inverse orbifold automorphism $g \in \operatorname{Aut}\left(V_{N}\right)$ is the unique one in $\operatorname{Aut}\left(V_{N}\right)$ up to algebraic conjugacy with certain properties (depending only on $V_{1}$ and $\zeta$ ) that allow us to conclude that $V \cong V_{N}^{\operatorname{orb}(g)}$.

Proposition 6.1. Let $\mathfrak{g}$ be one of the 70 non-zero Lie algebras on Schellekens' list and let $V$ be a strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \cong \mathfrak{g}$. Then there exists an inner automorphism $\zeta \in \operatorname{Aut}(V)$ such that $V^{\zeta}$ satisfies the positivity condition and the following holds:
(1) the order $n$ of $\zeta$ equals the order of the element in $\mathrm{O}(\Lambda)$ associated with the family of $\mathfrak{g}$ (see Table 1),
(2) $\zeta$ is of type 0 ,
(3) the conformal weights satisfy $\rho\left(V\left(\zeta^{i}\right)\right) \geq 1$ for all $i \neq 0(\bmod n)$, and
(4) $\operatorname{dim}\left(V_{1}^{\mathrm{orb}(\zeta)}\right)=\operatorname{dim}\left(\left(V_{N}\right)_{1}\right)$ for some Niemeier lattice $N$.

We find a total of 226 such inner automorphisms.
Proof. Since the case of abelian $\mathfrak{g}$ is trivial ( $V$ must be isomorphic to the Leech lattice vertex operator algebra $V_{\Lambda}$ by (DM04b), we may assume that $\mathfrak{g}$ is semisimple.

In Theorem 5.4 we realised each of the 70 non-zero Lie algebras $\mathfrak{g}$ on Schellekens' list as weight-one Lie algebra $\bar{V}_{1} \cong \mathfrak{g}$ of a concrete orbifold realisation $\bar{V}:=V_{N}^{\operatorname{orb}(\bar{g})}$ associated with a Niemeier lattice $N$ and a short automorphism $\bar{g}$ of order $n$.

For each of these orbifold constructions the inverse orbifold automorphism $\bar{\zeta} \in$ $\operatorname{Aut}(\bar{V})$ has order $n$, type 0 and satisfies $\bar{V}^{\operatorname{orb}(\bar{\zeta})}=V_{N}$. Moreover, $\bar{V} \bar{\zeta}$ satisfies the positivity condition. Since $\bar{g}$ has the property that $\rho\left(V_{N}\left(\bar{g}^{i}\right)\right) \geq 1$ for all $i \neq 0$ $(\bmod n)\left(\right.$ see proof of Proposition 5.9 , it even holds that $\rho\left(\bar{V}\left(\bar{\zeta}^{i}\right)\right) \geq 1$ for all $i \neq 0$ $(\bmod n)$. As a generalised deep hole (see Proposition 5.9) $\bar{g}$ satisfies the orbifold rank condition, which is equivalent to $\bar{\zeta}$ being inner (see Proposition 2.4).

Now, let $V$ be any strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \cong \mathfrak{g} \cong \bar{V}_{1}$. Then the lowest-order trace identity in Sch93 and the results of DM04a, DM06 imply that the full vertex operator subalgebra $\left\langle V_{1}\right\rangle$ of $V$ is isomorphic to $\left\langle\bar{V}_{1}\right\rangle$. Indeed, they are both isomorphic to the simple affine vertex operator algebra $L_{\hat{\mathfrak{g}}_{1}}\left(k_{1}, 0\right) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_{s}}\left(k_{s}, 0\right)$ (see Section 4 , with the levels $k_{i} \in \mathbb{Z}_{>0}$ determined by said trace identity. Moreover, it is shown in Sch93] that also the decomposition of $V$ into irreducible modules of $L_{\hat{\mathfrak{g}}_{1}}\left(k_{1}, 0\right) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_{s}}\left(k_{s}, 0\right)$ is uniquely determined by the Lie algebra structure of $V_{1}$, i.e. $V \cong \bar{V}$ as modules of $\left\langle V_{1}\right\rangle \cong\left\langle\bar{V}_{1}\right\rangle$. In other words, there is a linear isomorphism $\psi: \bar{V} \rightarrow V$ satisfying $\psi Y(a, z) b=Y(\psi a, z) \psi b$ for all $a \in\left\langle\bar{V}_{1}\right\rangle$ and $b \in \bar{V}$, restricting to a vertex operator algebra isomorphism $\left\langle\bar{V}_{1}\right\rangle \rightarrow\left\langle V_{1}\right\rangle$ and a Lie algebra isomorphism $\bar{V}_{1} \rightarrow V_{1}$.

Crucially, as an inner automorphism, $\bar{\zeta}$ is specified in terms of the action of $\bar{V}_{1}$ on $\bar{V}$ via the zero-mode. This allows us to transport the automorphism $\bar{\zeta}$ to an automorphism of $V$ via $\psi$. By Proposition 2.2 we may assume that $\bar{\zeta}=\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ for some $v$ in a choice $\overline{\mathcal{H}}$ of Cartan subalgebra of $V_{1}$, though this is not essential. Then $\mathcal{H}:=\psi(\overline{\mathcal{H}})$ is a Cartan subalgebra of $V_{1}$. We define $\zeta:=\psi \bar{\zeta} \psi^{-1}$, which is an (a priori linear) isomorphism of $V$. The identity $\zeta=\psi \mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \psi^{-1}=\mathrm{e}^{(2 \pi \mathrm{i})(\psi v)_{0}}$ (where $\psi v \in \mathcal{H} \subseteq V_{1}$ ) shows that $\zeta$ is an inner automorphism and hence a vertex operator algebra automorphism of $V$.

We stated above that $\psi Y(a, z) b=Y(\psi a, z) \psi b$ for all $a \in\left\langle\bar{V}_{1}\right\rangle$ and $b \in \bar{V}$. Here, the left $Y$ denotes the module vertex operator of $\left\langle\bar{V}_{1}\right\rangle$ on $\bar{V}$ and the right one that of $\left\langle V_{1}\right\rangle$ on $V$. For the orbifold construction we consider the irreducible $\bar{\zeta}^{i}$-twisted $\bar{V}$-modules $\bar{V}\left(\bar{\zeta}^{i}\right)$ for $i \in \mathbb{Z}_{n}$, which are also $\bar{\zeta}^{i}$-twisted $\left\langle\bar{V}_{1}\right\rangle$-modules (and analogously without the bar). As $\left\langle\bar{V}_{1}\right\rangle$ is a full vertex operator subalgebra of $\bar{V}$, the $L_{0}$-grading of $\bar{V}\left(\bar{\zeta}^{i}\right)$ is already determined by the structure as a $\bar{\zeta}^{i}$-twisted $\left\langle\bar{V}_{1}\right\rangle$-module (and again without the bar). On the other hand, the definition of the twisted module vertex operator $Y^{(v)}$ for inner automorphisms $\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}}$ in Li96] shows that $\psi Y^{(i v)}(a, z) b=Y^{(i \psi v)}(\psi a, z) \psi b$ for all $a \in\left\langle\bar{V}_{1}\right\rangle$ and $b \in \bar{V}\left(\bar{\zeta}^{i}\right)$. This implies that $\bar{V}\left(\bar{\zeta}^{i}\right)$ is isomorphic to $V\left(\zeta^{i}\right)$ as a $\bar{\zeta}^{i}$-twisted/ $\zeta^{i}$-twisted module of $\left\langle V_{1}\right\rangle \cong\left\langle\bar{V}_{1}\right\rangle$. In particular, they have the same $L_{0}$-grading.

We conclude that the automorphisms $\zeta \in \operatorname{Aut}(V)$ and $\bar{\zeta} \in \operatorname{Aut}(\bar{V})$ share the following properties:

- By definition, $\zeta$ and $\bar{\zeta}$ have the same order $n$.
- Since $\psi\left(\bar{V}_{1}^{\bar{\zeta}}\right)=V_{1}^{\zeta}, \zeta$ and $\bar{\zeta}$ have isomorphic fixed-point Lie subalgebras $V_{1}^{\zeta} \cong \bar{V}_{1}^{\bar{\zeta}} \cong\left(V_{N}^{\bar{g}}\right)_{1}$.
- For $i \in \mathbb{Z}_{n}$, the conformal weights of the twisted modules $V\left(\zeta^{i}\right)$ and $\bar{V}\left(\bar{\zeta}^{i}\right)$ are the same. Hence, $\zeta$ is also of type 0 and $V^{\zeta}$ satisfies the positivity condition so that the orbifold construction $V^{\text {orb }(\zeta)}$ exists.
- Moreover, $\rho\left(V\left(\zeta^{i}\right)\right)=\rho\left(\bar{V}\left(\bar{\zeta}^{i}\right)\right) \geq 1$ for all $i \neq 0(\bmod n)$.
- Finally, $\operatorname{dim}\left(V_{1}^{\text {orb }(\zeta)}\right)=\operatorname{dim}\left(\bar{V}_{1}^{\text {orb }(\bar{\zeta})}\right)=\operatorname{dim}\left(\left(V_{N}\right)_{1}\right)$ by the dimension formula (see Corollary 3.2).

Remark 6.2. Instead of proving the existence of the inner automorphisms $\zeta$ in $\operatorname{Aut}(V)$ in the above lemma indirectly by mimicking the inverse orbifold automorphisms of the short automorphisms $\bar{g} \in \operatorname{Aut}\left(V_{N}\right)$, one could also explicitly write down an appropriate element $v$ in $\mathcal{H}$, the Cartan subalgebra of $V_{1}$, and then show that $\zeta=\mathrm{e}^{(2 \pi \mathrm{i}) v_{0}} \in \operatorname{Aut}(V)$ has the desired properties (using the methods discussed in, e.g., EMS20b, Section 7), and we have done so for many of the possible Lie algebras $V_{1}$. However, we find the above approach to be more conceptual.

For most of the automorphisms $\zeta$ from Proposition 6.1 it is possible to determine the vertex operator algebra $V^{\operatorname{orb}(\zeta)}$ obtained in the orbifold construction:

Lemma 6.3. For 157 (at least one for each weight-one Lie algebra $\mathfrak{g}$ on Schellekens' list) of the 226 inner automorphisms $\zeta$ defined in Proposition 6.1 one can show that
(5) $V^{\mathrm{orb}(\zeta)} \cong V_{N}$ for some Niemeier lattice $N$.

Proof. Continuing where we left off in the proof of Proposition 6.1, we attempt to identify the orbifold Lie algebra $V_{1}^{\text {orb( }()}$. We know that $\operatorname{dim}\left(V_{1}^{\text {orb( }()}\right)=\operatorname{dim}\left(\left(V_{N}\right)_{1}\right)$ and that $V_{1}^{\zeta} \cong\left(V_{N}^{\bar{g}}\right)_{1}$ is the fixed-point Lie subalgebra of $V_{1}^{\text {orb( } \zeta)}$ under an automorphism of order dividing $n$. The possible fixed-point Lie subalgebras of a semisimple (or reductive) Lie algebra under finite-order automorphisms can be classified using Kac90, Chapter 8. In 157 of the 226 cases we consider, there is only one Lie algebra on Schellekens' list, namely $\left(V_{N}\right)_{1}$, that has dimension $\operatorname{dim}\left(\left(V_{N}\right)_{1}\right)$ and $\left(V_{N}^{\bar{g}}\right)_{1}$ as possible fixed-point Lie subalgebra under an automorphism of order dividing $n$. Hence $V_{1}^{\text {orb }(\zeta)} \cong\left(V_{N}\right)_{1}$.

Then, since the rank of $V_{1}^{\operatorname{orb}(\zeta)} \cong\left(V_{N}\right)_{1}$ equals the central charge $24, V^{\operatorname{orb}(\zeta)}$ is isomorphic to a lattice vertex operator algebra by [DM04b], and $V^{\operatorname{orb}(\zeta)} \cong V_{N}$ since the Niemeier lattices are uniquely determined by their root systems.

We now consider the inverse orbifold automorphism $g \in \operatorname{Aut}\left(V_{N}\right)$ corresponding to one of the inner automorphisms $\zeta \in \operatorname{Aut}(V)$. In all cases we can show that $g$ must be algebraically conjugate to the automorphism $\bar{g}$ whose inverse orbifold automorphism $\bar{\zeta}$ was mimicked by $\zeta$.

Lemma 6.4. All of the 226 short automorphisms $\bar{g}$ of the Niemeier lattice vertex operator algebras $V_{N}$ are uniquely specified up to algebraic conjugacy in $\operatorname{Aut}\left(V_{N}\right)$ by the following six properties:
(1) the order $n$ of $\bar{g}$,
(2) $\bar{g}$ having type 0 ,
(3) the extremality of $\bar{g}$,
(4) the fixed-point Lie subalgebra $\left(V_{N}^{\bar{g}}\right)_{1}$,
(5) the orbifold Lie algebra $\left(V_{N}^{\operatorname{orb}(\bar{g})}\right)_{1}$ and
(6) the dimension of the eigenspace $\left\{v \in\left(V_{N}\right)_{1} \mid \bar{g} v=\mathrm{e}^{(2 \pi \mathrm{i}) 1 / n} v\right\}$.

Proof. Let $\bar{g}$ be a short automorphism of a Niemeier lattice vertex operator algebra $V_{N}$ and let $g$ be an automorphism of the same vertex operator algebra sharing properties (1) to (6) with $\bar{g}$. Note that $g$ is a generalised deep hole by properties (2) to (5).

First, we show that $g$ is again a short automorphism. For this, by Theorem 5.10 it suffices to show that the order of the projection of $g$ to $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ also has order $n$. If $N$ is the Leech lattice $\Lambda$, it was shown in MS19 that any generalised deep hole must project to one of the 11 conjugacy classes in $\operatorname{Aut}\left(V_{\Lambda}\right) / K \cong \mathrm{O}(\Lambda)$ listed in Table 1. Since the rank of the fixed-point Lie subalgebra $\left(V_{N}^{g}\right)_{1}$ is determined by the projection to $H$, which moreover must have order dividing $n$, the assertion follows by inspecting the 11 possible Frame shapes.

Then, let $N$ be a Niemeier lattice other than the Leech lattice. Again, by inspecting the (up to) 25 possible Frame shapes that can occur in $\operatorname{Aut}\left(V_{N}\right) / K \cong H$ (see Table 3 in the appendix), we conclude that the order of the projection of $g$ to $H$ must have order $n$ except for possibly the Frame shape $3^{8}$ if $n=6$ and $\operatorname{rk}\left(\left(V_{N}^{g}\right)_{1}\right)=8$ and $4^{6}$ if $n=8$ and $\operatorname{rk}\left(\left(V_{N}^{g}\right)_{1}\right)=6$. By an explicit classification like in the proof of Theorem 5.10 the existence of these spurious cases can be ruled out.

Knowing that $g$ is again a short automorphism, we verify that every short automorphism $\bar{g}$ is unique up to algebraic conjugacy amongst all the short automorphisms with properties (1) to (6). Hence, $g$ is algebraically conjugate to $\bar{g}$.

Finally, we combine the above results to give a uniform proof of the uniqueness statement in Theorem 4.1.

Theorem 6.5 (Uniform Uniqueness). Let $\mathfrak{g} \neq\{0\}$ be a Lie algebra on Schellekens' list. Then there is a Niemeier lattice $N$ and a short automorphism $\bar{g} \in \operatorname{Aut}\left(V_{N}\right)$ such that any strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \cong \mathfrak{g}$ satisfies $V \cong V_{N}^{\operatorname{orb}(\bar{g})}$. In particular, the vertex operator algebra structure of $V$ is uniquely determined by the Lie algebra structure of $V_{1}$.

In total we find 157 such short automorphisms $\bar{g}$, at least one for each of the 70 non-zero Lie algebras $\mathfrak{g}$.

Proof. By Proposition 6.1 and Lemma 6.3 there are 157 inner automorphisms $\zeta$, at least one for each Lie algebra $\mathfrak{g}$ and defined on any strongly rational, holomorphic vertex operator algebra $V$ of central charge 24 with $V_{1} \cong \mathfrak{g}$, whose orbifold constructions are isomorphic to a Niemeier lattice vertex operator algebra $V_{N}$. Each $\zeta \in \operatorname{Aut}(V)$ was defined to resemble the inverse orbifold automor$\operatorname{phism} \bar{\zeta} \in \operatorname{Aut}\left(V_{N}^{\operatorname{orb}(\bar{g})}\right)$ corresponding to a short automorphism $\bar{g} \in \operatorname{Aut}\left(V_{N}\right)$ with $\left(V_{N}^{\operatorname{orb}(\bar{g})}\right)_{1} \cong \mathfrak{g}$.

We then consider the inverse orbifold automorphism $g \in \operatorname{Aut}\left(V_{N}\right)$ corresponding to $\zeta$ so that $V_{N}^{\operatorname{orb}(g)}=V$. By construction of $\zeta$, the automorphism $g$ shares with the corresponding short automorphism $\bar{g} \in \operatorname{Aut}\left(V_{N}\right)$ properties (1) to (6) listed in Lemma 6.4 Indeed, all of these properties follow from corresponding common properties of $\zeta$ and $\bar{\zeta}$.

We conclude that the so obtained 157 automorphisms $g$ of the Niemeier lattice vertex operator algebras $V_{N}$ are algebraically conjugate to $\bar{g}$ in $\operatorname{Aut}\left(V_{N}\right)$ so that the vertex operator algebra $V=V_{N}^{\mathrm{orb}(g)}$ is isomorphic to $V_{N}^{\mathrm{orb}(\bar{g})}$.

## Appendix. List of Short Orbifold Constructions

For the reader's convenience, in Table 3 we list the algebraic conjugacy classes of outer automorphisms in $H=\mathrm{O}(N) / W$ for the Niemeier lattices $N$.

If $N$ is not the Leech lattice, we realise $H$ as the subgroup $H_{\Delta} \subseteq \mathrm{O}(N)$ of the lattice automorphisms fixing a choice $\Delta$ of the simple roots of the root system of $N$. Then the Frame shapes of the elements of $H_{\Delta}$ have only non-negative exponents and also describe how $H$ permutes the choice of simple roots. A total of 25 different Frame shapes appear.

For the Leech lattice $\Lambda$ the root system is empty so that $H=\mathrm{O}(\Lambda)$. In total, $\mathrm{O}(\Lambda)$ has 70 algebraic conjugacy classes $\nu$ with $\operatorname{rk}\left(\Lambda^{\nu}\right)>0$ but, for the sake of brevity, in the table we restrict to those automorphisms whose Frame shapes have only non-negative exponents. Then, the same 25 Frame shapes appear (see Section 5.2.

Table 4 to Table 14 list the orbifold constructions associated with the 226 short automorphisms of the Niemeier lattice vertex operator algebras $V_{N}$ (see Theorem 5.4.

For each of the 11 genera of orbit lattices $L_{\mathfrak{g}}$, each corresponding to a conjugacy class in $\mathrm{O}(\Lambda)$ (see Table 1p, the rows represent the strongly rational, holomorphic vertex operator algebras $V$ of central charge 24 whose associated orbit lattice $L_{\mathfrak{g}}$, $\mathfrak{g}=V_{1}$, is in the selected genus (more than one vertex operator algebra for a given isomorphism class $L_{\mathfrak{g}}$ in genera D and J ). We list the number of the corresponding entry in [Höh17] and [Sch93] and the isomorphism type of the Lie algebra $\mathfrak{g}=V_{1}$ with the levels $k_{i}$ of $\left\langle V_{1}\right\rangle \cong L_{\hat{\mathfrak{g}}_{1}}\left(k_{1}, 0\right) \otimes \cdots \otimes L_{\hat{\mathfrak{g}}_{s}}\left(k_{s}, 0\right)$ separated by a comma.

The columns represent the different algebraic conjugacy classes $\nu$ in the outer automorphism groups $\operatorname{Aut}\left(V_{N}\right) / K \cong H \subseteq \mathrm{O}(N)$ for all Niemeier lattices $N$ (denoted by the number of the corresponding entry in Höh17] and by the root system) with the same Frame shape as the class in $\mathrm{O}(\Lambda)$.

The entry for a pair $(V, \nu)$ is the number of algebraic conjugacy classes of short automorphisms projecting to $\nu$ such that $V_{N}^{\operatorname{orb}(g)} \cong V$.

TABLE 3. Algebraic conjugacy classes of outer automorphisms of the Niemeier lattices $N$. For the Leech lattice only those whose Frame shapes have only non-negative exponents are listed.

| ${ }^{\text {Frame Shp. }}$ | A1 | A2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 | A11 A12 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $D_{24}$ | $D_{16} E_{8}$ | $E_{8}^{3}$ | $A_{24}$ |  | $A_{17} E_{7}$ | $D_{10} E_{7}^{2}$ | $A_{15} D_{9}$ | $D_{8}^{3}$ |  | $A_{11} D_{7} E_{6}$ | $E_{6}^{4}$ |  |
| A $1^{24}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |  |
| B $\quad 1{ }^{8} 2^{8}$ | . | . | 1 | . | . | 1 | 1 | 1 | 1 | . | 1 | 2 |  |
| C $\quad 1^{6} 3^{6}$ | . | . | . |  | . | . | . | . | . | . | . | 1 |  |
| D $2^{12}$ | . | . | . | 1 | 1 | . | . | . | . | 1 | . | . |  |
| E $1^{4} 2^{2} 4^{4}$ | . | . | . | . | . |  | . | . | . | . | . | . |  |
| F $\quad 1{ }^{4} 5^{4}$ | . | . | . | . | . | . | . | . | . | - | . | . |  |
| G $1^{2} 2^{2} 3^{2} 6^{2}$ | . | . | . | . | . | . | . | . | . | . | . | 1 |  |
| H $\quad 1{ }^{3} 7^{3}$ |  | . | . | . | . |  | . | . | . | . | . | . |  |
| I $1^{2} 2^{1} 4^{1} 8^{2}$ | . | . | . | . | . |  | . | . | . | . | . | . |  |
| J $\quad 2^{3} 6^{3}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| K $\quad 2^{2} 10^{2}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| $3^{8}$ | . | . | 1 | . | . | . | . | . | 1 |  | . |  |  |
| $2^{4} 4^{4}$ | . | . | . | . | . | . | . | . | . | . | . | 1 |  |
| $4^{6}$ | . | . | . | . | . | . | - | . | . | 1 | . | . |  |
| $4^{2} 8^{2}$ | . | . | . | . | . | . | . | . | . | . | . | 1 |  |
| $1^{2} 11^{2}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| $6^{4}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| $2^{1} 4^{1} 6^{1} 12^{1}$ |  | . | . | . |  |  | . | . | . | . | . |  |  |
| $1^{1} 2^{1} 7^{1} 14^{1}$ | . | . | . | . | . |  | . | . | . | . | . |  |  |
| $1^{1} 3^{1} 5^{1} 15^{1}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| $3^{1} 21^{1}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| $1^{1} 23^{1}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| $12^{2}$ |  | . | . | . | . | . | . | . | . | . | . |  |  |
| $4^{1} 20^{1}$ | . | . | . | . | . | . | . | . | . | . | . |  |  |
| $2^{1} 22^{1}$ | . | . | . | . | . | . | . | . | . | . | . | . |  |
| No. (A-K) | 1 | 1 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 2 | 5 |  |
| No. (all) | 1 | 1 | 3 | 2 | 2 | 2 | 2 | 2 | 3 | 3 | 2 | 7 |  |
| $N$ | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A20 | A21 | A22 | A23 | A24 |  |
| Frame Shp. | $A_{9}^{2} D_{6}$ | $D_{6}^{4}$ |  | $A_{7}^{2} D_{5}^{2}$ | $A_{6}^{4}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $A_{4}^{6}$ |  |  | $A_{1}^{24}$ | $\Lambda$ | No. |
| A $1^{24}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 24 |
| B $1^{8} 2^{8}$ | 1 | 1 | 1 | 3 |  | 2 | 2 | 1 | 2 | 1 | 1 | 1 | 24 |
| C $\quad 1^{6} 3^{6}$ | . | 1 | . | . | 1 | 1 | 2 | . | 1 | 1 | 1 | 1 | 10 |
| D $2^{12}$ | . | 1 | 2 | . | 1 | . | 1 | 2 | 1 | 2 | 1 | 1 | 15 |
| E $1^{4} 2^{2} 4^{4}$ | 1 | . | . | . | . | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 8 |
| F $\quad 1{ }^{4} 5^{4}$ | . | . | . | . | . | . | 1 | 1 | . | 1 | 1 | 1 | 5 |
| G $1^{2} 2^{2} 3^{2} 6^{2}$ | . | . | . | . | . | 1 | 2 | . | 1 | 1 | 1 | 1 | 8 |
| H $\quad 1{ }^{3} 7^{3}$ | . | . | . | . | . | . | . | . | 1 | . | 1 | 1 | 3 |
| I $1^{2} 2^{1} 4^{1} 8^{2}$ | . | . | . | . | . | 1 | . | . | . | 1 | 1 | 1 | 4 |
| J $\quad 2^{3} 6^{3}$ | . | . | . | . | 1 | . | . | . | 1 | 2 | . | 1 | 5 |
| K $\quad 2^{2} 10^{2}$ | . | . | . | . | . | . | . | 1 | . | 1 | 1 | 1 | 4 |
| $3^{8}$ | . | . | 1 | . | . | . | 1 | 1 | . | 1 | 1 | 1 | 8 |
| $2^{4} 4^{4}$ | . |  | . | 1 | . | . | 1 | . | 1 | 1 | 1 | 1 | 7 |
| $4^{6}$ | . | 1 | . | . | 1 | . | . | 1 | 1 | 1 | 1 | 1 | 8 |
| $4^{2} 8^{2}$ | . | . | . | . | . | . | . | . | 1 | 1 | . | 1 | 4 |
| $1^{2} 11^{2}$ | . | . |  | . | . | . | . | . | . | 1 | 1 | 1 | 3 |
| $6^{4}$ | . | . | 1 | . | . | . | 1 | 1 | . | 1 | 1 | 1 | 6 |
| $2^{1} 4^{1} 6^{1} 12^{1}$ | . | . | . | . | . | . | 1 | . | . |  | 1 | 1 | 3 |
| $1^{1} 2^{1} 7^{1} 14^{1}$ | . | . | . | . | . | . | . | . | 1 | . | 1 | 1 | 3 |
| $1^{1} 3^{1} 5^{1} 15^{1}$ | . | . | . | . | . | . | 1 | . | . | . | 1 | 1 | 3 |
| $3^{1} 21^{1}$ | . | . | . | . | . | . | . | . | . |  | 1 | 1 | 2 |
| $1^{1} 23^{1}$ | . | . | . | . | . | . | . | . | . | . | 1 | 1 | 2 |
| $12^{2}$ | . | . | . | . | . | . | - | 1 | . | 1 | 1 | 1 | 4 |
| $4^{1} 20^{1}$ | . | . | . | . | . | . | . | . | . | 1 | . | 1 | 2 |
| $2^{1} 22^{1}$ | . | . | . | . | . | . | . | . | . | 1 | . | 1 | 2 |
| No. (A-K) | 3 | 4 | 4 | 4 | 4 | 7 | 10 | 7 | 9 | 12 | 10 | 11 | 110 |
| No. (all) | 3 | 5 | 6 | 5 | 5 | 7 | 15 | 11 | 13 | 21 | 21 | 25 | 167 |

Table 4. Short orbifold constructions for genus A (124).

| $V_{N}^{\text {orb }(g)}$ |  | $V_{N}$ | A1 | A 2 | A3 | A4 | A5 | A6 | A7 | A8 | A9 | A10 | A11 | A12 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $D_{16} E_{8}$ | $E_{8}^{3}$ | $A_{24}$ | $D_{12}^{2}$ | $A_{17} E_{7}$ | $D_{10} E_{7}^{2}$ | $A_{15} D_{9}$ | $D_{8}^{3}$ | $A_{12}^{2}$ | $A_{11} D_{7} E_{6}$ | $E_{6}^{4}$ |
| A1 | 70 |  | $D_{24,1}$ | 1 |  | . |  |  | . | . | . | . | . |  | . |
| A2 | 69 | $D_{16,1} E_{8,1}$ | . | 1 | . | . | . | . | . | . | . | . | . | . |
| A3 | 68 | $E_{8,1}^{3}$ | . | . | 1 | . | . | . | . | . | . | . | . | . |
| A4 | 67 | $A_{24,1}$ | . | . |  | 1 |  | . | . |  | . | . | - | . |
| A5 | 66 | $D_{12,1}^{2}$ | . | . | . | . | 1 | . | . | . | . | . | . | . |
| A6 | 65 | $A_{17,1} E_{7,1}$ | . | . | . | . | . | 1 | - | . | . | . | . | . |
| A7 | 64 | $D_{10,1} E_{7,1}^{2}$ | . | . | . | . | . | . | 1 | . | . | . | . | . |
| A8 | 63 | $A_{15,1} D_{9,1}$ | . | . | . | . | . | . | . | 1 | . | . | . | . |
| A9 | 61 | $D_{8,1}^{3}$ | . | . | . | . | . | . | . | . | 1 | . | - | . |
| A10 | 60 | $A_{12,1}^{2}$ | . | . | . | . | . | . | - | . | . | 1 | . | . |
| A11 | 59 | $A_{11,1} D_{7,1} E_{6,1}$ | . | . | . | . | . | . | . | . | . | . | 1 | . |
| A12 | 58 | $E_{6,1}^{4}$ | . | . | . | . | . | . | . | . | . | . | - | 1 |
| A13 | 55 | $A_{9,1}^{2} D_{6,1}$ | . | . | . | . | . | . | . | . | . | . | - | . |
| A14 | 54 | $D_{6,1}^{4}$ |  | . | . | . | . | . | . |  |  | . |  | . |
| A15 | 51 | $A_{8,1}^{3}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| A16 | 49 | $A_{7,1}^{2} D_{5,1}^{2}$ | . | . | . | . | . | . | . | . | . | . | - | . |
| A17 | 46 | $A_{6,1}^{4}$ | . | . | . | . | . | . | . | . |  | . |  | . |
| A18 | 43 | $A_{5,1}^{4} D_{4,1}$ | . | . | . | . | . | . | . | - | . | . | - | . |
| A19 | 42 | $D_{4,1}^{6}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| A20 | 37 | $A_{4,1}^{6}$ | . | . | . | . | . | . | . |  |  | . |  | . |
| A21 | 30 | $A_{3,1}^{8}$ | . | . | . | . | . | . | . | . |  | . |  | . |
| A22 | 24 | $A_{2,1}^{12}$ | . | . | . | . | . | . | . | . |  | . |  | . |
| A23 | 15 | $A_{1,1}^{24}$ | . | . | . | . | . | . | . |  |  |  |  | . |
| A24 | 1 | $\Lambda$ | . | . | . | . | . | . | - |  |  | . | . | . |
|  |  | $V_{N}$ | A13 | A14 | A15 | A16 | A17 | A18 | A19 | A 20 | A21 | A 22 | A23 | A24 |
| $V_{N}^{\text {orb }}$ | (g) |  | $A_{9}^{2} D_{6}$ | $D_{6}^{4}$ | $A_{8}^{3}$ | $A_{7}^{2} D_{5}^{2}$ | $A_{6}^{4}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $A_{4}^{6}$ | $A_{3}^{8}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| A1 | 70 | $D_{24,1}$ | . | . |  | . |  | . | . |  |  | . | . | . |
| A2 | 69 | $D_{16,1} E_{8,1}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| A3 | 68 | $E_{8,1}^{3}$ | . | . | . | . | . | . | . | - | . | . | . | . |
| A4 | 67 | $A_{24,1}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| A5 | 66 | $D_{12,1}^{2}$ | . | . | . | . | . | . | . | . | . | - | . | . |
| A6 | 65 | $A_{17,1} E_{7,1}$ | . | . | . | . | . | . | . | . | . | - | . | . |
| A7 | 64 | $D_{10,1} E_{7,1}^{2}$ | . | . | . | . | . | . | . | - | . | . | - | . |
| A8 | 63 | $A_{15,1} D_{9,1}$ | . | . | . | . | . | . | . | . | . | . | . | . |
| A9 | 61 | $D_{8,1}^{3}$ | . | . | . | . | . | . | . | . | . | - | - | . |
| A10 | 60 | $A_{12,1}^{2}$ | . | . | . | - | . | - | . | - | . | - | - | . |
| A11 | 59 | $A_{11,1} D_{7,1} E_{6,1}$ | . | . | - | . | . | . | . | . | . | . | . | . |
| A12 | 58 | $E_{6,1}^{4}$ | . | . | . | . | . | . | . | . | . | . | - | . |
| A13 | 55 | $A_{9,1}^{2} D_{6,1}$ | 1 | . | . | . | . | . | . | . | . | . | . | . |
| A14 | 54 | $D_{6,1}^{4}$ | . | 1 | . | . | . | . | . | . | . | . | - | . |
| A15 | 51 | $A_{8,1}^{3}$ | . | . | 1 | . | . | . | . | . | . | . | . | . |
| A16 | 49 | $A_{7,1}^{2} D_{5,1}^{2}$ | . | . | . | 1 | . | . | - | . | . | . | . | . |
| A17 | 46 | $A_{6,1}^{4}$ | . | . | . | . | 1 | . | . | . | . | . | . | . |
| A18 | 43 | $A_{5,1}^{4} D_{4,1}$ | . | . | . | . | . | 1 | . | . | . | . | . | . |
| A19 | 42 | $D_{4,1}^{6}$ | . | - | - | - | . | . | 1 | . | . | - | . | - |
| A20 | 37 | $A_{4,1}^{6}$ | . | . | . | . | . | . | . | 1 | . | . | . | . |
| A21 | 30 | $A_{3,1}^{8}$ | . | . | . | . |  | . | . |  | 1 | . | . | . |
| A22 | 24 | $A_{2,1}^{12}$ | . | . | - | - | . | . | . | . |  | 1 | . | . |
| A23 | 15 | $A_{1,1}^{24}$ | . | . | . | . | . | . | . | . | . | . | 1 | . |
| A24 | 1 | $\Lambda$ | . | . | . | . | . | . | . | . | . | . | . | 1 |

Table 5. Short orbifold constructions for genus B $\left(1^{8} 2^{8}\right)$.

| $V_{N}^{\text {orb }(g)}$ |  |  | A3 | A6 | A7 | A8 | A9 | A11 | A12 |  | A13 | A14 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $E_{8}^{3} A_{17} E_{7} D_{10} E_{7}^{2} A_{15} D_{9}$ | $D_{8}^{3} A_{11} D_{7} E_{6}$ |  |  | $E_{6}^{4}$ |  | $D_{6}^{4}$ | $A_{8}^{3} A_{7}^{2} D_{5}^{2}$ |  |
| B1 | 62 |  |  |  | $B_{8,1} E_{8,2}$ |  | 1 |  | 1 | 1 |  | . |  |  |
| B2 | 56 | $B_{6,1} C_{10,1}$ |  | 1 |  | . | 1 | . | 1 |  |  | 1 | . |  |  |
| B3 | 52 | $C_{8,1} F_{4,1}^{2}$ | . | . | . | 1 | . | 1 | 1 |  | . | . |  | 1 |
| B4 | 53 | $B_{5,1} E_{7,2} F_{4,1}$ | . |  | 1 | . |  | 1 |  | 1 | . | 1 |  |  |
| B5 | 50 | $A_{7,1} D_{9,2}$ | . | 1 | . |  | 1 | . |  | . |  | . | 1 |  |
| B6 | 47 | $B_{4,1}^{2} D_{8,2}$ | . | . | . | 1 | 1 | . |  |  |  | 1 |  | 1 |
| B7 | 48 | $B_{4,1} C_{6,1}^{2}$ | . |  | . | . | . | 1 |  |  | 1 | . |  | 1 |
| B8 | 44 | $A_{5,1} C_{5,1} E_{6,2}$ | . | . | . |  | . | . |  | 1 | 1 | . |  |  |
| B9 | 40 | $A_{4,1} A_{9,2} B_{3,1}$ | . | . | . | . |  | . |  | . | . | . | 1 |  |
| B10 | 39 | $B_{3,1}^{2} C_{4,1} D_{6,2}$ | . | . | . | . | . | 1 | . | . | . | 1 | . | 1 |
| B11 | 38 | $C_{4,1}^{4}$ | . | . | . |  |  |  | 1 |  |  | . |  | 1 |
| B12 | 33 | $A_{3,1} A_{7,2} C_{3,1}^{2}$ | . | . | . |  |  | . | . | . | $\cdot$ | . |  |  |
| B13 | 31 | $A_{3,1}^{2} D_{5,2}^{2}$ | . | . | . | . | . | . | . | . | 1 | . |  |  |
| B14 | 26 | $A_{2,1}^{2} A_{5,2}^{2} B_{2,1}$ | . | . | . | . |  | . |  | . | . | . |  |  |
| B15 | 25 | $B_{2,1}^{4} D_{4,2}^{2}$ |  | . | . |  |  | . |  | . | . | . | . | 1 |
| B16 | 16 | $A_{1,1}^{4,6} A_{3,2}^{4}$ | . | . | . | . |  | . |  | . | . | . |  |  |
| B17 |  | $A_{1,2}^{16}$ |  |  |  |  |  | . |  | . | . |  |  |  |


| $\frac{V_{N}^{\text {orb }(g)}}{}$ |  |  | A16 | A16 | $\begin{array}{r} \mathrm{A} 18 \\ \hline A_{5}^{4} D_{4} \\ \hline \end{array}$ | A18 A19 |  | A19 A20 A21 |  |  | A21 A22 A23 |  | A 24$\Lambda$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $A_{7}^{2} D_{5}^{2}$ | $A_{7}^{2} D_{5}^{2}$ | $A_{5}^{4} D_{4}$ |  |  |  | $A_{4}^{6}$ | $A_{3}^{8}$ | $A_{3}^{8} A_{2}^{12} A_{1}^{24}$ |  |  |
|  |  | $B_{8,1} E_{8,2}$ | . |  |  |  |  |  |  |  | . |  |  |
| B2 | 56 |  | $B_{6,1} C_{10,1}$ | . |  |  |  |  |  |  |  | . | . |  |
| B3 | 52 | $C_{8,1} F_{4,1}^{2}$ |  |  |  |  |  | . | . | . | . | . |  |
| B4 | 53 | $B_{5,1} E_{7,2} F_{4,1}$ |  | 1 |  | . |  | . | . | . | . | . |  |
| B5 | 50 | $A_{7,1} D_{9,2}$ | 1 |  |  | . |  | . |  | . | . | . |  |
| B6 | 47 | $B_{4,1}^{2} D_{8,2}$ | . | 1 |  |  |  | 1 |  | . | . | . |  |
| B7 | 48 | $B_{4,1} C_{6,1}^{2}$ | . | . | 1 | . | . | . |  | . | . | . |  |
| B8 | 44 | $A_{5,1} C_{5,1} E_{6,2}$ | 1 | . |  | 1 | . | . | . | . | . | . |  |
| B9 | 40 | $A_{4,1} A_{9,2} B_{3,1}$ | . | 1 |  | 1 |  | . | 1 | . | . | . |  |
| B10 | 39 | $B_{3,1}^{2} C_{4,1} D_{6,2}$ | 1 | . |  | 1 | 1 | . | . | . | 1 | . |  |
| B11 | 38 | $C_{4,1}^{4}$ | . |  | 1 |  | . | . |  | 1 | . | . |  |
| B12 | 33 | $A_{3,1} A_{7,2} C_{3,1}^{2}$ | . | 1 | 1 | 1 | . | . |  | . | 1 | . |  |
| B13 | 31 | $A_{3,1}^{2} D_{5,2}^{2}$ | 1 | . | . | . | . | 1 | 1 | . | 1 | . |  |
| B14 | 26 | $A_{2,1}^{2} A_{5,2}^{2} B_{2,1}$ | . |  | . | 1 | . | . | 1 | . | 1 | 1 |  |
| B15 | 25 | $B_{2,1}^{4} D_{4,2}^{2}$ | . |  |  |  | 1 | 1 |  | 1 | 1 | 1 |  |
| B16 | 16 | $A_{1,1}^{4} A_{3,2}^{4}$ | . |  | 1 |  |  | . |  | . | 1 | 1 |  |
| B17 |  | $A_{1,2}^{16}$ |  |  |  |  |  |  |  | 1 | . | . | 1 |

Table 6. Short orbifold constructions for genus C $\left(1^{6} 3^{6}\right)$.

| $V_{N}^{\text {orb (g) }}$ |  | $V_{N}$ | A12 | A14 | A17 | A18 | A19 | A19 | A21 | A22 | A23 | A24 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $E_{6}^{4}$ | $D_{6}^{4}$ | $A_{6}^{4}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $D_{4}^{6}$ | $A_{3}^{8}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| C1 | 45 |  | $A_{5,1} E_{7,3}$ | 1 | 1 | 1 | 1 | . | . | . |  |  |  |
| C2 | 34 | $A_{3,1} D_{7,3} G_{2,1}$ | . | 1 | 1 | 1 |  | 1 | 1 |  |  |  |
| C3 | 32 | $E_{6,3} G_{2,1}^{3}$ | 1 | . | . | 1 | 1 | 1 |  | 1 |  |  |
| C4 | 27 | $A_{2,1}^{2} A_{8,3}$ | . |  | 1 | 1 |  |  | 1 | 1 |  |  |
| C5 | 17 | $A_{1,1}^{3} A_{5,3} D_{4,3}$ |  |  |  | 1 |  | 1 | 1 | 1 | 1 |  |
| C6 | 6 | $A_{2,3}^{6}$ |  |  |  | . | 1 | . | . | 1 | 1 | 1 |

TABLE 7. Short orbifold constructions for genus D (2 ${ }^{12}$ ).

| $V_{N}^{\text {orb }(g)}$ |  |  | A4 A10 A15 A17 A20 A22 |  |  |  | A5 A14 A19 A21 A23 A24 |  |  |  |  |  |  | A15 A20 A22 |  |  |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  |  | $A_{24} A_{12}^{2}$ | $A_{8}^{3}$ | $A_{6}^{4}$ | $A_{4}^{6} A_{2}^{12}$ | $D_{12}^{2}$ |  | $D_{6}^{4}$ | $D_{4}^{6}$ |  | $A_{1}^{24}$ | $\Lambda$ | $A_{8}^{3}$ | $A_{4}^{6}$ | ${ }_{4}^{6} A_{2}^{12}$ |
| D1a | 57 | $B_{12,2}$ | 1 | . |  |  |  | 1 | . | . | . |  |  |  |  |  |
| D1b | 41 | $B_{6,2}^{2}$ | 1 | . | . | . |  | . | 1 | . | . |  |  |  |  |  |
| D1c | 29 | $B_{4,2}^{3}$ | . . | 1 | . | . |  | . | . | 1 | . | . |  | . |  |  |
| D1d | 23 | $B_{3,2}^{4}$ | . . |  | 1 |  |  | . | . | . | 1 |  |  |  |  | . |
| D1e | 12 | $B_{2,2}^{6}$ | . . |  |  | 1 |  | . | . | . | . | 1 |  |  |  | . |
| D1f | 2 | $A_{1,4}^{12}$ | . | . | . | 1 |  | . | . | . | . |  | 1 | . |  | . |
| D2a | 36 | $A_{8,2} F_{4,2}$ | . . | - | . | . |  | . | . | . |  |  |  | 1 |  |  |
| D2b | 22 | $A_{4,2}^{2} C_{4,2}$ | . . | . | . |  |  | . | . | . |  |  |  |  |  | 1 |
| D2c | 13 | $A_{2,2}^{4} D_{4,4}$ |  |  |  |  |  | . | - |  |  |  |  |  |  | - 1 |

TABLE 8. Short orbifold constructions for genus $\mathrm{E}\left(1^{4} 2^{2} 4^{4}\right)$.

| $V_{N}^{\text {orb }(g)}$ |  | $V_{N}$ | A13 | A18 | A19 | A20 | A21 | A22 | A23 | A24 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $A_{9}^{2} D_{6}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $A_{4}^{6}$ | $A_{3}^{8}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| E1 | 35 |  | $A_{3,1} C_{7,2}$ | 1 | 2 | . | 1 | 1 |  |  |  |
| E2 | 28 | $A_{2,1} B_{2,1} E_{6,4}$ | 1 | . | 1 | 2 | 1 | 1 |  |  |
| E3 | 18 | $A_{1,1}^{3} A_{7,4}$ | . | 1 | . | 1 | 1 | 2 | 1 |  |
| E4 | 19 | $A_{1,1}^{2} C_{3,2} D_{5,4}$ |  | 1 | 1 | 1 | 3 | 1 | 2 |  |
| E5 | 7 | $A_{1,2} A_{3,4}^{3}$ |  |  | . |  | 1 | 1 | 2 | 1 |

Table 9. Short orbifold constructions for genus F $\left(1^{4} 5^{4}\right)$.

| $V_{N}^{\text {orb }(g)}$ |  | $V_{N}$ | A19 | A20 | A22 | A23 | A24 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $D_{4}^{6}$ | $A_{4}^{6}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| F1 | 20 |  | $A_{1,1}^{2} D_{6,5}$ | 1 | 1 | 1 | 1 |  |
| F2 | 9 | $A_{4,5}^{2}$ |  | 1 | 1 | 1 | 1 |

Table 10. Short orbifold constructions for genus $G\left(1^{2} 2^{2} 3^{2} 6^{2}\right)$.

| $V_{N}^{\operatorname{orb}(g)}$ |  | $V_{N}$ | A12 | A18 | A19 | A19 | A21 | A22 | A23 | A24 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $E_{6}^{4}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $D_{4}^{6}$ | $A_{3}^{8}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| G1 | 21 |  | $A_{1,1} C_{5,3} G_{2,2}$ | 1 | 3 | 2 | 1 | 2 | 2 | 2 |  |
| G2 | 8 | $A_{1,2} A_{5,6} B_{2,3}$ |  | 1 | 1 | 1 | 2 | 2 | 5 | 1 |

TABLE 11. Short orbifold constructions for genus $\mathrm{H}\left(1^{3} 7^{3}\right)$.

| $V_{N}^{\text {orb }(g)}$  <br>   |  | $V_{N}$ | A21 | A23 | A24 |
| :--- | :--- | :--- | ---: | ---: | ---: |
| H1 | 11 | $A_{6,7}$ | $A_{3}^{8}$ | $A_{1}^{24}$ | $\Lambda$ |

Table 12. Short orbifold constructions for genus I ( $1^{2} 2^{1} 4^{1} 8^{2}$ ).

| $V_{N}$ |  | A18 | A22 | A23 | A24 |  |
| :--- | :--- | ---: | ---: | ---: | ---: | ---: |
| $V_{N}^{\text {orb }(g)}$ |  | $A_{5}^{4} D_{4}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |  |
| I1 | 10 | $A_{1,2} D_{5,8}$ | 1 | 2 | 2 | 1 |

TABLE 13. Short orbifold constructions for genus J $\left(2^{3} 6^{3}\right)$.

| $V_{N}^{\text {orb }(g)}$ |  | $V_{N}$ | A17 | A22 | A21 | A24 | A22 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  |  | $A_{6}^{4}$ | $A_{2}^{12}$ | $A_{3}^{8}$ | $\Lambda$ | $A_{2}^{12}$ |
| J1a | 14 |  | $A_{2,2} F_{4,6}$ | 1 |  | 1 |  | 1 |
| J1b | 3 | $A_{2,6} D_{4,12}$ |  | 1 |  | 1 | 1 |

Table 14. Short orbifold constructions for genus $\mathrm{K}\left(2^{2} 10^{2}\right)$.

|  |  | $V_{N}$ | A 20 | A 22 | A 23 | A 24 |
| :--- | :--- | ---: | ---: | ---: | ---: | ---: |
| $V_{N}^{\mathrm{orb}(g)}$ |  | $A_{4}^{6}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |  |
| K 1 | 4 | $C_{4,10}$ | 1 | 1 | 1 | 1 |

The following five tables list the non-trivial powers of the 226 short automorphisms $g$ of the Niemeier lattice vertex operator algebras $V_{N}$.

For the five genera E, G, I, J and K of orbit lattices corresponding to a Frame shape of an element of composite order we list the non-trivial powers $g^{p}$ with prime exponent $p$.

The rows and columns are labelled as in Table 4 to Table 14 An entry $X i_{1}, \ldots, i_{k}$ for a power $g^{p}$ refers to the elements in the table for the genus associated with $g^{p}$ in the column corresponding to the same Niemeier lattice and the rows labelled by the orbifold constructions $X i_{1}, \ldots, X i_{k}$.

If the same Niemeier lattice $N$ is the label for more than one column for a power $g^{p}$, the extra row named $g^{p}$ identifies the column if there is a possible ambiguity.

Lastly, if there is more than one element $g^{p}$ for an entry specified by $X i_{\nu}$, an extra index $l$ refers to the $l$-th element in that entry (this applies to two cases of $g$ for genus I).

Table 15. Powers $g^{2}$ for genus $\mathrm{E}\left(1^{4} 2^{2} 4^{4}\right)$.

|  | A13 | A18 | A19 | A20 | A21 | A22 | A23 | A24 |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | $A_{9}^{2} D_{6}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $A_{4}^{6}$ | $A_{3}^{8}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| $g^{2}$ | . | 1 | 2 | . | 2 | . | . | . |
| E1 | B7 | B7,12 | . | B9 | B12 | . | . |  |
| E2 | B13 | . | B13 | B13,14 | B13 | B14 | . | . |
| E3 | . | B16 | . | B14 | B16 | B14,16 | B16 | . |
| E4 | . | B12 | B15 | B9 | B12,15,16 | B14 | B15,16 |  |
| E5 |  | . | . | . | B16 | B14 | B16,17 | B17 |

Table 16. Powers $g^{2}$ and $g^{3}$ for genus $G\left(1^{2} 2^{2} 3^{2} 6^{2}\right)$.

|  | A 12 | A 18 | A 19 | A 19 | A 21 | A 22 | A 23 | A 24 |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
|  | $E_{6}^{4}$ | $A_{5}^{4} D_{4}$ | $D_{4}^{6}$ | $D_{4}^{6}$ | $A_{3}^{8}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| $g^{2}$ | $\cdot$ | $\cdot$ | 2 | 1 | $\cdot$ | $\cdot$ | $\cdot$ | $\cdot$ |
| $g^{3}$ | 1 | 1 | 1 | 2 | 1 | $\cdot$ | $\cdot$ | $\cdot$ |
| G 1 | C 3 | $\mathrm{C} 3,3,5$ | $\mathrm{C} 3,5$ | C 3 | $\mathrm{C} 5,5$ | $\mathrm{C} 3,5$ | $\mathrm{C} 5,5$ | $\cdot$ |
|  | B 11 | $\mathrm{~B} 11,16,11$ | $\mathrm{~B} 15,15$ | B 15 | $\mathrm{~B} 11,15$ | $\mathrm{~B} 16,16$ | $\mathrm{~B} 15,16$ | $\cdot$ |
| G 2 | . | C 5 | C 5 | C 6 | $\mathrm{C} 5,5$ | $\mathrm{C} 5,6$ | $\mathrm{C} 5,5,6,6,6$ | C 6 |
|  | . | B 16 | B 15 | B 15 | $\mathrm{~B} 15,17$ | $\mathrm{~B} 16,16$ | $\mathrm{~B} 16,17,15,16,17$ | B 17 |

Table 17. Powers $g^{2}$ for genus I $\left(1^{2} 2^{1} 4^{1} 8^{2}\right)$.

|  | A 18 | A 22 | A 23 | A 24 |
| :--- | :--- | :--- | :--- | :--- |
|  | $A_{5}^{4} D_{4}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| I 1 | E 3 | $\mathrm{E} 3_{2}, 5$ | $\mathrm{E} 3,5_{2}$ | E 5 |

Table 18. Powers $g^{2}$ and $g^{3}$ for genus $\mathrm{J}\left(2^{3} 6^{3}\right)$.

|  | A17 | A22 | A21 | A24 | A22 |
| :--- | :--- | :--- | :--- | :--- | :--- |
|  | $A_{6}^{4}$ | $A_{2}^{12}$ | $A_{3}^{8}$ | $\Lambda$ | $A_{2}^{12}$ |
| J1a | C4 | $\cdot$ | C 4 | $\cdot$ | C 4 |
|  | D1d | $\cdot$ | D1d | $\cdot$ | D2c |
| J1b | $\cdot$ | C6 | $\cdot$ | C6 | C6 |
|  | . | D1f | . | D1f | D2c |

Table 19. Powers $g^{2}$ and $g^{5}$ for genus $\mathrm{K}\left(2^{2} 10^{2}\right)$.

|  | A20 | A22 | A23 | A24 |
| :--- | :--- | :--- | :--- | :--- |
|  | $A_{4}^{6}$ | $A_{2}^{12}$ | $A_{1}^{24}$ | $\Lambda$ |
| K1 | F2 | F2 | F2 | F2 |
|  | D1e | D1f | D1e | D1f |

## References

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[^0]:    ${ }^{\text {a }}$ Kansas State University, Manhattan, KS, United States of America.
    ${ }^{\mathrm{b}}$ Rutgers University, Piscataway, NJ, United States of America.
    ${ }^{\text {c }}$ Research Institute for Mathematical Sciences, Kyoto University, Kyoto, Japan.
    Email: gerald@monstrous-moonshine.de math@moeller-sven.de

