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Syzygies of Veronese Embeddings

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Abstract. We prove that the Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ with $n \geqslant 2$, $d \geqslant 3$ does not satisfy property N_p (according to Green and Lazarsfeld) if $p \geqslant 3d-2$. We make the conjecture that also the converse holds. This is true for $n = 2$ and for $n = d = 3$.

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Introduction

Let $\mathbb{P}^n$ be the projective n-space over an algebraically closed field of characteristic zero and let $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}: \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ be the Veronese embedding associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$. In order to understand the homogeneous ideal $\mathcal{I}$ of $\mathbb{P}^n$ in $\mathbb{P}^N$ as well as its syzygies, it is useful to study some properties about the minimal free resolution of $\mathcal{I}$.

M. Green and R. Lazarsfeld ([G2], [GL]) introduced the property N_p (Definition 1.3) for a complete projective nonsingular variety $X \hookrightarrow \mathbb{P}^N$ embedded in $\mathbb{P}^N$ with an ample line bundle L. When property N_p holds for every integer p, the resolution of $\mathcal{I}$ is 'as nice as possible'. M. Green proved in [G2], Theorem 2.2, that $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies N_p if $p \leqslant d$. L. Manivel ([M]) has generalized this result to flag manifolds. The rational normal curves (which are the Veronese embeddings of $\mathbb{P}^1$) satisfy $N_p\, \forall p$. C. Ciliberto showed us that the results of [G1] imply that $\varphi_{\mathcal{O}_{\mathbb{P}^2}(d)}$ with $d \geqslant 3$ satisfies N_p if $p \leqslant 3d-3$. This sufficient condition has been found also by C. Birkenhake in [B1] as a corollary of a more general result. Here we prove that this condition is also necessary (Theorem 3.1) and we formulate (for $n \geqslant 2$) the following conjecture:

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CONJECTURE.

$$\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} \textit{ satisfies } \mathrm{N_p} \Longleftrightarrow \begin{cases} \mathrm{n = 2, d = 2, \forall p,} \\ \mathrm{n \geqslant 3, d = 2, p \leqslant 5,} \\ \mathrm{n \geqslant 2, d \geqslant 3, p \leqslant 3d - 3.} \end{cases}$$

Our precise result is the following:

THEOREM. *The implication '$\Longrightarrow$' of the previous conjecture is true.*

Moreover, we remark that the implication '$\Longleftarrow$' of the previous conjecture is true in the cases $n = 2$ ([G1]), $n = d = 3$ ([G1]), $d = 2$ ([JPW]). This solves the Problem 4.5 of [EL] (raised by Fulton) in the first cases given by the projective plane and by the cubic embedding of the projective three-dimensional space.

We also remark that our conjecture could be overcome by the knowledge of the minimal resolution of the Veronese variety. This is stated as an open problem in [G2] (remark of Section 2). Our results can be seen as a step towards this problem.

The paper is organized as follows: in Section 1 we recall some definitions we will need later and we improve a known cohomological criterion for the property N_p. In Section 2 we prove our main results and in Section 3 we fit our results into the literature.

1. Notations and Preliminaries

Let V be a vector space of dimension $n + 1$ over an algebraically closed field $\mathbb{K}$ of characteristic 0 and let $\mathbb{P}^n = \mathbb{P}(V^*)$ the projective space associated to the dual space of V. Note that $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong S^d V \quad \forall d \geqslant 0$.

For any vector bundle E over $\mathbb{P}^n$ we will denote by $H^i(E)$ the ith cohomology group of E over $\mathbb{P}^n$ and by $E(t)$ the tensor product $E \otimes \mathcal{O}_{\mathbb{P}^n}(t)$

The following bundles will play a fundamental role in this paper:

DEFINITION 1.1. For any positive integer d, the line bundle $\mathcal{O}_{\mathbb{P}^n}(d)$ is generated by global sections $H^0(\mathcal{O}_{\mathbb{P}^n}(d)) \cong S^d V$ so that the evaluation map $ev: S^d V \otimes \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^n}(d)$ is surjective. Call E_d the kernel. Thus, the vector bundle E_d is defined by the exact sequence

$$0 \longrightarrow E_d \longrightarrow S^d V \otimes \mathcal{O}_{\mathbb{P}^n} \xrightarrow{ev} \mathcal{O}_{\mathbb{P}^n}(d) \longrightarrow 0. \tag{1.2}$$

It follows immediately from the definition that the bundle E_d has rank $N := rk_{E_d} = \binom{n+d}{n} - 1$ and first Chern class $c_1(E_d) = -d$.

Note that, if $d = 1$, (1.2) is the dualized Euler sequence so that

$$E_1 \cong \Omega^1_{\mathbb{P}^n}(1) \quad \text{and} \quad \bigwedge^q E_1 \cong \Omega^q_{\mathbb{P}^n}(q).$$

For any integer $d \geqslant 0$, we will denote by $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ the *Veronese embedding* $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ associated to the complete linear system $|\mathcal{O}_{\mathbb{P}^n}(d)|$ of dimension $N+1 := \binom{n+d}{n}$. Recall that if $[x_0 : \cdots : x_n]$ is a system of homogeneous coordinates on $\mathbb{P}^n$ and $[y_0 : \cdots : y_N]$ on $\mathbb{P}^N = \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^n}(d))^*)$, then $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ is the embedding:

$$[x_0 : \cdots : x_n] \hookrightarrow [x_0^d : x_0^{d-1}x_1 : \cdots : x_n^d].$$

With the above notation, let $S := \bigoplus_{k \geqslant 0} S^k(H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)))$ be the homogeneous coordinate ring of $\mathbb{P}^N$ and define the graded S-module $R := \bigoplus_{k \geqslant 0} H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(kd))$. Let

$$0 \to \oplus_j S(-j)^{b_{rj}} \to \cdots \to \oplus_j S(-j)^{b_{0j}} \to R \to 0$$

be a minimal free resolution of R with *graded Betti numbers* b_{ij}.

DEFINITION 1.3. For any integer $p \geqslant 0$ the embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)} : \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is said to satisfy *property* N_p if

$$b_{0j} = \begin{cases} 1 & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad b_{ij} = 0 \text{ for } j \neq i+1, \ \text{when } 1 \leqslant i \leqslant p.$$

Thus, N_0 means that $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}(\mathbb{P}^n)$ is projectively normal in $\mathbb{P}^N$; N_1 means that N_0 holds and the ideal I of $\mathbb{P}^n \hookrightarrow \mathbb{P}^N$ is generated by quadrics; N_2 means that, moreover, the module of syzygies among quadratic generators $Q_i \in I$ is spanned by the relations of the form $\sum L_i Q_i = 0$ where the L_i are *linear* polynomials; and so on.

Remark 1.4. Let $\mathcal{C} \hookrightarrow \mathbb{P}^d$ be the rational normal curve (of degree d) in $\mathbb{P}^d$. If V is a vector space of dimension 2, then $\mathcal{C} \cong \mathbb{P}(V^*) \hookrightarrow \mathbb{P}^d = \mathbb{P}(S^d V^*)$ is the image of the Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^1(d)}}\colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$.

It is well known (e.g. by using the Eagon–Northcott complex) that the sheaf ideal $\mathcal{I}$ of $\mathcal{C}$ in $\mathcal{O}_{\mathbb{P}^d}$ has the following resolution:

$$0 \to \mathcal{O}_{\mathbb{P}^d}(-d)^{\oplus b_d} \to \mathcal{O}_{\mathbb{P}^d}(-d+1)^{\oplus b_{d-1}} \to \cdots \to \mathcal{O}_{\mathbb{P}^d}(-2)^{\oplus b_2} \to \mathcal{I} \to 0,$$

where $b_k := (k-1)\binom{d}{k}$. So the Veronese embeddings of $\mathbb{P}^1$ satisfy $N_p \quad \forall p$.

From [B2], Remark 2.7, and [G1] we have the following cohomological criterion:

PROPOSITION 1.5. *The Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies property N_p if and only if*

$$H^1\left(\bigwedge^q E_d(jd)\right) = 0, \quad \textit{for } 1 \leqslant q \leqslant p+1 \quad \textit{and} \quad \forall j \geqslant 1. \qquad \diamond$$

We have the following cohomological criterion, which slightly improves the previous one (in fact $H^2(\bigwedge^q E_d) \simeq H^1(\bigwedge^{q-1} E_d(d))$).

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THEOREM 1.6. *The Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies property N_p if and only if $H^2(\bigwedge^q E_d) = 0$ for $1 \leqslant q \leqslant p+2$.*

The proof of Theorem 1.6 relies on the following proposition:

PROPOSITION 1.7. *If $H^2(\bigwedge^q E_d) = 0$ for $1 \leqslant q \leqslant k$, then $H^2(\bigwedge^q E_d(t)) = 0$ for $1 \leqslant q \leqslant k$ and $\forall t \geqslant 0$.*

Proof. Consider the two exact sequences:

$$0 \to \bigwedge^q E_d(t-1) \to \bigwedge^q E_d(t) \to \bigwedge^q E_d(t)|_{\mathbb{P}^{n-1}} \to 0, \qquad (*)$$

$$0 \to \bigwedge^q E_d(t-1) \to \bigwedge^q (S^d V) \otimes \mathcal{O}_{\mathbb{P}^n}(t-1) \to \bigwedge^{q-1} E_d(t+d-1) \to 0. \qquad (**)$$

The proof is by double induction on n and k. The statement is true for $n = 2$ (Serre duality) and for $k = 1$ (it follows immediately from (1.2)). From the cohomology sequence associated to $(**)$ with $t = 0$ and the inductive hypothesis on k we get $H^3(\bigwedge^q E_d(-1)) = 0$ for $1 \leqslant q \leqslant k$. Since

$$E_{d|\mathbb{P}^{n-1}} \cong \tilde{E}_d \oplus \mathcal{O}_{\mathbb{P}^{n-1}}^{\oplus \binom{n+d-1}{n}},$$

where $\tilde{E}_d$ is the vector bundle E_d over $\mathbb{P}^{n-1}$, the previous vanishing implies in the cohomology sequence associated to $(*)$ with $t = 0$ that the hypothesis of the proposition are true on $\mathbb{P}^{n-1}$. Hence, by induction on n, $H^2(\mathbb{P}^{n-1}, \bigwedge^q E_d(t)_{|\mathbb{P}^{n-1}}) = 0$ for $1 \leqslant q \leqslant k$ and $\forall t \geqslant 0$. From the cohomology sequence associated to $(*)$ with $q = k$ we get that the map $H^2(\mathbb{P}^n, \bigwedge^k E_d(t-1)) \to H^2(\mathbb{P}^n, \bigwedge^k E_d(t))$ is surjective $\forall t \geqslant 0$ and the thesis follows easily. □

Proof of Theorem 1.6. The implication '$\Longrightarrow$' is a consequence of Proposition 1.5. To prove the converse, we may apply Proposition 1.7 and then Proposition 1.5 again.

PROPOSITION 1.8. *If $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ satisfies N_p, then $\varphi_{\mathcal{O}_{\mathbb{P}^m}(d)}$ satisfies N_p $\forall m \leqslant n$.*

Proof. It follows by the remark of Section 2 of [G2] (which is an insight into representation theory). □

2. Necessary Conditions on Property N_p for the Veronese Embedding $\varphi_{\mathcal{O}_{\mathbb{P}}(d)}$

In this section we will prove the following theorem:

THEOREM 2.1. *The Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(d)}$ does not satisfy N_{3d-2} for $n \geqslant 2$, $d \geqslant 3$.*

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Proof. By Proposition 1.8, we can let $n = 2$. By Theorem 1.6 and Serre duality, it is enough to show that $H^0(\mathbb{P}^2, \bigwedge^K E_d(d-3)) \leqslant 0$ with $K := d(d-3)/2$. So the theorem will follow from the following lemma:

LEMMA 2.2. *The bundle $\bigwedge^q E_d(t)$ has a nonzero global section for $1 \leqslant q \leqslant N$; $q+1 \leqslant \binom{n+t}{n}$ and $t \geqslant 1$.*

Proof. The exact sequence $0 \to \bigwedge^q E_d \to \bigwedge^q S^d V \otimes \mathcal{O}_{\mathbb{P}^n} \to \bigwedge^{q-1} E_d(d) \to 0$ implies that

$$H^0\left(\bigwedge^q E_d(t)\right) = \operatorname{Ker}\left(\bigwedge^q S^d V \otimes S^t V \xrightarrow{\alpha_t} \bigwedge^{q-1} S^d V \otimes S^{t+d} V\right).$$

Now there is a Koszul complex

$$\to \bigwedge^{q+1} S^d V \otimes \mathcal{O}(t-d) \to \bigwedge^q S^d V \otimes \mathcal{O}(t) \xrightarrow{a_t} \bigwedge^{q-1} S^d V \otimes \mathcal{O}(t+d) \to$$

with $\alpha_t = H^0(a_t)$. For $t \geqslant d$, global sections of $\bigwedge^{q+1} S^d V \otimes \mathcal{O}(t-d)$ will therefore give sections of $\bigwedge^q E_d(t)$. In particular, for $d = t$, we get that for each family $s_0, \ldots, s_q$ of degree d polynomials,

$$\sum_{i=0}^{q} (-1)^i s_0 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_q \otimes s_i$$

is in the kernel of α_d. Now let $1 \leqslant t < d$. If we can factor $s_i = uw_i$ with u of degree $d - t$, then

$$\sum_{i=0}^{q} (-1)^i s_0 \wedge \cdots \wedge \hat{s}_i \wedge \cdots \wedge s_q \otimes w_i$$

must be in the kernel of α_t, and therefore defines a global section of $\bigwedge^q E_d(t)$. Thus, to get a nonzero section of $\bigwedge^q E_d(t)$, it suffices to find $q+1$ linearly independent polynomials of degree t, which is possible as soon as $q+1 \leqslant \binom{n+t}{n}$. □

Remark 2.3. The bundles $\bigwedge^q E_d$ are semistable (see [P], Proposition 5.6), so $H^0(\bigwedge^q E_d(t)) = 0$ if $\mu(\bigwedge^q E_d(t)) = t - (qd/N) < 0$. In particular,

$$H^0\left(\bigwedge^q E_d(t)\right) = 0 \quad \forall t \leqslant 0.$$

3. Conclusions

In this section we will fit our results into the literature. In particular, we will prove the following theorem:

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THEOREM 3.1. *Let d be an integer s.t. $d \geqslant 3$. Then the Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^2}(d)}\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^N$ satisfies property N_p if and only if $0 \leqslant p \leqslant 3d-3$. Moreover, if $d = 2$, the embedding $\varphi_{\mathcal{O}_{\mathbb{P}^2}(2)}\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ satisfies $N_p \quad \forall p$.*

We have the following proposition:

PROPOSITION 3.2 (M. Green, C. Birkenhake). *Let $d \geqslant 2$ and $p = \begin{cases} 3d-3 & \text{if } d \geqslant 3 \\ 2 & d=2 \end{cases}$. Then the complete Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^2}(d)}\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^N$ satisfies property N_p.*

Proof. See [B1], Corollary 3.2. The result follows from also applying Theorem 3.*b*.7 of [G1] (which says that the minimal resolution of a Veronese variety restricts to the minimal resolution of its curve hyperplane section) and Theorem 4.*a*.1 of [G1] (which says that a line bundle of degree $2g+1+p$ on a curve of genus g satisfies N_p). □

In the same way we get the following lemma:

LEMMA 3.3. *The Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^3}(3)}\colon \mathbb{P}^3 \hookrightarrow \mathbb{P}^{19}$ satisfies N_6.*

Proof. The curve hyperplane section of the image of the cubic Veronese embedding of $\mathbb{P}^3$ is the space curve complete intersection of two cubics embedded by $|\mathcal{O}_{\mathbb{P}^3}(3)|$ and it has genus 10. The result follows again applying Theorem 3.*b*.7 and Theorem 4.*a*.1 of [G1]. □

LEMMA 3.4. *The ideal $\mathcal{I}$ of $\varphi_{\mathcal{O}_{\mathbb{P}^2}(2)}(\mathbb{P}^2)$ in $\mathbb{P}^5$ has the following resolution:*

$$\mathcal{O} \to \mathcal{O}_{\mathbb{P}^5}(-4)^{\oplus 3} \to \mathcal{O}_{\mathbb{P}^5}(-3)^{\oplus 8} \to \mathcal{O}_{\mathbb{P}^5}(-2)^{\oplus 6} \to \mathcal{I} \to 0.$$

In particular, the Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^2}(2)}\colon \mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ satisfies $N_p \quad \forall p$.

Proof. Easy computation. □

Proof of Theorem 3.1. By Proposition 3.2 and Lemma 3.4, we just need to show that if $d \geqslant 3$, then property N_p does not hold for $p \geqslant 3d-2$. But this is exactly the bound coming from Theorem 2.1. □

When $d = 2$, the minimal free resolution of the quadratic Veronese variety is known from the work of Jozefiak, Pragacz and Weyman [JPW], in which they prove a conjecture made by Lascoux. As a corollary of the above paper, we have the following result (which agrees with our conjecture formulated in the introduction):

THEOREM 3.5. *The quadratic Veronese embedding $\varphi_{\mathcal{O}_{\mathbb{P}^n}(2)}\colon \mathbb{P}^n \hookrightarrow \mathbb{P}^N$ satisfies N_p if and only if $p \leqslant 5$ when $n \geqslant 3$ and $\forall p$ when $n = 2$.* □

The following nice characterization, probably well known, was found during discussions with E. Arrondo:

THEOREM 3.6. *The only (smooth) varieties in $\mathbb{P}^n$ such that N_p holds for every $p \geqslant 0$ are the quadrics, the rational normal scrolls and the Veronese surface in $\mathbb{P}^5$.*

Proof. Suppose X is a variety satisfying N_p for every $p \geqslant 0$, then $H^i(\mathcal{O}_X(t)) = 0$ for $t \geqslant 0$ and $1 \leqslant i \leqslant \dim X - 1$. Hence, from Theorem 3.$b$.7 in [G1], it follows that the minimal free resolution of X restricts to the minimal resolution of its generic curve section C. This implies that $H^1(\mathcal{O}_C) = 0$ and C is linearly normal, hence C is a rational normal curve. In particular, X has minimal degree and we get the result. □

We remark that the only Veronese varieties appearing in Theorem 3.6 are the rational normal curves and the Veronese surface in $\mathbb{P}^5$.

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