Systems with Finite Communication Bandwidth Constraints—Part I: State Estimation Problems

Wing Shing Wong and Roger W. Brockett

***Abstract*— In this paper, we investigate a state estimation problem involving finite communication capacity constraints. Unlike classical estimation problems where the observation is a continuous process corrupted by additive noises, there is a constraint that the observations must be coded and transmitted over a digital communication channel with finite capacity. This problem is formulated mathematically, and some convergence properties are defined. Moreover, the concept of a *finitely recursive coder-estimator sequence* is introduced. A new upper bound for the average estimation error is derived for a large class of random variables. Convergence properties of some coder-estimator algorithms are analyzed. Various conditions connecting the communication data rate with the rate of change of the underlying dynamics are established for the existence of stable and asymptotically convergent coder-estimator schemes.**

***Index Terms*—Finitely recursive coder-estimator sequence, hybrid systems, prefix code, state estimation.**

I. Introduction

Information theoretical issues are traditionally decoupled from the consideration of decision and control problems. Hence, if an engineer tries to design a classical feedback controller, the standard assumptions are that all information processing and data transmission required by the algorithm can be performed with zero delay and infinite precision. It is worth questioning the validity of these assumptions. Decoupling the communication aspects from the underlying dynamics of a system greatly simplifies the analysis and generally works well for classical models. However, as the application of decision and control theory spreads to new areas, this fundamental assumption deserves a careful re-examination. In some new system models, it is common to encounter situations where a single decision maker controls a large number of subsystems, and observation and control signals are sent over a communication channel with finite capacity and significant transmission delays. Neurobiological systems offer a basic paradigm. Another class of examples comes from social-economical systems where the state observation process is typically slow and the control requires some time to be implemented. Both of these characteristics can be viewed in the context of systems with finite communication constraints. A third class of examples is offered by remotely controlled systems. With the rapid growth of mobile communication systems, it is easy to envision commercial applications where a large number of mobile units need to be controlled remotely. Since the radio spectrum is limited, communication constraints for these systems are a real concern.

In this paper, a state estimation problem based on observations transmitted with finite communication capacity constraint is investigated. However, unlike classical estimation problems where the observation is a continuous process corrupted by additive noises, the condition here is that the observations must be coded and transmitted over a digital communication channel with finite capacity.

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W. S. Wong is with the Department of Information Engineering, the Chinese University of Hong Kong, Shatin, NT, Hong Kong (e-mail: wswong@ie.cuhk.edu.hk).

R. W. Brockett is with the Division of Applied Sciences, Harvard University, Cambridge, MA USA.

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The concepts of a *coder-estimator sequence* and a *finitely recursive coder-estimator sequence* are introduced in this paper. The latter concept is clearly motivated by the idea of finite-dimensional filters. Convergence issues related to these sequences are investigated in this paper. In particular, various necessary and sufficient conditions connecting the communication data rate with the rate of change of the underlying process are established for the existence of stable and asymptotically convergent coder-estimator schemes.

Our work here is connected with several well-developed theories. In the special case where the stochastic process is reduced to a static random variable, the problem is closely related to the vector quantization problem [9], and to a lesser degree, it is also related to the rate distortion theory [3] and statistical inference theory via compressed data [1], [18], [2]. There are also several previous research works which investigate various information-related aspects of decision and control systems. For example, the issue of finite precision of quantized systems is studied in [7] and [16], and the sampled-data control systems are well investigated (see the references in [10]). Recent papers [15], [4], and [11] are also related to our work here.

II. Basic Model

Consider a remotely located system with state represented by a stochastic process $\{X(t)\}_{t=0}^{\infty}$. We assume that the *a priori* distribution of the stochastic process is known and that its first and second moments are bounded for all time. The system is observed continuously at the remote location. However, the information processing element is not colocated with the observing element; so the observed data has to be transmitted over a communication channel before it can be processed.

Aspects of classical information theory, such as rate distortion theory, investigate how to encode and decode information from an independently identically distributed (*i.i.d.*) source so that the asymptotic distortion (or error) rate between the source and its quantized representation is minimized. The coding and decoding considered there is *nonrecursive* in the sense that the coding is based on a sequence of observations on an *i.i.d.* source. From the system estimation viewpoint, neither the *i.i.d.* assumption on the source state nor the nonrecursive nature of the coding and decoding schemes is appropriate. Typical models of interest to system scientists and engineers are of the form

$$\frac{dX}{dt} = f(X(t), u(t)) + \xi(t), \quad X(0) \text{ unknown} \tag{1}$$

where ξ is the perturbation or noise process independent of $X(t)$, and u is the control. The sampled states of such a system are highly correlated and are far from forming an *i.i.d.* source. Moreover, the natural dynamics of such a system may be unstable. Hence, the behavior may be drastically different from standard rate distortion models. A *nonrecursive* coding and decoding scheme is also unnatural for these problems, since typically estimations of the states are used in the feedback control of the systems.

In this paper, we investigate a class of estimation problems that can be viewed as extending the classical rate distortion investigation to system state estimation problems. Our focus differs from rate distortion theory in that a recursive coding and decoding scheme for a highly correlated source state is considered. Compared to classical state estimation theory, the observations in these problems are corrupted not only by additive observation noises but also by a communication channel with finite bandwidth.

III. Coder-Estimator Sequence

Let the stochastic process $\{X(t)\}_{t=0}^{\infty}$ denoting the state of a system be defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. The state is observed via an observation process, $Y(t)$, where

$$Y(t) = X(t) + V(t) \tag{2}$$

where $V(t)$ is a zero-mean observation noise process independent of $X(t)$, also defined on the probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $\mathcal{F}_t$ denote the σ-field generated by $\{Y(s), 0 \leq s \leq t\}$.

If the observed information is transmitted to an estimator with no delay and no loss in precision, that is, via a channel with infinite bandwidth, one can formulate a classical estimation problem to find the optimal estimate of X given Y. However, if the communication channel between the observer and the estimator has a finite communication bandwidth, then the resulting problem does not fit any of the classical models. Notice that even if there is no propagation delay, the finite communication bandwidth assumption automatically imposes a transmission delay on the problem. Hence, the analysis is inherently complicated.

Assume that the data rate of the communication channel is R bits per second. We assume that $\delta = 1/R$ represents the time to send one bit of information.

It is assumed that the observation data, once obtained, are encoded to form a codeword according to a predefined *prefix coding* scheme [6]. Moreover, the encoding is achieved by dedicated circuitry with zero processing delay. To achieve good performance, it may be necessary to vary the coding scheme from transmission to transmission. It is assumed that the transmitter and the receiver have an *a priori* agreement on an algorithm that enables both sides to decide which coding scheme is currently in use.

Let $\mathcal{B}^*$ represent the set of finite length strings of symbols from a binary alphabet. Denote by l the integer-valued function on $\mathcal{B}^*$ which evaluates the length of a codeword.

Definition 1: A coder sequence is a sequence of ordered pairs $\{(h_i, t_i)\}_{i=0}^{\infty}$ satisfying the following relations.

1) h_i is a function from Ω to $\mathcal{B}^*$ measurable with respect to $\mathcal{F}_{t_i}$.
2) For $i \geq 1$, $t_i = t_{i-1} + l(h_{i-1}(\omega))\delta$, where $t_0 = 0$.

Denote the ith codeword by c_i, that is

$$c_i = h_i(\omega). \tag{3}$$

In the special cases when *fixed length codewords* are used, the t_i's become deterministic.

Definition 2: A coder-estimator sequence is a sequence of triples, $\{(h_i, t_i, \hat{X}_i)\}_{i=0}^{\infty}$, where $\{(h_i, t_i)\}_{i=0}^{\infty}$ is a coder sequence and $\hat{X}_i$ is an estimator based on $\{c_0, \cdots, c_i\}$.

Notice that in this definition, the coding decision can be dependent on the whole past history of the observation process. Similarly, the estimator can be dependent on the whole sequence of past codewords. This flexibility of course comes with a computation and memory storage cost. Later on, we will address this issue and restrict our attention to coder-estimator sequences that are computationally efficient.

The performance of a coder-estimator sequence is captured by a related sequence of coder-estimator errors

$$\mathbf{E}\|\hat{X}_i - X(t_i)\|^2. \tag{4}$$

A natural question is whether a coder-estimator sequence is *stable* in the following sense.

Definition 3: A coder-estimator sequence is stable for X if there exists a finite constant C, so that for all i

$$\mathbf{E}\|\hat{X}_i - X(t_i)\|^2 \leq C. \tag{5}$$

In some special cases, a stable coder-estimator sequence may even converge in the following sense.

Definition 4: A coder-estimator sequence converges in quadratic mean for X if

$$\lim_{i \to \infty} \mathbf{E}\|\hat{X}_i - X(t_i)\|^2 = 0. \tag{6}$$

IV. Finitely Recursive Coder-Estimator Sequences

In the definition of a coder-estimator sequence, the coding functions and the estimates are based on the whole past history of the process X. The computation and storage demand for such a general coder-estimator sequence can be excessively high. From an efficiency viewpoint, it makes good sense to consider the following special class of coder-estimator sequences.

Definition 5: Let $\mathcal{D}$ be a direct sum of finitely many copies of the real numbers, $\Re$ and $\mathcal{B}^*$. A coder-estimator sequence is said to be finitely recursive if there exists a vector of finitely many auxiliary variables, A, taking values in $\mathcal{D}$, and functions F from $\mathcal{D} \oplus B^*$ to $\mathcal{D}$, $\mathcal{G}$ from $\mathcal{D} \oplus \Re$ to $\mathcal{B}^*$, and H from $\mathcal{D} \oplus \mathcal{B}^*$ to $\Re$, such that for all $i \geq 1$

$$\begin{cases} c_i = G(A_i, Y_{t_i}(\omega)) \\ \hat{X}_i(c_1, \cdots, c_i) = H(A_i, c_i) \\ A_{i+1} = F(A_i, c_i). \end{cases} \tag{7}$$

Notice that the admissible coding schemes are no longer dependent on all aspects of the past history of X. The estimator depends only on the auxiliary variables and the latest codeword. In our paradigm, the coding component evaluates the first and the third equations in (7) at time t_i, based on the observed process. The output codeword is then transmitted over a communication channel to the estimation component which then computes the estimate by means of the second and the third equations in (7).

V. Fixed Codeword Length Sequences

Coder-estimator sequences with fixed codeword length form an important sub-family. In this case, sampling times, t_i's, form a deterministic lattice with a spacing of $n\delta$, where n is the length of the codeword. As a result, one can reduce the state equation to

$$\begin{cases} X_{i+1} = f(X_i) + U_i \\ X_0 \text{ has a known p.d.f. } q \\ Y_{i+1} = X_{i+1} + V_{i+1} \end{cases} \tag{8}$$

where U_i represents the state perturbation process and V_i is the observation noise. Unless stated otherwise, in the rest of this paper, we assume that f is a continuous, Lipschitz function and the probability density function (p.d.f.) for X_0, U_i, V_i are all continuous in order to simplify the presentation.

In the case of a scalar linear system, the system reduces to

$$\begin{cases} X_{i+1} = \tau X_i + U_i \\ Y_{i+1} = X_{i+1} + V_{i+1} \end{cases} \tag{9}$$

where

$$\tau = e^{an\delta} \tag{10}$$

for some constant a.

VI. Important Classes of Coder-Estimator Sequences

For simplicity, we assume from now on that the state to be estimated is a scalar process. There are two natural examples of these fixed codeword length sequences: *mean sequences* and *equal-partition sequences*.

Mean Coder-Estimator Sequence and its Generalization: A mean coder-estimator sequence uses a 1-bit codeword per sample data. Define a 1-bit coder-estimator sequence in the following recursive manner: at sampling time t, define a 1-bit coding scheme on the observed signal, $Y(t)$, by means of the characteristic sets $[-\infty, \mathbf{E}Y(t))$ and $[\mathbf{E}Y(t), \infty]$. After receiving the transmitted codeword, a minimum variance estimator is used to estimate $X(t)$. Moreover, by accounting for the effects of the system dynamics, one can obtain a new distribution of Y at the next observation instant conditioned on the received codeword. The same procedure can then be repeated. To be more rigorous, construct a coder-estimator sequence by

$$A_0 = \mathbf{E}Y(0) \tag{11}$$

$$h_0(Y(0)) = \begin{cases} 1 & -\infty < Y(0) < A_0 \\ 0 & A_0 \le Y(0) < \infty \end{cases} \tag{12}$$

$$\hat{X}_0(c_0) = \mathbf{E}[X(0)|c_0]. \tag{13}$$

For $i \ge 1$, define

$$A_i = \mathbf{E}[Y(i\delta)|c_0, \cdots, c_{i-1}] \tag{14}$$

$$h_i(Y(i\delta)) = \begin{cases} 1 & -\infty < Y(i\delta) < A_i \\ 0 & A_i \le Y(i\delta) < \infty \end{cases} \tag{15}$$

$$\hat{X}_i(c_0, \cdots, c_i) = \mathbf{E}[X(i\delta)|c_0, \cdots, c_i]. \tag{16}$$

The mean coder-estimator sequence is characterized by the fact that the coding functions are defined by using the conditional mean of the observed process, Y, as a partition point. Define an n-bit generalized mean coder-estimator sequence to be any sequence with coding functions that partition the real line into 2^n intervals and use the conditional mean of the observed process as one of the $2^n - 1$ partition points.

Equal-Partition Coder-Estimator Sequences: This is a finitely recursive coder-estimator sequence. Define a *support interval* to be the smallest closed interval which contains the support of a function. Assume that the p.d.f. for X_0, U_i, and V_i for all i all have finite support intervals. Particularly, let the support interval for the p.d.f. of X_0, U_i, and V_i be $[-x_0, x_0]$, $[-u, u]$, and $[-v, v]$, respectively.

The coding function for the n-bit equal-partition coder-estimator sequence at sampling time $in\delta$ is defined by dividing the support interval of the p.d.f. of $Y(in\delta)$ conditioned on the codewords $c_0, \cdots, c_i$ into 2^n equal length, consecutive subintervals, $(\mathbf{S}_1, \cdots, \mathbf{S}_{2^n})$, and mapping the characteristic set $\mathbf{S}_i$ to the codeword i. The estimator is defined by the midpoint of the support interval of the p.d.f. of X conditioned on the received codewords. To define the equal-partition coder-estimator sequence more precisely, let

$$\begin{pmatrix} A_0 \\ B_0 \end{pmatrix} = \begin{pmatrix} -x_0 \\ x_0 \end{pmatrix}. \tag{17}$$

In general, for $i \ge 0$, define

$$\begin{aligned} U_i &= \max_{x \in [A_i, B_i]} f(x) + u + v \\ L_i &= \min_{x \in [A_i, B_i]} f(x) - u - v \end{aligned} \tag{18}$$

and denote by $M_j = j\dfrac{U_i - L_i}{2^n}$

$$h_i(Y(in\delta)) = j \quad \text{if } L_i + M_{j-1} \le Y(in\delta) < L_i + M_j \tag{19}$$

$$\hat{X}_i(c_i = j) = L_i + M_{j-\frac{1}{2}} \tag{20}$$

$$\begin{pmatrix} A_{i+1}(c_i = j) \\ B_{i+1}(c_i = j) \end{pmatrix} = \begin{pmatrix} \max(L_i + M_{j-1} - v, L_i + v) \\ \min(L_i + M_j + v, U_i - v) \end{pmatrix}. \tag{21}$$

VII. Basic Properties and Vector Quantization

Before one can analyze the performance of a general coder-estimator sequence, it is crucial to analyze the performance of a single coder-estimator step. Let X be a real-valued random variable with known measurable p.d.f. q, mean m, and variance σ^2. The observation Y is defined by

$$Y = X + V \tag{22}$$

where V is a zero mean observation noise with a known, measurable p.d.f. r. The observed information is coded by a coding scheme h; the received codeword is denoted by c.

A natural question concerns finding the optimal coding scheme and estimator g to minimize the error

$$\mathbf{E}|X - g(c)|^2. \tag{23}$$

For the special case where there is no observation noise, that is $V = 0$, this corresponds to the well-known vector quantization problem. The error function is typically referred to as the distortion measure. Define the coding scheme by partitioning the state-space into 2^n cells, $\{\mathbf{A}_i\}$, so that if the observed value falls in $\mathbf{A}_i$ it is represented by the value $\hat{X}_i$. The $\mathbf{A}_i$'s are called the characteristic sets of the coding function. The following properties are fundamental results from vector quantization theory [9].

- If the estimates, $\hat{X}_i$, are given, then the distortion is minimized by the Voronoi partition. The partition cells are polytopal.
- Given a state-space partition, the optimal estimate for each cell is chosen by minimizing the conditional expected distortion.

There is no known general algorithm for deriving the optimal coder-estimator pair, although there are locally optimal algorithms such as the Lloyd algorithm or simulated annealing based algorithms [9].

Even though the setting of our investigation is more general than vector quantization, in the rest of this paper, it is assumed that:

- all coding functions are defined by polytopal partition cells.

Lemma 1: Suppose the coding function is defined by the characteristic sets $(\mathbf{A}_1, \cdots, \mathbf{A}_M)$, and conditional mean estimator is used. Then, the estimation error is given by

$$\mathbf{E}\epsilon = \sigma^2 - \sum_{i=1}^{M} \frac{(m_i - mp_i)^2}{p_i} \tag{24}$$

where

$$\begin{aligned} p_i &= \text{prob}(Y \in \mathbf{A}_i) \\ m_i &= \mathbf{E}[X|Y \in \mathbf{A}_i]. \end{aligned} \tag{25}$$

Proof:

$$\begin{aligned} \mathbf{E}\epsilon &= \sum_{i=1}^{M} p_i \mathbf{E}[(X - \mathbf{E}[X|Y \in \mathbf{A}_i])^2 | Y \in \mathbf{A}_i] \\ &= \mathbf{E}X^2 - \sum_{i=1}^{M} p_i (\mathbf{E}[X|Y \in \mathbf{A}_i])^2 \\ &= \sigma^2 + m^2 - \sum_{i=1}^{M} \frac{m_i^2}{p_i} \\ &= \sigma^2 - \sum_{i=1}^{M} \frac{(m_i - mp_i)^2}{p_i}. \end{aligned}$$

□

For our study of coder-estimator sequences, it is important to obtain lower bounds for the error reduction capability of a coder-estimator step. A result is proven here for the mean coder-estimator step based on an interesting inverse Cauchy–Schwartz inequality and the following lemma.

Lemma 2: If the p.d.f. q is nondecreasing, then for $a \le b < c < \infty$, the following inequality holds:

$$(\int_b^c xq(x)dx - a\int_b^c q(x)dx)^2 \ge \frac{1}{2}\int_b^c q(x)dx \int_b^c (x-a)^2 q(x)dx. \tag{26}$$

Proof: The left-hand side is equal to

$$\begin{aligned}&\left(\int_b^c (x-a)q(x)dx\right)^2\\ &= \left(\int_{b-a}^{c-a} xq(x+a)dx\right)^2\\ &= (c-a)^2\left(\int_{b-a}^{c-a} \frac{x}{c-a}q(x+a)dx\right)^2\\ &\ge \int_{b-a}^{c-a} \frac{x}{c-a}q(x+a)dx \int_{b-a}^{c-a} x^2 q(x+a)dx\\ &\ge \frac{1}{c-b}\int_{b-a}^{c-a}\frac{x}{c-a}dx \int_{b-a}^{c-a} q(x+a)dx \int_{b-a}^{c-a} x^2 q(x+a)dx.\end{aligned}$$

The last step follows from an inequality due to Čebyšev [13, p. 39].

Hence

$$\left(\int_b^c (x-a)q(x)dx\right)^2 \ge \frac{b+c-2a}{2(c-a)}\int_b^c q(x)dx \int_b^c (x-a)^2 q(x)dx.$$

□

Theorem 1: Assume that q is piecewise concave, monotone nonincreasing or nondecreasing on N intervals, and $n = \lceil \log_2(N+1)\rceil$. Then there exists an n-bit generalized mean coder-estimator step that satisfies

$$\mathbf{E}\epsilon < \frac{3}{4}\sigma^2. \tag{27}$$

Proof: From the theorem assumption, it follows that there exists a generalized n-bit coding scheme so that each of the characteristic set is an interval and q is concave and monotonic on each interval. Moreover, the coding function also contains characteristic sets of the form $(a, m]$ and $(m, b]$ for some a and b. Label the characteristic sets by $\mathbf{A}_i$. Define

$$\begin{aligned}p_i &= \int_{\mathbf{A}_i} q(x)dx\\ m_i &= \int_{\mathbf{A}_i} xq(x)dx\\ \sigma_i^2 &= \int_{\mathbf{A}_i} (x-m)^2 q(x)dx.\end{aligned}$$

By Lemma 1

$$\mathbf{E}\epsilon = \sum_{i=1}^M \sigma_i^2 - \sum_{i=1}^M \frac{(m_i - mp_i)^2}{p_i}.$$

Hence, to establish the theorem, it is sufficient to show that

$$\left(\int_{\mathbf{A}_i} (x-m)q(x)dx\right)^2 \ge \frac{1}{4}\int_{\mathbf{A}_i}(x-m)^2 q(x)dx \int_{\mathbf{A}_i} q(x)dx.$$

Consider the characteristic set $\mathbf{A}_i$. By construction, it is either to the left or right of m. By symmetry argument, one can assume that $x \in \mathbf{A}_i$ implies that $x \ge m$. Notice that q is concave on $\mathbf{A}_i$, and it is either nonincreasing or nondecreasing on it. The two cases are considered separately.

Case 1: q is Nondecreasing on $\mathbf{A}_i$: Since q is a probability measure and nondecreasing, the subinterval $\mathbf{A}_i$ must be finite. By Lemma 2

$$\begin{aligned}&(\int_{\mathbf{A}_i} xq(x)dx - m\int_{\mathbf{A}_i} q(x)dx)^2\\ &\ge \frac{1}{2}\int_{\mathbf{A}_i}(x-m)^2 q(x)dx \int_{\mathbf{A}_i} q(x)dx.\end{aligned}$$

Case 2: q is Nonincreasing on $\mathbf{S}_i$: Since the square-root function is concave and increasing, it follows that $\sqrt{q}$ is concave and nonincreasing on $\mathbf{A}_i$. On the other hand, the function $x - m$ is nonnegative, concave, and increasing on $\mathbf{A}_i$. Suppose f and g are both concave, nonnegative on an interval $\mathbf{I}$, with f nonincreasing and g nondecreasing. Since

$$(f(x) - f(y))(g(x) - g(y)) \le 0$$

it follows that

$$f(x)g(y) + f(y)g(x) \ge f(x)g(x) + f(y)g(y).$$

Since

$$\begin{aligned}&f(\lambda x + (1-\lambda)y)g(\lambda x + (1-\lambda)y)\\ &\ge (\lambda f(x) + (1-\lambda)f(y))(\lambda g(x) + (1-\lambda)g(y))\\ &\ge \lambda f(x)g(x) + (1-\lambda)f(y)g(y)\end{aligned}$$

hence fg is concave. In particular, $(x-m)\sqrt{q}$ is concave on $\mathbf{A}_i$.

The Blaschke and Pick theorem [8] states that for nonnegative concave functions f and g defined on an interval $\mathbf{I}$ with Lebesque measure μ

$$\int_{\mathbf{I}} f^2 d\mu \int_{\mathbf{I}} g^2 d\mu \le 4\left(\int_{\mathbf{I}} fg d\mu\right)^2.$$

Let $f = \sqrt{q}$ and $g = (x-m)\sqrt{q}$, then it follows that

$$\int_{\mathbf{A}_i} qdx \int_{\mathbf{A}_i} (x-m)^2 qdx \le 4\left(\int_{\mathbf{A}_i}(x-m)qdx\right)^2.$$

Hence, the claimed inequality holds. □

VIII. Convergence Result for Mean Coder-Estimator Sequences

The mean coder-estimator sequence is a natural 1-bit sequence. In this section, its error-reduction performance is analyzed. In fact, the analysis covers the slightly more general class of generalized n-bit mean coder-estimator sequences.

Consider the system defined by (9). Assume that U_i and V_i are both discrete, zero-mean random variables with probability mass defined on $-N_s\epsilon, -(N_s-1)\epsilon, \cdots, N_s\epsilon$ and $-N_o\epsilon, -(N_o-1)\epsilon, \cdots, N_o\epsilon$, respectively. Denote their variance by σ_s^2 and σ_o^2, respectively. If $\tau > 1$ and if the system is unobserved, the error of the minimum variance estimation of the process X will diverge. A basic question is whether one can construct a coder-estimator sequence that is stable as defined in Section III.

Theorem 2: Suppose X_0 has mean m, variance σ_*, and a p.d.f. q that is concave and monotone nondecreasing or nonincreasing. If $n = \lceil \log_2[(N_o+1)(2N_o+2N_s+1)]\rceil + 1$, then there exists an n-bit generalized mean coder-estimator sequence which is stable for the process X if

$$a\delta = \frac{a}{R} < \frac{1}{2n}\ln\frac{4}{3}. \tag{28}$$

Proof: Assume the p.d.f. of X_i conditioned on the codewords $c_0, \cdots, c_i$ is piecewise concave and monotone nondecreasing on L_i intervals. Let P_i denote the corresponding set of $L_i - 1$ partition points. Define Q_i by $\{x + i\epsilon : x \in P_i, -(N_s + N_o) \le j \le N_s + N_o\}$. The order of this partition set is at most $(L_i - 1)(2N_s + 2N_o + 1)$.

From (9)

$$Y_{i+1} = \tau X_i + Z_{i+1}$$

where

$$Z_{i+1} = V_{i+1} + U_i$$

and Z_{i+1} is a discrete random variable with probability mass on $-(N_s + N_o)\epsilon, \cdots, (N_s + N_o)\epsilon$. It follows that the conditional distribution of Y_{i+1} conditioned on the codewords $c_0, \cdots, c_i, q_{i+1}$ is also piecewise concave and monotone nondecreasing on the interval defined by Q_i. Define R_i by adding the conditional mean of $\int x q_{i+1} dx$ to Q_i. R_i defines a partition of the real line with at most $(L_i - 1)(2N_s + 2N_o + 1) + 2$ intervals; (two of which are semi-infinite intervals.) By the concavity property, one can show that q_{i+1} must vanish on the two semi-infinite intervals. Hence, one can construct a $\lceil \log_2[(L_i - 1)(2N_s + 2N_o + 1)] \rceil$-bit generalized mean coder scheme by using the partition defined by R_i if one does not include the two semi-infinite intervals in the coding scheme.

After the $i + 1$th codeword is received, the p.d.f. of X_{i+1} conditioned on the codewords $c_0, \cdots, c_i, c_{i+1}$ is piecewise concave and monotonic on at most $2N_0 + 3$ consecutive intervals. By induction argument, it follows that if

$$n \ge \log_2[2(N_o + 1)(2N_o + 2N_s + 1)]$$

then there always exist n-bit generalized mean coding sequences so that all the conditioned p.d.f. of Y is piecewise concave and monotone nondecreasing on the characteristic sets.

At the end of each coded observation, there are $M = 2^n$ possible outcomes. Hence, one can trace the history of an n-bit coder-estimator sequence by an M-nary tree as depicted in Fig. 1. The depth of the graph corresponds to the time at which the coding-estimation takes place. The nodes of the graph are labeled by a variable length string beginning with a "$*$" and followed by the sequence of codewords received in order to reach that state. Also associated with each node is a value, denoted by $\sigma^2_{*i_0,\cdots,i_j}$, which represents the variance of $X(jn\delta)$ conditioned on the sequence of codewords, $(i_0, \cdots, i_j)$, that leads to that node. Associated with each node is a value, denoted by $p_{i_0,\cdots,i_j}$, which represents the probability of reaching that node. Notice that

$$\sum_{i_j \in \mathcal{Z}_M = \{1,\cdots,M\}} p_{i_0,\cdots,i_{j-1},i_j} = p_{i_0,\cdots,i_{j-1}}.$$

From (9), it follows that the conditional variance of $X(kn\delta)$ conditioned on the codewords $i_0, \cdots, i_k$ is equal to $\tau^2 \sigma^2_{*i_0,\cdots,i_{k-1}} + \sigma_o^2$. By construction, the conditional distribution of $X(kn\delta)$ satisfies the conditions of Theorem 1. Hence

$$\sum_{i_k \in \mathcal{Z}_M} p(i_k | i_0, \cdots, i_{k-1}) \sigma^2_{*i_0,\cdots,i_k} \le \frac{3}{4}\left(\tau^2 \sigma^2_{*i_0,\cdots,i_{k-1}} + \sigma_o^2\right).$$

Hence

$$\begin{aligned} &\mathbf{E}|\hat{X}_k - X(kn\delta)|^2 \\ &= \sum_{i_0 \in \mathcal{Z}_M} \cdots \sum_{i_k \in \mathcal{Z}_M} p_{i_0,\cdots,i_{k-1},i_k} \sigma^2_{*i_0,\cdots,i_{k-1},i_k} \\ &\le \frac{3}{4}\left(\tau^2 \sum_{i_0 \in \mathcal{Z}_M} \cdots \sum_{i_{k-1} \in \mathcal{Z}_M} p_{i_0,\cdots,i_{k-1}} \sigma^2_{*i_0,\cdots,i_{k-1}} + \sigma_o^2\right) \\ &< \left(\frac{3}{4}\tau^2\right)^{k+1} \sigma_*^2 + 3\sigma_o^2. \end{aligned}$$

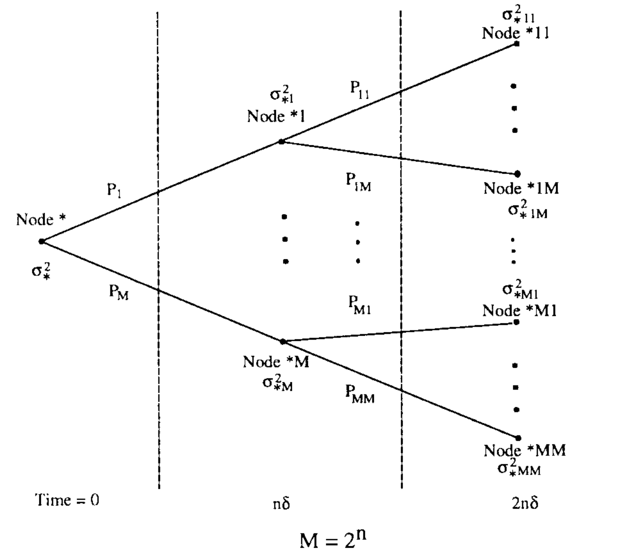

Fig. 1. Illustration of the assignment with different size networks.

It follows that the coder-estimator sequence is stable for X if (28) holds. □

Corollary 1: If $U_i = V_i = 0$, then the 1-bit mean coder-estimator sequence converges in the quadratic mean if

$$a\delta < \frac{1}{2} \ln \frac{4}{3}. \tag{29}$$

IX. Convergence Property of Equal-Partition Sequences

Consider a slightly more general version of the system defined by (9)

$$\begin{cases} X_{i+1} = f_i(X_i) + U_i \\ X_0 \text{ has a known p.d.f. } q \\ Y_{i+1} = X_{i+1} + V_{i+1} \end{cases} \tag{30}$$

where f_i is a Lipschitz function satisfying the condition

$$|f_i(x) - f_i(y)| \le C_f |x - y| \tag{31}$$

for all i, x, and y.

Assume that the p.d.f. of X_0, U_i,, and V_i have finite support intervals, $[-x_0, x_0]$, $[-u, u]$, and $[-v, v]$, respectively. The performance of an n-bit equal-partition coder-estimator sequence is analyzed here for such a system.

Theorem 3: The n-bit equal-partition coder-estimator sequence is stable if

$$a\delta \log_2 e < 1. \tag{32}$$

Proof: Let $S_{i_0,\cdots,i_j}$ represent the length of the support interval of the p.d.f. of X_j conditioned on receiving the codewords $i_0, \cdots, i_j$.

It follows from the Lipschitz assumption on f that the p.d.f. of X_{j+1} conditioned on the codewords $c_0, \cdots, c_j$ has a support interval with length not greater than $C_f S + 2u$, and the support interval of Y_{j+1} conditioned on the same set of codewords has length not greater than $C_f S + 2u + 2v$. Hence, each of the characteristics set is an interval with length not greater than $(C_f S + 2u + 2v)/2^n$. If the $i + 1$th codeword is received, X_{i+1} must lie within $+v$ or $-v$ of the associated characteristic set. Hence

$$S_{i_0,\cdots,i_{j+1}} \leq (\tau S_{i_0,\cdots,i_j} + 2u + 2v)/2^n + 2v.$$

Let

$$T_j = \max_{i_0,\cdots,i_j} S_{i_0,\cdots,i_j}.$$

Then

$$T_{j+1} \leq (\tau T_j + 2u + 2v)/2^n + 2v.$$

The sequence T_j is bounded if $\tau < 2^n$. Since the variance of X conditioned on the codewords $i_0, \cdots, i_j$ is bounded by T_j^2, the theorem, therefore, holds. □

Remark: If $V_i = U_i = 0$, then the equal-partition coder-estimator sequence converges in the quadratic mean if the inequality on τ holds.

X. Conclusion

In this paper, a new class of state estimation problems with communication bandwidth constraints is proposed. These problems couple the issue of estimation with the issue of information communication. Although the estimation problem investigated here is by itself quite simple, it serves to illustrate the complexity and the intricacy of these finite communication bandwidth problems. Extension of this work to more sophisticated estimation problems and feedback control problems will be reported in subsequent papers.

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Nonovershooting and Monotone Nondecreasing Step Responses of a Third-Order SISO Linear System

Shir-Kuan Lin and Chang-Jia Fang

***Abstract*— This paper presents the necessary and sufficient conditions for a third-order single-input/single-output linear system to have a nonovershooting (or monotone nondecreasing) step response. If the transfer function of an overall system has real poles, a necessary and sufficient condition is found for the nonovershooting (or monotone nondecreasing) step response. In the case of complex poles, one sufficient condition and two necessary conditions are obtained. The resulting conditions are all in terms of the coefficients of the numerator of the transfer function. Simple calculations can be used to check a system for the nonovershooting (or monotone nondecreasing) step response. Another feature is that the conditions in terms of pole-zero configurations can be easily derived from the present results.**

***Index Terms*—Linear system, PID controller, step response.**

I. Introduction

The controller design for a third-order linear system has been drawing the attention of many researchers for several decades [1]–[6] because a conventional dynamic plant controlled by a proportional-integral-derivative (PID) controller turns out to be a third-order system. It was pointed out [7] that not only poles but also zeros significantly characterize the step response of a transfer function. Recently, the focus is on the pole-zero relations for the step response without overshoot and undershoot. Note that a step response has no undershoot in the whole history if and only if it is a monotone nondecreasing step response. The number of undershoot times (or local extrema) in the step response has been widely discussed for a strictly proper transfer function with only real poles and real zeros [8]–[11]. A special case of this theme is the initial undershoot [12], which is actually an old result [11], [13]–[15]. Incidentally, another kind of old result for the monotone nondecreasing step response was also repeatedly reported in the recent works, which will be explained in the following paragraph. For a single-input/single-output (SISO) discrete-time system, the linear programming approach formulated by the l^1 theory [16] or simple coefficient relations [17] can be used to design a minimum overshoot controller.

On the other hand, many works [18]–[24] were devoted to finding explicit conditions for a nonovershooting and a monotone nondecreasing step response. The condition proposed in [18] is in terms of the

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The authors are with the Department of Control Engineering, National Chiao Tung University, Hsinchu 30050, Taiwan ROC (e-mail: sklin@cc.nctu.edu.tw).

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