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Systematic Risk in Recovery Rates - An Empirical Analysis of U.S. Corporate Credit Exposures — Source link

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Abstract

This paper presents an analytical and empirical analysis of a parsimonious model framework that accounts for a dependence of bond and bank loan recoveries on systematic risk. We extend the single risk factor model by assuming that the recovery rates also depend on this risk factor and follow a logit–normal distribution. The results are compared with those of two related models, suggested in Frye (2000) and Pykhtin (2003), which pose the assumption of a normal and a log–normal distribution of recovery rates.

We provide estimators of the parameters of the asset value process and their standard errors in closed form. For the parameters of the recovery rate distribution we also provide closed–form solutions of a feasible maximum–likelihood estimator for the three models.

The model parameters are estimated from default frequencies and recovery rates that were extracted from a bond and loan database of Standard&Poor's. We estimate the correlation between recovery rates and the systematic risk factor and determine the impact on economic capital.

Furthermore, the impact of measuring recovery rates from market prices at default and from prices at emergence from default is analysed. As a robustness check for the empirical results of the maximum–likelihood estimation method we also employ a method–of–moments.

Our empirical results indicate that systematic risk is a major factor influencing recovery rates. The calculation of a default–weighted recovery rate without further consideration of this factor may lead to downward–biased estimates of economic capital.

Recovery rates measured from market prices at default are generally lower and more sensitive to changes of the systematic risk factor than are recovery rates determined at emergence from default. The choice between these two measurement methods has a stronger impact on the expected recovery rates and the economic capital than introducing a dependency of recovery rates on systematic risk in the single risk factor model.

Keywords: asset correlation, New Basel Accord, recovery rate, LGD, recovery correlation, single risk factor model

JEL Classification: G 21, G 33, C 13

Non–technical Summary

This paper analyses three credit risk models that account for systematic risk in recovery rates of bonds or loans. The systematic risk is driven by a single unobservable factor, similar to the single risk factor model that was used to derive the risk weight function of the internal ratings based approach in Basel II. The three models differ in the distributional assumption for the recovery rates. The model parameters are estimated from default rates and recovery rates that were extracted from a bond and loan database of Standard&Poor's.

The following main conclusions can be drawn.

- The empirical analyses indicate that systematic risk is a major factor influencing recovery rates of bonds and loans. Ignoring it may lead to downward–biased estimates of economic capital.
- Measuring recovery rates just after default or at emergence from default seems to have a stronger impact on recovery rates and the economic capital than does extending the single risk factor model to capture systematic risk in recovery rates. Recovery rates measured at default are generally lower and more sensitive to changes of the systematic risk factor than are recovery rates at emergence from default.

Nichttechnische Zusammenfassung

Diese Arbeit untersucht drei Kreditrisikomodelle, die ein systematisches Risiko in den Erlösquoten von Anleihen oder Buchkrediten berücksichtigen. Das systematische Risiko wird von einem einzelnen, unbeobachtbaren Faktor getrieben. Die drei Modelle stellen Erweiterungen des Ein–Faktor–Modelles dar, welches für die Risikogewichtsfunktionen des auf internen Ratings basierenden Ansatzes von Basel II verwendet wurde. Sie unterscheiden sich in ihren Verteilungsannahmen für die Erlösquoten. Die Modellparameter werden aus Ausfallraten und Erlösquoten geschätzt, die einer Buchkredit- und Anleihedatenbank von Standard&Poor's entnommen wurden.

Die Untersuchungen liefern die folgenden wesentlichen Ergebnisse:

- Die empirischen Analysen geben Hinweise, dass das systematische Risiko ein wichtiger Einflussfaktor für die Erlösquoten ist. Die Vernachlässigung dieser Risikokomponente kann zu einer Unterschätzung des benötigten Eigenkapitals führen.
- Die Berechnungsmethode für die Erlösquoten, d.h. Marktpreise bei Ausfall oder Erlösquoten am Ende der Abwicklungsphase mittels Diskontierung der erhaltenen Zahlungen, hat einen größeren Einfluss auf die Schätzergebnisse als die Erweiterung des Ein–Faktor–Modelles durch Berücksichtigung des systematischen Risikos. Erlösquoten, die unmittelbar nach dem Ausfallereignis bestimmt werden, sind im Allgemeinen niedriger und reagieren sensitiver auf Änderungen des systematischen Faktors als Erlösquoten, die am Ende der Abwicklungsphase bestimmt werden.

Contents

1. Introduction

The purpose of this paper is to analyse analytically and empirically the dependence of bank loan recoveries on systematic risk that is driven by the business cycle. Furthermore, it explores the impact on the economic capital calculation if systematic risk is ignored.

The new model that is put forward in this paper and two other models which serve as benchmarks build on the assumption of a single systematic risk factor. This assumption has already been widely applied in earlier studies.[1] It also forms the foundation of the risk weight function in the internal ratings based (IRB) approach of the revised Framework (Basel II) that sets out the future minimum capital requirements for banks.[2] The model for the IRB risk weights focuses only on systematic risk in the default rate. It implicitly assumes that default rate and recovery rate are independent. We refer to it as the *classic* one–factor model. The three *extended* models which are analysed in this paper incorporate a non–zero correlation between default rates and recovery rates that is driven by systematic risk. Note that the loss given default (LGD) can be directly transformed into a recovery rate R by the relation $R = 1 - LGD$. For convenience, we prefer the term 'recovery rate' and we use LGD instead only for the calculation of economic capital, following common terminology.

An infinitely granular loan portfolio is a critical assumption for applying the one–factor model to determine economic capital. This assumption will, in principle, be fulfilled for large, internationally well–diversified banks for which the single risk factor can be interpreted as a proxy for the "world business cycle".[3] This justifies the use of Standard&Poor's Credit Pro database, which is geared towards large international firms.

This paper makes the following four contributions: first, it considers different extensions of the classic one–factor model which account additionally for systematic risk in recovery rates.

Second, closed form solutions are provided for the maximum–likelihood estimators of the asset correlation and the probability of default and their standard errors as well as for the parameters of the recovery rate distribution.

[1] See, for example, Schönbucher (2000) and Belkin and Suchower (1998).
[2] See Basel Committee on Banking Supervision (2004).
[3] New empirical evidence of the existence of an international business cycle is provided in Kose et al. (2003).

Third, as an empirical contribution, the correlations of default rates and recovery rates with the systematic risk factor are estimated based on Standard&Poor's Credit Pro database.

Fourth, we explore the consequences for economic capital if the classic one–factor model is applied instead of the extended models that also incorporate systematic risk in recovery rates.

The results in this paper are submitted to several robustness checks. We explore the effect of replacing the assumption of logit–normally distributed recovery rates by a normal and a log–normal distribution. In addition to maximum likelihood, a method–of–moments estimation is also carried out. The estimation results are compared for recovery rates based on market prices shortly after default and recovery rates that are determined at emergence from default or bankruptcy.

This paper builds on the work by Frye (2000) and Pykhtin (2003) who have applied models with systematic risk in recovery rates which are also based on the one–factor assumption. We extend their work by proposing a new model that additionally incorporates the restriction that recovery rates are bounded between 0% and 100%.

The assumption of the extended models that default rates and recovery rates are influenced by the same systematic risk factor has been questioned by Altman et al. (2003). The authors argue that GDP is not significant as a regressor in a multivariate model for recovery rates. In addition to this empirical argument, other factors like collateralisation and seniority are also expected to have an impact on recovery rates, although not necessarily on their correlation with the systematic risk factor. However, our assumption of a single systematic risk factor that drives default rates and recovery rates is motivated primarily by the desire to have a tractable model that incorporates systematic risk in recovery rates. Even this parsimonious model represents an important extension of credit risk models used in practice, for example, CreditMetrics and CreditRisk$^+$,[4] that do not account for this source of risk.

This paper is structured as follows. Section 2 reviews a selection of related literature. In section 3 we set up the model framework and describe the estimation methods. Section 4 is devoted to a description of the database and a descriptive analysis of the data. The empirical results are presented in section 5. This includes estimates of the model parameters, a comparison of different measurement methods for the recovery rates and an analysis of

[4]See Credit Suisse Financial Products (1997) and Gupton et al. (1997).

the expected recovery rates and the sensitivity of the recovery rate to the systematic risk factor. In section 6, implications of considering systematic risk in the determination of economic capital are analysed. Section 7 provides a summary and conclusions.

2. Empirical literature on systematic risk in recovery rates

Recovery rates and their dependence on business cycle effects have been the object of various empirical studies. A selection of recent studies is provided in the following section.

Asarnow and Edwards (1995) carried out a long–term empirical study on recovery rates which covers a time period of 24 years from 1970 to 1993. They analysed 830 commercial and industrial loans and 89 structured loans from Citibank's US borrowers that were classified as doubtful or non–accrual. Loss in the event of default is measured as the shortfall of the contractual cash–flow and subsequently discounted at the contractual lending rate. Asarnow and Edwards find a time–stable non–linear uptrend of this variable that seems to be independent of macroeconomic factors with average values of 34.79% for commercial and industrial loans and 12.75% for structured loans. The relatively small loss for the latter is explained by a closer monitoring of this loan type.

Altman and Brady (2002) and Altman et al. (2003) explore the impact of supply and demand for securities of defaulted companies on recovery rates. Their studies cover the years from 1982 to 2001 with recoveries measured as the first market price after default has occurred. The analysis is performed with an aggregate annual recovery rate as the dependent variable, which is defined as the market–value weighted average recovery of all corporate bond defaults, computed from approximately 1,000 bonds. The authors claim that the performance of the macroeconomy is of secondary importance if another measure of supply such as the aggregate bond default rate is used. For the univariate model incorporating bond default rates as explanatory variables, they find that they can explain about 60% of the variation in average annual recovery rates. However, the inclusion of supply–side indicators, such as the total amount of high yield bonds outstanding for a particular year or the Altman–NYU Salomon Center Index of Defaulted Bonds, increases the explanatory power to nearly 90% of the variation. An open issue is why macroeconomic variables such as GDP growth which have been found to influence aggregate default rates seem to have a much lower explanatory power for recovery rates than the supply–side variables.

A study by Gupton et al. (2000) explores the recovery rates of 181 bank loans to 121 large US borrowers which defaulted in the time interval between 1989 and 2000. Secondary market price quotes of bank loans one month after default are used as a proxy for actual recoveries on defaulted loans. The authors find evidence of a positive correlation between LGD for defaulted senior secured bank loans and senior secured public debt which exists even for different market instruments and non–overlapping sets of defaulters. They infer that economic factors that might influence recoveries have an impact on both kinds of securities. Compared with the results by Asarnow and Edwards (1995), they find that the mean LGD is higher, between 30.5% and 47.9%, depending on the type of debt. They also observe that LGD is determined by several factors other than systematic risk, such as the availability of collateral, the length of the workout period, and the number of outstanding loans to a defaulted borrower.

In Frye (2003), 960 securities are analysed that defaulted between 1983 and 2001. They are separated into 49 debt types according to seniority and other contract characteristics and the LGD is a contemporaneous estimate of the loss to the lender. Frye finds that the LGD is generally higher in high–default years, which indicates the presence of a systematic effect. Separating the time interval into the subsets "bad years" and "good years", he claims that LGD, both on a granular and on an aggregate level, reacts sensitively to the state of the economy. The effect, however, is stronger on the granular level. In addition, bad years seem to have a stronger impact on instruments with lower LGD than on those with higher LGD.

Hu and Perraudin (2002) present evidence for a negative correlation between recovery rates and default rates. Their regression analysis of 958 defaulted bonds between 1971 and 2000 reveals a negative correlation between quarterly recovery rates and default rates over the entire sample period. This negative correlation increases after 1982. After their initial regression analysis, Hu and Perraudin standardise the observed recovery rates, measured as the ratio of the market value of the bonds and the unpaid principal one month after default, to account for the time–variation in the sample of recoveries and obtain a *filtered* set of recoveries. The comparison of the unfiltered and the filtered recovery rates indicates that the actual correlation on the granular level is downward–biased by the additional volatility that results from the fact that the sample evolves over time.

Acharya et al. (2003) study the empirical determinants of the variability of recovery rates. Their analyses are based on Standard&Poor's Credit Pro database with LGD measured in two ways, from prices at default and from prices at emergence. The authors identify the condition of the industry sector of the defaulted firm as an important driver of the recovery rate. Poor industry liquidity affects the recovery at emergence but not the recovery rates measured at default.

Summarising the results of previous studies, there is broad agreement that default rates and the business cycle are correlated. The results for a potential correlation between business cycle indicators and recovery rates are mixed. Whereas Asarnow and Edwards (1995) and Altman and Brady (2002) observe only a weak dependence of recovery rates on macroeconomic variables, the work by Gupton et al. (2000) and Frye (2003) suggests that recovery rates are more closely linked to the business cycle. Apart from a potential influence by the macroeconomy, several contract–specific factors, for example, seniority and collateral, also seem to affect recovery rates. However, the explanatory power of some of the analysed factors, for example, the business sector, is highly controversial.

3. Model and estimation procedure

3.1. Model setup

Two different sets of assumptions have been posed in the literature in order to justify the use of a one–factor model to capture systematic risk in default rates and recovery rates. One framework is based on the assumption of a *homogenous* loan portfolio. A loan portfolio is usually considered as homogenous if the distribution of its loss vector that collects losses of the individual facilities is *exchangeable*, that is invariant under permutations of its components.[5] A more general framework has been suggested by Gordy (2001). He has shown that an economic capital charge for a loan portfolio can still be determined as the sum of the capital charges for single loans if there is only one systematic risk factor *and* if the portfolio is "infinitely granular".

Following Frye (2000) and for ease of presentation, we assume in the following homogeneity of the portfolios under consideration. However, the results can be extended to the more general framework of an infinitely granular portfolio which allows a more general interpretation of our results.

[5] See, for example, Frey and McNeil (2003).

The classic one–factor model that is the basis of the following analyses is well described in the literature.[6] The innovation of this paper is the extension of the model by additionally accounting for systematic risk in recovery rates. We analyse three possible extensions of which two build on the work by Frye (2000) and Pykhtin (2003).

For completeness we summarise the key characteristics of the classic one–factor model. Let A_j denote the innovation in the asset value index of firm j. According to (1) it depends on changes of a systematic risk factor, denoted by X, and a firm–specific (idiosyncratic) risk factor $\breve{Z}_j$:

$$A_j = \sqrt{\rho}\, X + \sqrt{1-\rho}\, \breve{Z}_j. \tag{1}$$

As X and $\breve{Z}_j$ are assumed to be independently standard normally distributed, A_j is also standard normally distributed. By assumption, the $\breve{Z}_j$ are pairwise uncorrelated.

The parameter ρ measures the *asset correlation* between the innovations in the asset values of any pair of firms which equals the square of the correlation with the single systematic risk factor. It is restricted to the interval $[0,\ 1]$ and assumed to be constant for all firms and across all time periods. Conditional on X, the innovations in the asset values of two firms are uncorrelated. The higher ρ is, the stronger is the firm's asset value exposed to fluctuations in the business cycle.

Firm j defaults if and only if its asset value A_j falls below an exogenous threshold. Let PD denote the unconditional probability of default in a certain time horizon of usually one year.[7] Then the default threshold is given by $\Phi^{-1}(PD)$, where Φ^{-1} denotes the inverse of the cumulative distribution function of the standard normal distribution.

The classic one–factor model is extended below by accounting for systematic risk in recovery rates under three different assumptions for the distribution of default rates. All three models have a parsimonious structure which facilitates their implementation. Only the reference model meets the requirement that recovery rates are usually bounded between 0% and 100%.[8]

Following a proposal by Schönbucher[9], the recovery rate is modelled in the first (reference) model as a logit transformation of a normally distributed random variable Y_j. The

[6] See, for example, Gordy (2001), Schönbucher (2000) or Belkin and Suchower (1998).

[7] We can drop the firm–specific index j for the probability of default PD because of the homogeneity assumption for the portfolio.

[8] In practice these bounds can be violated to some extent, for example, if recoveries exceed the outstanding exposure.

[9] See Schönbucher (2003), pp 147–150.

recovery rate $R\left(Y_j(X)\right)$ follows a logit–normal distribution defined as follows

$$\begin{aligned} Y_j(X) &= \mu + \sigma\sqrt{\omega}\,X + \sigma\sqrt{1-\omega}\,Z_j \\ R\left(Y_j(X)\right) &= \frac{\exp\left(Y_j(X)\right)}{1+\exp\left(Y_j(X)\right)}, \end{aligned} \tag{2}$$

where X and Z_j are independent standard normally distributed. The parameter ω is restricted to the interval $[0,\,1]$. We demand that PD, μ, σ and ω, like ρ, are constant for all firms and across all time periods. We further assume that the Z_j are pairwise uncorrelated cross–sectionally.

The second and the third extended model are taken from the literature and used for benchmarking purposes. The second model follows Frye (2000) who suggests a normal distribution of the unconditional recovery rates. The recovery rate $\check{R}_j\left(X\right)$ of obligor j, conditional on X, is given by

$$\check{R}_j\left(X\right) = \check{\mu} + \check{\sigma}\sqrt{\check{\omega}}\,X + \check{\sigma}\sqrt{1-\check{\omega}}\,\check{Z}_j, \tag{3}$$

where $\check{Z}_j$ denotes the idiosyncratic risk that follows a standard normal distribution. We pose the same assumptions for $\check{Z}_j$ and $\check{\omega}$ as for Z_j and ω in (2). A presentational advantage of the normal distribution assumption is that the parameters $\check{\mu}$ and $\check{\omega}$ have a straightforward interpretation as *mean recovery* and *recovery correlation*. A drawback is that in this model, recovery rates are neither bounded from below nor from above.

In the third model the recovery rate is log–normally distributed. Following Pykhtin (2003) the recovery rate $\tilde{R}_j\left(X\right)$ is defined as follows

$$\tilde{R}_j\left(X\right) = \exp\left(\tilde{\mu} + \tilde{\sigma}\sqrt{\tilde{\omega}}\,X + \tilde{\sigma}\sqrt{1-\tilde{\omega}}\,\tilde{Z}_j\right). \tag{4}$$

We pose the same assumptions for $\tilde{Z}_j$ and $\tilde{\omega}$ as for Z_j and ω in (2). The log–normal distribution may be more realistic than the normal distribution because recovery rates are strictly non–negative and because it has a thicker tail.[10]

[10]See, for example, Van de Castle and Keisman (1999).

3.2. Estimation procedure

In all three extended models parameter estimation is carried out in two steps: in the first step, we estimate the parameters of the asset value process, ρ and PD, and in the second step the three parameters of the recovery rate distribution. The model parameters are estimated by maximum likelihood and also by a method–of–moments as a robustness check.

In the first step we estimate the parameters PD and ρ. From (1) follows for the conditional default probability $p(x)$, given $X = x$

$$p(x) = \Phi\left[\frac{\Phi^{-1}(PD) - \sqrt{\rho}\,x}{\sqrt{1-\rho}}\right]. \tag{5}$$

The parameter estimation is based on observed default rates in periods $1, \ldots, T$, in our case on a yearly basis. We assume that there is neither autocorrelation in the systematic risk factor X_t nor in the idiosyncratic risk factors $Z_{j,t}$, $\breve{Z}_{j,t}$, $\check{Z}_{j,t}$ and $\tilde{Z}_{j,t}$.[11]

The default rate of a loan portfolio containing N_t borrowers in time period t converges for $N_t \to \infty$ to the conditional default probability $p(x_t)$, given $X_t = x_t$.[12] The probability density of the default frequency DF_t is given by

$$f(DF_t; PD, \rho) = \sqrt{\frac{1-\rho}{\rho}} \exp\left[-\frac{\gamma^2 + (1-2\rho)\,\delta_t^2 - 2\sqrt{1-\rho}\,\gamma\,\delta_t}{2\rho}\right], \tag{6}$$

where $\gamma = \Phi^{-1}(PD)$ and $\delta_t = \Phi^{-1}(DF_t)$.

The maximum of the log–likelihood function

$$LL(PD, \rho; DF_1, \ldots, DF_T) = \sum_{t=1}^{T} \ln\left(f(PD, \rho; DF_t)\right) \tag{7}$$

can be determined analytically[13]

[11] The assumption of no autocorrelation in X_t is justified if the PD is a point–in–time estimate and X_t pure random noise. However, assuming that PD is constant over time, we expect that X_t will be subject to autocorrelation in the real world. Nevertheless, we demand a constant PD to ensure that the parameter estimation is feasible.

[12] Assuming that the portfolio is large and well diversified, this is implied by the law of large numbers, see for example, Bluhm et al. (2003), pp 70–71.

[13] We thank Dirk Tasche for suggesting this closed form solution.

$$\hat{\rho}^{ml} = \frac{Var[\delta]}{1 + Var[\delta]}, \tag{8}$$

$$\widehat{PD}^{ml} = \Phi\left(\frac{\overline{\delta}}{\sqrt{1 + Var[\delta]}}\right), \tag{9}$$

where $\overline{\delta} = \frac{1}{T}\sum_{t=1}^{T}\delta_t$ and $Var[\delta] = \frac{1}{T}\sum_{t=1}^{T}\delta_t^2 - \left(\overline{\delta}\right)^2$.

A closed–form solution of the asymptotic Cramér-Rao lower bound for the standard deviation of the estimators is derived in the appendix.

To compare our estimates with Frye (2000), the PD is alternatively estimated in the first step by the mean default frequency $\overline{DF}$ and ρ is estimated separately by maximum likelihood conditional on $\overline{DF}$.

In the second step, the parameters of the recovery rate distribution are estimated. We exploit the fact that the inferred systematic risk factor $\hat{x}_1, \ldots, \hat{x}_T$ for the time intervals $1, \ldots, T$ can be inferred from the estimates $\widehat{PD}^{ml}$ and $\hat{\rho}^{ml}$, and the observed default frequencies $DF_1, \ldots, DF_T$. Three different models are analysed, in which the distribution of the recovery rates is defined by (2), (3), and (4).

In the first model that serves as reference, recovery rates follow a logit–normal distribution as in (2). Let D_t denote the number of defaults in period t. Given observations $r_1, \ldots, r_T$ of the recovery rates, the maximum–likelihood $(ML-)$estimation of the model parameters μ, σ and ω involves maximising the log–likelihood function

$$LL(\mu, \sigma, \omega; r_1, \ldots, r_T, D_1, \ldots, D_T) = \sum_{t=1}^{T} \ln\left(\sqrt{\frac{D_t}{2\pi\sigma^2(1-\omega)\, r_t^2\,(1-r_t)^2}} \exp\left(-\frac{D_t\left(\ln\left(\frac{r_t}{1-r_t}\right) - \mu - \sqrt{\omega}\sigma\hat{x}_t\right)^2}{2\sigma^2\ (1-\omega)}\right)\right). \tag{10}$$

An analytical solution for the maximum of (10) cannot be determined because the polynomials resulting from the first–order–conditions are of fifth and higher order. Searching numerically for the maximum may provide spurious results because the underlying model

$$Y(\hat{x}_t) = \mu + \sigma\sqrt{\omega}\hat{x}_t + \sigma\sqrt{1-\omega}\, Z_t$$

is poorly identified.[14]

The problem is solved by a *feasible* $ML-$estimation that involves two steps. In the first step σ is estimated by the historical volatility σ_{hist} of the transformed default rates $Y_j = \ln\left(\frac{r_j}{1-r_j}\right)$. The parameters μ and ω are estimated in the second step conditional on σ_{hist}.[15]

The first–order conditions of (10) with regard to μ and ω can be solved analytically

$$\mu^{ml} = \frac{\sum_t D_t \left(\ln\left(\frac{r_t}{1-r_t}\right) - \sigma_{hist}\sqrt{\omega}\hat{x}_t\right)}{\sum_t D_t}. \tag{11}$$

μ^{ml} can be interpreted as the default–weighted average difference between the standardized observed recoveries and the share of these recoveries that is explained by systematic risk.

ω^{ml} is given as the second power of the solution to the following third-order polynomial and can be computed using Cardano's formula

$$z^3 - s_{x,\bar{r}} \cdot z^2 - (1 - s_x^2 - s_r^2) \cdot z = s_{x,\bar{r}}, \tag{12}$$

where $s_{x,\bar{r}} = \frac{1}{T}\sum_t D_t \hat{x}_t \left(\frac{\ln\left(\frac{r_t}{1-r_t}\right) - \mu^{ml}}{\sigma_{hist}}\right)$, $s_x^2 = \frac{1}{T}\sum_t D_t \hat{x}_t^2$ and $s_r^2 = \frac{1}{T}\sum_t D_t \left(\frac{\ln\left(\frac{r_t}{1-r_t}\right) - \mu^{ml}}{\sigma_{hist}}\right)^2$.[16]

The second model assumes that unconditional recovery rates are normally distributed according to (3). Maximum likelihood estimates $\breve{\mu}^{ml}$, $\breve{\sigma}^{ml}$ and $\breve{\omega}^{ml}$ for $\breve{\mu}$, $\breve{\sigma}$ and $\breve{\omega}$ are determined from observed recovery rates and conditional on $\hat{x}_1, \ldots, \hat{x}_T$.

[14]A model is defined as "poorly identified" if its Hessian matrix is nearly singular for certain combinations of parameter values (see Davidson and MacKinnon (1993), pp 181–185). In such a case $ML-$estimates for μ, σ and ω are highly unstable, which is confirmed by Monte–Carlo simulations.

[15]Even though σ_{hist} is an estimate of the standard deviation of recoveries *given* a certain value of X for each time period, one can easily show that for the true parameters σ and ω, the equation $\sigma_{hist}^2 = \sigma^2\left[1 - \omega\left(1 - \sigma_{X_{hist}}^2\right)\right]$ holds. Considering that the true value of ω lies in the interval $[0,\, 1]$, we can determine a confidence interval for σ. The parameter estimates of μ and ω in section 5 are not affected by letting σ vary within this confidence interval.

[16]Substituting μ^{ml} in (12) by (11), we obtain a third–order polynomial that only depends on $\sqrt{\omega^{ml}}$ but which we do not show here for ease of presentation.

From (3) follows for the distribution of the conditional recovery rate in period t

$$\check{R}(\hat{x}_t) \sim N\left(\check{\mu} + \check{\sigma}\sqrt{\check{\omega}}\hat{x}_t, \check{\sigma}^2(1-\check{\omega})\right).$$

Because $\check{R}(\hat{x}_t)$ is normally distributed, the log–likelihood function is given by

$$LL(\check{\mu}, \check{\sigma}, \check{\omega}; r_1, \ldots, r_T, D_1, \ldots, D_T) = \sum_{t=1}^{T} \ln\left(\sqrt{\frac{D_t}{2\pi\check{\sigma}^2(1-\check{\omega})}} \cdot \exp\left(-\frac{D_t\left(r_t - \check{\mu} - \check{\sigma}\sqrt{\check{\omega}}\hat{x}_t\right)^2}{2\check{\sigma}^2(1-\check{\omega})}\right)\right), \quad (13)$$

where $r_1, \ldots, r_T$ denote the observed recovery rates. Because the model is poorly identified, we determine $\check{\sigma}_{hist}$ as a proxy for the actual volatility and then determine the maximum of (13). The analytical solutions for $\check{\mu}$ and $\check{\omega}$ are given by (11) and (12) with $\ln\left(\frac{r_t}{1-r_t}\right)$ substituted by r_t.

In the third model, the assumption of normally distributed recoveries is replaced by the assumption of Pykhtin (2003) that the recovery rate $\tilde{R}(\hat{x}_t)$ is log–normally distributed

$$\tilde{R}(\hat{x}_t) \sim \Lambda\left(\tilde{\mu} + \tilde{\sigma}\sqrt{\tilde{\omega}}\hat{x}_t, \tilde{\sigma}^2(1-\tilde{\omega})\right),$$

where Λ denotes the cumulative distribution function of the log–normal distribution.

The joint log–likelihood function equals

$$LL(\tilde{\mu}, \tilde{\sigma}, \tilde{\omega}; r_1, \ldots, r_T, D_1, \ldots, D_T) = \sum_{t=1}^{T} \ln\left(\sqrt{\frac{D_t}{2\pi r_t^2\tilde{\sigma}^2(1-\tilde{\omega})}} \exp\left(-\frac{D_t\left(\ln(r_t) - \tilde{\mu} - \tilde{\sigma}\sqrt{\tilde{\omega}}\hat{x}_t\right)^2}{2\tilde{\sigma}^2(1-\tilde{\omega})}\right)\right). \quad (14)$$

Because the model is poorly identified, the two–step approach with $\tilde{\sigma}_{hist}$ as a proxy for $\tilde{\sigma}$ is used again to determine the maximum of (14). The analytical solution is given by (11) and (12) with $\ln\left(\frac{r_t}{1-r_t}\right)$ substituted by $\ln(r_t)$.

As a robustness check, the parameters of the analysed models are also determined by a method–of–moments ($MM-$) estimator. For the parameters of the asset value process, the default probability PD is estimated by the average default frequency $\overline{DF}$. The parameter ρ can be determined numerically from the following equation for the variance of the default rates in which Φ_2 denotes the cumulative distribution function of the bivariate normal distribution

$$Var\left[DF\right] = \Phi_2\left(\Phi^{-1}\left(\overline{DF}\right), \Phi^{-1}\left(\overline{DF}\right), \rho\right). \tag{15}$$

The parameters of the distribution of the recovery rates are also estimated by a method–of–moments. For the normal distribution, $\breve{\mu}$ and $\breve{\sigma}_{hist}\sqrt{\breve{\omega}}$ are estimated by the average recovery rate and the sample correlation between the recovery rates and the systematic risk factor. For the logit–normal and the log–normal distribution, we employ the same methodology after transforming recoveries to normally distributed values.

4. Data

4.1. Database

The data source for default frequencies and recovery rates is Standard&Poor's Credit Pro database. Information on recovery rates is quoted from Acharya et al. (2003) who use the same data source. Our study considers recovery rate and default rate observations for 18 years, from January 1982 to December 1999. The original data set contains recovery information up to 2001 but the last two years typically constitute the workout period. Therefore, including them in our sample would yield biased results for the recovery rates at emergence.

The data source consists of two parts, the S&P bond database and the Portfolio Management Data (PMD) database, which are described briefly below.[17]

The extract from the S&P bond database contains information on 379 defaulted companies. At the issuer level, the database provides company names and industry codes; at the issue level, bond names, coupons, seniority rankings, issue sizes in dollars, prices at default and default dates. The price at default is defined as the last recorded trading price at the end of the month in which the default event occurred. The S&P database contains only information about bonds and no collateral information. Recoveries are measured in two ways: first, by market prices at the end of the month in which the default occurred and, second, by prices at emergence. The total number of 645 default instances exceeds

[17] This description is based on Acharya et al. (2003) and Standard&Poor's (2003).

the number of firm defaults because firms generally have more than one type of bond outstanding at the time of default.[18]

The PMD database includes aggregate recovery information about 1,540 bank loans, high yield bonds and other debt instruments with a total of over US$ 100 billion from 300 non-financial, public and private US companies that have defaulted since 1987. The overlap of the S&P and the PMD database consists of approximately 400 facilities. In addition to the information from the S&P database, the PMD database specifically identifies collateral for each secured instrument, including all assets, inventory and/or receivables, real estate, equipment, non–current assets and other. The data was obtained by S&P from bankruptcy documents, reorganisation and disclosure statements, Securities and Exchange Commission filings, press articles, press releases and their internal rating studies on the issuers. Recoveries are measured as the price at emergence from default discounted by the high yield index for the period between default and emergence.

To explore the impact of the measurement method of the recovery rate, we perform the estimation procedure based on market prices shortly after default and also based on recovery rates at emergence. To this purpose we exclude from the merged PMD loan and S&P bond data set those instruments for which no prices at emergence could be observed, the total of borrower defaults equals 465 and the total of defaulted instruments equals 1511. This sample, in which bank loans constitute approximately 24% of all defaults, is the basis for all analyses with recovery rates based on prices at emergence. For the S&P bond database only market prices are available .

We emphasise that two conceptual problems occur in practice for recovery rates measured from prices at emergence from default. First, prices at emergence sometimes exceed 100% which should be the upper limit for recovery rates.[19] It is not clear how these recoveries should be treated in practice. In our case, however, we observe only average annual recoveries which are always between 0% and 100%. Second, the rate at which prices at emergence should be discounted is not unique. While discounting at the initial coupon rate will clearly offer arbitrage opportunities,[20] the appropriate selection of the discount rate also depends on the question of the extent to which recovery rates are subject to

[18]The standard default definition automatically classifies all outstanding bonds as defaulted if default occurs for one bond.

[19]The main reason for this inconsistency is that legal settlements involve price and valuation uncertainties for both parties.

[20]Take two bonds A and B with equal price at emergence P and time between default and emergence but different coupon rates i, say $i_A > i_B$. Then the discounted price of bond A is lower than that of bond B. Short-selling one bond B and buying one bond A will yield a risk–less profit.

systematic risk during the workout period. In the presence of systematic risk in this period, discounting at a risk–free interest rate is not justified. A solution could be to use instead a high–yield bond index as discount rate, following Acharya et al. (2003).

Default rate information, including annual default frequencies and the total number of defaults, was taken from Standard&Poor's (2003). They report this information from 1981 to 2002 for corporate defaults both across industry sectors and across countries, only excluding public–sector and sovereign issuers.

Because of the structure of the available data set it is not possible to test for effects within different industry sectors or rating classes. The results in Acharya et al. (2003) indicate that this would be a promising extension of our work. They find that industry sector, pre–default rating and capital structure have a significant impact on recovery rates.

4.2. Descriptive analysis

Figure 1 presents the time series of default rates and recovery rates. The latter are measured from market prices shortly after default and at emergence. The graphs indicate a possible negative correlation between default rate and recovery rate. Also apparent are notable differences that arise from the two measurement methods of recovery rates.

We find that in the entire time interval from 1982 to 1999, default rates were more volatile than recovery rates in terms of a higher relative standard deviation, determined as the ratio of the standard deviation of the time series and its mean. This ratio is 75% for default rates, and 16% or 20% for recovery rates, dependent on their measurement method.

A descriptive analysis of the default and recovery rates yields that from 1982 to 1999, the mean default rate equals 1.24% with a standard deviation of 0.7% while the mean recovery rate, measured from market prices, amounts to 45.6% with a standard deviation of 8.5%.

A stylised fact from previous empirical studies[21] is a negative correlation between default rates and recovery rates. In Table 1, we show the test results for three different tests regarding the correlation between default rates and recovery rates, measured from market prices at default and at emergence from default. Our results imply that for recovery

[21] See, for example, Altman et al. (2003), Frye (2003) or Acharya et al. (2003).

Figure 1. Recovery and Default Rates

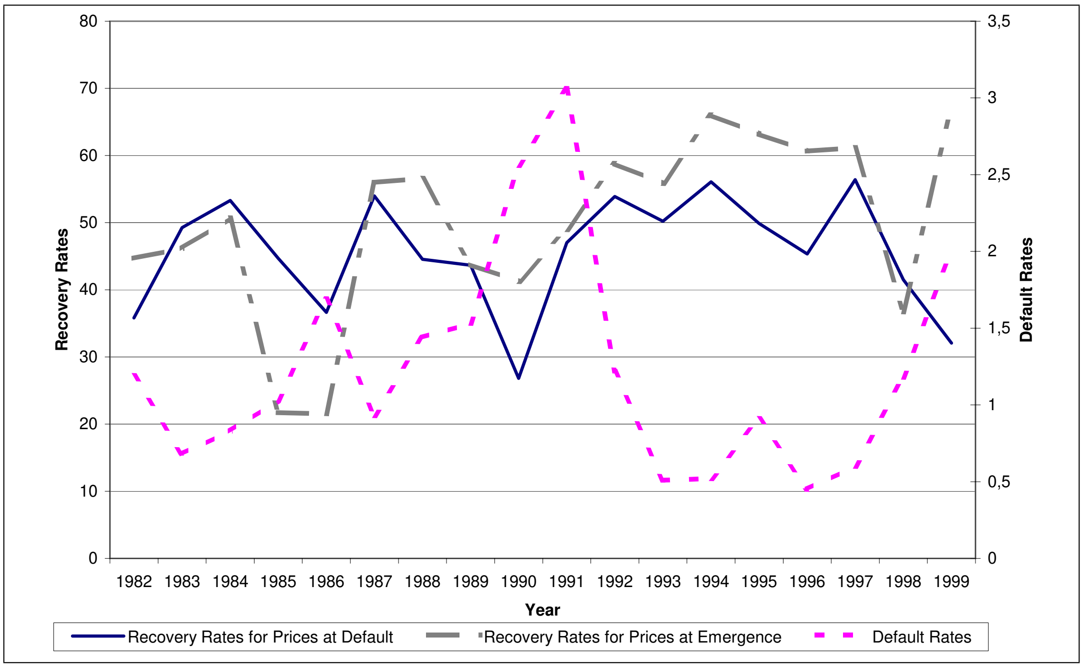


Table 1
Significance Tests of Correlation between Default Rates and Recovery Rates

The table gives the estimated correlation between default frequencies and recovery rates and p–values of the t–tests for significance. Recovery rates are determined at default (R_{pd}) and at emergence (R_{pe}). The fourth column shows results for a one–period lag of default rates. All values are given as a percentage.

Test		R_{pd}	R_{pe}	$R_{pe_{lead}}$
Pearson	correlation	-61.36	-25.73	-41.25
	p–value	0.68	30.26	9.99
Spearman	p–value	0.41	11.77	1.97
Kendall	p–value	0.39	13.11	4.22

rates determined from market prices at default, correlation with default rates is negative and significant at the 99% level. If, however, we measure recovery rates from prices at emergence from default, the correlation is less significant and its absolute value is smaller.

Altman and Brady (2002) observe that recovery rates are influenced by the supply and demand for defaulted securities. Following this argument we would expect a negative correlation between current default rates and recovery rates in later periods. This agrees with the lower p–values in the t–tests of the correlation, that are based on default rates with a one–period time lag and observed recovery rates. We will explore this issue further in section 5.1.

According to Table 1, we find a statistically significant negative correlation between aggregate annual default rates and recovery rates measured from prices at default. These results are in line with most empirical studies including Gupton et al. (2000), Hu and Perraudin (2002), Altman et al. (2003) and Acharya et al. (2003).

Next, two tests of model specification are carried out to determine which distributional assumptions are adequate for recovery rates. The test results are presented in Table 2. The first is the Shapiro–Wilk test for normality.[22] This test is carried out for recovery rates defined as price at default (R_{pd}) and as price at emergence (R_{pe}). For prices at default, p–values of 20.5% and 27.2% compared with 5.5% indicate that the logit–normal and the normal distribution may be more adequate assumptions than the log–normal. For prices at emergence, the p–values are more than 50% lower, but still the normal and the logit–normal distribution seem to be more adequate than the log–normal.

The second test regarding the distribution of recoveries is a Jarque–Bera test for normality.[23] As expected, the p–values for all three distributional assumptions are higher than for the Shapiro–Wilk test. For log recoveries, the p–values are much smaller than in the other two cases which confirms the results of the Shapiro–Wilk test.

The results of these specification tests slightly favour the normal distribution assumption over the logit–normal distribution. However, the logit–normal model may still be preferable on the grounds that it has the desirable property to restrict recovery rates to the interval between 0% and 100%. This additional structural element may make parameter estimation more efficient. Nevertheless, in section 5 we estimate and analyse expected recoveries and recovery correlations under all three distributional assumptions.

[22]Shapiro–Wilk tests give highly precise results for small data samples. High p–values imply that the hypothesis of an underlying normal distribution cannot be rejected, see for example, Davidson and MacKinnon (1993), pp 80–81.

[23]Jarque–Bera type tests use skewness and kurtosis coefficients, see for example, Cromwell et al. (1994).

Table 2
Tests of a Normal Distribution of Recovery Rates

Normality tests are performed on recovery rates measured as market prices at default and as prices at emergence. For the logit–normal distribution, recovery rates R are transformed by $\ln\left(\frac{R}{1-R}\right)$. For the log–normal distribution, recovery rates are transformed by $\ln(R)$.

measurement method	recoveries from market prices			recoveries from emergence prices		
distribution	logit–normal	normal	log–normal	logit–normal	normal	log–normal
Shapiro-test p–value	0.2046	0.2719	0.0549	0.0531	0.0980	0.0026
Jarque-Bera-test p–value	0.4092	0.4861	0.1897	0.2364	0.3493	0.0289

5. Empirical results

5.1. Maximum likelihood estimation of asset correlation and PD

In the first step, the parameters of the asset value process, ρ and PD, are estimated. For the entire time period between 1982 and 1999, we estimate an asset correlation ρ of approximately 4% and a (probability of default) PD of 1%. The estimate of the asset correlation is broadly in line with findings by Dietsch and Petey (2004) and Duellmann and Scheule (2003) for small and medium–sized corporate borrowers. Whereas the estimates of PD are very similar for both estimation methods, the $MM-$estimate of ρ is 10% higher than the $ML-$estimate. The parameters of the recovery rate distribution are estimated conditional on the $ML-$estimates of ρ and PD. The conclusions from the later analyses are robust against this decision.

Next, we infer the changes of the systematic risk factor X_t from the estimates of ρ and PD and the observed default rates. Figure 2 relates the inferred changes of the systematic risk factor, $\hat{x}_t$, to the annual changes in the US gross domestic product (GDP)[24] which we use as a proxy for the world business cycle. Altman et al. (2003) and Acharya et al. (2003) find that the GDP is a superior proxy for systematic risk compared with, for example, changes in the S&P 500 stock index or other market–based indicators. Figure 2 shows a co–movement between the inferred changes of the systematic risk factor and the

[24]The data comes from the International Monetary Fund's World Economic Outlook Database.

Table 3

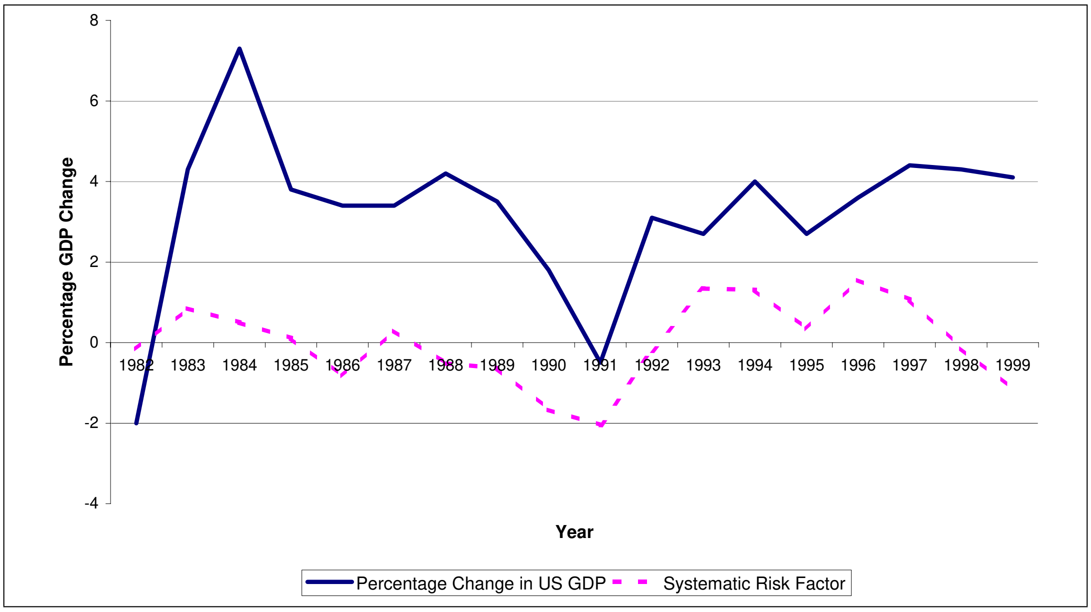
$ML-$Estimates of Asset Correlation and Default Probability

The estimates are determined by maximum likelihood and using an average default frequency as an estimate of PD. Estimates by a method of moments are also given. All estimates are given as a percentage.

parameter	$ML-$estimate	for $PD = \overline{DF}$	MM–estimate
ρ	4.06	4.07	4.47
(standard error)	(1.30)	(1.30)	(0.2)
PD	1.23	1.24	1.24
(standard error)	(0.16)	$(\|\cdot\| < 10^{-6})$	$(\|\cdot\| < 10^{-6})$

annual changes in the US GDP. Changes in the systematic risk factor reflect the 1990/1991 recession and the less volatile movements of the business cycle since 1993.[25]

Figure 2. Changes in US GDP and in the Systematic Risk Factor between 1982 and 1999



In order to investigate systematic risk in recovery rates, we test for correlation between the inferred changes of the systematic risk factor and the observed recovery rates. The test

[25]For an extensive description of the business cycle mechanism, see Filardo (1997) and Christiano and Fitzgerald (1998).

Table 4
Tests of Correlation between Recovery Rates and the Systematic Risk Factor

The table gives the p–values for the correlation tests given as a percentage. Small p–values imply that we can reject the hypothesis that correlation equals zero. The fourth column shows results for a one–period time lag of the systematic risk factor.

Test	R_{pd}	R_{pe}	$R_{pe_{lead}}$
Pearson	0.38	17.42	6.96
Kendall	0.39	13.11	4.2
Spearman	0.4	11.8	1.9

results are presented in Table 4. We can reject the hypothesis that the recovery rates based on prices at default are uncorrelated with the systematic risk factor on a 99.5% confidence level. The results are less clear–cut for recovery rates measured from prices at emergence. If we test for correlation between recovery rates and the systematic risk factor of the same time period, the p–values are much higher than for prices at default. If, however, we use lead values of the systematic risk factor, the p–values decrease significantly. This could be explained by the fact that prices at emergence are usually determined between 12 and 18 months after default. Recovery rates will therefore be influenced by the level of systematic risk over the whole workout period.

The next section focuses on the estimation of the recovery rate distribution under different distributional assumptions. These estimations are carried out conditional on the estimates of ρ and PD in Table 3 and the inferred changes $\hat{x}_t$ of the systematic risk factor.

5.2. Estimation results for recovery rates for a logit–normal, a normal and a log–normal distribution

The following section compares $ML-$parameter estimates under three different distributional assumptions. It also includes results from an $MM-$estimation as a robustness check. Whereas the model parameter μ cannot be directly compared across the three extended models, the expected recovery rate is a variable that allows such a comparison for the recovery level. The parameter ω is also not directly comparable across the models. In order to analyse differences in recovery correlation, we focus on the sensitivity of recovery rates to systematic risk that is analysed in detail in section 5.4.

For logit–normally distributed recovery rates, the $ML-$estimates of μ and ω are given in the second column of Table 5. The expected recovery rate equals 45.57% which is 0.05 percentage points lower than the average recovery rate for the entire time interval. The $ML-$estimate of ω equals 13.48%.

Table 5
Parameter Estimates Under Logit–normally Distributed Recovery Rates

The estimates are determined for logit–normally distributed recovery rates by maximum likelihood (ML) and a method of moments (MM). Volatility is fixed at $\sigma_{hist} = 35.37\%$. Recovery rates are measured by market prices at default. All estimates are given as a percentage.

parameter	$ML-$estimate	$MM-$estimate
μ	-18.32	-18.32
(standard error)	$(\lvert \cdot \rvert < 10^{-9})$	(1.97)
ω	13.48	n.a.
(standard error)	$(\lvert \cdot \rvert < 10^{-9})$	n.a.
$\sigma\sqrt{\omega}$	13.03	22.25
$E[R]$	45.57	45.57

One of the most intuitive estimation procedures (and therefore most widely used in practice) is the method of moments. We compare the $ML-$estimates of μ and σ to those implied by matching the first and second moment of the default rate to their sample values.[26] While the $MM-$estimate of μ is close to the $ML-$estimate, we find that $\sigma\sqrt{\omega}$ is 71% higher.

For normally distributed recovery rates in the spirit of Frye (2000) the $ML-$estimates of expected recovery and recovery correlation are given in the second column of Table 6. The expected recovery rate $\breve{\mu}^{ml}$ equals 43.81% while the $ML-$estimate of $\breve{\omega}^{ml}$ equals 9.98%. Frye (2000) finds similar levels of asset correlation and expected recovery rate for the time period from 1983 to 1997, but his estimate of 3% for $\breve{\omega}^{ml}$ differs from our estimate of 10%. There are two potential explanations for this difference.

First, owing to the use of different data sets the recovery rates are more volatile than ours. A higher value of $\breve{\sigma}$ will imply a lower $\breve{\omega}$ for a fixed total recovery correlation which has a similar value as in Frye (2000).

[26]See, for example, Bluhm et al. (2003).

Table 6
Parameter Estimates Under Normally Distributed Recovery Rates

The estimates are determined for normally distributed recovery rates by maximum likelihood (ML), a method of moments (MM) and also by ML but conditional on the average default frequency $\overline{DF}$ as a proxy for PD. Volatility is fixed at $\breve{\sigma}_{hist} = 8.45\%$. Recovery rates are measured by market prices at default. All estimates are given as a percentage.

parameter	$ML-$estimate	$ML-$estimate with $PD = \overline{DF}$	$MM-$estimate
$\breve{\mu}$ $(= E[\tilde{R}])$	43.81	43.78	45.62
(standard error)	(0.34)	(0.34)	(0.47)
$\breve{\omega}$	9.98	9.99	n.a.
(standard error)	(1.66)	(1.66)	n.a.
$\breve{\sigma}\sqrt{\breve{\omega}}$	2.67	2.67	5.31

The second potential explanation is that Frye proposes a long–term average of default rates, $\overline{DF}$, as a proxy for the true probability of default while we compute a $ML-$estimate for PD. Replicating his approach yields nearly the same results as the original $ML-$estimation. This can be seen by comparing the second and the third column of Table 6. We conclude that the observed differences in the correlation estimates cannot be explained by this methodological difference.

Although the preliminary tests that we performed with the recovery data in section 4.2 do not support the hypothesis of an underlying log–normal distribution, we compare the associated estimates as a robustness check to those obtained from a logit–normal and a normal distribution assumption. The $ML-$estimates for log–normally distributed recovery rates are given in Table 7.

Comparing the expected recovery rates in Tables 5, 6 and 7, we find that for logit–normal recovery rates, the expected recovery rate of 45.57% is higher than for the other two distributional assumptions which lead to similar estimates (43.81% and 43.68%). The estimates of ω are close to each other in Tables 6 and 7 but differ from the case of the logit–normal distribution in Table 5. Because we cannot compare the estimates of ω directly for the different distributions, we focus instead in section 5.4 on the recovery sensitivity as a measure of recovery correlation for all three extended models.

Table 7
Parameter Estimates under Log–normally Distributed Recovery Rates

The estimates are determined by maximum likelihood (ML) and a method of moments (MM). Volatility is fixed at $\tilde{\sigma}_{hist} = 20.45\%$. Recovery rates are measured by market prices at default. All estimates are given as a percentage.

parameter	$ML-$estimate	MM–estimate
$\tilde{\mu}$	-84.90	-80.33
(standard error)	(0.82)	(1.14)
$\tilde{\omega}$	9.51	n.a.
(standard error)	(1.59)	n.a.
$\check{\sigma}\sqrt{\tilde{\omega}}$	6.31	12.82
$E\left[\tilde{R}\right]$	43.68	45.73

5.3. Impact of the recovery definition

In the following section estimates from recovery rates based on market prices at default are compared with those from recovery rates determined at emergence. With these new recovery rates the analyses from section 5.2 are repeated. The estimates of ρ and PD are taken from section 5.1 because they are independent of the measurement of recovery. Therefore, we only report changes in μ, σ and ω.

The parameter estimates for the three extended models are given in Table 8. We compare first the estimates for the logit–normal model. The expected recovery rate increases from 45.57% to 49.72%, which is a relative increase of 9.11%, compared with the case of recovery rates inferred from market prices at default in Table 5. An intuitive explanation for this result is that bank loans, which are only included in this second set of recovery rates, generally yield higher recoveries than bonds, which has also been observed by Van de Castle and Keisman (1999). Historical volatility of the transformed recovery rates $\ln\left(\frac{R_j}{1-R_j}\right)$ is higher at 58.49%, which signifies a relative increase of 65%, and ω consecutively falls to 2.75%.

Assuming a normal distribution, we find that the expected recovery rate increases to 53.05% which is a relative increase of 21% compared with Table 6. The volatility of the recovery rates increases to 13.50%, a relative increase of 60%, and the recovery correlation falls by about 39% to 6.04%. These two effects roughly cancel each other out in

Table 8
Parameter Estimates for Recovery Rates Based on Prices at Emergence

The estimates are determined by maximum likelihood for logit–normally, normally and log–normally distributed recovery rates using the sample estimates σ_{hist}, $\breve{\sigma}_{hist}$ and $\tilde{\sigma}_{hist}$ as proxies for the volatility parameters. The recovery rate is measured at emergence from default. ML–estimates are given as a percentage.

parameter	logit–normal	normal	log–normal
σ, $\breve{\sigma}$, $\tilde{\sigma}$	58.49	13.50	33.45
μ, $\breve{\mu}$, $\tilde{\mu}$	-1.21	53.05	-65.58
(standard error)	$(\lvert \cdot \rvert < 10^{-10})$	(0.37)	(0.92)
ω, $\breve{\omega}$, $\tilde{\omega}$	2.75	6.04	3.75
(standard error)	$(\lvert \cdot \rvert < 10^{-11})$	(0.89)	0.72
$E[R]$,$E[\breve{R}]$,$E[\tilde{R}]$	49.72	53.05	54.89

their impact on the square root of the *total recovery correlation*, $\breve{\sigma}_{hist}\sqrt{\breve{\omega}}$. The square root of the total recovery correlation, 3.32%, is close to the value of 2.67% in Table 6.

For completeness we also analyse the parameter estimates of a log–normal distribution of recoveries.[27] In this case, the expected recovery rate in Table 8 is slightly higher than for the normal distribution. The higher total recovery correlation, however, can quickly neutralise this effect on the default loss when systematic risk increases.

Overall, the estimates of the expected recovery rates are 9% - 26% higher for prices at emergence than for market prices at default. This finding is broadly consistent with Acharya et al. (2003) who observe that recovery rates at emergence are 20 percentage points higher than at default. This result could be explained by a risk premium in market prices at default that accounts for systematic risk in recovery payments during the work-out period. This effect would decrease market prices at default relative to recoveries at emergence that build on observed cash flows. However, the prices at emergence have been derived by discounting at a high–yield bond rate that may already incorporate a risk premium. The increase in the expected recovery rate is smallest for the logit–normal model, which appears not to be as sensitive to the choice of the measurement method as the other

[27] After taking logarithms, a Shapiro-Wilk test yields a p–value of 0.26% and a Jarque-Bera test results in a p–value of 2.89%. The hypothesis of a log–normal distribution can thus be rejected at the 97% confidence level.

two models.

Nevertheless, drawing inference from results for two different measurement methods of recovery rates is a delicate issue because the estimates are based on two different databases. The overlap between both databases is rather small because only 399 out of 1,540 facilities from the PMD database are also included in the S&P database. As a consequence, the observed differences may partly derive from differences in samples. The observation of lower recovery rates based on market prices at default is consistent with the argument in Altman et al. (2003) that investors in distressed assets cannot absorb an increased supply of defaulted bonds in high–default years. Therefore, liquidity dries up and market prices can be too low compared with actual recoveries in later periods.

In summary, we observe quite similar estimates for the mean recovery rate under all three distributional assumptions for recovery rates at emergence. They differ, however, between 9% and 26% from the expected recovery rates that are estimated from market prices at default. How far differences in the recovery correlation transfer to differences in economic capital is explored in section 6. In the next section, the sensitivity of the recovery rates to the systematic risk factor is analysed for the three models defined by (2), (3), and (4).

5.4. Sensitivity of recovery rates to systematic risk

From a risk management perspective it is interesting to compare the three extended models in terms of the sensitivity of the recovery rate to changes in the systematic risk factor. This sensitivity is defined here as the first derivative of the recovery rate with regard to the single risk factor. Table 9 lists the sensitivities for the three models, which are derived from (2), (3), and (4).

In the reference model that assumes a logit–normal distribution of recovery rates, the sensitivity depends not only on the product $\sigma\sqrt{\omega}$ but also in a non–linear way on the recovery rate. For normally distributed recovery rates, the sensitivity is fully described by the total recovery correlation $\breve{\sigma}\sqrt{\breve{\omega}}$. Only in this model, the sensitivity is independent of the level of systematic risk, and relative changes in recovery rates are a linear function of changes in X. For log–normally distributed recovery rates, the sensitivity linearly depends on the level of the recovery rate.

The sensitivities in Table 9 depend on the level of the systematic risk factor X. Therefore, we compare the sensitivities of the three models for a range of realistic values of X, based

Table 9
Sensitivity of the Recovery Rate to the Systematic Risk Factor

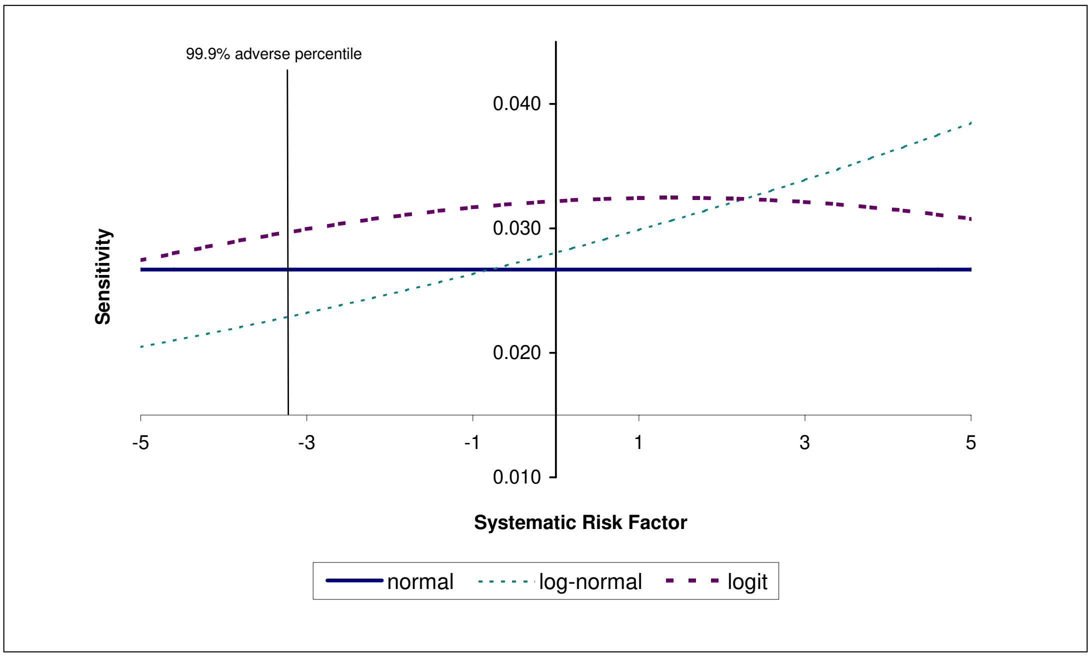
This table lists the sensitivity, defined as the first derivative of the recovery rate with regard to the single risk factor, for the logit–normal, the normal and the log–normal model.

distribution of recovery rates	sensitivity $\frac{\partial R}{\partial X}$
logit–normal	$\sigma\,\sqrt{\omega}\,\frac{R(Y)}{1+\exp(Y)}$
normal	$\breve{\sigma}\sqrt{\breve{\omega}}$
log–normal	$\tilde{\sigma}\,\sqrt{\tilde{\omega}}\,\tilde{R}\,(X)$

on the parameter estimates from sections 5.2 and 5.3. Given the estimates in Tables 5, 6, and 7, the sensitivity is plotted in Figure 3 as a function of the systematic risk factor X. Especially relevant from a risk management perspective are high absolute values of X in the negative domain. For reference we have marked the 99.9% percentile that is also used in the risk weight functions of the IRB approach of Basel II.

Figure 3. Sensitivity of Recovery Rates to the Systematic Risk Factor, Estimated from Market Prices at Default

This figure shows the sensitivity to the systematic risk factor for logit–normally, normally, and log–normally distributed recovery rates.



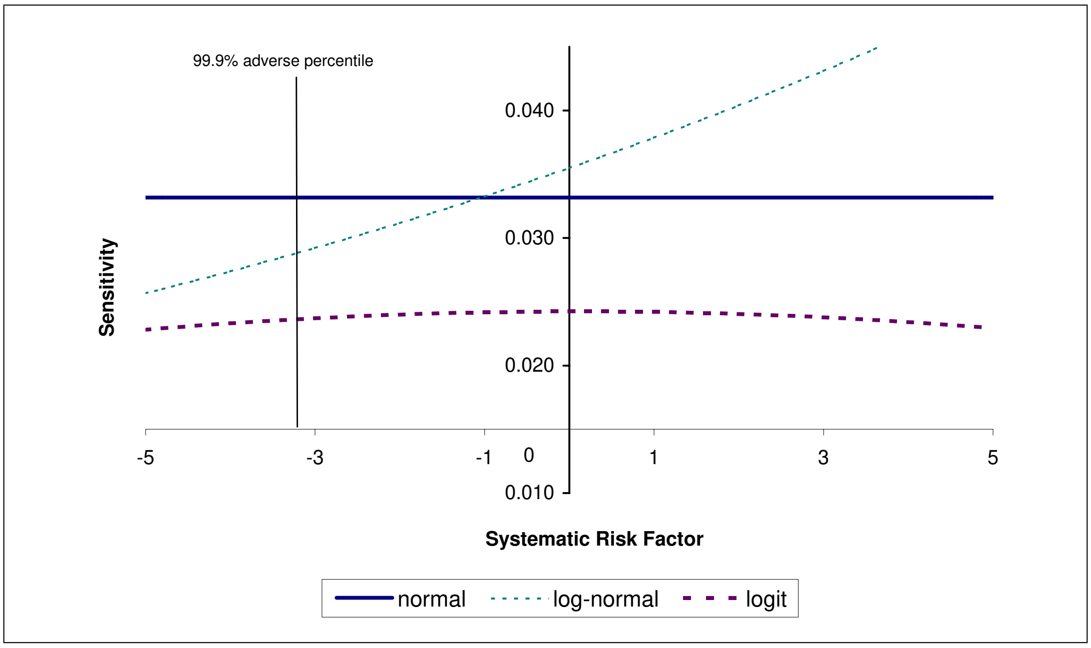
In the case of normally distributed recovery rates, sensitivity is always equal to 0.0267. For the logit–normal distribution, the sensitivity is larger for values of X between -5.6 and 8.4 which covers the 99.9% percentile of X. Therefore, the logit–normal distribution is more risk–sensitive in terms of systematic risk for the range of X that is relevant from a risk management perspective.

The log–normal distribution yields the least systematic risk–sensitive estimate of the three models if X is smaller than -0.8 which is the most relevant range from a risk–management perspective. Between -0.8 and 2.3, it produces sensitivity values between the other two models, and for values of X greater than 2.3, it yields the most risk–sensitive estimates.

Figure 4 shows the sensitivity to the systematic risk factor if the parameters are estimated from prices at emergence, instead of market prices at default as in Figure 3.

Figure 4. Sensitivity of Recovery Rates to the Systematic Risk Factor, Estimated from Prices at Emergence

This figure shows the sensitivity to the systematic risk factor for logit–normally, normally, and log–normally distributed recovery rates.



In this case the ordering between the three models in terms of their risk sensitivity changes from the case in Figure 3. For the adverse percentile of 99.9% the logit–normal model has a lower risk sensitivity than the other two models. The value of the sensitivity, 0.0237, is closer to the previous one in Figure 3 than for the other two models. In the neighbourhood of the 99.9% percentile the logit–normal model appears to be more stable than the other two models if recovery rates are measured from prices at emergence instead of market prices at default.

6. Implications for economic capital

Bank loans are subject to two different sources of credit risk: borrower–specific risk that can be controlled or even neutralised by diversification and systematic risk. As in the model underlying the IRB risk weights of Basel II, economic capital is determined below assuming that the bank loan portfolio is fully diversified and that economic capital is only held for systematic credit risk.[28] Whereas the effect of systematic risk on default rates has been widely explored, literature on the effects of systematic risk on recovery rates is scarce.[29] The presence of systematic risk increases losses from two directions: firstly, through a higher default frequency and, secondly, through a higher loss rate in default. Both effects have to be taken into account when computing economic capital.

Below, we use the variable $LGD\,(= 1 - R)$ instead of the recovery rate R, following common terminology in the literature on economic capital. Figure 5 shows the probability density function of the LGD for a logit–normal, a normal, and a log–normal distribution of the recovery rates and given the parameter estimates in Tables 5, 6 and 7 for market prices at default.

For the logit–normal distribution, the maximum likelihood is reached at an LGD of 54.5% which agrees with the ML–estimate for μ in Table 5. The maximum likelihood for normally distributed LGD is reached at an LGD of 56.2% which is consistent with our estimate for $\check{\mu}^{ml}$ in Table 6. Based on the assumption of log–normally distributed recovery rates, the $LGD-$density function is slightly skewed towards the right and shows the typical fat–tailed behaviour; the maximum likelihood is reached at an LGD of 56.5%, which is approximately what we found for normally distributed LGD.

[28] An explicit treatment of idiosyncratic risk has been removed from the minimum capital requirements of Basel II when the granularity adjustment, put forward in the second Consultative Paper of the Basel Committee, was abandoned. Nevertheless, this second source of risk also warrants attention from supervisors.

[29] See section 2.

Figure 5. Density of LGD for Average Systematic Risk

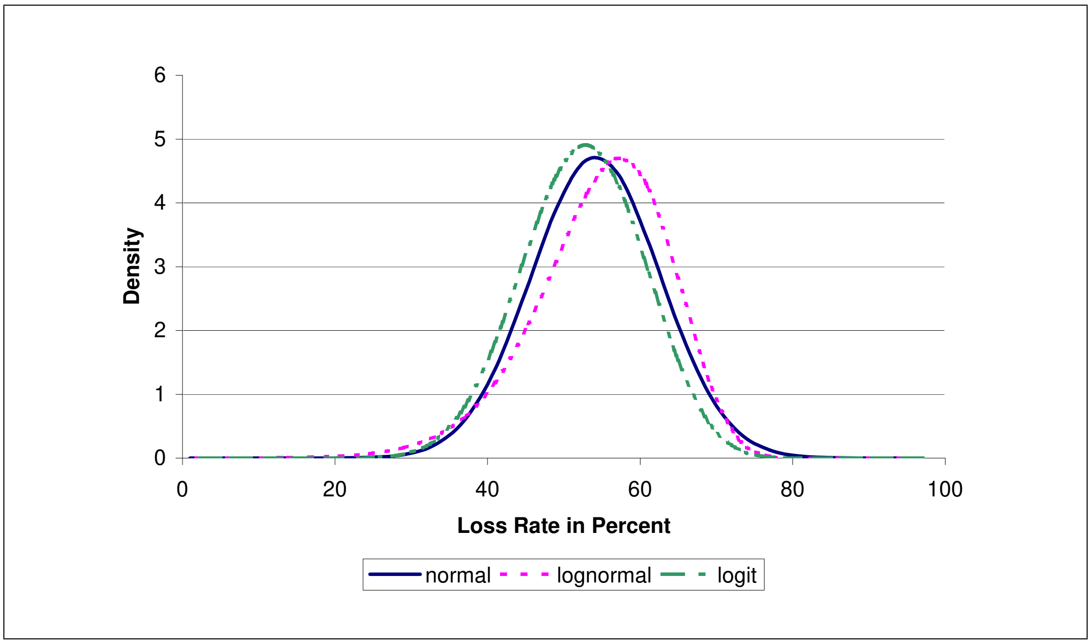
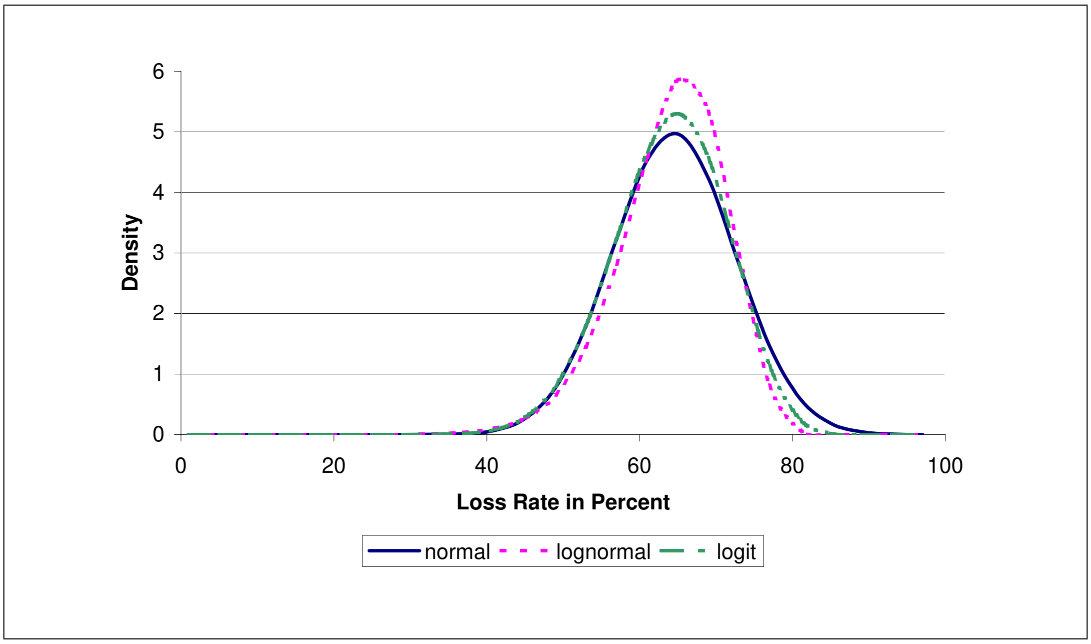


Figure 6 shows the LGD densities conditional on a 99.9% percentile of the systematic risk factor. The differences between the three models emerge more clearly than in Figure 5 in which X was close to zero. In the adverse tail of the distribution, that means for percentiles of 80% and beyond, the values of the density function of the log–normal model are closer to those of the logit–normal model than to those of the normal model. This is due to the fact that the logit–normal and the log–normal have fatter tails than the normal distribution. However, the values of the density function for extreme percentiles like 99.9% are close to zero in all three models, so in these cases the differences may have only a negligible impact on the economic capital. This issue is explored in the following analysis.

After having determined the $LGD-$density function, we can compute economic capital (EC), which is defined here as the 99.9% percentile of the portfolio loss distribution. For the entire section, we assume that the loan exposure at default equals 100 so that all results for the economic capital can be interpreted in percentage terms.

Since the conditional LGD distribution of the portfolio is known and, conditional on a realization of X, the probability of default and the loss given default are independent, we can infer the economic capital at the 99.9% level for the normal and the log–normal

Figure 6. Density of LGD for the 99.9% Percentile of the Systematic Risk factor



distribution:[30]

$$EC_{99.9\%} = 100 \cdot E\left[PD|X = \Phi^{-1}(0.001)\right] \cdot E\left[LGD|X = \Phi^{-1}(0.001)\right] \qquad (16)$$

$$E\left[PD|X = \Phi^{-1}(0.001)\right] = p(-3.09)$$

$$E\left[LGD|X = \Phi^{-1}(0.001)\right] = \begin{cases} \left(1 - \breve{\mu} + \breve{\sigma}\sqrt{\breve{\omega}} \cdot 3.09\right) \\ \qquad \text{if normal distribution} \\ \left(1 - \exp\left(\tilde{\mu} - \tilde{\sigma}\sqrt{\tilde{\omega}} \cdot 3.09 + \frac{\tilde{\sigma}^2(1-\tilde{\omega})}{2}\right)\right) \\ \qquad \text{if log–normal distribution.} \end{cases}$$

If, however, economic capital is calculated as in the classic model, we neglect the systematic risk term in $E\left[LGD|X = -3.09\right]$ and economic capital is simply the product of the conditional default probability and the unconditional expected LGD. The estimates for the logit–normal, the normal and the log–normal distribution are given in Tables 10 and 11. The conditional default probability is determined from equation (5) with the $ML-$estimates of PD and ρ taken from Table 3.

[30]The expected value of a logit–normally distributed variable with parameters μ and σ^2 generally has no closed–form solution and we apply instead numerical integration.

In order to compare estimates of economic capital with and without systematic risk in LGD, expected LGD in the classic model is calculated by maximum likelihood with the recovery correlation set to zero. The results for LGD based on the price at default are given in Table 10. Another procedure that can be found in applications of the classic model is the use of an equally weighted or a default–weighted average LGD. In both cases, LGD is treated as a parameter in the calculation of economic capital. The IRB approach of Basel II requires that banks' $LGD-$estimates cannot be less than the long–run default–weighted average loss rates given default.[31] Default–weighted $LGD-$estimates are provided in the last column of Table 10. Here, they are calculated as average $LGDs$ weighted by the ratio of the number of defaults in each period and the total number of defaults over all periods.[32]

Table 10
Economic Capital without Systematic Risk in LGD, Based on Market Prices at Default

Expected values of LGD are inferred from $ML-$estimates. LGD is measured as $1-R_{pd}$ where R_{pd} equals the market price at default. All estimates are given as a percentage.

parameter	logit–normal	normal	log–normal	default–weighted
conditional PD	4.76			
mean LGD	54.43	56.19	56.32	57.65
economic capital	2.59	2.67	2.68	2.74

Table 11 provides estimates of economic capital that incorporate systematic risk in LGD. The differences in economic capital that arise from different distributional assumptions of the LGD are small (close to 1%). Allowing for correlation between the LGD and the systematic risk factor in the extended model, however, leads to estimates of economic capital which are between 14% and 17% higher than in Table 10.

In the classic model, the calculation of default–weighted $LGDs$ yields an expected loss of 57.65% and an economic capital of 2.74%, given in Table 10. As a consequence, the economic capital increases between 2% and 6% compared with the use of an equally weighted mean LGD. Even though this calculation produces a more conservative esti-

[31]See paragraph 468 of the revised Framework, Basel Committee on Banking Supervision (2004).

[32]We employ this weighting method instead of pooling observed $LGDs$ over time and calculating the average LGD because single–obligor $LGDs$ are not available to us.

Table 11
Economic Capital with Systematic Risk in LGD, Based on Market Prices at Default

All quantiles are computed at the 99.9% level. LGD is measured as $1 - R_{pd}$ where R_{pd} equals the market price at default. All estimates are given as a percentage.

parameter	logit–normal	normal	log–normal
conditional PD	4.76		
stress LGD	63.87	64.44	64.12
economic capital	3.04	3.07	3.05

mate, we see that the extended models still result in by 12% higher estimates of economic capital than in the case of a default–weighted LGD.

For our second data set where recoveries are measured at emergence, the results are presented in Tables 12 and 13. Otherwise the structure of the two tables follows Tables 10 and 11. The estimates of ρ and PD are necessarily the same because they are unaffected by the $LGD-$modelling.

Table 12 shows that in the absence of systematic risk in LGD, the mean LGD and the economic capital vary approximately by 10% between the three extended models. However, in Table 13, which incorporates an adverse systematic risk scenario, the three approaches lead to estimates of economic capital that differ by less than 4%. Therefore, for a 99.9% level of the systematic risk factor, the differences between the models are smaller than for unstressed LGDs.

Table 12
Economic Capital without Systematic Risk in LGD, Based on Recoveries at Emergence

Expected values of LGD are inferred from $ML-$estimates. LGD is measured as $1 - R_{em}$ where R_{em} denotes the recovery rate at emergence. All estimates are given as a percentage.

parameter	logit–normal	normal	log–normal	default–weighted
conditional PD	4.76			
mean LGD	50.28	46.95	45.11	48.89
economic capital	2.39	2.23	2.15	2.33

Table 13
Economic Capital with Systematic Risk in LGD, Based on Recoveries at Emergence

All quantiles are computed at the 99.9% level. LGD is measured as $1 - R_{em}$ where R_{em} denotes the recovery rate at emergence. All estimates are given as a percentage.

parameter	logit–normal	normal	log–normal
conditional PD	4.76		
stress LGD	57.19	57.21	55.16
economic capital	2.72	2.72	2.63

Comparing the results in Tables 12 and 13 reveals that under the assumptions of a logit–normal, a normal and a log–normal distribution of the recovery rates, economic capital is between 14% and 22% higher in the extended models, in which LGD depends on the systematic risk factor.

Finally, we compare the results for economic capital based on recovery rates inferred from market prices at default (Tables 10 and 11) and from prices at emergence (Tables 12 and 13). In all three models, economic capital is lower for recovery rates measured at emergence than for market prices at default. The differences are between 8% and 25% without systematic risk in LGD and between 12% and 16% for an adverse 99.9% percentile of the systematic risk factor. In both cases the LGD estimates at emergence from default are used as the reference basis.

7. Summary and conclusions

This paper extends the classic one–factor credit risk model of Gordy (2001) by also considering systematic risk in recovery rates. We put forward an extended, but still parsimonious model and compare it with the two models of Frye (2000) and Pykhtin (2003). The three extended models have the same structure but differ in the distributional assumption for the recovery rate: a logit–normal, a normal and a log–normal distribution.

As a theoretical contribution we provide closed form solutions for $ML-$estimators of the parameters default probability PD and asset correlation ρ in the classic one–factor model, their standard errors, and also for the parameters of the recovery rate distribution in the three extended models.

Our empirical contribution comprises a comparison of the three extended models, based on time series of default rates and recovery rates from Standard&Poor's Credit Pro database. The data set includes bond and loan default information of US corporate obligors in the time period between 1982 and 1999.

The estimate of the asset correlation between two borrowers is 4% and in line with previous results in the literature where this parameter was also estimated from default rates. The inferred changes of the systematic risk factor show a high degree of co–movement with US GDP changes.

Three distributional assumptions are tested for the recovery rates: a logit–normal distribution, a normal distribution and a log–normal distribution. In standard specification tests, the logit–normal distribution and the normal distribution are found to explain the observed recovery rates better.

Estimates of the expected recovery rate, based on market prices at default, are between 44% and 46%, independent of the distributional assumption for the recovery rate and independent of the estimation method. However, if estimated from recovery rates at emergence, the expected recovery rate increases between 9% and 26%, depending on the distributional assumption.

We analyse the correlation effect in the three extended models by comparing the sensitivity of the recovery rate to the systematic risk factor. This sensitivity is defined as the first derivative. Its level and also the ranking of the three models depend on the systematic risk factor.[33] However, depending on the measurement of recovery rates from market prices or at emergence, the ordering of the three models in terms of their sensitivity to the systematic risk factor changes. For an adverse 99.9% percentile of the systematic risk factor, the logit–normal model is more stable in the sense that the sensitivity to the systematic risk factor depends less on the question of whether market prices at default or at emergence are used. This result together with the implicit bounding of the recovery rates to the interval from 0% to 100% advocates, in our view, the application of the logit–normal model.

From the perspective of a credit risk manager, the following three results are considered

[33]It is only in the special case of normally distributed recovery rates that the sensitivity is independent of the systematic risk factor.

to be most important.

1. Incorporating systematic risk in recovery rates leads to a significantly higher economic capital charge. For an adverse 99.9% percentile of the systematic risk factor, economic capital would increase between 11% and 12% relative to an unstressed, default–weighted average LGD, based on market prices at default. This result could be interpreted that the treatment of LGD in the Third Consultative Document[34] may not fully capture systematic risk in LGD. However, for several reasons this result should be treated more as an indicative finding than as a conclusion based on empirical evidence. One reason is that it was derived for a particular corporate bond and loan data set that may substantially differ from an actual credit portfolio of an investor or a bank. Another reason is that the dependence on systematic risk of mortgages and other retail loans may be quite different from that of corporate exposures.
2. We observe notable differences between an LGD that is based on market prices after default and an LGD that is determined at emergence. In our example, these differences lead to deviations in economic capital in the range of 12%–16%, depending on the distributional assumption for recovery rates in the extended model. This may have implications for the level playing field if competing institutes measure LGD differently. Furthermore, it complicates the validation of LGD estimates because a comparison across institutes may provide distorted results. At this point it is important to emphasise that this result should be taken only as an initial indication because the datasets that are used for comparing the two measurement methods of LGD only partially overlap. This may explain the observed differences to an unknown extent.
3. The estimates of economic capital when systematic risk is accounted for are relatively stable with respect to the distributional assumptions of LGD because they vary by less than 4%. This result is reassuring if systematic risk has to be modelled and the most appropriate distribution cannot be identified from empirical data.

The following two aspects call for further research and can provide useful extensions of our analyses. In our model, systematic risk in recovery rates is captured by a single unobservable stochastic variable. A step forward would be a breakdown of this factor by identifying individual economic factors that influence recovery rates. For this purpose borrowers with similar characteristics, for example, contract specifics (Neto de Carvalho

[34] See Basel Committee on Banking Supervision (2003).

and Dermine (2003)), firm specifics (Acharya et al. (2003)), and industry specifics, should be separated and the recovery correlations could then be estimated independently.

Analysing what causes recovery rates derived from market prices at default to be so different from recovery rates derived from prices at emergence from default is a second interesting aspect requiring further research, especially as these differences have a strong impact on the economic capital.

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Appendix

Asymptotic standard errors of the ML–estimates of asset correlation ρ and PD

Below, we derive the Cramér-Rao lower bounds for the standard deviation of the ML–estimators of ρ and PD.

For this purpose, the Hessian matrix of the log–likelihood function of the default rates,

$$\begin{aligned} LL &= \sum_{t=1}^{T} \ln\left(f\left(PD, \rho; DF_t\right)\right) \\ &= \frac{T}{2}\ln\left(\frac{1-\rho}{\rho}\right) - \frac{T\gamma^2}{2\rho} + \frac{\sqrt{1-\rho}}{\rho}\,\gamma\,\underbrace{\sum_{t=1}^{T}\delta_t}_{=s_1} + \frac{2\rho-1}{2\rho}\underbrace{\sum_{t=1}^{T}\delta_t^2}_{=s_2}, \end{aligned}$$

has to be determined.

From the closed–form solutions for $\hat{\rho}^{ml}$ and $\hat{\gamma}^{ml}$ in (8) and (9) follows for the second derivatives of LL with regard to ρ and PD

$$\begin{aligned} \frac{\partial^2 LL}{\partial \rho^2} &= \frac{T}{2}\left(\frac{1}{\left(\hat{\rho}^{ml}\right)^2} - \frac{1}{\left(1-\hat{\rho}^{ml}\right)^2}\right) - \frac{T\left(\hat{\gamma}^{ml}\right)^2 + s_2}{\left(\hat{\rho}^{ml}\right)^3} + \hat{\gamma}^{ml}\, s_1\, \frac{3\left(\hat{\rho}^{ml}\right)^2 - 12\hat{\rho}^{ml} + 8}{4\left(\hat{\rho}^{ml}\right)^3\left(1-\hat{\rho}^{ml}\right)^{\frac{3}{2}}} \\ &= -\frac{\overbrace{s_1^4 + 2T^2\left(s_2+T\right)^2 - s_1^2 T\left(3s_2 + 4T\right)}^{=s_3}}{4T^3}\,\frac{\left(1+Var\left[\delta\right]\right)^2}{Var\left[\delta\right]^2} \\ \frac{\partial^2 LL}{\partial\rho\,\partial PD} &= \frac{\partial^2 LL}{\partial PD\,\partial\rho} = \frac{1}{\left(\hat{\rho}^{ml}\right)^2}\frac{1}{\phi\left(\Phi^{-1}\left(PD\right)\right)}\left(T\Phi^{-1}\left(PD\right) + s_1\frac{\hat{\rho}^{ml}-2}{2\sqrt{1-\hat{\rho}^{ml}}}\right) \\ &= -\frac{T}{2}\,\frac{\overline{\delta}}{\phi\left(\hat{\gamma}^{ml}\right)}\,\frac{\left(1+Var\left[\delta\right]\right)^{3/2}}{Var\left[\delta\right]} \\ \frac{\partial^2 LL}{\partial PD^2} &= -\frac{T}{\left(\hat{\rho}^{ml}\right)^2\,\phi\left(\Phi^{-1}\left(PD\right)\right)}\left(1 + \Phi^{-1}\left(PD\right)^2 + \overline{\delta}\sqrt{1-\hat{\rho}^{ml}}\,\Phi^{-1}\left(PD\right)\right) \\ &= -\frac{T}{\phi\left(\hat{\gamma}^{ml}\right)^2}\,\frac{1+Var\left[\delta\right]}{Var\left[\delta\right]}. \end{aligned}$$

The inverse of the negative Hessian matrix of the log–likelihood function LL in $\left(\hat{\rho}^{ml}, \hat{\gamma}^{ml}\right)$ produces the Fisher matrix F, which is defined as the inverse of the information matrix I at this point[35]

$$H_{LL} = \left(\begin{array}{cc} \frac{\partial^2 LL}{\partial \rho^2} & \frac{\partial^2 LL}{\partial \rho \, \partial \gamma} \\ \frac{\partial^2 LL}{\partial \gamma \, \partial \rho} & \frac{\partial^2 LL}{\partial \gamma^2} \end{array} \right), F_{LL} = \left(\begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right).$$

The entries of F_{LL} are as follows

$$\begin{aligned} f_{11} &= -\frac{h_{22}}{-h_{12}^2 + h_{11}h_{22}} \\ &= \frac{2Var\left[\delta\right]^2}{T\left(1+Var\left[\delta\right]\right)^4} \\ f_{12} &= f_{21} = \frac{h_{12}}{-h_{12}^2 + h_{11}h_{22}} \\ &= -\frac{s_1\, \phi\left(\gamma^{ml}\right)\; Var\left[\delta\right]^2}{T^2\left(1+Var\left[\delta\right]\right)^{7/2}} \\ f_{22} &= -\frac{h_{11}}{-h_{12}^2 + h_{11}h_{22}} \\ &= \frac{s_3\, \phi\left(\gamma^{ml}\right)^2\; Var\left[\delta\right]}{2T^5\left(1+Var\left[\delta\right]\right)^3}. \end{aligned}$$

The asymptotic Cramér-Rao lower bound for the standard deviation of the estimates can be determined as the square root of the diagonal elements of F_{LL}

$$\eta_{\hat{\rho}^{ml}} = \sqrt{\frac{2}{T}} \frac{Var\left[\delta\right]}{\left(1+Var\left[\delta\right]\right)^2}$$

$$\eta_{\hat{P}D^{ml}} = \sqrt{\frac{s_3\, Var\left[\delta\right]}{2T^5}} \frac{\phi\left(\gamma^{ml}\right)}{\left(1+Var\left[\delta\right]\right)^{3/2}}.$$

[35]Because the estimates are asymptotically consistent, the Cramér-Rao theorem ensures that the asymptotic covariance matrix will always exceed the inverse of I. We use the estimator $I_{\left(\hat{\rho}^{ml}, \hat{\gamma}^{ml}\right)}$ as an approximation for I.

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