Systemic Risk and International Portfolio Choice*

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Abstract

Returns on international equities are characterized by jumps; moreover, these jumps tend to occur at the same time across countries leading to *systemic risk*. In this paper, we evaluate whether systemic risk reduces substantially the gains from international diversification. First, in order to capture these stylized facts, we develop a model of international equity returns using a multivariate system of jump-diffusion processes where the arrival of jumps is simultaneous across assets. Second, we determine an investor's optimal portfolio for this model of returns. Third, we show how one can estimate the model using the method of moments. Finally, we illustrate our portfolio optimization and estimation procedure by analyzing portfolio choice across a riskless asset, the US equity index, and five international indexes. Our main finding is that, while systemic risk affects the allocation of wealth between the riskless and risky assets, it has a small effect on the composition of the portfolio of only-risky assets, and reduces marginally the gains to a US investor from international diversification: for an investor with a relative risk aversion of 3 and a horizon of one year, the certainty-equivalent cost of ignoring systemic risk is of the order $1 for every $1000 of initial investment. These results are robust to whether the international indexes are for developed or emerging countries, to constraints on borrowing and shortselling, and to reasonable deviations in the value of the parameters around their point estimates; the cost increases with the investment horizon and decreases with risk aversion.

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1 Introduction

Returns on international equities are characterized by jumps; moreover, these jumps tend to occur at the same time across countries. Our objective in this paper is to evaluate the gains from international diversification in the presence of *systemic risk*, defined as the risk from infrequent events that are highly correlated across a large number of assets.

Evidence on jumps in international equity returns is provided by Jorion (1988), Akgiray and Booth (1988), Bates (1996), and Bekaert, Erb, Harvey and Viskanta (1998). In addition, Duffee, Kupiec and White (1992) report that the incidence of abnormally high daily returns has increased in the recent past: twenty of the fifty largest postwar daily percentage movements occurred in the past decade, compared with only five in the 60s and nine in the 70s.

More importantly, these price drops are highly correlated across countries.[1] Work documenting that the correlations between international equity returns tend to be higher in periods of high market volatility or following large downside moves includes Speidell and Sappenfield (1992), Odier and Solnik (1993), Erb, Harvey and Viskanta (1994), Longin and Solnik (1995), Longin and Solnik (1998), Karolyi and Stulz (1996), De Santis and Gerard (1997), Bekaert, Erb, Harvey and Viskanta (1998), Ang and Bekaert (2000), and Ang and Chen (2000). However, the finding of large gains from international portfolio diversification in the early literature, for instance, Grubel (1968), Levy and Sarnat (1970), Lessard (1973), and Solnik (1974), relies on these correlations being low. The focus of our work is to understand whether the benefit from international diversification implied by traditional models has been reduced substantially because of systemic risk.[2]

A variety of explanations have been offered for systemic risk and contagion.[3] Engle, Ito and Lin (1990) provide evidence that connected shocks across international stock markets are largely the result of information transmission. They test for whether changes in volatility in one market cause changes in volatility in other markets, too. They liken this to a "meteor

[1]For example, on March 12, 2001, Nasdaq dropped by 6.3%, S&P 500 by 4.3%, Nikkei by 3%, and FTSE by 2%. Similarly, world equity markets fell in lockstep on October 27, 1997, when the drop from the 12-month peak was 9.2% in Britain, 35.4% in Hong Kong, 21.3% in Japan, 12.1% in Australia, 10.7% in Mexico, 27.9 % in Brazil, and 9.1% in the United States (*BusinessWeek*, November 10, 1997; *Economist*, November 1st-7th, 1997, p. 84). Other events with large correlated price drops include the Debt crises of 1982 , the Mexican crisis in December 1994, and the Russian crisis in August 1998. See Rigobon (2000) for a more complete list of dates with large market moves.

[2]The aim of our paper is not to explain the observed bias in portfolios towards home assets; for a review of the "home-bias" literature, see Stulz (1995) and Lewis (1999).

[3]For papers on the measurement of contagion in financial markets, we refer the reader to Forbes and Rigobon (1998), Rigobon (2000) and Bae, Karolyi and Stulz (2000).

shower" as opposed to localized volatility persistence within a market, which they call a "heat wave." Their paper finds strong evidence in favor of the meteor shower hypothesis, and the model for asset returns that we specify is consistent with their findings. King and Wadhwani (1990) investigate the October 1987 crash to determine why all the markets moved together despite dissimilar economic circumstances. The paper argues that a "mistake" in one market can be transmitted to other markets by means of contagion. Harvey and Huang (1991) find that large shocks in the foreign exchange futures markets are caused by macro-economic news announcements, with announcements regarding the US economy appearing to be the most influential. Other models have proposed shocks to liquidity (for instance, Allen and Gale (2000)) and the cross-hedging of macroeconomic risks (Kodres and Pritsker (1998)) as explanations for financial contagion.[4]

Systemic risk has become a matter of serious concern to policymakers, regulators of financial institutions and managers of mutual-funds and hedge-funds. One measure of its importance in the portfolio context is the large amount invested in international assets: for example, US investment in international equity funds is about 360 billion dollars (*Economist*, November 1st-7th, 1997), international equity flows are in excess of $1.5 trillion per year, and, cross-border equity flows exceed 20 percent of total world equity trading (Sorensen, Mezrich and Thadani (1993)). The existing literature has typically studied two aspects of systemic risk. First, there are papers trying to understand the causes of systemic risk events. The second area of study is the regulation of banking institutions in order to manage this risk.[5] The focus of our work on portfolio choice distinguishes it from these two strands of the literature on systemic risk.

One can also distinguish our work from the literature on portfolio choice with *idiosyncratic* jumps in returns, for example, Aase (1984), Jeanblanc-Picque and Pontier (1990), and Shirakawa (1990). In contrast to these theoretical models, our motivation is to understand the effect of *systemic* jumps on portfolio selection, and we provide a method for estimating the returns-model empirically, and for implementing the model based on these estimates. In contrast to the *static* model in Chunhachinda, Dandapani, Hamid and Prakash (1997), where polynomial goal programming is used to examine the effect of skewness on portfolio choice by

[4]Calomiris (1995) provides a survey of the literature on crises; Bandt and Hartmann (2000) and Dow (2000) survey the literature on systemic risk. See also Claessens (2000) and the papers for a World Bank conference on contagion at the web site: http://www.worldbank.org/research/interest/confs/past/papersfeb3-4/papers.htm.

[5]The Group of Ten working committee (1992) underscored the importance of managing systemic risk given the increasing concentration of deals amongst a few traders, and the impact of price, credit and liquidity risk on the stability of the banking system in the U.S.

assuming a utility function defined over the moments of the distribution of returns, our model is dynamic with preferences given by a standard constant relative risk averse utility function, and in our model the effect of skewness (and higher moments) arises because of jumps in the returns process rather than being introduced explicitly through the utility function.[6]

Our work is also related to two portfolio selection models with regime switching. In the first paper, Chow, Jacquier, Kritzman and Lowry (1999) discuss in a *static* setting how one can extend the Markowitz mean-variance analysis to allow for outliers in returns. They propose estimating two covariance matrixes, one for "good" times and another for "bad" times. Then, the optimal portfolio is selected on the basis of a blend of these covariance matrixes for the two regimes, where the weight attached to each covariance matrix reflects the investor's view about the likelihood of each regime. The second paper, Ang and Bekaert (2000), embeds an international portfolio choice problem in a dynamic model with a regime-switching data-generating process. Two regimes are considered, which correspond to a "normal" regime with low correlations and a "down-turn" regime with higher correlations. In their setup regimes can be persistent and their paper includes an analysis of portfolio choice when the short interest rate and earnings yields predict returns. The framework in Ang and Bekaert, however, cannot accommodate intermediate consumption, admits only a numerical solution even in the absence of intermediate consumption, and is difficult to estimate when there are more than two regimes or three risky assets.

In contrast to Ang and Bekaert, we develop a theoretical framework along the lines of Merton (1971); because our model nests the well-understood Merton model as a special case, it allows one to interpret cleanly the effect of systemic jumps. Also, we provide explicit analytic (though approximate) expressions for the optimal portfolio weights that allow for comparative statics with respect to the parameters driving systemic jumps. Moreover, our model can handle intermediate consumption (the solution to the portfolio problem stays the same), and can be estimated and implemented for any number of assets. While in the main text of the paper we consider only a simple IID environment, (i) we argue in Section 5.2.6 that this is sufficient to show that the effect of systemic jumps will not be large even in the presence of regime shifts, and (ii) we show in Section A.2 of the appendix how our model can be extended to allow for persistence in jumps. Our framework can also allow for predictability

[6]For early work on how skewness influences portfolio choice, see Samulelson (1970), Tsiang (1972) and Kane (1982); Kraus and Litzenberger (1976) shows the implications for equilibrium prices of a preference for positive skewness, while Kraus and Litzenberger (1983) derives the sufficient conditions on return distributions to get a three moment (mean, variance, skewness) capital asset pricing model; Harvey and Siddique (2000) provides an empirical test of the effect of skewness on asset prices.

in returns and for other state variables, just as they are incorporated in Merton (1971), but since this is tangential to our main objective, we do not include these features in the model we present.[7]

We now discuss the literature related to the estimation of jump-diffusion processes. The estimation approach we use relies on exploiting the link between the characteristic function and the Kolmogorov backward differential equation to obtain the moments and the cross-moments of the returns processes. Because it allows us to obtain the moments of returns in closed form, this estimation technique is both simple and easy to implement. The link between the characteristic function and the Kolmogorov backward differential has also been exploited in Chacko and Viceira (1999) and Singleton (1999), where they express the moments themselves in terms of the characteristic function. A brief discussion of some of the alternative approaches for estimating jump-diffusion processes follows, and a more extensive comparison of in the broader context of estimating continuous time processes is contained in Chacko (1999). Honore (1998) discusses the problems of using maximum-likelihood techniques to estimate jump-diffusion processes, in particular, that in the absence of restrictions on parameter estimates the likelihood function is unbounded and hence the MLE does not exist. Restricting the mean size of jumps in returns to be zero, Andersen, Benzoni and Lund (1998) and Chernov, Gallant, Ghysels and Tauchen (2000) use the Efficient Method of Moments, which combines Simulated Method of Moments, where simulated sample moments are matched to the theoretical moments, with a preliminary step where the unknown transition function is approximated using a semi-non-parametric Hermite expansion with a Gaussian leading term to generate moment conditions. Schaumburg (2000) extends to Levy processes the approach of Ait-Sahalia (1998), where a fully parameteric approximation of the transition function is used to alleviate the numerical problems encountered in solving for the transition density function which, typically, does not have an explicit form. Johannes, Kumar and Polson (1998), on the other hand, estimate also the jump times and sizes by treating these as latent variables in order to create a hierarchical model which is then estimated using Monte Carlo Markov Chain Methods; Eraker, Johannes and Polson (2000) extends this to account for estimation risk and to incorporates prior information.[8]

[7]Liu (1998) considers the Merton portfolio problem when the investment opportunity set is stochastic and provides closed-form solutions in several instances; when a closed-form solution is not available, one can use the perturbation method advocated in Kogan and Uppal (2000) to get closed-form approximate solutions or numerical methods as proposed in Brandt (1998).

[8]Comon (2000), on the other hand, studies the problem of estimating the parameters of rare events in financial markets in the context of a Bayesian model where the investor "learns" about these events over time using a continuous-time filter.

We conclude this section by summarizing the major contributions of our work. On the theoretical front, we provide a mathematical model of security returns that captures the stylized facts about international equity returns described above. We do this by modeling security returns as jump-diffusion processes where jumps across assets are systemic (occur simultaneously), though the size of the jump is allowed to differ across assets. Next, we derive the optimal portfolio weights for this model of returns. On the application side, we show how one can use the method of moments to estimate the model. Then, we illustrate the portfolio optimization and estimation procedure by analyzing a portfolio allocated across a riskfree asset, the US equity index, and five international equity indexes. We consider two sets of international indexes: the first for developed countries, and the second for emerging countries. We find that while systemic risk leads to a shift in the allocation of wealth from risky assets to the riskless asset, it has only a small effect on the allocation across the different risky assets, and it reduces only marginally the benefits to a US investor from international diversification. Moreover, the effect of systemic risk, measured as the increase in wealth required to compensate an agent who ignores this risk and is investing \$1000 for one year, is \$1 for the developed-country indexes and \$0.06 for the emerging-country indexes. This cost increases with horizon and risk tolerance. These results are robust to the choice of data set, to constraints on borrowing and shortselling, and to reasonable deviations in the parameter values around their point estimates.

The rest of the paper is organized as follows. In Section 2, we develop a model of asset returns that captures systemic risk. In Section 3, we derive the optimal portfolio weights when asset returns have a systemic-jump component. In Section 4, we describe our data, show how one can derive the moments for the returns in the presence of systemic risk, and use the method of moments to estimate the parameters of the returns processes. In Section 5, we calibrate the portfolio model to the estimated parameters in order to compare the portfolio weights of an investor who accounts for systemic risk and an investor who ignores systemic risk. We conclude in Section 6. The major results of the paper are given in propositions and the proofs for these propositions, along with extensions of the model, are presented in the appendix.

2 Asset returns with systemic risk

In this section, we develop a model of asset prices that allows for systemic jumps, and compare it to a pure-diffusion model without jumps. The two features of the data that we wish our returns-model to capture are (i) large changes in asset prices, and (ii) a high degree of correlation across these changes. To allow for large changes in returns we introduce a jump component in prices; to model these jumps as being systemic, we assume that this jump is common across all assets, though the distribution of the jump size is allowed to vary across assets.

We start by describing the standard continuous-time process that is typically assumed for asset returns:

$$\frac{dS_n}{S_n} = \hat{\alpha}_n dt + \hat{\sigma}_n dz_n, \ n = 1, \ldots, N, \tag{1}$$

with

$$E_t\left[\frac{dS_n}{S_n}\right] = \hat{\alpha}_n dt \tag{2}$$

$$E_t\left[\left(\frac{dS_n}{S_n}\right) \times \left(\frac{dS_m}{S_m}\right)\right] = \hat{\sigma}_{nm} dt = \hat{\sigma}_n \hat{\sigma}_m \hat{\rho}_{nm} dt, \tag{3}$$

where S_n is the price of asset n, N is the total number of risky assets being considered for the portfolio, and the correlation between the shocks dz_n and dz_m is denoted by $\hat{\rho}_{nm} dt = E(dz_n \times dz_m)$. We will denote the $N \times N$ matrix of the covariance terms arising from the diffusion components by $\hat{\mathbf{\Sigma}}$, with its typical element being $\hat{\sigma}_{nm} \equiv \hat{\sigma}_n \hat{\sigma}_m \hat{\rho}_{nm}$. We adopt the convention of denoting vectors and matrices with boldface characters in order to distinguish them from scalar quantities; parameters of the pure-diffusion returns process, and other quantities related to the pure-diffusion model, are denoted with a "hat" over the variable.

To allow for the possibility of infrequent but large changes in asset returns,[9] we extend the specification in equation (1) by introducing a jump-component to the process for returns, as in Merton (1976):

$$\frac{dS_n}{S_n} = \alpha_n dt + \sigma_n dz_n + (\tilde{J}_n - 1)\, dQ_n(\lambda_n), \ n = 1, \ldots, N, \tag{4}$$

where Q_n is a Poisson process with intensity λ_n, and $(\tilde{J}_n - 1)$ is the random jump amplitude that determines the percentage change in the asset price if the Poisson event occurs. We assume that the diffusion shock, the Poisson jump, and the random variable J_n are independent and that $J_n \equiv \ln(\tilde{J}_n)$ has a a normal distribution with mean μ_n and variance γ_n^2 implying

[9] In contrast to systemic risk, *systematic* risk refers to correlation between assets and a common factor, but does not require that the size of this correlation be large or that the correlated changes be infrequent.

that the distribution of the jump-size is asset specific. These assumptions imply that the price of the asset cannot be negative, and this can be seen explicitly from the stochastic process written in integral form:

$$S_n(t) = S_n(0)\exp\left[y_n(t) + \sum_{k=1}^{Q_n(t)} J_{nk}\right],$$

where $y_n(t) \sim N\left[(\alpha_n - \frac{1}{2}\sigma_n^2)t,\, \sigma_n^2 t\right]$, $J_{nk} \sim N\left[\mu_n, \gamma_n^2\right]$, and $Q_n(t) \sim \text{Poisson}\,(\lambda_n t)$.

Given our desire to model the large changes in prices as occurring at the same time across the risky assets, we specialize the above model by assuming that the arrival of jumps is coincident across all assets: $dQ_n(\lambda_n) = dQ_m(\lambda_m) = dQ(\lambda)$, $\forall n = \{1,\ldots,N\}$, $m = \{1,\ldots,N\}$, so that:

$$\frac{dS_n}{S_n} = \alpha_n dt + \sigma_n dz_n + (\tilde{J}_n - 1)\,dQ(\lambda),\; n = 1,\ldots,N, \tag{5}$$

with

$$E_t\left[\frac{dS_n}{S_n}\right] = \alpha_n dt + \alpha_n^{\mathsf{J}} dt, \tag{6}$$

$$E_t\left[\left(\frac{dS_n}{S_n}\right)\times\left(\frac{dS_m}{S_m}\right)\right] = \sigma_{nm} dt + \sigma_{nm}^{\mathsf{J}} dt, \tag{7}$$

Thus, for the process in (5), the total expected return in equation (6) has two components: one part coming from the diffusion process, α_n and the other, denoted α_n^{J}, from the jump process. The total covariance between dS_n and dS_m, given in (7), also arises from two sources: the covariance between the diffusion components of the returns, $\sigma_{nm} \equiv \sigma_n\sigma_m\rho_{nm}$, and the covariance between the jump components, σ_{nm}^{J}.

We denote the $N\times N$ matrix of the covariance terms arising from the diffusion components by $\mathbf{\Sigma}$, with its typical element being $\sigma_{nm} \equiv \sigma_n\sigma_m\rho_{nm}$. We assume that the jump-size is perfectly correlated across assets; as we shall see in Section 5, this turns out to be a conservative assumption, and it has the further advantage that it reduces the number of parameters to be estimated. The $N \times N$ matrix containing the covariation arising from the jump terms is denoted by $\mathbf{\Sigma}^{\mathsf{J}}$. Explicit expressions for α_{nm}^{J} and σ_{nm}^{J} in terms of the parameters of the underlying returns processes, $\{\lambda, \mu_n, \gamma_n\}$, are derived in Section 4.1 and are given in (32) and (33).

In our experiment, we wish to compare the portfolio of an investor who models security returns using the pure-diffusion process in (1), with that of an investor who accounts for systemic risk by using the jump-diffusion process in (5) but matches the first two moments of returns. Thus, we need to choose the parameters of the jump-diffusion processes in such a way that the first two moments for this process given in equations (6)–(7) match exactly the first two moments of the pure-diffusion returns process in equations (2)–(3). Even though it is straightforward to see how one can do this, because this result is important for understanding our experiment we highlight it in a proposition.

Proposition 1 *In order that the first and second moments from the jump-diffusion process match the corresponding moments from the pure-diffusion process, we set, for* $n, m = \{1, \ldots, N\}$,

$$\alpha_n = \hat{\alpha}_n - \alpha_n^{\mathsf{J}}, \tag{8}$$

$$\sigma_{nm} = \hat{\sigma}_{nm} - \sigma_{nm}^{\mathsf{J}}. \tag{9}$$

One interpretation of the above *compensation* of the parameters is that the investor using the jump-diffusion returns process takes the total expected return on the asset, $\hat{\alpha}_n$, and the covariance, $\hat{\sigma}_{nm}$, and subtracts from them α_n^{J} and σ_{nm}^{J} respectively, with the understanding that this will be added back through the jump term, $(\tilde{J}_n - 1)dQ(\lambda)$. In this way, she reduces the expected return and covariance coming from the diffusion terms in order to offset exactly the contribution of the jump.

Even though the unconditional expected return and covariance under the compensated jump-diffusion process will match those from the pure-diffusion process, the two processes will not lead to identical portfolios. This is because the jump also introduces skewness and kurtosis into the returns process (see equations (30) and (31) on page 17). In the next section, we analyze the difference between the portfolio of an investor who allows for systemic jumps in returns and an investor who ignores this effect.

3 Portfolio selection in the presence of systemic risk

In this section, we formulate and solve the portfolio selection problem when returns are given by the jump-diffusion process in (5). Given that financial markets are incomplete in the presence of jumps of random size, we determine the optimal portfolio weights using stochastic dynamic programming rather than the martingale pricing approach.

Our modeling choices are driven by the desire to develop the simplest possible framework in which one can examine the portfolio selection problem in the presence of systemic risk. Hence, we work with a model that has a constant investment opportunity set; an extension of this model to the case where the investment opportunity set is changing over time, via shifts in the likelihood of systemic jumps, is discussed in the appendix. Also, we choose to model the portfolio problem in continuous time because of the analytical convenience this affords, the results would be very similar if one considered a discrete-time setting, as can be seen by the expression for the optimal portfolio below, and from the fact that in our simple setting the portfolio policy is constant over time. Finally, we describe the model in the context of international portfolio selection, but the model applies to any set of securities with appropriate returns processes.

3.1 Optimal portfolio weights

We consider a US investor who wishes to maximize the expected utility from terminal wealth,[10] W_T, with utility being given by: $U(W) = \frac{W^\eta}{\eta}$, where $\eta < 1, \eta \neq 0$, so that constant relative risk aversion is equal to $1 - \eta$.[11] The investor can allocate funds across $n = \{0, 1, \ldots, N\}$ assets: a riskless asset denominated in US dollars ($n = 0$), a risky US equity index ($n = 1$), and risky foreign equity indexes, $n = \{2, \ldots, N\}$.

The price process for the riskless asset, S_0, is

$$dS_0 = rS_0 dt, \tag{10}$$

where r is the instantaneous riskless rate of interest, which is assumed to be constant over time. The stochastic process for the price of each equity index (in dollar terms),[12] with a common jump term is as given in equation (5), which is restated below:

$$\frac{dS_n}{S_n} = \alpha_n dt + \sigma_n dz_n + (\tilde{J}_n - 1)\, dQ(\lambda),\ n = 1, \ldots, N,$$

[10]We do not consider intermediate consumption since it has no effect on the optimal portfolio weights in our model.

[11]For the case where $\eta = 0$, the utility function is given by $\ln W$.

[12]The dollar return on a foreign equity index includes the return on currency and the return on the international equity index in local-currency terms. For the process for international equity returns, one could model separately the equity return in local-currency terms and the return on currency. We do not do this because it complicates the notation without adding any insights.

with α_n and σ_{nm} defined in equations (8) and (9).

Denoting the proportion of wealth invested in asset n by $w_n, n = \{1, \ldots, N\}$, the investor's problem at t can be written as:

$$V(W_t, t) \equiv \max_{\{w_n\}} E\left[\frac{W_T^\eta}{\eta}\right], \tag{11}$$

subject to the dynamics of wealth

$$\frac{dW_t}{W_t} = [\mathbf{w}'\mathbf{R} + r]\, dt + \mathbf{w}'\boldsymbol{\sigma}\mathbf{dZ}_t + \mathbf{w}'\mathbf{J}_t dQ(\lambda), \quad W_0 = 1, \tag{12}$$

where $\mathbf{w}$ is the $N \times 1$ vector of portfolio weights for the N risky assets, $\mathbf{R} \equiv \{\alpha_1 - r, \ldots, \alpha_N - r\}'$ is the excess-returns vector, $\boldsymbol{\sigma}\boldsymbol{\sigma}' \equiv \boldsymbol{\Sigma}$, $\mathbf{dZ}$ is the vector of diffusion shocks, and $\mathbf{J}$ is the vector of random jump amplitudes for the N assets.

Using the standard approach to stochastic dynamic programming and the appropriate form of Ito's Lemma for jump-diffusion processes, one can obtain the following Hamilton-Jacobi-Bellman equation:

$$\begin{aligned} 0 \;=\; \max_{\{\mathbf{w}\}} \Bigg\{ & \frac{\partial V(W_t,t)}{\partial t} + \frac{\partial V(W_t,t)}{\partial W} W_t [\mathbf{w}'\mathbf{R} + r] + \frac{1}{2}\frac{\partial^2 V(W_t,t)}{\partial W^2} W_t^2\, \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\ & + \lambda E\left[V\left(W_t + W_t\mathbf{w}'\mathbf{J}, t\right) - V(W_t,t)\right] \Bigg\}, \end{aligned} \tag{13}$$

where the terms on the first line are the standard terms when the processes for returns are continuous, and the term on the second line accounts for jumps in returns.

We guess (and verify) that the solution to the value function is of the following form:

$$V(W_t, t) = A(t)\frac{W_t^\eta}{\eta}. \tag{14}$$

Expressing the jump term using this guess for the value function (details are in the proof for the proposition), and simplifying the resulting differential equation, we get an equation that is independent of wealth:

$$0 \;=\; \max_{\{\mathbf{w}\}} \left\{ \frac{1}{A(t)}\frac{dA(t)}{dt} + \eta [\mathbf{w}'\mathbf{R} + r] + \frac{1}{2}\eta(\eta - 1)\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda E\left[(1 + \mathbf{w}'\mathbf{J})^\eta - 1\right] \right\}. \tag{15}$$

Differentiating the above with respect to $\mathbf{w}$, one gets the following result.

Proposition 2 *The optimal portfolio weights, in the presence of systemic risk, are given by the solution to the following system of N nonlinear equations:*

$$\mathbf{0} = \mathbf{R} + (\eta - 1)\boldsymbol{\Sigma}\mathbf{w} + \lambda E\left[\mathbf{J}\left(1 + \mathbf{w}'\mathbf{J}\right)^{\eta-1}\right], \forall t. \tag{16}$$

Note that (16) gives only an implicit equation for the portfolio weights, $\mathbf{w}$. Thus, to determine the magnitude of the optimal portfolio weights one needs to solve this equation numerically, which we do in Section 5. But one can understand many of the essential insights of the systemic-risk portfolio model by considering a series expansion of the non-linear term in (16):[13]

$$(1 + \mathbf{w}'\mathbf{J})^{\eta-1} \quad \simeq \quad 1 + (\eta - 1)\mathbf{w}'\mathbf{J} + \frac{(\eta-1)(\eta-2)}{2}(\mathbf{w}'\mathbf{J})^2. \tag{17}$$

Using the above expansion to simplify the nonlinear term in (16) leads to a quadratic equation for the portfolio weights, whose solution is given in the following proposition. In the proposition, we also present the solution for the case where there is a single risky asset ($N = 1$), which is simpler than that for the case where $N > 1$ and allows us to understand the effect of systemic jumps on portfolio weights.

Proposition 3 *The explicit (but approximate) expression for the vector of portfolio weights for an investor modeling returns using a jump-diffusion model is:*

$$\mathbf{w} \quad \simeq \quad \left((1-\eta)\hat{\boldsymbol{\Sigma}} + \left\{(1-\eta)\left(2(\eta-2)\lambda\hat{\mathbf{R}}E[\mathbf{J}'\mathbf{J}\mathbf{J}'] + (1-\eta)\hat{\boldsymbol{\Sigma}}^2\right)\right\}^{1/2}\right)^{-1} 2\hat{\mathbf{R}}.$$

For the case where there is a single risky asset ($N = 1$), the above simplifies to:

$$w \quad \simeq \quad \frac{2\,\hat{R}}{(1-\eta)\,\hat{\sigma}^2 + \sqrt{(1-\eta)\,\left(2\,(\eta-2)\,\lambda\,\mu\,(3\,\gamma^2+\mu^2)\,\hat{R} + (1-\eta)\,\hat{\sigma}^4\right)}}. \tag{18}$$

In contrast to the above solution, an investor who ignores the possibility of systemic jumps and assumes the standard model where price processes are multivariate diffusions without jumps will choose the portfolio weights given by the familiar Merton (1971) expression below.

[13]We employ the Taylor series expansion of the form: $f(a+x) = f(a) + f'(a)\frac{x}{1!} + f''(a)\frac{x^2}{2!} + \ldots$, where $f(1 + \mathbf{w}'\mathbf{J}) = (1 + \mathbf{w}'\mathbf{J})^{\eta-1}$. Note that $\mathbf{w}'\mathbf{J}$ is a scalar. These derivations, and all those that follow, were done using *Mathematica*; all the numerical work and also the estimation was done in *Mathematica* and verified independently in *Matlab*. The computer code in both languages is available upon request.

Corollary 1 *The weights chosen by an investor who assumes that returns are described by the pure-diffusion process in equation (1) are*

$$\hat{\mathbf{w}} \;=\; \frac{1}{1-\eta}\hat{\mathbf{\Sigma}}^{-1}\hat{\mathbf{R}}. \tag{19}$$

or, in the univariate case

$$\hat{w} \;=\; \frac{\hat{R}}{(1-\eta)\hat{\sigma}^2}. \tag{20}$$

The difference between the portfolio of the investor who accounts for systemic jumps, w, and that of an investor who ignores this feature of the data and chooses portfolio $\hat{w}$ can be understood by comparing the expression in (18) to that in (20). First, we note that w in (18) reduces to $\hat{w}$ in (20) when there are no jumps ($\lambda = 0$). Second, the term $\lambda\,\mu\,\left(3\,\gamma^2 + \mu^2\right)$ in the denominator of w represents the skewness of the returns process (this is derived in Proposition 5). This term is not present in the expression for $\hat{w}$ because this portfolio weight is derived assuming a pure-diffusion returns process which has no skewness (or kurtosis). Third, the sign of $w - \hat{w}$ depends on the sign of μ, which indicates whether skewness is negative or positive. For the case where $\mu < 0$ implying that skewness is negative, $w < \hat{w}$ indicating that the investor who accounts for systemic risk will invest less in the risky asset relative to the investor who ignores systemic risk.

Thus, the difference between w and $\hat{w}$ arises from the higher moments ignored in (20). Because we have considered only the first three terms in the Taylor's expansion of the nonlinear term, only skewness appears explicitly in the approximate expression for the optimal portfolio weight; if one included higher order terms in the expansion, then the expression would also reflect the higher moments of the distribution.[14]

The above discussion shows also how our model is related to that of Chunhachinda, Dandapani, Hamid and Prakash (1997), who use polynomial goal programming in a single-period model to examine the effect of skewness on portfolio choice by assuming a utility function defined over the moments of the distribution of returns. In contrast, we work with the standard constant relative risk averse utility function that is commonly used to examine optimal portfolio selection and instead modify the returns process to allow for the possibility of skewness and higher moments.

[14]The expression in (18) also seems to suggest that when $\mu = 0$ the portfolio weight w reduces to $\hat{w}$; this is only because we are considering just the first three terms of the expansion; it is not true when additional terms are included, in particular the fourth term, which captures the kurtosis of the distribution.

3.2 Comparative statics for the portfolio weights

In this section, we describe the comparative static results for the portfolio weights with respect to the three parameters controlling the systemic-jump component of returns.

Proposition 4 *The comparative static results with respect to $\{\lambda, \mu, \gamma\}$ are as follows. Defining a as,*

$$a \equiv (1-\eta)\left[2\,(\eta-2)\,\lambda\,\mu\left(3\,\gamma^2+\mu^2\right)\hat{R}+(1-\eta)\,\hat{\sigma}^4\right], \tag{21}$$

we have:

- *The relation between the portfolio weight w and the jump-intensity parameter, λ, depends on the sign of μ; if μ is negative (indicating that returns are negatively skewed), then the investment in the risky asset declines with λ:*

$$\frac{\partial w}{\partial \lambda} = \frac{2\,(-2+\eta)\,(-1+\eta)\,\mu\,(3\,\gamma^2+\mu^2)\,\hat{R}^2}{\sqrt{a}\left(\sqrt{a}+(1-\eta)\,\hat{\sigma}^2\right)^2} \tag{22}$$

- *The portfolio weight w is positively related to μ, the parameter that controls skewness; hence, as μ becomes more negative, investment in the risky asset declines:*

$$\frac{\partial w}{\partial \mu} = \frac{6\,(-2+\eta)\,(-1+\eta)\,\lambda\,(\gamma^2+\mu^2)\,\hat{R}^2}{\sqrt{a}\left(\sqrt{a}+(1-\eta)\,\hat{\sigma}^2\right)^2} \tag{23}$$

- *The relation between w and γ, the volatility of the jump, depends on the sign of μ; if μ is negative, then the investment in the risky asset declines with γ:*

$$\frac{\partial w}{\partial \gamma} = \frac{12\,\gamma\,(-2+\eta)\,(-1+\eta)\,\lambda\,\mu\,\hat{R}^2}{\sqrt{a}\left(\sqrt{a}+(1-\eta)\,\hat{\sigma}^2\right)^2}. \tag{24}$$

These comparative static results are illustrated in Figures 1 and 2. Both figures are drawn for the case where there are two risky assets with equal $\hat{\alpha}_n$ and $\hat{\sigma}_n$. Each figure plots four quantities: (i) the total investment in the two risky assets by an investor using a pure diffusion process; (ii) the total investment in the two risky assets by an investor who recognizes the possibility of systemic jumps; (iii) the investment in the first risky asset as a percentage of the total investment in risky assets for the systemic investor, and (iv) the investment in the second risky asset as a percentage of the total investment in risky assets for the systemic investor. Note that because $\hat{\alpha}_n$ and $\hat{\sigma}_n$ are the same for the two risky assets, the diffusion investor holds each one in equal proportion. Also, by construction, the portfolio of the diffusion investor is independent of the parameters for the systemic jump, $\{\lambda, \{\mu_1, \mu_2\}, \{\gamma_1, \gamma_2\}\}$. In the figures, these parameters are assigned values which match the average of the values estimated for the six indexes in the developed-country dataset, reported in the last column of Panel A of Table 3.[15]

Figures 1 provides the comparative static results with respect to the jump-intensity parameter, λ. In the top panel of this figure, $\mu_2 = 2\mu_1 < 0$, implying that the second asset is more negatively skewed than the first, while $\gamma_1 = \gamma_2 > 0$. In the lower panel, $\mu_1 = \mu_2 < 0$, while $\gamma_2 = 2\gamma_1 > 0$, implying that the second asset has a higher kurtosis than the first. In both panels, as systemic jumps are more frequent (an increase in λ), the investor reduces her total investment in risky assets, which is always less than that of the diffusion investor. Moreover, there is a shift in the composition of the risky-asset portfolio away from the asset that is more strongly affected by the systemic jump either because of a more negative μ (upper panel) or because of a larger γ (lower panel).

Figure 2 illustrates the effect of the expected jump size parameter, $\mu_2 < 0$ in the first panel, while the second panel studies the effect of $\gamma_2 > 0$, the parameter for the volatility of the jump size. From the upper panel, we see that the total investment in the risky assets is always less than what it is for the pure-diffusion case, and this difference increases as μ_2 becomes more negative. Also, the relative investment in the two assets is the same when $\mu_2 = \mu_1$ (where the two dashed lines cross) but the asset with the more negative μ has less invested in it.

The lower panel of Figure 2 shows the effect of the parameter for the volatility of the jump size for the second asset, γ_2. Again, the investor who accounts for systemic risk always invests less in risky assets relative to an investor who ignores this risk, and the relative investment in the two risky assets depends on the relative magnitude of γ: when $\gamma_2 = \gamma_1 = 0.10$, then each asset is held in equal proportion; for $\gamma_2 > \gamma_1$, the relative weight in the second asset is less than that in the first asset.

[15]The parameters for these figures are described in detail in Section 4.

3.3 Certainty equivalent cost of ignoring systemic risk

Above, we have compared the optimal portfolio weights for an investor who accounts for systemic jumps in returns and the investor who ignores this feature of the data. In this section, we compare the certainty equivalent cost of following the sub-optimal portfolio strategy. The objective of this exercise is to express in dollar terms the cost of ignoring systemic risk.

In order to quantify the cost of ignoring systemic jumps, we compute the additional wealth needed to raise the expected utility of terminal wealth under the suboptimal portfolio strategy to that under the optimal strategy. In this comparison, we denote by CEQ the additional wealth that makes lifetime expected utility under $\hat{\mathbf{w}}$, the portfolio policy that ignores systemic risk, equal to that under the optimal policy, $\mathbf{w}$. Using the notation $V(W_t, t; \mathbf{w}_i)$, $\mathbf{w}_i = \{\mathbf{w}, \hat{\mathbf{w}}\}$, to denote explicitly which portfolio weights are used to compute the value function, the compensating wealth, CEQ, is computed as follows:

$$V\Big((1+\text{CEQ})W_t, t; \hat{\mathbf{w}}\Big) \;=\; V\Big(W_t, t; \mathbf{w}\Big). \tag{25}$$

Then, from equations (14) and (25), we have

$$A(t;\hat{\mathbf{w}})\left[\frac{1}{\eta}\Big((1+\text{CEQ})W_t\Big)^{\eta}\right] \;=\; A(t;\mathbf{w})\left[\frac{1}{\eta}W_t^{\eta}\right],$$

which implies that

$$\text{CEQ} \;=\; \left[\frac{A(t;\mathbf{w})}{A(t;\hat{\mathbf{w}})}\right]^{1/\eta} - 1, \tag{26}$$

where, from the proof for Proposition 2,

$$A(t;\mathbf{w}_i) \;\equiv\; e^{\left(\eta\left[\mathbf{w}_i'\mathbf{R}+r\right]+\frac{1}{2}\eta(\eta-1)\mathbf{w}_i'\mathbf{\Sigma}\mathbf{w}_i+\lambda E\left[\left(1+\mathbf{w}_i'\mathbf{J}\right)^{\eta}-1\right]\right)(T-t)}. \tag{27}$$

The effect of systemic risk on portfolio weights and lifetime expected utility is studied in Section 5, after the estimation of the parameters for the returns processes, which we describe in the section below.

4 Estimating the model parameters

In order to evaluate the effect of systemic risk on portfolio choice, we would like to calibrate the jump-diffusion returns model to reasonable parameter values. We explain our estimation procedure in this section, which is divided into three parts. In the first part we derive the moments for security returns in the presence of systemic risk, in the second part we describe the data, and in the third, we use the method of moments to estimate the parameters of the returns processes.

4.1 Deriving the unconditional moments for returns

To derive the unconditional moments of the returns processes in (5) we exploit the relation between the characteristic function and the Kolmogorov backward equation.

Start by defining the continuously compounded return on a particular asset by $dx_n \equiv d\ln S_{nt}$, $\alpha_n^* \equiv \alpha_n - (1/2)\sigma_n^2$, $n = \{1, \ldots, N\}$. Let the characteristic function be denoted by $F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N)$, where s_n, are the Fourier-transform parameters for each stock index. The stochastic process for which the moments are derived is as follows:

$$dx_n = \alpha_n^* dt + \sigma_n dz_n + (\tilde{J}_n - 1)dQ(\lambda).$$

The Kolmogorov backward differential equation for $F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N)$ is

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\rho_{nm}\sigma_n\sigma_m\frac{\partial^2 F}{\partial x_n \partial x_m} + \sum_{n=1}^{N}\alpha_n^*\frac{\partial F}{\partial x_n} \\ &\quad +\lambda\, E\Big[F(x_1 + J_1, \ldots, x_N + J_N, \tau; s_1, \ldots, s_N) - F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N)\Big]. \end{aligned}$$

The solution to the above differential equation leads to the following characteristic function.

Lemma 1 *The characteristic function for the returns process with systemic jumps is*

$$F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N) = \exp\left[i\sum_{n=1}^{N} s_n x_n + C(\tau)\right],$$

where

$$\begin{aligned} C(\tau) &= \left[-\frac{1}{2}\sum_{n=1}^{N}\sigma_n^2 s_n^2 - \sum_{n\neq m}\rho_{nm}\sigma_n\sigma_m s_n s_m + \sum_{n=1}^{N}\alpha_n^* i s_n + \lambda M\right]\tau, \\ M &\equiv E\left(\exp\left[i\sum_{n=1}^{N} s_n J_n\right] - 1\right) \\ &= \exp\left[i\sum_{n=1}^{N}\mu_n s_n - \frac{1}{2}\sum_{n=1}^{N} s_n^2\gamma_n^2 - \sum_{n\neq m} s_n s_m \gamma_n \gamma_m\right] - 1. \end{aligned}$$

Then, to get the k^{th} (non-central) moment for asset n, we use: $\frac{1}{i^k}\left(\frac{\partial^k F}{\partial s_n^k}\right)_{s=0}$, from which one can get the central moments using the standard relation between central and non-central moments. Cross-moments such as covariance and co-skewness are also obtained from the characteristic function by taking partial derivatives with respect to both assets in the cross-moment.

Proposition 5 *The expressions for the moments of the continuously compounded returns are the following: for* $n, m = \{1, \ldots, N\}$,

$$\text{Mean} = t\left(\alpha_n - \frac{1}{2}\sigma_n^2 + \lambda\mu_n\right), \tag{28}$$

$$\text{Co-variance} = t\left[\sigma_{nm} + \lambda\left(\mu_n\mu_m + \gamma_n\gamma_m\right)\right] \tag{29}$$

$$\text{Co-skewness} = \frac{t\,\lambda\left[2\mu_n\gamma_n\gamma_m + \mu_m\left(\mu_n^2 + \gamma_n^2\right)\right]}{\text{Variance}_n\text{Variance}_m^{1/2}}, \tag{30}$$

$$\text{Excess kurtosis} = \frac{t\ \lambda\left(3{\gamma_n}^4 + 6{\gamma_n}^2{\mu_n}^2 + {\mu_n}^4\right)}{\text{Variance}_n^2}. \tag{31}$$

Comparing the mean and covariance for the jump-diffusion processes considered above, with those for the pure-diffusion processes ($\lambda = 0$), one gets the following:

$$\alpha_n^{\mathsf{J}} = \lambda\mu_n, \tag{32}$$

$$\sigma_{nm}^{\mathsf{J}} = \lambda\left(\mu_n\mu_m + \gamma_n\gamma_m\right). \tag{33}$$

With this compensation in equations (8) and (9), the expected returns and covariances will be the same under the jump-diffusion and pure-diffusion processes.

4.2 Description of the data

The analysis in this paper is from the point of view of a US investor. To ensure that our results are not sensitive to the choice of data, we consider two data sets—one for developed countries and the other for emerging economies. Both sets of data are from Datastream Inc.

The data for the developed countries consists of the month-end US$ values of the equity indexes for the period January 1982 to February 1997 for the United States (US), United Kingdom (UK), Switzerland (SW), Germany (GE), France (FR), and Japan (JP). The data for emerging economies is for the period January 1980 to December 1998, and consists of the beginning-of-month value of the equity index for the USA, Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA). To distinguish the two sets of data, we abbreviate the countries in the developed-economy dataset with two characters and the emerging-economy dataset with three characters.

Table 1 reports the descriptive statistics for the continuously compounded monthly return on index j in U.S. dollars, r_{it}, which is defined as the ratio of the log of the index value at time t and its lagged value:

$$r_{jt} = \ln \left[\frac{V_{jt}}{V_{j,t-h}} \right],$$

where V_{jt} is the US$ value of the index at time t. Examining first the moments for developed economies, we observe from Panel A of Table 1 that the excess kurtosis of returns is substantially greater than that for normal distributions (in the table, we report kurtosis in excess of 3, which is the kurtosis for the normal distribution). The excess kurtosis in the data ranges from 0.87 for France to 7.22 for the US. For the data on emerging economies, as one would expect, the excess kurtosis is much greater, ranging from 3.77 for Thailand to 9.18 for Mexico. All twelve kurtosis estimates are significant. There are two possible reasons for the kurtosis: (i) when the multivariate return series is not stationary, the mixture of distributions results in kurtosis; (ii) if the returns are characterized by large shocks, then the outliers inject kurtosis.

The second feature of the data is that skewness of the return for all the developed-market indexes is negative, and for the emerging-country indexes it is more strongly negative, except for Argentina, where it is insignificantly different from zero. The negative skewness is a well-known feature of equity index time series over this time period (1982-97). Within this period, there were several large negative shocks to the markets contributing to the negative skewness: for instance, the market crash of October 1987, and the outbreak of the Gulf war in August 1990, the Mexican crisis in December 1994, and the Russian crisis in August 1998.[16]

Table 2 reports the covariances and correlations between the returns on the international equity indexes. The correlations for the developed countries range from a low of 0.33 between the US and Japan, to a high of 0.68 between Germany and Switzerland. The average correlation between the equity markets for developed countries is 0.51. For the emerging countries, the correlations range from the very low 0.05 between Hong Kong and Argentina to 0.55 be-

[16]The negative skewness is also a consequence of the fact that volatility tends to be higher when returns are negative than when they are positive.

tween Singapore and the US. The average correlation for the emerging countries is only 0.31 which, as one would expect, is much lower than that for the developed countries.

4.3 Estimating the parameters using the method of moments

For the benchmark case of pure-diffusion process in equation (1), the parameters to be estimated are $\{\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\Sigma}}\}$, with the moment conditions available being the ones in equations (2) and (3). From these moment conditions we see that $\{\hat{\boldsymbol{\alpha}}, \hat{\boldsymbol{\Sigma}}\}$ can be estimated directly from the means and the covariances of the data series.

For the jump-diffusion process, the parameters to be estimated are $\{\lambda, \boldsymbol{\alpha}, \boldsymbol{\Sigma}, \boldsymbol{\mu}, \boldsymbol{\gamma}\}$, from the moment conditions in equations (28)–(31). In our experiment we wish to match the means and covariances of the jump-diffusion processes to those from the pure-diffusion process; that is, we want to set $\boldsymbol{\alpha} = \hat{\boldsymbol{\alpha}} - \lambda\boldsymbol{\mu}$ and $\boldsymbol{\Sigma} = \hat{\boldsymbol{\Sigma}} - \lambda(\boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\gamma}\boldsymbol{\gamma}')$. Thus, we need to estimate only λ, and the 6×1 vectors $\boldsymbol{\mu}$ and $\boldsymbol{\gamma}$ (a total of 13 parameters) to match the 6×1 kurtosis and 6×6 co-skewness conditions for a total of 42 moment conditions.[17] We choose these 13 parameters to minimize the squared deviation of the 42 moment conditions from their values implied by the data. Once we have these parameters, we can obtain $\boldsymbol{\alpha}$ by subtracting $\lambda\boldsymbol{\mu}$ from $\hat{\boldsymbol{\alpha}}$ and $\boldsymbol{\Sigma}$ by subtracting $\lambda(\boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\gamma}\boldsymbol{\gamma}')$ from $\hat{\boldsymbol{\Sigma}}$.

Table 3 reports the parameter estimates obtained using the method of moments. From Panel A of this table we see that for the developed countries, the estimated value of $\lambda = 0.0501$, and this is significantly different from 0. The estimated λ of 0.0501 indicates that on average the chance of a jump in any month is about 5%, or one jump is expected every 20 months. Our estimate of λ is lower than that in studies estimating the likelihood of a jump in the returns series of a single index, which is typically of the order 0.20; the reason is that in our model λ measures the likelihood of a *systemic* jump rather than an idiosyncratic jump. The average expected jump size across countries is -0.0483 while the volatility of the jump size is 0.1006.

From Panel B of Table 3, we see that the estimated value of λ in the data for emerging markets is 0.0138, lower than that for the return indexes of developed countries. This is not surprising given that linkages between emerging countries are much weaker than those for the developed countries considered in our sample. Observe, however, from the comparison of

[17] Note that coskewness between asset n and m is different from that between m and n; thus, the co-skewness matrix contains 6 skewness terms on the diagonal, and 30 unique co-skewness terms.

parameter estimates in Panel A for the developed countries to those for emerging markets in Panel B, that the average jump size (μ) is more than double for emerging markets—which reflects the higher skewness in this data set. Also, the average volatility of jumps (γ) is three times higher for emerging markets, and this reflects the higher kurtosis in this data. Thus, even though systemic jumps are less frequent for emerging countries than for developed countries, their expected (absolute) size and volatility are much larger.

To measure how well the estimated parameters do at matching the moments of the data, we use the estimated parameters and the moment conditions in (30)–(31) to reconstruct the skewness and kurtosis measures.[18] Comparing these reconstructed moments with the estimated moments, we see that the model does quite well in matching the kurtosis in the data but is less successful in matching the skewness. Looking at the averages, the kurtosis is matched almost exactly for both developed and emerging countries while the magnitude of skewness from the model is greater than that in the data for the developed countries (Panel A), and smaller for emerging countries (Panel B). Because the moments are not matched exactly, we will evaluate in Section 5.2 the sensitivity of our results to these parameter estimates .

5 Calibrating the effect of systemic risk

In this section, we evaluate the effect of systemic risk using the estimated values for the parameters of the returns process. In the first part of this section, we determine the optimal portfolio weights, $\mathbf{w}$, by calibrating the portfolio model described in Section 2 to the parameters reported in Table 3. We then compare these weights to those of an investor who ignores systemic risk, $\hat{\mathbf{w}}$. To evaluate whether the difference in these portfolio policies are substantial, we compute CEQ, which is the additional wealth required to raise the lifetime utility of the investor who ignores systemic risk to the level of the investor who accounts for this. In the second part of the section, we evaluate the robustness of our results to the choices we have made in undertaking our experiment.

5.1 Portfolio weights and certainty-equivalent cost

In our calibration exercise, the parameters we use for the returns process are those reported in Table 3. In addition to this, we need to specify the riskfree rate and the agent's relative

[18] The means and covariances are matched exactly by construction. The results on the comparison of coskewness from the model to that in the data are not reported but are similar to those for skewness.

risk aversion. We assume that the monthly riskless interest rate for the US investor is 0.005, which is the close to the average US one-month riskless interest rate in our data, and we set the parameter of relative risk aversion $1 - \eta$ to be 3.0.

With these parameter values, we solve numerically the first-order conditions in Proposition 2 to obtain the optimal portfolio weights for an investor who accounts for systemic risk, $\mathbf{w}$. We also compute $\hat{\mathbf{w}}$, the weights of the investor who ignores systemic jumps and models returns as a pure-diffusion process. In addition to these portfolio weights, we also report the composition of the portfolio consisting of *only* risky assets, which can be obtained by dividing each individual weight by the total investment in risky assets. These weights are given by $\mathbf{w}/(\mathbf{w}'\mathbf{1})$ for the systemic-jump case and $\hat{\mathbf{w}}/(\hat{\mathbf{w}}'\mathbf{1})$ for the pure-diffusion case.

Table 4 reports the weights for developed-country indexes in Panel A and for the emerging countries in Panel B. Observe that for both developed and emerging countries, the *total* investment in risky assets (last row, columns 2 and 3) is larger for the investor who ignores systemic risk: for developed countries the investment in risky assets is 1.1409 for the investor who ignores systemic risk as opposed to only 0.9857 for the investor who accounts for this; for emerging countries, the investment in risky assets for the pure-diffusion case is 1.0477 instead of the optimal 1.0190. Thus, as one would expect, accounting for systemic risk leads an investor to reduce the overall investment in risky assets.

To address the question of whether systemic risk implies that US investors should hold a greater proportion of US assets and diversify less internationally, we study the last two columns of the first row, which report the percentage of total risky-asset investment that is in US assets. For the developed countries (Panel A), we see that the US share in risky assets for the investor who accounts for systemic jumps is 0.7892 versus 0.7700 for the investor who ignores this effect; for the emerging-country returns (Panel B), the share of the USA in the risky-asset portfolio is again higher for the investor who accounts for systemic risk: 1.4619 as opposed to 1.4516. Hence, while systemic risk induces the investor to hold relatively more of the US asset in the risky-asset portfolio, the effect is not large.

That the effect of systemic jumps on foreign investment is not substantial can also be observed from the weights for the individual countries. Examining first Panel B of Table 4, we see that the individual weights in the last two columns are not very different for any of the emerging countries. In the case of developed countries, the weights for US, Japan, and France differ very little across the last two columns, while the difference is larger in the weights for Germany, Switzerland and the UK.

To evaluate whether the effect on lifetime utility of the portfolio strategy that accounts for systemic risk relative to the strategy that ignores this effect, we compute the quantity CEQ, defined in equation (26), which measures the additional wealth required at $t = 0$ to raise the lifetime utility of the investor following the sub-optimal portfolio strategy, $\hat{\mathbf{w}}$, to the level of the investor choosing the portfolio $\mathbf{w}$, which accounts for systemic risk. This quantity depends on the time-horizon of the investor and we report it for horizons of one to five years.

From Panel A of Table 5, we see that for the case of developed economies, and for the base case relative risk aversion of 3 that was considered in Table 4, CEQ is equal to 0.001 for a horizon of one year, and increases to 0.005 for a horizon of five years. That is, for an investor with an initial wealth of \$1000, the cost of ignoring systemic risk is \$1 if the horizon is 1 year and \$5 for a horizon of five years. For the case of emerging economies, reported in Panel B of Table 5, the CEQ is even smaller: for an initial wealth of \$1000, and relative risk aversion of 3, the cost of ignoring systemic risk ranges from \$0.06 for a horizon of one year to \$0.32 for a five-year horizon. The magnitude of CEQ indicates that the effect of systemic risk on the lifetime utility of the investor is not large. The other numbers in this table are discussed below, in Section 5.2.2.

5.2 Verifying the robustness of results

In this subsection, we examine whether the results reported above are sensitive to (i) the choice of data, (ii) the assumed degree of relative risk aversion ($1-\eta = 3$); (iii) the assumption about unrestricted borrowing and shortselling, (iv) the estimates of expected returns ($\hat{\alpha}$), which are notoriously difficult to estimate precisely, and (v) the estimates for $\{\lambda, \boldsymbol{\mu}, \boldsymbol{\gamma}\}$, the parameters driving the systemic jump. The main conclusion of this robustness exercise is that none of these factors materially change the conclusions drawn from Table 4 or the CEQ cost implied by these portfolio weights. We discuss each robustness check below.

5.2.1 Sensitivity to choice of dataset

In our discussion, we have considered two datasets: one for developed countries and the other for emerging countries. Comparing the parameter estimates for the returns process for these two sets of data, we see that systemic jumps are more likely across developed markets than for the emerging markets considered in our sample, though the expected jump size and volatility are larger for emerging countries. Our results on portfolio choice are broadly similar across these two datasets: the effect of systemic risk on portfolio weights is not large, and thus, the initial wealth required to compensate for systemic risk is also small, especially in the case of emerging countries.

5.2.2 Sensitivity to level of risk aversion

The conclusions we have drawn so far rely on the portfolio weights reported in Table 4, where the investor was assumed to have a relative risk aversion $(1-\eta)$ equal to 3. There is a large literature trying to evaluate the appropriate level of risk aversion for the representative investor, and the suggestions for the appropriate level of this parameter range from 2 to 60. In Table 5, we report the CEQ for risk aversion ranging from 1 to 10, and we see that as risk aversion increases the effect of systemic risk decreases. Thus, the base case value of 3 that we have considered is a conservative one, and our results would be even stronger for higher levels of risk aversion. The intuition for why the impact of systemic risk drops with risk aversion is straightforward: as risk aversion increases, the investor holds a smaller proportion of wealth in the risky assets; hence, the exposure to systemic risk, and its effect on CEQ, is smaller.

To get an idea about the *rate of change* in the CEQ as risk aversion and the horizon change, in Figure 3 also plot CEQ against relative risk aversion (RRA) and the investment horizon. The figure confirms that as RRA increases beyond 1, CEQ drops very quickly. Thus, the CEQ is large only for very low values of risk aversion; while these low values of risk aversion may be inappropriate for individual investors, perhaps they are less unreasonable for hedge funds.

5.2.3 Sensitivity to constraints on shortselling and borrowing

The portfolios we have examined in Table 4 were unconstrained, but in practice an investor may face constraints on borrowing or shortselling risky assets. There are two ways we could incorporate these constraints. One approach would be to constrain the numerical optimization program. Alternatively, one could extend the theoretical results in Proposition 2 to explicitly allow for constraints; this is done in Section A.1.7 of the appendix.

In Table 6, we report the optimal portfolio when borrowing and shortsales are prohibited. As in the unconstrained case, we find that there is only a small difference between the optimal portfolio $\mathbf{w}$ that accounts for systemic jumps, and $\hat{\mathbf{w}}$, the portfolio where this risk is ignored. For both developed and emerging countries, the difference is smaller than it was for the case where portfolios were unconstrained. Thus, with constraints on shortsales, the result of which is to reduce extreme portfolio positions, the effect of systemic jumps on portfolio weights is marginal. The CEQ is also significantly smaller: at the five-year horizon, it is 0.00001 for developed economies and 0.00060 for emerging economies.

5.2.4 Sensitivity to estimates of expected returns

Another concern is that the portfolio weights computed in Table 4 could potentially be sensitive to the estimate of the expected return, leading to large imbalances in the weights assigned to risky assets. There is a large literature discussing the problems in estimating expected returns, the extreme portfolios generated by this, and ways to reduce it.[19] To examine the sensitivity of our results to the estimates of expected returns, we recompute the portfolio weights by *averaging* the estimates of expected returns across all assets and using this average as a proxy for the expected return on all the assets. Thus, all assets have the same expected return so that the weights are no longer driven by differences in expected returns.

In Table 7, we report the portfolio weights when the expected return is set to be equal across all risky assets. As in Table 4, we find that the total investment in risky assets is smaller for the investor who accounts for systemic risk: 0.9726 rather than 1.1096 in the case of developed countries and 0.0673 instead of 0.0676 for emerging countries. Moreover, the individual country weights show the same pattern as in Table 4: for the developed-country portfolio, the biggest difference is in the US weight followed by that for Switzerland. For the emerging-market portfolio, the difference in weights across $\mathbf{w}$ and $\hat{\mathbf{w}}$ is even smaller than in the case of developed countries. The CEQ numbers are smaller than those for the weights corresponding to Table 4: for the developed countries they range from 0.0002 for a one-year horizon to 0.0011 for a five-year horizon. Hence, these results suggests that it is unlikely that our results are an artifact of imprecise estimation of expected returns.

5.2.5 Sensitivity to estimates of the parameters for the jump processes

Above, we have examined the robustness of our results to the estimates of expected returns. In this section, we explore the effect of the estimates for the three new parameters introduced by our jump-diffusion model (relative to the standard diffusion model): λ, which dictates the

[19] An early reference to this this problem is given in Jorion (1985) and Dumas and Jacquillat (1990). Green and Hollifield (1992) provide a good discussion of this problem, and Connor (1997) proposes a nice solution to it. Pástor (1999) proposes using information on the asset pricing framework to improve the portfolio selection process. The implications of this for bias in international portfolios towards domestic assets is explored in Gorman and Jorgensen (1997) and Britten-Jones (1999). A general discussion of small sample problems and proposed solutions, in the context of portfolio selection, can be found in Michaud (1998).

frequency of jumps, μ, which measures the expected size of jumps, and γ, which measures the variance of the jump size. In this analysis, relative risk aversion is assumed to be 3 and the riskfree rate is equal to 0.005 per month, which matches the assumptions for the base case considered in Table 4.

In order to make it possible to use figures so that we can report the weights for a wide range of parameter values, we consider a situation where an investor has to choose across one riskfree asset and only two risky assets. Moreover, the two risky assets are assumed to be symmetric, with parameter values for their returns process being the average of the values estimated for the six developed-country indexes given in Panel A of Table 3. Using these averages as the base case, in our experiment we evaluate in Figures 4-6 the portfolio weights and CEQ cost for a range of values for λ, μ and γ. Each of these figures has two panels: in the first, we report the optimal portfolio weights of an investor who accounts for systemic jumps (solid line in figure) and an investor who ignores this (flat dotted line in figure), and in the second panel we plot the CEQ corresponding to these portfolio weights.

In Figure 4, we vary the jump-intensity parameter from 0 to 0.25, which is five times its estimate value of 0.05. To gauge whether this range is broad enough, note that a λ of 0.25 implies that there are about 3 systemic jumps each year, and corresponds to skewness and kurtosis that are five times their value in the data. The figure shows that as λ increase, the difference between the two portfolio strategies increases. For $\lambda = 0.10$, which is double its estimated value, the difference in portfolio weights is 0.03 and the CEQ for an initial investment of \$1000 is only \$1 for an investor with a horizon of 5 years, and even smaller for shorter horizons. For the extreme value of $\lambda = 0.25$, the difference in the two portfolio weights is 0.06; that is, a pure-diffusion investor would invest 0.34 in each of the risky assets and 0.32 $(= 1 - 2 \times 0.34)$ in the riskfree asset, while an investor who accounts for systemic risk would invest only 0.28 in the risky assets and 0.44 $(= 1 - 2 \times 0.28)$ in the riskfree asset. The CEQ in this extreme case, for an investor with a horizon of one year is \$1, and for a horizon of five years is \$5. Thus, we conclude that small deviations from the estimated value of λ will not have a large effect on our conclusions about portfolio weights and the corresponding CEQ cost.[20]

Figure 5 considers the effect of μ, the parameter for the expected jump size that determines the sign for the skewness of returns. The average of this parameter in the data for developed

[20]This also allows one to get an idea of the effect on portfolios in a regime-switching model, where λ is stochastic and fluctuates between a high value and a low value. In such a model, if for instance λ in the high state is 0.1 and in the low state is 0, then the effect on the unconditional portfolio and CEQ will not be large.

countries is about -0.05 (Panel A of Table 3) and we allow this to vary from 0 to -0.15, three times its estimated value. A $\mu = -0.15$ implies that skewness is 4.8 times its estimated magnitude and kurtosis is 7.1 times what it is in the data. For $\mu = -0.10$, the difference in portfolio weights is about 0.03, and for $\mu = -0.15$, we find that the difference in the portfolio weights is 0.06; The effect on the CEQ is more sensitive to μ than it was to λ: for the extreme case where $\mu = -0.15$ and the investor has a horizon of 5 years, CEQ = \$6 for an initial investment of \$1000; for a more reasonable level of $\mu = -0.10$, CEQ is equal to \$1.80 for an investor with a 5-year horizon and only \$0.40 for an investor with a 1-year horizon.

Finally, in Figure 6 we vary the volatility of the jump size, γ, from 0 to 0.20, which is twice its estimated value and corresponds to 3.8 times the estimated value of skewness and 11.8 times the estimate of kurtosis. The effect of this parameter on the portfolio weights and CEQ cost is smaller than that of λ and μ. For $\gamma = 0.20$, the difference in the portfolio weights of a systemic and pure-diffusion investors is about 0.05, and the CEQ is \$5 for an investor with a five-year horizon and \$0.90 for an investor with a one-year horizon.

We conclude that our findings about the effect of systemic risk on the optimal portfolio composition and on CEQ are robust to reasonable deviations from the estimated values of the parameters. However, for parameter values that are very different from the estimated ones and for horizons much longer than the ones considered the effect of systemic risk could be important.

5.2.6 Sensitivity to assumption about IID returns

The returns process that we have developed in the paper is one that is IID; in particular, there is no persistence in jumps. We could extend this model to allow for persistence in systemic jumps, and we show in Section A.2 of the appendix how one can do this by making the arrival rate of jumps, λ, stochastic. For instance, the arrival rate of jumps could be either high (λ_H) or low (λ_L), and the persistence in jumps can then be captured by varying the rate at which λ switches from one state to another.

The reason why we have chosen to describe the simpler model in the main text of the paper is because persistence in jumps does not have a major effect on our results. To see this, denote the optimal portfolio in the regime where the likelihood of jumps is high by $\mathbf{w}(\lambda_H)$ and in the regime where jumps are low by $\mathbf{w}(\lambda_L)$. Assume also that λ_H and λ_L are such that the value in the current model assumed for λ is equal to $(\lambda_H + \lambda_L)/2$. Now, from the sensitivity results with respect to λ reported in Figure 4 and discussed in Section 5.2.5, we know that increasing the value of λ does not change substantially the CEQ; that is, if $\lambda_L = 0$ so that $\lambda_H = 2\lambda$, then the CEQ computed for the case where returns are *always* in state $\lambda_H = 0.10$ is not very different from that considered in the base case with $\lambda = 0.05$. Thus, even if the high-jump regime had extreme persistence, the effect of systemic risk on CEQ would not be very different from that reported for the base case.[21]

6 Conclusion

Returns on international equities are characterized by jumps occurring at the same time across countries leading to return distributions that are fat-tailed and negatively skewed. We develop a model of asset returns to capture these empirical properties, and then show how an investor would choose an optimal portfolio when returns have these features. We also describe how one can estimate such a model using the method of moments. We apply the proposed method to determine the weights for a portfolio allocated over a riskless asset, an equity index for the US, and five international equity indexes. We consider two sets of international indexes: one for developed countries and the other for emerging countries.

The main result from our analysis is that the portfolio of an investor who ignores systemic risk and matches only the first two moments of the returns process to the data is close to that of an investor who explicitly accounts for systemic jumps. For the case where the model is calibrated to returns on stock indexes of developed countries the economic cost of ignoring systemic risk, measured as the additional amount of wealth needed to make an investor who ignores this risk as well off as an investor who accounts for it, is about \$1 for every \$1000 invested. This cost is even smaller when we calibrate our model to returns on stock indexes for emerging countries. Thus, systemic risk reduces only slightly the gains from international diversification implied by the standard portfolio models.

[21] Of course, it is true that portfolio weights will differ across regimes; that is, there will be large differences between $\mathbf{w}(\lambda_H)$ and $\mathbf{w}(\lambda_L)$. However, the source of this difference is not the increased likelihood of systemic jumps, but rather that in the λ_H regime the overall *volatility* of returns is higher than that in the λ_L regime.

A Appendix

This first part of this appendix contains the proofs for all the propositions in the paper. The second part of the appendix presents extensions of the basic model considered in the main text of the paper.

A.1 Proofs for main results

A.1.1 Proof for Proposition 1

Equating the expressions in (6) and (7) to those for the pure-diffusion returns process in equations (2) and (3) gives the result. ■

A.1.2 Proof for Proposition 2

Simplifying the jump term in the Bellman equation (13) using the conjecture that the value function is of the form $V(W_t, t) = A(t)\frac{W^\eta}{\eta}$, we get:

$$\begin{aligned}
\lambda E\left[V(W_t + W_t\mathbf{w}'\mathbf{J}, t) - V(W_t, t)\right] &= \lambda E\left[V(W_t\left[1 + \mathbf{w}'\mathbf{J}\right], t) - V(W_t, t)\right] \\
&= \lambda\frac{A(t)W_t^\eta}{\eta}E\left[(1 + \mathbf{w}'\mathbf{J})^\eta - 1\right] \\
&= \lambda V(W_t, t)E\left[(1 + \mathbf{w}'\mathbf{J})^\eta - 1\right]. \qquad \text{(A1)}
\end{aligned}$$

After substituting (A1) into (13), one obtains the following:

$$\begin{aligned}
0 = \max_{\{\mathbf{w}\}} \Bigg\{ &\frac{\partial V(W_t, t)}{\partial t} + \frac{\partial V(W_t, t)}{\partial W}W_t\left[\mathbf{w}'\mathbf{R} + r\right] + \frac{1}{2}\frac{\partial^2 V(W_t, t)}{\partial W^2}W_t^2\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \\
&+ \lambda V(W_t, t)E\left[(1 + \mathbf{w}'\mathbf{J})^\eta - 1\right]\Bigg\}. \qquad \text{(A2)}
\end{aligned}$$

Substituting the functional form of the value function into (A2) gives equation (15). Differentiating this equation gives the result in the proposition.

To identify $A(t)$, we start by evaluating (15) at the optimal portfolio weights, $\mathbf{w}$, which implies:

$$\frac{1}{A(t)}\frac{dA(t)}{dt} = -\kappa, \qquad \text{(A3)}$$

where

$$\kappa \equiv \eta \left[\mathbf{w}'\mathbf{R} + r \right] + \frac{1}{2}\eta(\eta-1)\mathbf{w}'\mathbf{\Sigma}\mathbf{w} + \lambda E \left[(1+\mathbf{w}'\mathbf{J})^{\eta} - 1 \right]. \quad \text{(A4)}$$

Integrating then gives

$$A(t) = ae^{-\kappa t},$$

where a is the constant of integration. Using the boundary condition that

$$A(T) = ae^{-\kappa T} = 1$$

then implies that

$$a = e^{\kappa T}$$

so that

$$A(t) = e^{\kappa(T-t)}$$

and the value function is

$$V(W_t, t) = e^{\kappa(T-t)} \frac{W_t^{\eta}}{\eta}, \quad \text{(A5)}$$

with κ defined in equation (A4). ■

A.1.3 Proof for Proposition 3

Using the expansion in (17), the nonlinear term in (16) reduces to:

$$\lambda E \left[\mathbf{J}(1+\mathbf{w}'\mathbf{J})^{\eta-1} \right] \simeq \lambda E[\mathbf{J}] + (\eta-1)\lambda E[\mathbf{J}(\mathbf{w}'\mathbf{J})] + \lambda \frac{(\eta-1)(\eta-2)}{2} E[\mathbf{J}(\mathbf{w}'\mathbf{J})^2]. \quad \text{(A6)}$$

Substituting (A6) in (16) leads to the following (system of) quadratic equations for the portfolio weights:

$$\mathbf{0} = \mathbf{R} + (\eta-1)\mathbf{\Sigma}\mathbf{w} + \lambda E[\mathbf{J}] + (\eta-1)\lambda E[\mathbf{J}(\mathbf{w}'\mathbf{J})] + \lambda \frac{(\eta-1)(\eta-2)}{2} E[\mathbf{J}(\mathbf{w}'\mathbf{J})^2]. \quad \text{(A7)}$$

Solving the quadratic equation (A7), and eliminating one root by noting that it does not reduce to the solution in Corollary 1 in the absence of jumps ($\lambda = 0$), leads to the expression for the portfolio weight in the proposition. ■

A.1.4 Proof for Proposition 4

The comparative static results follow from differentiating the expression for portfolio weights in equation (18), and noting that $\eta < 1$, and $\lambda > 0$.

A.1.5 Proof for Lemma 1

The characteristic function, $F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N)$, is identified from the Kolmogorov backward equation which is:

$$\begin{aligned} \frac{\partial F}{\partial t} &= \frac{1}{2}\sum_{n=1}^{N}\sum_{m=1}^{N}\rho_{nm}\sigma_n^2\sigma_m^2\frac{\partial^2 F}{\partial x_n \partial x_m} + \sum_{n=1}^{N}\alpha_n^*\frac{\partial F}{\partial x_n} \\ &+\lambda E\left[F(x_1 + J_1, \ldots, x_N + J_N, \tau; s_1, \ldots, s_N) - F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N)\right]. \end{aligned}$$

We guess that the solution is of the form:

$$F(x_1, \ldots, x_N, \tau; s_1, \ldots, s_N) = \exp\left[i\sum_{n=1}^{N} s_n x_n + C(\tau)\right]. \tag{A8}$$

This guess can be verified by substituting the conjecture into the differential equation, which gives:

$$\frac{\partial C}{\partial \tau} = -\frac{1}{2}\sum_{n=1}^{N}\sigma_n^2 s^2 - \sum_{n \neq m}\rho_{nm}\sigma_n\sigma_m s_n s_m + \sum_{n=1}^{N}\alpha_n^* i s_n + \lambda M,$$

where

$$M = E\left(\exp\left[i\sum_{n=1}^{N} s_n J_n\right] - 1\right),$$

with the boundary condition: $\exp[i\sum_{n=1}^{N} s_n x_n] = F(x_1, \ldots, x_N, 0; s_1, \ldots, s_N)$. Solving the ordinary differential equation for $C(\tau)$ by integrating proves the result. ■

A.1.6 Proof for Proposition 5

Using the characteristic function identified in Lemma 1, we get the k^{th} (non-central) moment for asset n, by evaluating the following:

$$\frac{1}{i^k}\left(\frac{\partial^k F}{\partial s_n^k}\right)_{s=0}.$$

Central moments are then derived using the standard relation between central and non-central moments. Cross-moments, such as covariance and co-skewness, are also obtained from the characteristic function by taking partial derivatives with respect to the two assets in the cross-moment. ■

A.1.7 The optimal portfolio in the presence of constraints

In this section, we consider the portfolio selection problem in the presence of constraints: the standard constraints on short-selling and borrowing

1. Short-selling constraint restricts short positions in any of the portfolio choice assets:
$$\mathbf{w} \geq \mathbf{0}, \forall t.$$
2. Borrowing constraint imposed to prevent leveraging the portfolio:
$$\mathbf{w}_t'\mathbf{1} \leq 1, \forall t,$$
which implies that the investment in the riskless asset will always be greater than or equal to zero.

With these constraints added on, the objective function is:

$$\max_{\mathbf{w}} \left\{ \eta \left(\mathbf{w}'\mathbf{R} + r - \frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w} \right) + \frac{1}{2}\eta^2 \mathbf{w}'\mathbf{\Sigma}\mathbf{w} + \lambda_t E\left[(1 + \mathbf{w}'\mathbf{J})^{\eta} - 1 \right] \right.$$
$$\left. + \left(1 - \mathbf{w}'\mathbf{1}\right)\phi + \mathbf{w}'\boldsymbol{\xi} \right\}, \qquad \text{(A9)}$$

where $\{\phi, \boldsymbol{\xi}\}$ are Lagrange multipliers, with the standard complimentary slackness conditions:

$$1 - \mathbf{w}'\mathbf{1} \geq 0, \quad \phi = 0 \quad \text{or} \quad 1 - \mathbf{w}'1 = 0, \ \phi \geq 0 \qquad \text{(A10)}$$
$$w_i \geq 0, \quad \xi_i = 0 \quad \text{or} \quad w_i = 0, \ \xi_i \geq 0, \quad \forall i = \{1, \ldots, N\}. \qquad \text{(A11)}$$

The first-order condition for the above problem give us the following result.

Proposition A1 *In the presence of borrowing and shortselling constraints, the optimal portfolio is given by the solution to:*

$$\mathbf{0} = \mathbf{R} + (\eta - 1)\mathbf{\Sigma}\mathbf{w} + \lambda E\left[\mathbf{J}(1 + \mathbf{w}'\mathbf{J})^{\eta - 1} \right] - \mathbf{1}\phi + \boldsymbol{\xi},$$

subject to the complimentary slackness conditions in equations (A10) and (A11).

Proof: To solve the portfolio problem when the investor is constrained from borrowing and shortselling, we will work directly with the objective function in (11) along with the result in (A12), rather than using stochastic dynamic programming Merton (1971).[22] We start by considering the problem in the absence of constraints, and then in the second step analyze the effect of the constraints.

Step 1: The unconstrained problem reformulated

The solution to the stochastic differential equation in (12) is

$$W_T = W_0 \exp\left\{\int_0^T \left(\mathbf{w}'\mathbf{R} + r - \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right) dt + \int_0^T \mathbf{w}'\boldsymbol{\Sigma}\mathbf{dZ}_t + \int_0^T \mathbf{w}'\mathbf{J}\, dQ(\lambda)\right\}. \tag{A12}$$

In the lemma below, we show that the optimization problem in (11) is equivalent to maximizing the expression in (A13), which is in a much more convenient form. The equivalence is shown by exploiting the independence of the jump and the diffusion innovations and using results on moment generating functions. The expression in (A13) is independent of time because of our assumption that the parameters of the return distribution are constant over time, and the assumption of power utility, which makes the optimal weights independent of the level of wealth.[23]

Lemma A1 *The problem in (11) subject to (12) can be restated as:*

$$\begin{aligned}&\max_{\mathbf{w}} E_0\left[\frac{W_T^\eta}{\eta}\right] \\ &= \max_{\mathbf{w}}\left[\eta\left(\mathbf{w}_t{}'\mathbf{R} + r - \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right) + \frac{1}{2}\eta^2\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} + \lambda E\left[(1+\mathbf{w}'\mathbf{J})^\eta - 1\right]\right], \forall t. \end{aligned} \tag{A13}$$

Proof: Using the result in equation (A12), the objective function can be re-expressed as:

$$\max_{\mathbf{w}} E_0\left[\frac{W_T^\eta}{\eta}\right] = \max_{\mathbf{w}} E\frac{1}{\eta}\left\{\exp(\eta Y_D)\right\} E\left\{\exp(\eta Y_J)\right\}, \tag{A14}$$

[22] For the analysis of leverage constraints in a more general setting, see Cvitanic and Karatzas (1992), Tepla (1999), and Xu and Shreve (1992a,b).

[23] The stationarity of the problem can be seen directly also from the dynamic programming formulation given in the text.

where Y_D and Y_J are defined as

$$\begin{aligned} Y_D &\equiv \int_0^T \left[\mathbf{w}'\mathbf{R} + r - \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right] dt + \int_0^T \mathbf{w}'\boldsymbol{\Sigma}\mathbf{dZ}, \\ Y_J &\equiv \int_0^T \mathbf{w}'\mathbf{J} dQ(\lambda). \end{aligned}$$

We now simplify the above expression in two steps: first, we consider the diffusion terms, and then the jump terms. Considering only the diffusion terms, the maximand reduces to:

$$E\left\{\exp(\eta Y_D)\right\} = \left\{\exp\left[\eta\left\{\int_0^T \left(\mathbf{w}'\mathbf{R} + r - \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right) dt + \int_0^T \mathbf{w}'\boldsymbol{\Sigma}\mathbf{dZ}\right\}\right]\right\}.$$

Given our assumption of power utility and a constant investment opportunity set,[24] the maximand is independent of the level of wealth and can be solved instant-by-instant. Moreover, because the diffusion terms are distributed normally, we get:

$$\begin{aligned} E\left\{\exp(\eta Y_D)\right\} &= \left\{\exp\left(\eta E(Y_D) + \frac{1}{2}\eta^2 \mathrm{Var}(Y_D)\right)\right\} \\ &= \exp\left[\eta\left(\mathbf{w}'\mathbf{R} + r - \frac{1}{2}\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right) + \frac{1}{2}\eta^2\mathbf{w}'\boldsymbol{\Sigma}\mathbf{w}\right]. \qquad \text{(A15)} \end{aligned}$$

Considering the jump term, $E\left\{\exp(\eta Y_J)\right\}$, in (A14), notice that this is just the moment-generating function for the random variable Y_J with parameter η. Define a random variable $X_k = \mathbf{w}_k'\mathbf{J}_k$, for $k = 1, \ldots, n$. Thus, Y_J is a sum of iid random variables $X_1 + \ldots + X_n$ where n follows a Poisson distribution. Hence, Y_J at each point in time can be described by a compound Poisson process. Using results on generating functions in Karlin and Taylor (1975) we see that the generating function $E\left\{\exp(\eta Y_J)\right\}$ can be written as the generating function of a Poisson distributed random variable with a parameter that itself is the generating function for random variable X_k. Writing the generating function of a random variable x with parameter η as $g_x(\eta) = E[\exp(\eta x)]$, we have:

$$\begin{aligned} E\left\{\exp(\eta Y_J)\right\} &= \exp[-\lambda + \lambda g_x(\eta)] \\ &= \exp[-\lambda + \lambda E[\exp(\eta x)]] \\ &= \exp[-\lambda + \lambda E[(e^x)^\eta]] \\ &\approx \exp[-\lambda + \lambda E[(1+x)^\eta]] \end{aligned}$$

[24]This can be seen explicitly in the dynamic programming formulation presented above.

$$
\begin{aligned}
&= \exp[\lambda(E[(1+\mathbf{w}'\mathbf{J})^{\eta}]-1)] \\
&= \exp[\lambda E[(1+\mathbf{w}'\mathbf{J})^{\eta}-1]]. \qquad \text{(A16)}
\end{aligned}
$$

Combining this with the pure-diffusion terms in (A15), gives the jump-diffusion problem at each point in time:

$$
\max_{\mathbf{w}} \quad \frac{1}{\eta}\exp\left[\eta\left(\mathbf{w}'\mathbf{R}+r-\frac{1}{2}\mathbf{w}'\mathbf{\Sigma}\mathbf{w}\right)+\frac{1}{2}\eta^2\mathbf{w}'\mathbf{\Sigma}\mathbf{w}+\lambda E\left[(1+\mathbf{w}'\mathbf{J})^{\eta}-1\right]\right]. \qquad \text{(A17)}
$$

Optimizing this is equivalent to maximizing the expression in the lemma. ■

Differentiating the expression obtained in Lemma A1 with respect to **w** allows us to obtain the optimal portfolio weights, which are the same as those from solving equation (16).

Step 2: The constrained problem

With the constraints on shortselling and borrowing added on, the objective function in (A13) is equivalent to a Lagrangian problem, with the simplifying steps being the same as the ones for Lemma A1. This objective function is given in (A9), along with the standard complimentary slackness conditions: (A10) and (A11). Differentiating (A9) with respect to **w**, subject to equations (A10) and (A11), gives the result. ■

A.2 Extended model with regime shifts in jump intensity

In the framework developed in the main text, jumps in returns are transitory. However, if one wanted a model where the likelihood of jumps was stochastic over time, one could extend the basic model to allow for regimes in the intensity of jumps. This model, similar in flavor to that of Ang and Bekaert (2000), would improve the fit of the returns process; however, the essential insight that systemic risk has only a small effect on the gains from international diversification does not change. Thus, we have chosen to present the simpler model in the text, and the model with regimes is presented in this appendix.

In a model where λ_t is stochastic rather than constant, stock returns are given by:

$$\frac{dS_n}{S_n} = \alpha_n dt + \sigma_n dz_n + (\tilde{J}_n - 1)\, dQ(\lambda_t),\ n = 1, \ldots, N. \tag{A18}$$

We will restrict attention to the case where there are only two regimes for λ, $\{\lambda_L = 0, \lambda_H = 2\bar{\lambda}\}$, equally spaced around the value $\bar{\lambda}$. This implies that in one regime there are no jumps at all (a pure-diffusion world), and in the other jumps arrive with intensity $2\bar{\lambda}$. The process for λ is assumed to be given by:

$$d\lambda_t = 2\left(\bar{\lambda} - \lambda_t\right) dQ^{\lambda}(\nu), \tag{A19}$$

where Q^{λ} is a Poisson process (distinct from Q), ν is the intensity of Q^{λ}, and $\bar{\lambda}$ is the mean of λ, which in our setting is equal to $\left(\frac{\lambda_H + \lambda_L}{2}\right)$.

The investor's problem at t can be written as:

$$V(W_t, \lambda_t, t) \equiv \max_{\{w_n\}} E\left[\frac{W_T^{\eta}}{\eta}\right], \tag{A20}$$

subject to the dynamics of wealth

$$\frac{dW_t}{W_t} = \left[\mathbf{w}_t'\mathbf{R} + r\right] dt + \mathbf{w}_t'\boldsymbol{\sigma}\mathbf{dZ}_t + \mathbf{w}_t'\mathbf{J}_t dQ(\lambda_t), \quad W_0 = 1, \tag{A21}$$

The Hamilton-Jacobi-Bellman equation.

$$0 \;=\; \max_{\{\mathbf{w}\}} \left\{ \frac{\partial V(W_t, \lambda_t, t)}{\partial t} + \frac{\partial V(W_t, \lambda_t, t)}{\partial W} W_t \left[\mathbf{w}_t'\mathbf{R} + r\right] + \frac{1}{2}\frac{\partial^2 V(W_t, \lambda_t, t)}{\partial W^2} W_t^2\, \mathbf{w}_t'\boldsymbol{\Sigma}\mathbf{w}_t \right.$$

$$+ \lambda_t E\left[V\left(W_t + W_t \mathbf{w}_t' \mathbf{J}, \lambda_t, t\right) - V(W_t, \lambda_t, t)\right]$$

$$\left. + \nu\left[V\left(W_t, \lambda_t + 2(\bar{\lambda} - \lambda_t), t\right) - V(W_t, \lambda_t, t)\right] \right\}, \tag{A22}$$

where the terms on the first line are the standard terms when the processes for returns are continuous, the term on the second line accounts for jumps in returns, and the term on the last line accounts for jumps in λ.

We guess (and verify below) that the solution to the value function is of the following form: $V(W_t, \lambda_t, t) = A(\lambda_t, t)\frac{W^\eta}{\eta}$. Expressing the jump terms using this guess for the value function and simplifying the resulting differential equation, we get an equation that is independent of wealth:

$$\begin{aligned} 0 \quad = \quad \max_{\{\mathbf{w}_t\}} & \left\{ \frac{1}{A(\lambda_t, t)} \frac{\partial A(\lambda_t, t)}{\partial t} + \eta\left[\mathbf{w}_t' \mathbf{R} + r\right] + \frac{1}{2}\eta(\eta - 1)\mathbf{w}_t' \mathbf{\Sigma} \mathbf{w}_t \right. \\ & \left. + \lambda_t E\left[(1 + \mathbf{w}_t' \mathbf{J})^\eta - 1\right] + \nu\left[\frac{A(2\bar{\lambda} - \lambda_t, t) - A(\lambda_t, t)}{A(\lambda_t, t)}\right] \right\}. \end{aligned} \tag{A23}$$

Differentiating the above with respect to $\mathbf{w}_t$, one gets the following result (proof not included).

Proposition A2 *The optimal portfolio weights, in the presence of systemic risk, are given by the solution to the following system of N nonlinear equations:*

$$\mathbf{0} = \mathbf{R} + (\eta - 1)\mathbf{\Sigma}\mathbf{w}(\lambda_t) + \lambda_t E\left[\mathbf{J}\left(1 + \mathbf{w}'(\lambda_t)\mathbf{J}\right)^{\eta - 1}\right], \forall t, \tag{A24}$$

where, to indicate the dependence on λ, the portfolio weight is expressed as a function of λ_t.

Observe that ν does not appear in the above equation; this is because the change in λ is not correlated with changes in stock returns. Note also that now the difference between the optimal weights, $\mathbf{w}(\lambda_t)$, and the weights of an investor who uses a pure-diffusion process for returns, $\hat{\mathbf{w}}(\bar{\lambda})$ arises from *two* sources: one, the pure-diffusion investor does not condition her portfolio on the current value of $\lambda_t = \{\lambda_H, \lambda_L\}$; two, the pure-diffusion investor ignores the effect of systemic jumps. As discussed in Section 5.2.6, the unconditional portfolio $\mathbf{w}(\bar{\lambda})$ is quite close to that of the pure-diffusion investor, $\hat{\mathbf{w}}$; however, the conditional portfolios, $\mathbf{w}(\lambda_H)$ and $\mathbf{w}(\lambda_L)$, are quite different from one another.

Table 1: Descriptive statistics for equity returns—univariate

Panel A of this table gives the first four moments of the monthly returns in US dollar terms for the developed-country indexes and Panel B gives the same information for the US and five emerging markets. The data for the developed countries is for the period January 1982 to February 1997, and comprises of 182 observations of month-end values of the equity indexes for the US, UK, Japan (JP), Germany (GE), Switzerland (SW) and France (FR). The data for emerging economies consists of 227 observations of the beginning-of-month value of the equity indexed for the USA, Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA) for the period January 1980 to December 1998. The numbers in the table are discussed in Section 4.2 on page 17.

Panel A: Developed countries—Moments

	US	UK	JP	GE	SW	FR	Avg.
Mean	0.0102	0.0084	0.0080	0.0120	0.0102	0.0113	0.0100
Standard Deviation	0.0420	0.0567	0.0697	0.0576	0.0515	0.0611	0.0564
Variance	0.0018	0.0032	0.0049	0.0033	0.0027	0.0037	0.0033
Skewness	-1.1648	-0.4623	-0.0508	-0.2308	-0.6382	-0.4325	-0.4966
significance level	0.0000	0.0115	0.7815	0.2074	0.0004	0.0181	
Excess Kurtosis	7.2236	1.9212	0.8754	2.9546	5.5405	1.5780	3.3489
significance level	0.0000	0.0000	0.0180	0.0000	0.0000	0.0000	

Panel B: Emerging countries—Moments

	USA	ARG	HKG	MEX	SNG	THA	Avg.
Mean	0.0104	0.0040	0.0076	0.0034	0.0065	0.00004	0.0053
Standard Deviation	0.0414	0.2153	0.1026	0.1437	0.0772	0.1037	0.1140
Variance	0.0017	0.0464	0.0105	0.0206	0.0060	0.0108	0.0160
Skewness	-1.1353	0.1187	-1.4163	-2.0224	-0.7684	-0.6077	-0.9719
significance level	0.0000	0.4681	0.0000	0.0000	0.0000	0.0002	
Excess Kurtosis	6.1823	6.2377	6.9388	9.1851	4.8603	3.7800	6.1974
significance level	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	

Table 2: Descriptive statistics for equity returns—multivariate

Panel A give the covariances and the correlations between US dollar returns for the the developed-country indexes and Panel B gives this for the emerging-country indexes. The data for the developed countries is for the period January 1982 to February 1997, and comprises of 182 observations of month-end values of the equity indexes for the US, UK, Japan (JP), Germany (GE), Switzerland (SW) and France (FR). The data for emerging economies consists of 227 observations of the beginning-of-month value of the equity indexed for the USA, Argentina (ARG), Hong Kong (HKG), Mexico (MEX), Singapore (SNG), and Thailand (THA) for the period January 1980 to December 1998. The numbers in the table are discussed in Section 4.2 on page 17.

Panel A: Developed countries—Covariances (normal) *and Correlations (italics)*

	US	UK	JP	GE	SW	FR	Avg. correl.
US	0.0018	*0.5750*	*0.3348*	*0.4274*	*0.5472*	*0.5173*	
UK	0.0013	0.0032	*0.4468*	*0.4916*	*0.5593*	*0.5507*	
JP	0.0009	0.0017	0.0049	*0.3815*	*0.4442*	*0.4765*	
GE	0.0010	0.0016	0.0015	0.0033	*0.6873*	*0.6593*	
SW	0.0011	0.0016	0.0015	0.0020	0.0027	*0.6161*	*0.5142*
FR	0.0013	0.0019	0.0020	0.0023	0.0019	0.0037	

Panel B: Emerging countries—Covariances (normal) *and Correlations (italics)*

	USA	ARG	HKG	MEX	SNG	THA	Avg. correl.
USA	0.0017	*0.1039*	*0.4051*	*0.3586*	*0.5519*	*0.3445*	
ARG	0.0009	0.0464	*0.0580*	*0.2167*	*0.0842*	*0.1286*	
HKG	0.0017	0.0012	0.0105	*0.2475*	*0.5479*	*0.4347*	
MEX	0.0021	0.0067	0.0036	0.0206	*0.3543*	*0.2972*	
SNG	0.0017	0.0014	0.0043	0.0039	0.0060	*0.5291*	*0.3109*
THA	0.0014	0.0028	0.0046	0.0044	0.0042	0.0108	

Table 3: Parameter estimates for the returns processes

This table reports estimates of the parameters for the multivariate system of jump-diffusion asset returns, $\{\lambda, \boldsymbol{\mu}, \boldsymbol{\gamma}\}$ obtained by minimizing the square of the difference between the moment conditions in equations (30)–(31) and the moments implied by the data. Panel A gives the estimates for developed-country return indexes and Panel B gives the estimates for emerging economies. In addition to the parameter estimates, the table reports the reconstructed moments which are obtained by substituting the parameters estimated into equations (30)–(31), which are then compared to the moments of the data. The numbers in the table are discussed in Section 4.3 on page 19.

Panel A: Developed countries

	US	UK	JP	GE	SW	FR	Avg.
λ	0.0501						0.0501
μ	-0.0660	-0.0797	0.0043	-0.0344	-0.0466	-0.0675	-0.0483
γ	0.0914	0.0792	0.1075	0.1167	0.1185	0.0902	0.1006
Skewness: reconstructed	-1.3160	-0.5496	0.0222	-0.3782	-0.7567	-0.4291	-0.5679
Skewness: in data	-1.1648	-0.4623	-0.0508	-0.2308	-0.6382	-0.4325	-0.4966
Ex. kurtosis: reconstructed	7.2148	1.9182	0.8540	2.9662	5.5474	1.5872	3.3480
Ex. kurtosis: in data	7.2236	1.9212	0.8754	2.9546	5.5405	1.5780	3.3489

Panel B: Emerging countries

	USA	ARG	HKG	MEX	SNG	THA	Avg.
λ	0.0138						0.0138
μ	-0.1280	0.2292	-0.2295	-0.2631	-0.1576	-0.1107	-0.1099
γ	0.0919	0.7179	0.3001	0.4929	0.2068	0.3009	0.3518
Skewness: reconstructed	-1.0434	0.5085	-0.9507	-0.9806	-0.7272	-0.3903	-0.5973
Skewness: in data	-1.1353	0.1187	-1.4163	-2.0224	-0.7684	-0.6077	-0.9719
Ex. kurtosis: reconstructed	6.1980	6.2009	6.9511	9.1905	4.8729	3.7623	6.1960
Ex. kurtosis: in data	6.1823	6.2377	6.9388	9.1851	4.8603	3.7800	6.1974

Table 4: Portfolio weights—base case

This table gives the portfolio weights for an investor who chooses investments in six equity indexes and the riskless asset to maximize expected utility of terminal wealth. The first two columns of weights give (a) the optimal weights, $\mathbf{w}$, for an investor who accounts for systemic jumps; and, (b) the weights $\hat{\mathbf{w}}$, for an investor who ignores systemic jumps and assumes a pure-diffusion process for returns. For these two sets of weights, the last two columns of the table give the composition of the risky-asset portfolio, which is obtained by dividing the weight for each index by the total investment in risky assets. We assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. The weights are reported for two cases: in Panel A, for a portfolio diversified across equity indexes of developed countries; in Panel B for a portfolio diversifies across indexes for emerging countries. The weights reported in the table are discussed in Section 5.1 on page 20.

Panel A: Developed countries

Country	Systemic $\mathbf{w}$	Diffusion $\hat{\mathbf{w}}$	Risky-asset portfolio $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	$\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$
US	0.7779	0.8784	0.7892	0.7700
UK	-0.3424	-0.3255	-0.3474	-0.2853
JP	-0.0598	-0.0726	-0.0607	-0.0637
GE	0.5319	0.5417	0.5396	0.4748
SW	-0.0346	0.0028	-0.0351	0.0025
FR	0.1128	0.1161	0.1144	0.1018
Riskless	0.0143	-0.1409	0	0
Total in risky assets	0.9857	1.1409	1	1

Panel B: Emerging countries

Country	Systemic $\mathbf{w}$	Diffusion $\hat{\mathbf{w}}$	Risky-asset portfolio $\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	$\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$
USA	1.4896	1.5209	1.4619	1.4516
ARG	0.0033	-0.0019	0.0032	-0.0018
HKG	0.0474	0.0519	0.0465	0.0495
MEX	-0.1070	-0.1046	-0.1050	-0.0999
SNG	-0.1258	-0.1267	-0.1235	-0.1209
THA	-0.2885	-0.2919	-0.2831	-0.2786
Riskless	-0.0190	-0.0477	0	0
Total in risky assets	1.0190	1.0477	1	1

Table 5: Certainty equivalent (CEQ) cost of ignoring systemic risk

This table gives the CEQ for an investor who chooses investments in six equity indexes and the riskless asset to maximize expected utility of terminal wealth. The CEQ measures the additional initial wealth, per dollar of investment, in order to raise the utility of an investor who ignores systemic risk to the level of an investor who recognizes this risk. The table reports the CEQ for different investment horizons and for different levels of relative risk aversion, RRA, which is equal to $1 - \eta$. The CEQ are reported for two cases: in Panel A, for a portfolio diversified across equity indexes of developed countries; in Panel B for a portfolio diversifies across indexes for emerging countries. The riskless interest rate is assumed to be 0.005 per month. Details of how this CEQ is computed are given in Section 3.3; a discussion of the numbers reported in the table can be found in Section 5.1 on page 20.

Panel A: Developed countries

	Investment horizon				
RRA $(1 - \eta)$	1 year	2 years	3 years	4 years	5 years
1	0.010765	0.021645	0.032643	0.043759	0.054995
2	0.002126	0.004257	0.006392	0.008531	0.010675
3 (base case)	0.001013	0.002027	0.003043	0.004059	0.005076
4	0.000637	0.001274	0.001912	0.002550	0.003188
5	0.000457	0.000913	0.001370	0.001827	0.002285
6	0.000353	0.000706	0.001060	0.001413	0.001767
7	0.000287	0.000573	0.000860	0.001147	0.001434
8	0.000241	0.000481	0.000722	0.000963	0.001204
9	0.000207	0.000414	0.000622	0.000829	0.001036
10	0.000182	0.000363	0.000545	0.000727	0.000909

Panel B: Emerging countries

	Investment horizon				
RRA $(1 - \eta)$	1 year	2 years	3 years	4 years	5 years
1	0.000639	0.001278	0.001918	0.002558	0.003198
2	0.000133	0.000265	0.000398	0.000531	0.000664
3 (base case)	0.000064	0.000128	0.000192	0.000256	0.000320
4	0.000040	0.000081	0.000121	0.000161	0.000202
5	0.000029	0.000058	0.000087	0.000116	0.000145
6	0.000022	0.000045	0.000067	0.000090	0.000112
7	0.000018	0.000037	0.000055	0.000073	0.000091
8	0.000015	0.000031	0.000046	0.000061	0.000077
9	0.000013	0.000026	0.000040	0.000053	0.000066
10	0.000012	0.000023	0.000035	0.000046	0.000058

Table 6: Portfolio weights with borrowing and shortsales constraints

This table gives the portfolio weights for an investor facing borrowing and short-sale constraints and who chooses investments in six equity indexes and the riskless asset to maximize expected utility of terminal wealth. The first two columns of weights give (a) the optimal weights, $\mathbf{w}$, for an investor who accounts for systemic jumps; and, (b) the weights $\hat{\mathbf{w}}$, for an investor who ignores systemic jumps and assumes a pure-diffusion process for returns. For these two sets of weights, the last two columns of the table give the composition of the risky-asset portfolio, which is obtained by dividing the weight for each index by the total investment in risky assets. We assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1-\eta$, is 3. The riskless interest rate is 0.005 per month. The weights are reported for two cases: in Panel A, for a portfolio diversified across equity indexes of developed countries; in Panel B for a portfolio diversifies across indexes for emerging countries. The weights reported below are discussed in Section 5.2.3 on page 23.

Panel A: Developed countries

Country	Systemic	Diffusion	Risky-asset portfolio	
	$\mathbf{w}$	$\hat{\mathbf{w}}$	$\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	$\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$
US	0.5515	0.5547	0.5515	0.5547
UK	0	0	0	0
JP	0	0	0	0
GE	0.4425	0.4338	0.4425	0.4338
SW	0	0	0	0
FR	0.0059	0.0115	0.0059	0.0115
Riskless	0	0	0	0
Total in risky assets	1.0000	1.0000	1	1

Panel B: Emerging countries

Country	Systemic	Diffusion	Risky-asset portfolio	
	$\mathbf{w}$	$\hat{\mathbf{w}}$	$\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	$\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$
USA	0.9442	1.0000	1.0000	1.0000
ARG	0	0	0	0
HKG	0	0	0	0
MEX	0	0	0	0
SNG	0	0	0	0
THA	0	0	0	0
Riskless	0.0558	0	0	0
Total in risky assets	0.9442	1.0000	1	1

Table 7: Portfolio weights assuming identical expected returns

In order to account for estimation error, in this table the expected return is set to be the same across all assets—based on the average reported in Table 1. As before, the portfolio weights reported are for an investor who chooses investments in six equity indexes and the riskless asset to maximize expected utility of terminal wealth. The first two columns of weights give (a) the optimal weights, $\mathbf{w}$, for an investor who accounts for systemic jumps; and, (b) the weights $\hat{\mathbf{w}}$, for an investor who ignores systemic jumps and assumes a pure-diffusion process for returns. For these two sets of weights, the last two columns of the table give the composition of the risky-asset portfolio, which is obtained by dividing the weight for each index by the total investment in risky assets. We assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. The weights are reported for two cases: in Panel A, for a portfolio diversified across equity indexes of developed countries; in Panel B for a portfolio diversifies across indexes for emerging countries. The weights reported in the table below are discussed in Section 5.2.4 on page 24.

Panel A: Developed countries

Country	Systemic	Diffusion	Risky-asset portfolio	
	$\mathbf{w}$	$\hat{\mathbf{w}}$	$\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	$\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$
US	0.6258	0.7145	0.6435	0.6439
UK	0.0219	0.0362	0.0225	0.0326
JP	0.1267	0.1159	0.1303	0.1045
GE	0.1641	0.1732	0.1688	0.1561
SW	0.1008	0.1342	0.1037	0.1209
FR	-0.0668	-0.0643	-0.0687	-0.0580
Riskless	0.0274	-0.1096	0	0
Total in risky assets	0.9726	1.1096	1	1

Panel B: Emerging countries

Country	Systemic	Diffusion	Risky-asset portfolio	
	$\mathbf{w}$	$\hat{\mathbf{w}}$	$\frac{\mathbf{w}}{\mathbf{w}'\mathbf{1}}$	$\frac{\hat{\mathbf{w}}}{\hat{\mathbf{w}}'\mathbf{1}}$
USA	0.0676	0.0677	1.0043	1.0016
ARG	0.0013	0.0013	0.0191	0.0192
HKG	-0.0002	-0.0002	-0.0036	-0.0024
MEX	-0.0022	-0.0021	-0.0326	-0.0315
SNG	-0.0015	-0.0015	-0.0230	-0.0228
THA	0.0024	0.0024	0.0357	0.0358
Riskless	0.9327	0.9324	0	0
Total in risky assets	0.0673	0.0676	1	1

Figure 1: Relative weights with respect to jump intensity

This figure shows the total proportion of wealth invested in risky assets by an investor who accounts for "systemic jumps" (solid line) and an investor who ignores this and models returns as a "pure diffusion" (dotted line), and the proportion invested in the risky assets *relative* to the total investment in risky assets (the two dashed lines). The model considered is one where there are only *two* risky assets. The parameters for the returns processes for both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table 3, with the following exceptions: λ, is allowed to range from 0 to 0.25, and in the first panel the mean jump size of the second asset is set equal to twice the average in the data, and in the second panel, the variance of the jumps size for the second asset is set to be twice the average in the data. As before, we assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. This figure is discussed in Section 3.2 on page 13.

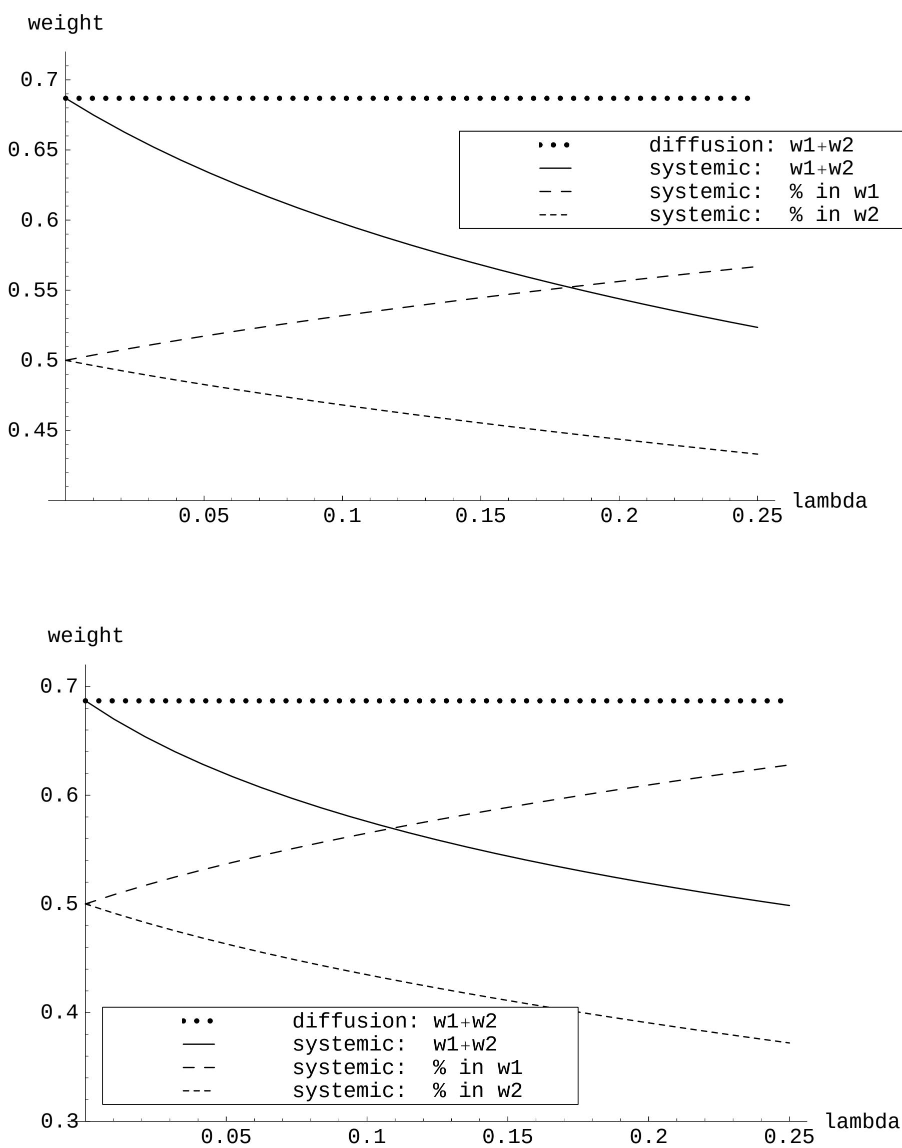


Figure 2: Relative weights with respect to mean and volatility of jump size

This figure shows the total proportion of wealth invested in risky assets by an investor who accounts for "systemic jumps" (solid line) and an investor who ignores this and models returns as a "pure diffusion" (dotted line), and the proportion invested in the risky assets *relative* to the total investment in risky assets (the two dashed lines). The model considered is one where there are only *two* risky assets. The parameters for the returns processes for both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table 3 with the following exceptions: in the first panel the mean jump size of the second asset is allowed to range from 0 to -0.15, and in the second panel the volatility of the jumps size for the second asset is allowed to range from 0 to 0.20. As before, we assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. This figure is discussed in Section 3.2 on page 13.

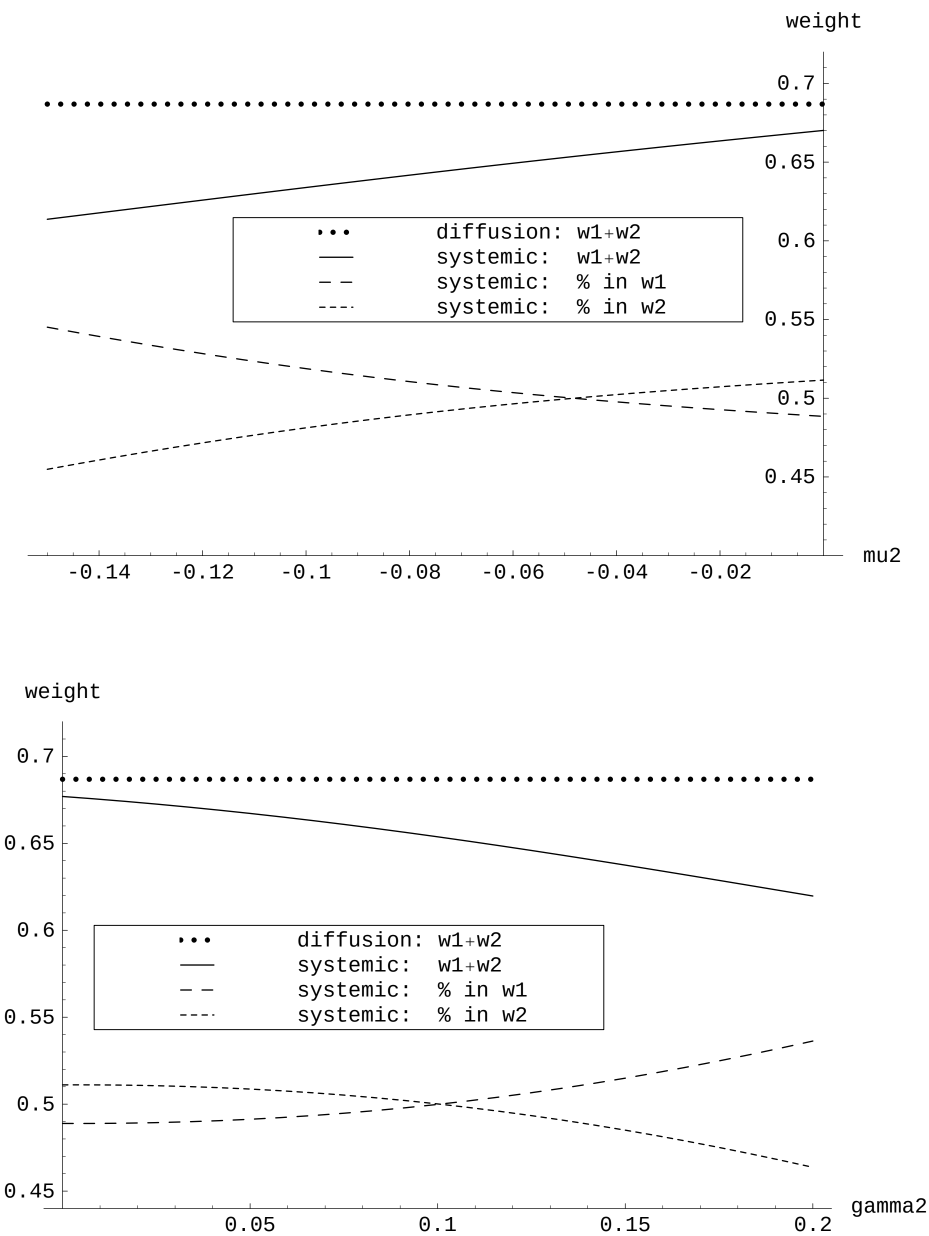


Figure 3: CEQ cost for different time horizons and levels of risk aversion

In this figure, we plot the certainty equivalent (CEQ) cost for different levels of relative risk aversion $(1-\eta)$ and time horizon, measured in years. The CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make her as well off as the investor who accounts for systemic risk. The CEQ reported in this figure is for portfolios diversified across developed countries This figure is discussed in Section 5.2.2 on page 23.

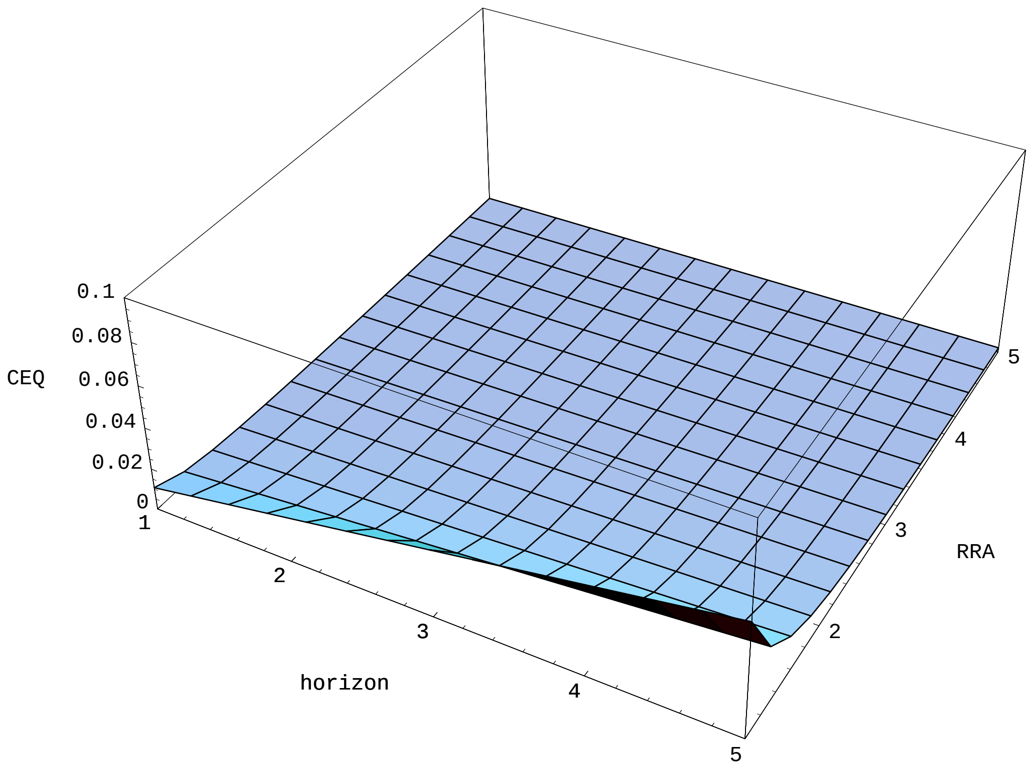
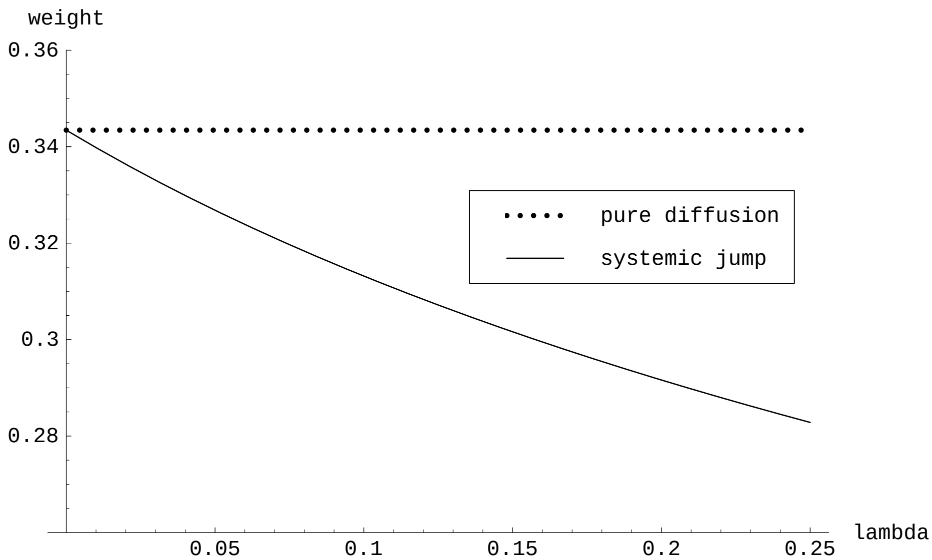


Figure 4: Portfolio weights and CEQ with respect to jump intensity

The first panel of the figure gives the portfolio of an investor who accounts for "systemic jumps" (solid line) and an investor who ignores this and models returns as a "pure diffusion" (dotted line). The case considered is one where there are only *two* risky assets. The parameters for the returns processes for both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table 3, with the exception of λ, which is allowed to range from 0 to 0.25, corresponding to skewness and kurtosis ranging from 0 to 5 times their estimates in the data. The second panel shows the corresponding CEQ for these two portfolios, where CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make him as well off as the investor who accounts for systemic risk. As before, we assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. This figure is discussed in Section 5.2.5 on page 24.



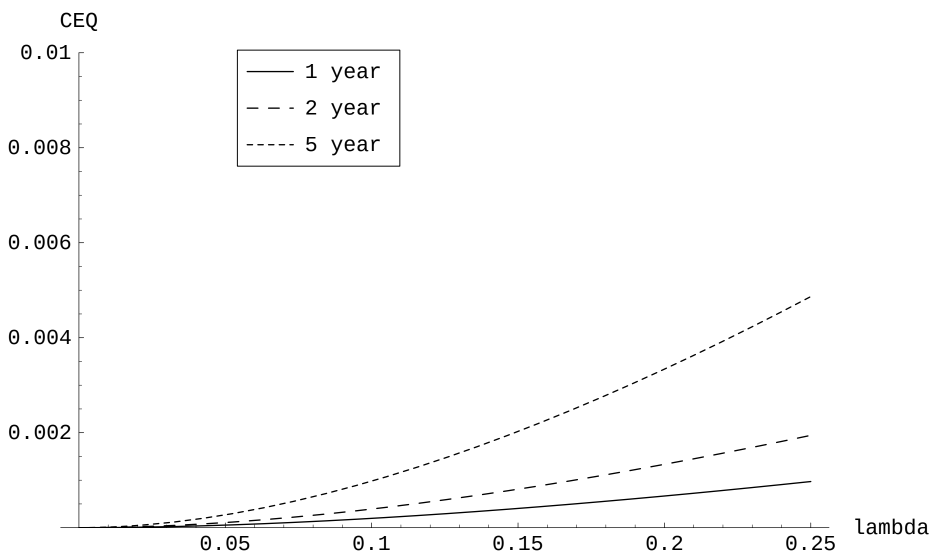
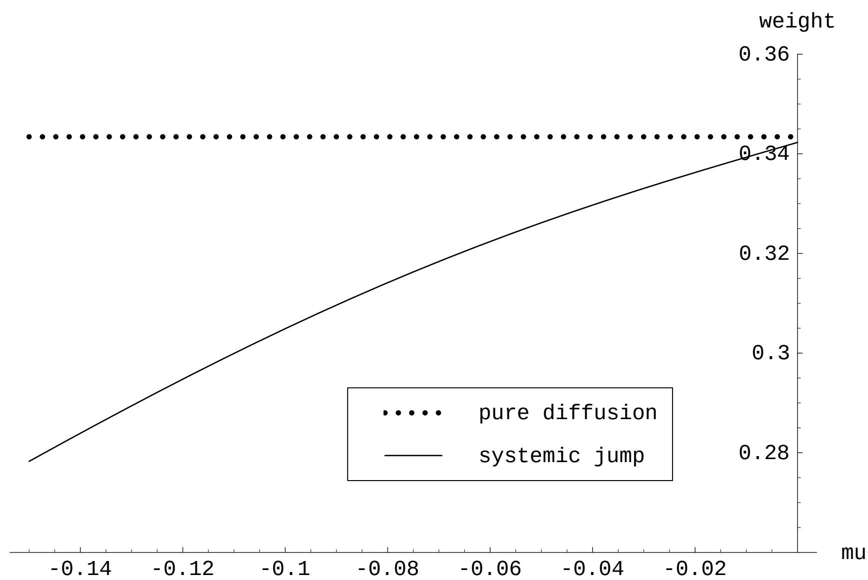


Figure 5: Portfolio weights and CEQ with respect to mean of jump size

The first panel of the figure gives the portfolio of an investor who accounts for "systemic jumps" (solid line) and an investor who ignores this and models returns as a "pure diffusion" (dotted line). The case considered is one where there are only *two* risky assets. The parameters for the returns processes on both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table 3, with the exception of μ, which is allowed to range from 0 to -0.15, corresponding to skewness ranging from 0 to 4.8 times its estimate in the data, and kurtosis ranging from 0 to 7.1 times its estimated value. The lower plot shows the corresponding CEQ for these two portfolios, where CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make him as well off as the investor who accounts for systemic risk. As before, we assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. This figure is discussed in Section 5.2.5 on page 24.



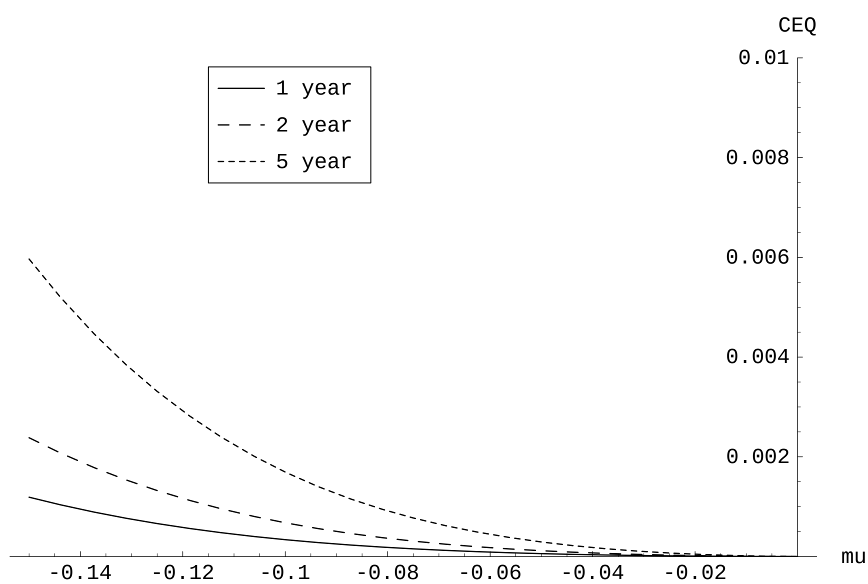
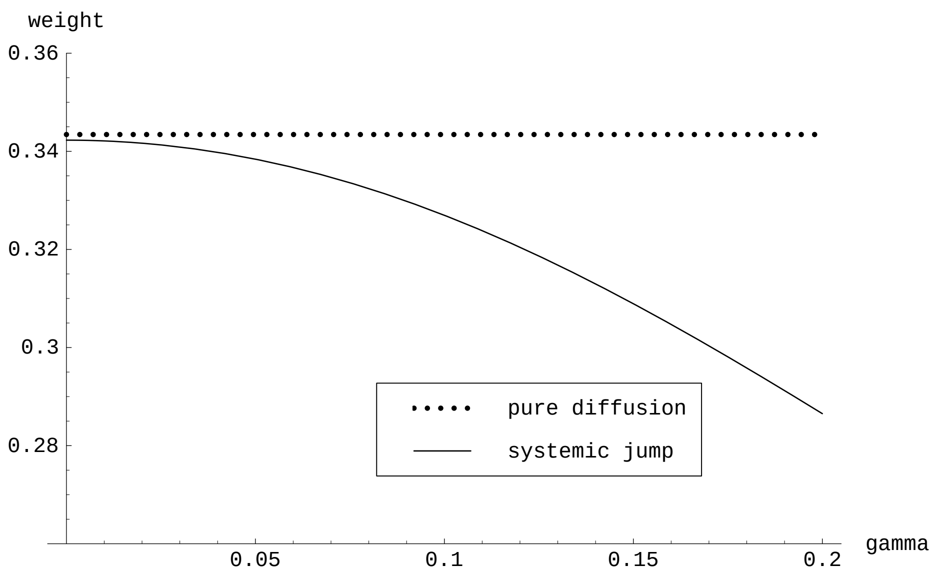
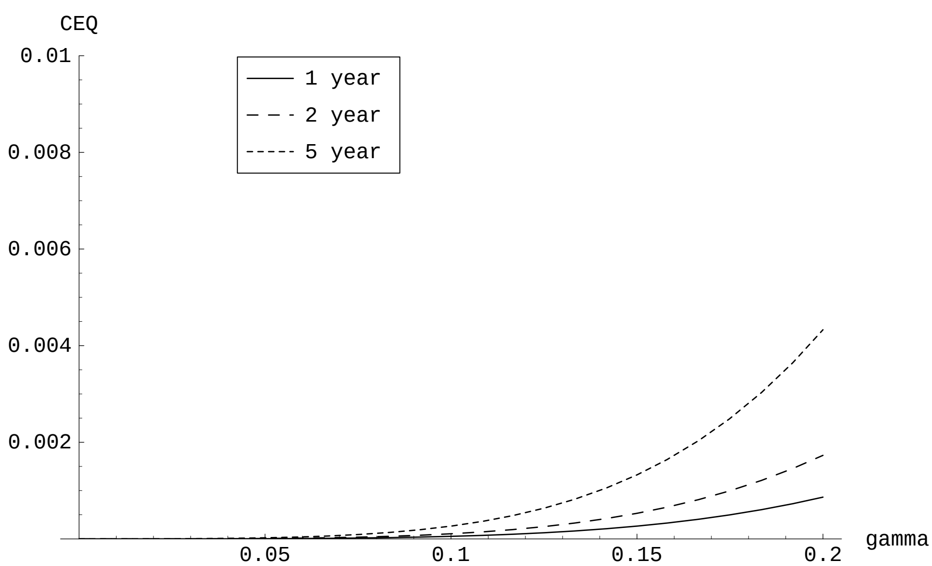


Figure 6: Portfolio weights and CEQ with respect to variance of jump size

The first panel of the figure gives the portfolio of an investor who accounts for "systemic jumps" (solid line) and an investor who ignores this and models returns as a "pure diffusion" (dotted line). The case considered is one where there are only *two* risky assets. The parameters for the returns processes for both assets are calibrated to the *average* of the estimates for developed countries, reported in the last column of Panel A of Table 3, with the exception of γ, which is allowed to range from 0 to 0.20, corresponding to skewness ranging from 0 to 3.8 times its estimate in the data, and kurtosis ranging from 0 to 11.8 times its estimated value. The lower plot shows the corresponding CEQ for these two portfolios, where CEQ measures the percentage of initial wealth that needs to be given to the investor who ignores systemic risk to make him as well off as the investor who accounts for systemic risk. As before, we assume that $\eta = -2$ implying that the investor's parameter of relative risk aversion, $1 - \eta$, is 3. The riskless interest rate is 0.005 per month. This figure is discussed in Section 5.2.5 on page 24.





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