SZLENK INDICES AND UNIFORM HOMEOMORPHISMS

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ABSTRACT. We prove some rather precise renorming theorems for Banach spaces with Szlenk index ω_0. We use these theorems to show the invariance of certain quantitative Szlenk-type indices under uniform homeomorphisms.

1. INTRODUCTION

Classical results about super-reflexive Banach spaces include Enflo-Pisier's renorming theorem ([5],[23]) and Heinrich-Mankiewicz theorems on uniform and Lipschitz homeomorphisms [10], from which it follows in particular that the moduli of uniform convexity or uniform smoothness of super-reflexive spaces is an invariant for uniform homeomorphisms. This work is an attempt to obtain similar results in the frame of non super-reflexive spaces. We will in particular relate, in a quantitative way, the Szlenk index with the existence of equivalent UKK^*-renormings of the space. These results extend and improve results in [16]. We will also show that the quantitative dependence on ϵ of the Szlenk index (when finite) is an invariant under uniform homeomorphisms.

We now turn to a detailed description of our results. In Section 2, the *convex Szlenk index* Cz is introduced and compared with the usual Szlenk index Sz. Precise duality formulas, somewhat related to duality between Orlicz spaces, are established which relate the "c_0-like" behavior of a Banach space X with the "l^1-like" behavior of its dual X^* (Proposition 2.8). Trees and tree-maps are introduced in Section 3 as a tool to translate estimates on the Szlenk index into geometrical language. Section 4 is devoted to renormings. It was recently shown [16] that if $\mathrm{Sz}(X) \leq \omega_0$ then X has an equivalent UKK^*-renorming of some power type. We prove more precise results. Trees are an operative tool in the proof of the main renorming theorem (Theorem 4.7) which provides the optimal relation between the behavior of the convex Szlenk index for any given value of $\tau > 0$ and the existence of a 2-equivalent norm with the UKK^* property for this value of τ. In Theorem 4.8 we improve the result of [16] mentioned above by giving a precise bound on the power-type. Note however that there is a small loss on the exponent in Theorem 4.8. Such a loss is unavoidable, as

shown by the reflexive example from [16]. This also shows Theorem 4.7 cannot be improved to give a simultaneous renorming which works for all $\tau > 0$. Summability of the Szlenk index is shown (Theorem 4.10) to be equivalent to the existence of a constant K with $\tau.Cz(X,\tau) < K$ for any $\tau \in (0,1)$. Note that Corollary 4.6 asserts that the indices Cz and Sz have the same power-type behavior, at least for $p > 1$. Section 5 presents applications of these results to uniform homeomorphisms. The main result of this section (Theorem 5.3), whose proof uses the Gorelik principle (Proposition 5.1), asserts in a quantitative way that the existence of equivalent UKK^* norms is invariant under uniform homeomorphisms. The main application of this result (Theorem 5.5) is that the convex Szlenk index is quantitatively invariant under uniform homeomorphisms. This invariance is naturally obtained by a combination of Theorem 5.3 with the renorming Theorem 4.7. Note that, although the class of spaces with separable dual is not stable under uniform homeomorphisms [24], the class of spaces with "very separable" dual (that is, of spaces with Szlenk index ω_0) is by the above stable under uniform homeomorphisms. It follows also (Theorem 5.6) that a Banach space which is uniformly homeomorphic to c_0 is an isomorphic predual of l^1 with summable Szlenk index. We do not know whether a predual of l^1 with summable Szlenk index is isomorphic to c_0. Our last application (Theorem 5.8) concerns quotients of l^p for $p \in (2, \infty)$.

Some of the results of this work have been announced in [7].

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2. The Szlenk index and properties of norms

We first recall the definition of the Szlenk index and the Szlenk derivation. Suppose X is a separable infinite-dimensional Banach space and K is a weak*-compact subset of X^*. If $\epsilon > 0$ we let $\mathcal{V}$ be the set of all weak*-open subsets V of X^* such that diam $V \cap K \leq \epsilon$ and the $\epsilon-$interior $\iota_\epsilon K = K \setminus \cup\{V : V \in \mathcal{V}\}$. (The set $\iota_\epsilon K$ is often denoted K'_ϵ as in [18], [17]). We then define $\iota_\epsilon^\alpha K$ for any ordinal α by $\iota_\epsilon^{\alpha+1} K = \iota_\epsilon \iota_\epsilon^\alpha K$ and $\iota_\epsilon^\alpha K = \cap_{\beta<\alpha} \iota_\epsilon^\beta K$ if α is a limit ordinal.

We denote by B_{X^*} the closed unit ball of X^*. We then define $Sz(X,\epsilon)$ (or $Sz(\epsilon)$ if no confusion can arise) to be the least countable ordinal α so that $\iota_\epsilon^\alpha B_{X^*} = \emptyset$, if such an ordinal exists. Otherwise we will put $Sz(X, \epsilon) = \omega_1$. The *Szlenk index* is defined by $Sz(X) = \sup_{\epsilon>0} Sz(X, \epsilon)$. We recall that $Sz(X) < \omega_1$ if and only if X^* is separable.

Note that $\mathrm{Sz}(X, \epsilon) \geq \epsilon^{-1}$ if $\epsilon > 0$, and compactness requires that $\mathrm{Sz}(X, \epsilon)$ is not a limit ordinal. Thus $\mathrm{Sz}(X) = \omega_0$ is equivalent to $\mathrm{Sz}(X, \epsilon) < \omega_0$ for every $\epsilon > 0$, where ω_0 denotes the first limit ordinal.

We also introduce an alternative *convex Szlenk index*. If K is compact and convex we may define $\hat{\iota}_\epsilon K = \overline{\mathrm{co}}\, \iota_\epsilon K$. Then $\mathrm{Cz}(X, \epsilon)$ and $\mathrm{Cz}(X)$ are defined as before, using instead this derivation. Obviously $\mathrm{Cz}(X, \epsilon) \geq \mathrm{Sz}(X, \epsilon)$. On the hand, $\mathrm{Cz}(X) < \omega_1$ is equivalent to the separability of X^*; this follows easily from the weak*-dentability of weak*-compact sets in separable duals.

Henceforward we will only be interested in cases when $\mathrm{Sz}(X, \epsilon)$ and $\mathrm{Cz}(X, \epsilon)$ are finite. It will therefore be useful to adopt the convention that $\mathrm{Sz}(X, \epsilon) = \infty$ if $\mathrm{Sz}(X, \epsilon) \geq \omega_0$ and similar convention for $\mathrm{Cz}(X, \epsilon)$.

Following [16], we will say that X admits a *summable Szlenk index* if there exists a constant K so that $\sum_{i=1}^{n} \epsilon_i \leq K$ whenever $\iota_{\epsilon_1} \cdots \iota_{\epsilon_n} B_{X^*} \neq \emptyset$.

The following lemma is proved in [17], p.57, or [18]:

Lemma 2.1. *If $0 < \epsilon, \eta \leq 1$, then $Sz(X, \epsilon\eta) \leq Sz(X, \epsilon)Sz(X, \eta)$.*

Note that this implies that $\mathrm{Sz}(X) = \omega_0$ if and only if $\mathrm{Sz}(X, \epsilon) < \infty$ for any fixed $0 < \epsilon < 1$.

Another immediate consequence of this is that if $\mathrm{Sz}(X) = \omega_0$ then

$$\lim_{\epsilon \to 0} \frac{\log \mathrm{Sz}(X, \epsilon)}{|\log \epsilon|} = p$$

exists where $1 \leq p < \infty$. We will call $p = p_X$ the *Szlenk power type* of X. We also have that if $\delta > 0$ then $\mathrm{Sz}(X, \epsilon) \leq C\epsilon^{-p-\delta}$ for some suitable constant C so that $\mathrm{Sz}(X, \epsilon)$ grows at a power rate. In fact we can also define p_X as the infimum of all q so that $\epsilon^q \mathrm{Sz}(X, \epsilon)$ is bounded.

Next we note

Lemma 2.2. *If $0 < \epsilon < 1$ and $n \in \mathbb{N}$ are such that $n\epsilon \leq 1$ then:*
(1) $Sz(X, \epsilon) - 1 \geq n(Sz(X, n\epsilon) - 1)$.
(2) $Cz(X, \epsilon) - 1 \geq n(Cz(X, n\epsilon) - 1)$.

Proof. Note that if $j < \mathrm{Sz}(X, n\epsilon)$ then for any $m \geq 2$,

$$(m-1)B_{X^*} + \iota_{n\epsilon}^j B_{X^*} \subset \iota_{n\epsilon}^j(mB_{X^*})$$

Hence $(m-1)B_{X^*} \subset \iota_{n\epsilon}^l mB_{X^*}$ where $l = \mathrm{Sz}(X, n\epsilon) - 1$. Iterating we obtain $\iota_{n\epsilon}^{nl}(nB_{X^*}) \neq \emptyset$ and so $\mathrm{Sz}(X, \epsilon) \geq nl + 1$. This implies the result for the Szlenk index and the convex version is similar.

$\square$

We will also need the following elementary fact:

Lemma 2.3. *Suppose X and Y are isomorphic Banach spaces. Then if $d = d(X, Y)$ we have*

$$Sz(X, d\epsilon) \leq Sz(Y, \epsilon) \leq Sz(X, d^{-1}\epsilon)$$

and

$$Cz(X, d\epsilon) \leq Cz(Y, \epsilon) \leq Cz(X, d^{-1}\epsilon).$$

There is one important advantage of the convex Szlenk index which is established in the following lemma. Note that part (2) shows that if the convex Szlenk index is $O(\tau^{-1})$, then it is actually summable.

Lemma 2.4. *Suppose X is a separable Banach space, $0 \leq \epsilon_k \leq 1$ for $1 \leq k \leq N$ and $s = \sum_{i=1}^{N} \epsilon_i$.*
(1) If $\iota_{\epsilon_1} \cdots \iota_{\epsilon_N} B_{X^*} \neq \emptyset$ and $0 < \tau \leq \frac{s}{2N}$

$$\sum_{\substack{k \geq 0 \\ 2^k \tau \leq 1}} 2^k (Sz(X, 2^k \tau) - 1) \geq \frac{s}{2\tau}.$$

(2) If $\hat{\iota}_{\epsilon_1} \cdots \hat{\iota}_{\epsilon_N} B_{X^*} \neq \emptyset$ and $0 < \tau \leq s/(4N)$,

$$Cz(X, \tau)) - 1 \geq \frac{s}{8\tau}.$$

Proof. For the first part, let us write $\alpha_i \leq \epsilon_i \leq \beta_i$ where $\alpha_i = 2^{k_i}\tau$ and $\beta = 2^{k_i+1}\tau$ with k_i a nonnegative integer, or $\alpha_i = 0$ and $\beta_i = \tau$. Then $\beta_i = \tau + \sum_{0 \leq j \leq k_i} 2^j \tau$. Now observe that $|\{j : \epsilon_j \geq 2^k \tau\}| < \text{Sz}(X, 2^k \tau)$ and so

$$s \leq N\tau + \sum_{\substack{k \geq 0 \\ 2^k \tau \leq 1}} 2^k \tau (\text{Sz}(X, 2^k \tau) - 1).$$

This immediately gives the conclusion of (1).

For the second part, first suppose C is a weak*-compact and convex subset of X^*. We will argue that if $\epsilon > 0$ and $k \in \mathbb{N}$ then $\hat{\iota}_{2k\epsilon} C \subset \hat{\iota}_{\epsilon}^k C$. Indeed suppose $x^* \in \iota_{2k\epsilon} C$. Then there is a sequence $x_n^* \in C$ with $\|x_n^* - x^*\| \geq k\epsilon$ and $x_n^* \to x^*$ weak*. Now for any $n_1 < n_2 < n_3 \cdots < n_k$ we have $\frac{1}{k}(x_{n_1}^* + \cdots + x_{n_k}^*) \in C$. Letting $n_k \to \infty$ we obtain that $\frac{1}{k}(x^* + x_{n_1}^* + \cdots + x_{n_{k-1}}^*) \in \iota_{\epsilon} C$. Letting $n_{k-1} \to \infty$ and repeating we obtain that $x^* \in \iota_{\epsilon}^k C$. In particular we observe that $\hat{\iota}_{2k\epsilon} C \subset \hat{\iota}_{\epsilon}^k C$.

Now suppose $\epsilon_1, \epsilon_2, \cdots, \epsilon_N > 0$ and $\sum_{k=1}^{N} \epsilon_k = s$. Suppose $\tau \leq s/4N$ and let $m_k = [\epsilon_k/2\tau]$. Then

$$\hat{\iota}_{\epsilon_1} \cdots \hat{\iota}_{\epsilon_N} B_{X^*} \subset \hat{\iota}_{\tau}^{m_1 + \cdots + m_N} B_{X^*}.$$

Now

$$m_1 + \cdots + m_N = \sum_{\epsilon_k \geq 2\tau} \left[\frac{\epsilon_k}{2\tau}\right] \geq \frac{1}{4\tau} \sum_{\epsilon_k \geq 2\tau} \epsilon_k.$$

But $\tau \leq \frac{s}{4N}$ yields $\sum_{\epsilon_k \leq 2\tau} \epsilon_k \leq \frac{s}{2}$. Hence $m_1 + \cdots + m_N \geq \frac{s}{8\tau}$. $\square$

Let us now turn to renormings. We need the following elementary lemma:

Lemma 2.5. *Suppose X is an infinite-dimensional Banach space with separable dual and that (x_n^*) is a sequence in X^* with $\lim_{n\to\infty} x_n^* = 0$ weak*. Then there is a sequence (x_n) in X with $\|x_n\| \le 1$, $\lim_{n\to\infty} x_n = 0$ weakly and*

$$\liminf \langle x_n, x_n^* \rangle \ge \frac{1}{2} \liminf \|x_n^*\|.$$

Proof. We observe that by Lemma 2.3 of [15] we have

$$\liminf_{n\to\infty} d(x_n^*, F) \ge \frac{1}{2} \liminf_{n\to\infty} \|x_n^*\|$$

if F is any finite-dimensional subspace of X^*. Hence, since X^* is separable we find an increasing sequence (not necessarily strictly increasing) of finite-dimensional subspaces (F_n) so that $\cup F_n$ is dense in X^* and $\liminf d(x_n^*, F_n) \ge \frac{1}{2} \liminf_{n\to\infty} \|x_n^*\|$.

There exist $x_n \in X$ with $\|x_n\| = 1$, $x_n \in F_n^\perp$ and $\langle x_n, x_n^* \rangle > d(x_n^*, F_n) - \frac{1}{n}$ and this gives the conclusion.

$\square$

Proposition 2.6. *. Suppose X is a separable Banach space and $0 < \sigma, \tau < 1$. Consider the following statements:*

(1) *If $x^* \in X^*$ with $\|x^*\| = 1$, $\lim x_n^* = 0$ weak* with $\lim_{n\to\infty} \|x_n^*\| = \tau$ then*

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \ge 1 + \sigma\tau.$$

(2) *X^* is separable and if $x \in X$ with $\|x\| = 1$, $\lim x_n = 0$ weakly with $\lim \|x_n\| = \sigma$ then*

$$\limsup_{n\to\infty} \|x + x_n\| \le 1 + \sigma\tau.$$

(3) *X^* is separable and if $x^* \in X^*$ with $\|x^*\| = 1$, $\lim x_n^* = 0$ weak* with $\lim_{n\to\infty} \|x_n^*\| = 6\tau$ then*

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \ge 1 + \sigma\tau.$$

Then $(1) \Rightarrow (2) \Rightarrow (3)$. Furthermore (1) implies that

$$Sz(X, 2\tau) \le Cz(X, 2\tau) \le \sigma^{-1}\tau^{-1} + 1.$$

(In particular if $2\tau < 1$ then $Sz(X) = \omega_0$.)

Proof. Note that (1) implies that $\iota_{2\tau} B_{X^*} \subset (1 - \sigma\tau) B_{X^*}$ and this immediately yields $Cz(X, 2\tau) \le \sigma^{-1}\tau^{-1} + 1$. Then the last statement follows from Lemma 2.1.

First assume (1) holds and x, x_n are chosen as in (2). It is enough to show it is impossible that $\lim_{n\to\infty} \|x + x_n\| > 1 + \sigma\tau$. Suppose this holds. Then we can

pick $y_n^* \in B_{X^*}$ with $\lim_{n\to\infty} \langle x + x_n, y_n^* \rangle > 1 + \sigma\tau$. Passing to a subsequence we can suppose y_n^* converges weak* to some $x^* \in B_{X^*}$ and then put $x_n^* = y_n^* - x^*$. We can assume that $\lim_{n\to\infty} \|x_n^*\| = \theta$ exists.

If $\theta \leq \tau$ then

$$\lim_{n\to\infty} \langle x + x_n, x^* + x_n^* \rangle \leq \langle x, x^* \rangle + \sigma\tau \leq 1 + \sigma\tau.$$

On the other hand if $\theta > \tau$ then we have, using the convexity of the norm,

$$1 = \liminf_{n\to\infty} \|x^* + x_n^*\| \geq \|x^*\| + \sigma\theta.$$

Hence $\|x^*\| \leq 1 - \sigma\theta$ and so

$$\lim_{n\to\infty} \langle x + x_n, x^* + x_n^* \rangle \leq 1 - \sigma\theta + \sigma\theta = 1.$$

This gives us the required contradiction.

Assume (2) holds. Suppose that x^*, x_n^* are chosen as in (3). It will be enough to show the conclusion for some subsequence.

Then, given $\epsilon > 0$ we can choose $x \in X$ with $\|x\| = 1$ and $x^*(x) > 1 - \epsilon$. Now by Lemma 2.5, we can, by passing to a subsequence, assume there exist a weakly null sequence $x_n \in X$ with $\|x_n\| = 1$ and $\liminf x_n^*(x_n) \geq 3\tau$. Now

$$\liminf \langle x + \sigma x_n, x^* + x_n^* \rangle \geq 1 - \epsilon + 3\sigma\tau$$

and so

$$1 - \epsilon + 3\sigma\tau \leq (1 + \sigma\tau) \liminf \|x^* + x_n^*\|.$$

Hence letting $\epsilon \to 0$ we have

$$\liminf \|x^* + x_n^*\| \geq \frac{1 + 3\sigma\tau}{1 + \sigma\tau} \geq 1 + \sigma\tau.$$

$\square$

Proposition 2.7. *Let X be a separable Banach space not containing a copy of ℓ_1, and suppose $0 < \sigma < 1$ and $0 < \tau < \frac{1}{2}$. Suppose that whenever $x \in X$ with $\|x\| = 1$ and $\lim x_n = 0$ weakly with $\lim_{n\to\infty} \|x_n\| = \sigma$ then*

$$\limsup_{n\to\infty} \|x + x_n\| \leq 1 + \sigma\tau.$$

Then X^* contains no proper norming subspaces and hence is separable.

Proof. Suppose $x^{**} \in X^{**}$ is of norm one and $\{x^* \in X^* : x^{**}(x^*) = 0\}$ is a norming subspace of X^*. Then by the Odell-Rosenthal theorem [22] there is a weakly Cauchy sequence (x_n) in X with $\|x_n\| = 1$ and $x_n \to x^{**}$ weak*.

If $m \in \mathbb{N}$ then, for any $\epsilon > 0$, and for each $n > m$ we can choose $e^* \in X^*$ with $\|e^*\| = 1$, $x^{**}(e^*) = 0$ and $e^*(x_m + \frac{1}{2}\sigma x_n) > \|x_m + \frac{1}{2}\sigma x_n\| - \epsilon$. Then we can find $k(n) > n$ so that $e^*(\sigma x_{k(n)}) < \epsilon$. We conclude that

$$\|x_m + \frac{1}{2}\sigma(x_n - x_{k(n)})\| > \|x_m + \frac{1}{2}\sigma x_n\| - 2\epsilon.$$

Letting $n \to \infty$, since $(x_n - x_{k(n)}) \to 0$ weakly we have

$$\limsup_{n\to\infty} \|x_m + \frac{1}{2}\sigma x_n\| \leq 1 + \sigma\tau + 2\epsilon.$$

Thus

$$\|x_m + \frac{1}{2}\sigma x^{**}\| \leq 1 + \sigma\tau.$$

Then $m \to \infty$ we have $1 + \frac{1}{2}\sigma \leq 1 + \sigma\tau$ which is a contradiction. $\square$

At this point we introduce some terminology. Let f, g be continuous monotone increasing functions on $[0,1]$ which satisfy $f(0) = g(0) = 0$. We will say that f *C-dominates* g if $f(\tau) \geq g(\tau/C)$ for every $0 \leq \tau \leq 1$. We will say that f, g are *C-equivalent* if f C-dominates g and g C-dominates f.

For any such monotone increasing function f we denote by f^* its dual Young's function i.e.

$$f^*(s) = \sup\{st - f(t) : \ 0 \leq t \leq 1\}.$$

Notice that if f C-dominates g then g^* C-dominates f^*. Note also that f^* is a convex function.

Now if X is any separable Banach space we define for $0 \leq \sigma \leq 1$, $\rho(\sigma) = \rho_X(\sigma)$ to be the least constant so that

$$\limsup_{n\to\infty} \|x + x_n\| \leq 1 + \rho_X(\sigma)$$

whenever $\|x\| = 1$, $\lim x_n = 0$ weakly and $\limsup_{n\to\infty} \|x_n\| \leq \sigma$. We define $\theta(\tau) = \theta_X(\tau)$ for $0 \leq \tau \leq 1$ to be the greatest constant so that

$$\liminf_{n\to\infty} \|x^* + x_n^*\| \geq 1 + \theta_X(\tau)$$

whenever $x^*, x_n^* \in X^*$, $\|x^*\| = 1$, $\lim x_n^* = 0$ weak* and $\liminf_{n\to\infty} \|x_n^*\| \geq \tau$.

We then define $\varphi(\sigma)$ by $\varphi(\sigma) = \inf\{\rho_Y(\sigma) : \ d(X,Y) \leq 2\}$ and $\psi(\tau) = \sup\{\theta_Y(\tau) : \ d(X,Y) \leq 2\}$.

We can now summarize Proposition 2.6 and Proposition 2.7:

Proposition 2.8. *Let X be a separable Banach space not containing ℓ_1. Then:*

(1) *Each of the functions $\rho(t)/t$, $\theta(t)/t$, $\varphi(t)/t$ and $\psi(t)/t$ is monotone increasing on $(0,1)$.*

(2) *θ is 4-equivalent to ρ^* and ψ is 4-equivalent to φ^*.*

(3) *ρ is 8-equivalent to θ^* and φ is 8-equivalent to ψ^**

(4) *If $0 < t \leq 1$ then*

$$Cz(X,t) \leq 1 + \frac{1}{\psi(t/4)}.$$

Proof. (1) follows trivially from convexity considerations.

(2) It follows from Proposition 2.7 that if X^* is not separable, then $\rho^*(\tau) = 0$ for all $0 \leq \tau \leq \frac{1}{2}$ and then it is obvious that θ 2-dominates ρ^*. If X^* is separable, we pick (x_n^*) in X^* such that $\liminf_{n\to\infty} \|x_n^*\| \geq \tau$ and

$$\liminf_{n\to\infty} \|x^* + x_n^*\| = 1 + \theta_X(\tau)$$

By Lemma 2.5 there is a weakly null sequence (x_n) with $\limsup_{n\to\infty} \|x_n\| \leq \sigma$ and $\lim x_n^*(x_n) = \sigma\tau/2$. It follows easily that

$$1 + \sigma\tau/2 \leq (1 + \rho(\sigma))(1 + \theta(\tau))$$

so that

$$\sigma\tau/2 \leq \rho(\sigma) + \theta(\tau) + \rho(\sigma)\theta(\tau).$$

Now since $\theta(\tau) \leq 1$ we obtain

$$\theta(\tau) \geq \sigma\tau/2 - 2\rho(\sigma) \geq 2\rho^*(\tau/4).$$

Hence θ 4-dominates ρ^*.

The same considerations show that ρ 4-dominates θ^*, φ 4-dominates ψ^* and ψ 4-dominates φ^*.

Now if $0 < \tau < 1$ pick $\sigma = 2\theta(\tau/2)/\tau$. Then by Proposition 2.6 we have $\rho(\sigma) \leq \frac{1}{2}\sigma\tau$. Hence

$$\theta(\tau/2) = \frac{1}{2}\sigma\tau \leq \sigma\tau - \rho(\sigma) \leq \rho^*(\tau).$$

Thus ρ^* 2-dominates θ. The proof for ψ in place of θ and φ in place of ρ is similar.

(3) We deduce from (2) that θ^* is 4-equivalent to ρ^{**}. Next let

$$\tilde{\rho}(t) = \int_0^t \frac{\rho(\tau)}{\tau} d\tau.$$

Then $\rho(t/2) \leq \tilde{\rho}(t) \leq \rho(t)$. Hence since $\tilde{\rho}$ is convex we have $\rho^{**}(t) \geq \tilde{\rho}(t) \geq \rho(t/2)$. Hence ρ is 2-equivalent to ρ^{**}. The argument for φ and ψ is similar.

(4) This is an immediate deduction from Proposition 2.6 and Lemma 2.3.   □

3. TREES AND TREE-MAPS

Consider the set $\mathcal{F}\mathbb{N}$ of all finite subsets of $\mathbb{N}$ with the following partial order. If $a = \{n_1, n_2, \cdots, n_k\}$ where $n_1 < n_2 < \cdots < n_k$ and $b = \{m_1, m_2 \cdots, m_l\}$ where $m_1 < m_2 < \cdots < m_l$, then $a \leq b$ if and only if $k \leq l$ and $m_i = n_i$ where $1 \leq i \leq k$ (i.e. a is an initial segment of b.) We say that b is a *successor* of a if $|b| = |a| + 1$ and $a \leq b$; the collection of successors of a is denoted by $a+$.

If $a \neq \emptyset$ then $a-$ denotes the unique predecessor of a i.e. a is a successor of $a -$. Let S be a subset of $\mathcal{F}\mathbb{N}$. We will say that S is a *full tree* if we have

(1) $\emptyset \in S$.
(2) Each $a \in S$ has infinitely many successors in S.
(3) If $a \in S$ and $\emptyset \neq a \in S$ then $a- \in S$.

It is easy to see that any full tree is isomorphic as an ordered set to $\mathcal{F}\mathbb{N}$. If S is any full tree we will say that a sequence $\beta = \{a_n\}_{n=0}^{\infty}$ is a *branch* of S if $a_n \in S$ for all n, $a_0 = \emptyset$ and a_{n+1} is a successor of a_n for all $n \geq 0$.

Now let V be a vector space. We define a *tree-map* to be a map $a \mapsto x_a$ defined on a full tree S with the properties that $x_\emptyset = 0$ and for every branch β the set $\{a : x_a \neq 0 : a \in \beta\}$ is finite. Given any tree-map we define a *height function* h which assigns to each a a countable ordinal; to do this we define $h(a) = 0$ if $x_b = 0$ for $b \geq a$ and then inductively $h(a)$ is defined by $h(a) \leq \eta$ if and only if $h(b) < \eta$ for every $b > a$. The *height* of the tree-map is defined to be $h(\emptyset)$. Note that the tree-map $a \mapsto x_a$ has finite height $m \leq n$ if and only if $x_a = 0$ whenever $|a| > n$. For a recent work on trees in Banach spaces we refer to [2].

The following easy lemma is a restatement of the fact that certain types of games (which are not used in this paper) are determined.

Lemma 3.1. *Suppose $(x_a)_{a \in S}$ is a tree-map and that A is any subset of V. Then either there is a full tree $T \subset S$ so that $\sum_{a \in \beta} x_a \in A$ for every branch $\beta \subset T$ or there is a full tree $T \subset S$ so that $\sum_{a \in \beta} x_a \notin A$ for every branch $\beta \subset T$.*

Proof. For each countable ordinal η we define a subset B_η of $\{a \in S : h(a) = \eta\}$ as follows. If $\eta = 0$ let $a \in B_0$ if $\sum_{b \leq a} x_a \in A$. Then inductively if $h(a) = \eta$ we say $a \in B_\eta$ if a has infinitely many successors b with $b \in B_{h(b)}$. Let $B = \cup_\eta B_\eta$. If $\emptyset \in B$ then an easy induction argument produces a full tree $T \subset S$ with $\sum_{a \in \beta} x_a \in A$ for every $\beta \subset T$. Otherwise the set $T = S \setminus B$ is a full tree with the property that $\sum_{a \in \beta} x_a \notin A$ for every $\beta \subset T$. $\square$

We now consider tree-maps with values in a Banach space X.

Lemma 3.2. *Suppose $(x_a)_{a \in S}$ is a bounded tree-map in X of finite height n. Then, given $\delta > 0$ we can find a full tree $T \subset S$ and $\epsilon_1, \cdots, \epsilon_n \geq 0$ so that if $a \in T$ and $|a| = k \leq n$ then $\epsilon_k \leq \|x_a\| \leq \epsilon_k + \delta$.*

Proof. One easy way to prove this is to consider $V = \mathbb{R}^n$ with canonical basis $e_1, \cdots, e_n$ and then the tree-map $u_a = \|x_a\| e_k$ if $|a| = k \leq n$ and $u_a = 0$ if $|a| > n$. The lemma follows from the Heine-Borel theorem and repeated applications of Lemma 3.1. $\square$

If τ is a topology on X (e.g. the weak topology or for dual spaces the weak*-topology) we say that a tree-map $(x_a)_{a \in S}$ is $\tau-null$ if for every $a \in S$ the set $\{x_b\}_{b \in a+}$ is a τ-null sequence.

Lemma 3.3. *Suppose X is a Banach space and that $(x_a)_{a \in S}$ is a weakly null tree-map in X and $(x_a^*)_{a \in S}$ is a weak*-null tree map in X^*. Then for any $\delta > 0$ there is a full tree $T \subset S$ so that for any branch $\beta \subset T$ we have*

$$|\langle \sum_{a \in \beta} x_a, \sum_{a \in \beta} x_a^* \rangle - \sum_{a \in \beta} \langle x_a, x_a^* \rangle| \leq \delta.$$

Proof. Let T be the set of a so that if $b < c \leq a$ then $|\langle x_b, x_c^* \rangle|, |\langle x_c, x_b^* \rangle| \leq |c|^{-1} 2^{-|c|-1} \delta$. It is not difficult to see that this is a full tree and that if β is a branch in this tree then

$$|\langle \sum_{a \in \beta} x_a, \sum_{a \in \beta} x_a^* \rangle - \sum_{a \in \beta} \langle x_a, x_a^* \rangle| = |\sum_{b < a} \sum_{a \in \beta} \langle x_a, x_b^* \rangle + \langle x_b, x_a^* \rangle| \leq \delta.$$

$\square$

Proposition 3.4. *Let X be a separable Banach space. In order that*

$$\iota_{\epsilon_1} \iota_{\epsilon_2} \cdots \iota_{\epsilon_n} B_{X^*} \neq \emptyset$$

it is necessary that there exists a weak-null tree-map $(x_a^*)_{a \in S}$ in X^* with $\|x_a^*\| \geq \frac{1}{4} \epsilon_{|a|}$ for $1 \leq |a| \leq n$ and so that $\|\sum_{a \in \beta} x_a^*\| \leq 1$ for every branch β, and sufficient that there exists a weak*-null tree-map $(x_a^*)_{a \in S}$ in X^* with $\|x_a^*\| \geq \epsilon_{|a|}$ for $1 \leq |a| \leq n$ and so that $\|\sum_{a \in \beta} x_a^*\| \leq 1$ for every branch β.*

Proof. First assume $\iota_{\epsilon_1} \iota_{\epsilon_2} \cdots \iota_{\epsilon_n} B_{X^*} = K \neq \emptyset$. Then

$$0 \in \frac{1}{2} K + \frac{1}{2} B_{X^*} \subset \iota_{\epsilon_1/2} \cdots \iota_{\epsilon_n/2} B_{X^*}$$

Now there exists a sequence x_k^* converging to 0 weak* so that $\|x_k^*\| \geq \frac{1}{4} \epsilon_1$ and $x_k^* \in \iota_{\epsilon_2/2} \iota_{\epsilon_3/2} \cdots \iota_{\epsilon_n/2} B_{X^*}$ since otherwise there is a weak*-open neighborhood of 0 relative to $\iota_{\epsilon_2/2} \iota_{\epsilon_3/2} \cdots \iota_{\epsilon_n/2} B_{X^*}$ of diameter less than $\epsilon_1/2$. Then for each $k \in \mathbb{N}$ we find a sequence $(x_{k_1,k_2}^*)_{k_2 > k_1}$ so that $\lim_{k_2} x_{k_1,k_2}^* = 0$ weak*, $\|x_{k_1,k_2}^*\| \geq \frac{1}{4} \epsilon_2$ and $x_{k_1}^* + x_{k_1,k_2}^* \in \iota_{\epsilon_3/2} \cdots \iota_{\epsilon_n/2} B_{X^*}$. This procedure can then be iterated to define x_a^* if $|a| \leq n$. Setting $x_a^* = 0$ if $|a| > n$ we obtain the desired tree-map.

The converse is equally easy. Obviously if $(x_a^*)_{a \in S}$ is the given tree-map then we have $\|\sum_{a \leq b} x_a^*\| \leq 1$ for all b. It then follows inductively that $\sum_{b \leq a} x_b^* \in \iota_{\epsilon_{k+1}} \iota_{\epsilon_{k+2}} \cdots \iota_{\epsilon_n} B_{X^*}$ whenever $|a| = k$. Setting $k = 0$ gives the result. $\square$

4. UKK^*-RENORMINGS

Suppose X is a separable Banach space. If $\sigma > 0$ we define $N = N(\sigma)$ to be the least integer N so that there exists a weakly null tree-map $(x_a)_{a \in S}$ in

X of height $N+1$ such that $\|x_a\| \leq \sigma$ for every $a \in S$ and $\|\sum_{a \in \beta} x_a\| > 1$ for every branch β. (We put $N(\sigma) = \infty$ if no such integer exists.)

Notice that $N(\sigma) > \sigma^{-1} - 1$.

Remark. It follows from Lemma 3.1 that if $k \leq N(\sigma)$ then for every weakly null tree-map $(x_a)_{a \in S}$ of height k and such that $\|x_a\| \leq \sigma$ for all $a \in S$ there is a full tree $T \subset S$ so that $\|\sum_{a \in \beta} x_a\| \leq 1$ for every branch.

We refer to [4] and references therein for the uniform Kadec-Klee property and its dual version. Here we denote by UKK^* what was named there weak-star UKK. Also, we consider it as a property of a Banach space which can be checked on its dual, rather than a property of a dual space which refers to a given predual. Our main reason for introducing the quantity $N(\sigma)$ is to obtain a renorming of the space X with an approximate UKK^*-condition.

Theorem 4.1. *Suppose X is a separable Banach space. Then for any $\sigma > 0$ if $N(\sigma) < \infty$ there is a norm $|\cdot|$ on X satisfying $\frac{1}{2}\|x\| \leq |x| \leq \|x\|$ and*

$$\limsup_{n \to \infty} |x + x_n| \leq 1 + \frac{1}{N(\sigma)}$$

whenever $|x| = 1$ and (x_n) is a sequence satisfying $\lim_{n \to \infty} |x_n| = \frac{1}{2}\sigma$ and $\lim_{n \to \infty} x_n = 0$ weakly.

Proof. Define $f_0(x) = \|x\|$ and then for $k > 0$ define $f_k(x)$ to be the infimum of all $\lambda > 0$ so that whenever $(x_a)_{a \in S}$ is a weakly null tree-map of height k with $\|x_a\| \leq \sigma$ for all $a \in S$ then there is a full subtree $T \subset S$ so that $\|x + \sum_{a \in \beta} x_a\| \leq \lambda$ for every branch β. We observe first that $(f_k(x))_{k=0}^{\infty}$ is an increasing sequence, and that $f_k(x) = f_k(-x)$. Next notice that

$$|f_k(x) - f_k(y)| \leq \|x - y\|$$

by an elementary calculation, which we omit. We also observe that $f_N(x) \leq \|x\| + 1$ where $N = N(\sigma)$.

We next claim that f_k is convex. Indeed let $u = tx + (1-t)y$, where $0 < t < 1$. Suppose $\lambda > f_k(x)$ and $\mu > f_k(y)$. Let $(x_a)_{a \in S}$ be any weakly null tree-map of height k with $\|x_a\| \leq \sigma$ for all $a \in S$.

Then we can find a full subtree $T_1 \subset S$ so that for every branch β we have

$$\|x + \sum_{a \in \beta} x_a\| \leq \lambda$$

and then a full subtree $T_2 \subset T_1$ so that for every branch $\beta \subset T_2$

$$\|y + \sum_{a \in \beta} x_a\| \leq \mu.$$

Obviously for every branch $\beta \subset T_2$

$$\|u + \sum_{a \in \beta} x_a\| \leq t\lambda + (1 - t)\mu$$

so that $f_k(u) \leq t\lambda + (1 - t)\mu$.

Next we note that if $\|x_n\| \leq \sigma$ and $\lim_{n \to \infty} x_n = 0$ weakly then

$$\limsup f_k(x + x_n) \leq f_{k+1}(x)$$

for all $k \geq 0$. Indeed for $k = 0$ this is obvious. If $k > 0$ assume that $\lambda < \limsup_{n \to \infty} f_k(x + x_n)$. By passing to a subsequence we can suppose $\lambda < f_k(x + x_n)$ for every n. Then for each n there is a weakly null tree-map $(y_a^{(n)})_{a \in S_n}$ of height k so that $\|y_a^{(n)}\| \leq \sigma$ for all $a \in S_n$ and

$$\|x + x_n + \sum_{a \in \beta} y_a^{(n)}\| > \lambda$$

for every branch $\beta \subset S_n$.

Now let T be the tree consisting of all sets $\{m_1, \cdots, m_l\}$ where $m_1 < m_2 < \cdots < m_l$ such that if $l > 1$ then $\{m_2, \cdots, m_l\} \in S_{m_1}$. We define a weakly null tree-map of height $k + 1$ by

$$z_{m_1, \cdots, m_l} = \begin{cases} x_{m_1} & \text{if } l = 1 \\ y_{m_2, \cdots, m_l}^{(m_1)} & \text{if } l > 1. \end{cases}$$

Then for every branch $\beta \subset T$ we have

$$\|x + \sum_{a \in \beta} z_a\| > \lambda$$

so that $f_{k+1}(x) \geq \lambda$. This implies our claim.

Now let us set $g(x) = \frac{1}{N} \sum_{k=0}^{N-1} f_k(x)$. Then g is convex, and $\|x\| \leq g(x) \leq \|x\| + 1$. Further if (x_n) is weakly null with $\|x_n\| \leq \sigma$ for all n then

$$\limsup_{n \to \infty} g(x + x_n) \leq \frac{1}{N} \sum_{k=1}^{N} f_k(x)$$

(4.1)
$$\leq g(x) + \frac{1}{N}(f_N(x) - f_0(x))$$

$$\leq g(x) + \frac{1}{N}.$$

Let $|\cdot|$ be the Minkowski functional of the set $\{x : g(x) \leq 2\}$. Then it is clear that $\frac{1}{2}\|x\| \leq |x| \leq \|x\|$. Suppose $|x| = 1$ and (x_n) is a weakly null sequence with $|x_n| \leq \frac{1}{2}\sigma$. Then $\|x_n\| \leq \sigma$. Thus

$$\limsup_{n \to \infty} g(x + x_n) \leq g(x) + \frac{1}{N} \leq 2 + \frac{1}{N}.$$

Now $g(0) < 1$ and so from the convexity of g it follows that

$$\limsup_{n\to\infty} g(\frac{N}{N+1}(x + x_n)) < 2$$

whence

$$\limsup_{n\to\infty} |x + x_n| \leq 1 + \frac{1}{N}.$$

$\square$

Theorem 4.2. *Suppose X is a separable Banach space. Suppose $\sigma > 0$ is such that $N(\sigma) = \infty$. Then, for any $\epsilon > 0$ there is a norm $|\cdot|$ on X satisfying $\frac{1}{2}\|x\| \leq |x| \leq \|x\|$ and*

$$\limsup_{n\to\infty} |x + x_n| \leq 1 + \epsilon$$

whenever $|x| = 1$ and (x_n) is a sequence satisfying $\lim_{n\to\infty} |x_n| = \frac{1}{2}\sigma$ and $\lim_{n\to\infty} x_n = 0$ weakly.

Proof. The proof is almost identical to the proof of the preceding Theorem 4.1, except that one considers $g_m(x) = \frac{1}{m}\sum_{k=0}^{m-1} f_k(x)$ for arbitrarily large choices of m. We omit the details. $\square$

Notice that the preceding two theorems allow us to say in the language of Section 2 that N^{-1} 2-dominates φ where $N^{-1}(\sigma) = (N(\sigma))^{-1}$.

We now turn to the problem of relating the function $N(\sigma)$ to the convex Szlenk index.

Lemma 4.3. *Suppose $0 < \sigma < 1$. If $N = N(\sigma)$ there exist $0 < \epsilon_1, \cdots, \epsilon_{N+1} < 1$ so that $\sum_{k=1}^{N+1} \epsilon_k > \frac{1}{3}\sigma^{-1}$ and $\iota_{\epsilon_1}\iota_{\epsilon_2}\cdots\iota_{\epsilon_{N+1}} B_{X^*} \neq \emptyset$.*

Proof. Suppose $(x_a)_{a\in S}$ is a weakly null tree-map of height $N+1$ with $\|x_a\| \leq \sigma$ and so that $\|\sum_{a\in\beta} x_a\| > 1$ for every branch β.

Fix $\delta > 0$ so that $(2N+3)\delta < \frac{1}{3}$. For any $a \in S$ with $|a| = N+1$ we choose y_a^* with $\|y_a^*\| = 1$ and $\langle\sum_{b\leq a} x_b, y_a^*\rangle = \|\sum_{b\leq a} x_b\|$. If $|a| > N$ set $y_a^* = y_b^*$ where $|b| = N$ and $b \leq a$. We now define y_a^* by backwards induction so that for each a, y_a^* is a weak*-cluster point of $\{y_b^* : b \in a+\}$. It is then easy to apply induction to produce a full tree $T \subset S$ so that $\lim_{b\in a+} y_b^* = y_a^*$.

Now let $x_a^* = y_a^* - y_{a-}^*$ when $|a| \geq 1$, and $x_\emptyset^* = 0$ so that (x_a^*) is a weak*-null tree-map in X^* of height N. Let $y^* = y_\emptyset^*$ so that $\|y^*\| \leq 1$. We thus have on every branch β of T that

$$\|y^* + \sum_{a\in\beta} x_a^*\| = 1$$

and so

$$\|\sum_{a\in\beta} x_a^*\| \leq 2.$$

Now using Lemma 3.2 we can pass to a further full tree $T_1 \subset T$ so that for suitable $\epsilon_1, \cdots, \epsilon_{N+1}$ we have $2\epsilon_k \leq \|x_a^*\| \leq 2\epsilon_k + \delta$ if $|a| = k$ for $1 \leq k \leq N$. Then by Proposition 3.4 we have that $\iota_{\epsilon_1} \cdots \iota_{\epsilon_{N+1}} B_{X^*} \neq \emptyset$.

Next it is clear that we can pass to a further full subtree T_2 so that for every a we have $|y^*(x_a)| < \delta$ (since (x_a) is weakly null). Finally we use Lemma 3.3 to produce a full tree $T_3 \subset T_2$ so that for any branch β

$$|\langle \sum_{a \in \beta} x_a, \sum_{a \in \beta} x_a^* \rangle - \sum_{a \in \beta} \langle x_a, x_a^* \rangle| \leq \delta.$$

Now for any branch β in T_3 we have

$$\| \sum_{a \in \beta} x_a \| = \langle \sum_{a \in \beta} x_a, y^* + \sum_{a \in \beta} x_a^* \rangle.$$

Hence

$$1 < \| \sum_{a \in \beta} x_a \| \leq (N+2)\delta + \sum_{a \in \beta} \langle x_a, x_a^* \rangle$$

(4.2)
$$\leq (2N+3)\delta + 2\sigma \sum_{k=1}^{N+1} \epsilon_k.$$

Thus we have $\sum_{k=1}^{N+1} \epsilon_k \geq \frac{1}{3}\sigma^{-1}$.

$\square$

Theorem 4.4. *Suppose X is a separable Banach space containing no copy of ℓ_1. Let $H(\tau) = (Cz(X, \tau) - 1)^{-1}$ for $0 \leq \tau < 1$. Then there is a universal constant $C \leq 19200$ so that $N(\sigma)^{-1}$ is C-equivalent to $\varphi(\sigma)$ and $H(\tau)$ is C-equivalent to $\psi(\tau)$.*

Proof. Suppose $0 < \sigma < 1$. Then by Lemma 2.4 and the preceding Lemma 4.3 we have that if $\tau \leq \frac{1}{12}\sigma^{-1}(N+1)^{-1}$ then

$$Cz(X, \tau) - 1 \geq \frac{1}{24\sigma\tau}.$$

Thus

$$H(\tau) \leq 24\sigma\tau.$$

Then for any σ if $0 < \tau \leq \frac{25}{12}\sigma^{-1}(N(\sigma/25)+1)^{-1}$ we have

$$H^*(\sigma) \geq \sigma\tau - \frac{24}{25}\sigma\tau = \frac{1}{25}\sigma\tau.$$

Hence

$$H^*(\sigma) \geq \frac{1}{12}(N(\sigma/25)+1)^{-1} \geq \frac{1}{24}(N(\sigma/25))^{-1}.$$

Since H^* is convex this implies that H^* 600-dominates N^{-1}.

Now by Proposition 2.8 we have that H 4-dominates ψ and hence ψ^* 4-dominates H^*. Thus φ C-dominates N^{-1} with $C \leq 19200$. By the remarks after Theorem 4.2 this means that φ and N^{-1} are C-equivalent.

Now recall that $H(n\tau) \geq nH(\tau)$ for $n \in \mathbb{N}$ by Lemma 2.2.

Thus for $s \geq 1$ we have $H(st) \geq \frac{1}{2}sH(t)$. It follows easily that H is 2-equivalent to a function H_1 with the property that $H_1(t)/t$ is increasing, which is then 2-equivalent to a convex function. Hence H is 4-equivalent to a convex function and hence also to H^{**}.

Now H^* 600-dominates N^{-1} and hence 1200-dominates φ. Thus H^{**} is 1200-dominated by φ^* and thus $4800-$dominated by ψ. Hence H is 19200-dominated by ψ and so H is 19200-equivalent to ψ. $\hfill\square$

Theorem 4.5. *Suppose X is a separable Banach space not containing ℓ_1. Then there is a universal constant $C < 10^6$ so that if $0 < \tau \leq 1$,*

$$Cz(X, \tau) \leq \sum_{\substack{k \geq 0 \\ 2^k \tau/C \leq 1}} 2^k Sz(X, 2^k \tau/C).$$

Proof. Let

$$K(\tau) = \left(\sum_{\substack{k \geq 0 \\ 2^k \tau \leq 1}} 2^k (\mathrm{Sz}(X, 2^k \tau) - 1) \right)^{-1}.$$

Then arguing as with H we have that K is 4-equivalent to K^{**}.

We next note that by Lemma 4.3 and Lemma 2.4 if $0 < \sigma < 1$ we have that if $0 < \tau \leq \frac{1}{6}\sigma^{-1}(N(\sigma) + 1)^{-1}$ then

$$K(\tau) \leq 6\sigma\tau.$$

Reasoning as above gives

$$K^*(\sigma) \geq \frac{1}{6}(N(\sigma/7) + 1)^{-1} \geq \frac{1}{12}N(\sigma/7)^{-1}.$$

Hence K^* 100-dominates N^{-1}. By Theorem 4.4 this implies that K is C-dominated by H for a suitable absolute constant $C \leq 600.100.4.4 < 10^6$. This clearly yields the result. $\hfill\square$

Our next Corollary follows easily from Theorem 4.5.

Corollary 4.6. *Suppose X is a separable Banach space with $Sz(X) = \omega_0$. Suppose $p > 1$ and*

$$\sup_{0 < \epsilon < 1} \epsilon^p Sz(X, \epsilon) < \infty.$$

Then

$$\sup_{0 < \epsilon < 1} \epsilon^p Cz(X, \epsilon) < \infty.$$

We are now in position to state our main renorming theorems. The first one simply restates Theorem 4.4 :

Theorem 4.7. *Suppose X is a separable Banach space with $Sz(X) = \omega_0$. Then there exists an absolute constant $C \leq 19200$ such that for any $0 < \tau < 1$ there is 2-equivalent norm $|\cdot|$ on X so that if $x^*, x_n^* \in X^*$ satisfy $|x^*| = 1$, $|x_n^*| = \tau$ and $\lim_{n \to \infty} x_n^* = 0$ weak* then*

$$\liminf_{n \to \infty} |x^* + x_n^*| \geq 1 + \frac{1}{Cz(X, \tau/C)}.$$

It is clear that Theorem 4.7 is in a sense a best possible result. Notice that we do not have a simultaneous renorming which works for all $0 < \tau < 1$. However we can combine these norms to give a single norm which has the UKK^* property with a nearly optimal modulus. Note that it is clear that any space X which has an equivalent UKK^* norm satisfies $Sz(X) = \omega_0$.

Theorem 4.8. *Suppose X is a separable Banach space with $Sz(X) = \omega_0$. Let p_X be the Szlenk power type of X. Then for any $q > p_X$ there is an equivalent norm $|\cdot|$ on X and a constant $c > 0$ so that if $0 < \tau \leq 1$ and $x^*, x_n^* \in X^*$ satisfy $|x^*| = 1$, $|x_n^*| = \tau$ and $\lim_{n \to \infty} x_n^* = 0$ weak* then*

$$\liminf_{n \to \infty} |x^* + x_n^*| \geq 1 + c\tau^q.$$

Proof. Fix $p_X < r < q$. In this case we have that $Sz(X, \epsilon) \leq C\epsilon^{-r}$ for some constant C. By Corollary 4.6 we have a similar estimate $Cz(X, \epsilon) \leq C\epsilon^{-r}$. Hence, for a suitable constant $c_1 > 0$, for each $k \in \mathbb{N}$ there is a norm $|\cdot|_k$ on X which is 2-equivalent to the original norm and such that if $|x^*|_k = 1$, $|x_n^*|_k = 2^{-k}$ with x_n^* weak*-null then

$$\liminf_{n \to \infty} |x^* + x_n^*|_k \geq 1 + c_1 2^{-rk}.$$

Now define the (dual) norm $|\cdot|$ on X^* by

$$|x^*| = \sum_{k=1}^{\infty} 2^{(r-q)k} |x^*|_k.$$

This clearly defines an equivalent dual norm on X^*. Thus there is a uniform constant B so that for every k we have $B^{-1}|x^*| \leq |x^*|_k \leq B|x^*|$. Suppose $|x^*| = 1$ and $|x_n^*| = \tau$ with (x_n^*) weak*-null. Pick $k \in \mathbb{N}$ so that $2^{-k} \leq B^{-2}\tau \leq 2^{1-k}$. Then $|x^*|_k \leq B$ and $|x_n^*|_k \geq B^{-1}\tau \geq 2^{-k}|x^*|_k$. Hence

$$\liminf |x^* + x_n^*|_k \geq |x^*|_k(1 + c_1 2^{-rk}).$$

This implies that

$$\liminf |x^* + x_n^*| \geq 1 + c_2 2^{-qk} \geq 1 + c_3 \tau^q$$

for suitable $c_2, c_3 > 0$.

$\square$

In general we do not know whether the functions $Cz(X,\tau)$ and $Sz(X,\tau)$ are equivalent. However in certain cases, which include for instance super-reflexive spaces but also James' quasi-reflexive space J ([19]), they are equivalent:

Theorem 4.9. *Suppose X is a separable Banach space with $Sz(X) = Sz(X^*) = \omega_0$. Then there is a constant C (depending only on X) so that $Cz(X,\tau) \leq Sz(X,\tau/C)$.*

Proof. We begin by noting that X^{**} is separable. From Theorem 4.7 and the fact that $Cz(X^*,\tau) < \infty$ for all $\tau > 0$ it follows that we can replace the original norm with an equivalent norm on $X \subset X^{**}$ so that there exists $\delta > 0$ with the property that if $\|x\| = \|x_n\| = 1$ and (x_n) is weakly null then

(4.3) $$\liminf \|x + x_n\| > 1 + \delta.$$

Using this property we make an estimate of $N(\sigma)$. If $N = N(\sigma) < \infty$ there is weakly null tree-map $(x_a)_{a \in \mathcal{F}\mathbb{N}}$ of height $N + 1$ so that $\|x_a\| \leq \sigma$ for every $a \in S$ but $\|\sum_{a \in \beta} x_a\| > 1$ on every branch. We define a second tree-map $(y_a)_{a \in \mathcal{F}\mathbb{N}}$ by $y_a = x_a$ if $|a| \leq N + 1$ and $y_a = x_b$ if $N + 1 < |a| = k \leq 2(N+1)$ and $b = \{m_{N+2}, \cdots, m_k\}$ if $a = \{m_1, \cdots, m_k\}$. If $|a| > 2(N+1)$ then $y_a = 0$. It follows easily from (4.3) that there is a full subtree S so that on every branch we have $\sum_{a \in \beta} > 1 + \delta$. In particular $N((1+\delta)^{-1}\sigma) \leq 2N(\sigma) + 1 \leq 3N(\sigma)$.

Iterating we have $N(\sigma) < \infty$ for all σ. The estimate on N clearly implies an estimate of the type

$$N(\lambda\sigma) \leq C\lambda^{-p}N(\sigma) \qquad 0 < \lambda, \sigma < 1.$$

where $1 < p < \infty$.

By Theorem 4.4 and Proposition 2.8 we obtain a dual estimate $Cz(X, \lambda\tau) \geq C\lambda^{-q}Cz(X,\tau)$ where $\frac{1}{p} + \frac{1}{q} = 1$. Note that $q > 1$.

We now use Theorem 4.5. First note that for a suitable constant C_1 we have

$$\sum_{\substack{k \geq 0 \\ 2^k\tau \leq 1}} 2^k Sz(X, 2^k\tau) \leq \sum_{\substack{k \geq 0 \\ 2^k\tau \leq 1}} 2^k Cz(X, 2^k\tau)$$

(4.4)
$$\leq C \sum_{\substack{k \geq 0 \\ 2^k\tau \leq 1}} 2^{k(1-q)} Cz(X, \tau)$$

$$\leq C_1 Cz(X, \tau).$$

Recall we also have an estimate from Theorem 4.5

$$Cz(X, \tau) \leq \sum_{\substack{k \geq 0 \\ 2^k\tau/C \leq 1}} 2^k Sz(X, 2^k\tau/C_2).$$

Now pick k_0 so that for every τ we have

$$Cz(X, 2^{-k_0}C_2\tau) \geq (1 + 2^{k_0}C_1)Cz(X, \tau).$$

This is possible by the growth condition on $\mathrm{Cz}(X, \tau)$ as $\tau \to 0$. Then

$$\mathrm{Cz}(X, \tau) \leq \mathrm{Cz}(X, 2^{-k_0} C_2 \tau) - 2^{k_0} C_1 \mathrm{Cz}(X, \tau)$$

$$(4.5) \qquad \leq \sum_{\substack{k \geq 0 \\ 2^{k-k_0}\tau \leq 1}} 2^k \mathrm{Sz}(X, 2^{k-k_0}\tau) - \sum_{\substack{k \geq 0 \\ 2^k\tau \leq 1}} 2^{k+k_0} \mathrm{Sz}(X, 2^k\tau)$$

$$\leq \sum_{k=0}^{k_0-1} 2^k \mathrm{Sz}(X, 2^{k-k_0}\tau)$$

$$\leq 2^{k_0} \mathrm{Sz}(X, 2^{-k_0}\tau).$$

This implies the Theorem. $\qquad \square$

Note that the above proof shows that if X is separable and $\mathrm{Sz}(X) = \mathrm{Sz}(X^*) = \omega_0$ then $N(\sigma) < \infty$ for any $\sigma > 0$. Hence by Theorem 4.10 below, the Szlenk index of such spaces is not summable.

We conclude this section with a characterization of spaces with summable Szlenk index (see also Theorem 4.2). Recall that by Lemma 2.4, condition (iii) below is equivalent to the summability of the convex Szlenk index Cz. Note that Sz and Cz are also equivalent for the spaces which satisfies the conditions of the next theorem.

Theorem 4.10. *Let X be a separable Banach space. The following assertions are equivalent:*
(i) X has summable Szlenk index.
(ii) There exists $\sigma > 0$ so that $N(\sigma) = \infty$.
(iii) There exists $K > 0$ so that for any $0 < \tau < 1$, $Cz(X, \tau) \leq K\tau^{-1}$.
(iv) For any function $f : (0, 1) \to (0, 1)$ which satisfies $\lim_{\tau \to 0} \tau^{-1} f(\tau) = 0$, there is an equivalent norm $|\cdot|$ on X and a constant $c > 0$ so that if $0 < \tau < 1$ and $x^*, x_n^* \in X$ satisfy $|x^*| = 1$, $|x_n^*| = \tau$ and $\lim_{n \to \infty} x_n^* = 0$ weak* then

$$\liminf_{n \to \infty} |x^* + x_n^*| \geq 1 + cf(\tau).$$

Proof. We first note that (iii) $\Rightarrow$ (i) follows directly from the second part of Lemma 2.4.

Next we prove that (iv) $\Rightarrow$ (iii). Assume that (iii) does not hold. Let

$$f(\tau) = \sqrt{\frac{\tau}{\mathrm{Cz}(X, \tau) - 1}}.$$

Then we have $\lim_{\tau \to 0} \tau^{-1} f(\tau) = 0$. Thus assuming (iv) we conclude an estimate $\mathrm{Cz}(X, \tau) \leq C(f(\tau))^{-1}$ which gives a contradiction.

Next we note that (i) $\Rightarrow$ (ii) is immediate from Lemma 4.3.

It remains therefore to show that (ii) $\Rightarrow$ (iv). Note first that we can assume that $\tau^{-1} f(\tau)$ is a monotone increasing function. It will be sufficient to prove the result with $f(\tau)$ replaced by $f(\tau/4)$.

We note that by Theorem 4.4 and Proposition 2.8 we have that $\psi(\tau) \geq c\tau$ for some $c > 0$. Hence if $0 < c_1 < c$ there is, for each $k \in \mathbb{N}$, a 2-renorming $|\cdot|_k$ of X so that $\frac{1}{2}\|x\| \leq |x|_k \leq \|x\|$ for $x \in X$ and

$$\liminf |x^* + x_n^*|_k \geq 1 + c_1 2^{-k}$$

whenever $|x^*|_k = 1$ and (x_n^*) is a weak*-null sequence in X^* with $|x_n^*|_k = 2^{-k}$. Now using the same idea as in the preceding proof we define a dual norm on X^* by

$$|x^*| = \frac{1}{2f(\frac{1}{2})} \sum_{k=1}^{\infty} \left(2^k f(2^{-k}) - 2^{k+1} f(2^{-k-1})\right) |x^*|_k.$$

It follows from our definition that $|\cdot|$ is well-defined and 2-equivalent to the original norm. Now suppose $x^*, x_n^* \in X^*$ are such that $|x^*| = 1$, $|x_n^*| = \tau$ and (x_n^*) is weak*-null. Pick $j \in \mathbb{N}$ so that $2^{-j} \leq \frac{1}{4}\tau \leq 2^{1-j}$. Then if $k \geq j$ we have $|x^*|_k \leq 2$ and $|x_n^*|_k \geq \frac{1}{2}\tau \geq 2^{-j}|x^*|_k$.

Hence we obtain by convexity

$$\liminf |x^* + x_n^*|_k \geq |x^*|_k (1 + c_1 2^{-j}).$$

Summing over k we have

$$\liminf |x^* + x_n^*| \geq 1 + c_1 2^{-j} \sum_{k=j}^{\infty} (2^k f(2^{-k}) - 2^{k+1} f(2^{-k-1}))|x^*|_k.$$

Hence for a suitable constant $c_2 > 0$ we have

$$\liminf |x^* + x_n^*| \geq 1 + c_2 f(\tau/4).$$

This completes the proof. $\qquad\square$

Remarks. Spaces with summable Szlenk index are considered in [16] where it is shown that the original Tsirelson space is a reflexive example. These spaces are very close to being subspaces of c_0 which are characterized by the existence of an equivalent Lipschitz-UKK*-norm (cf. [15], [6]) i.e. a norm such that

$$\liminf \|x^* + x_n^*\| \geq 1 + c\tau$$

whenever $\|x^*\| = 1$ and (x_n^*) is a weak*-null sequence with $\|x_n^*\| = \tau$. The "lack of isotropy" of the Szlenk derivation when applied to the Tsirelson space seems to be responsible for the difference between summability of the Szlenk index and embeddability into c_0.

5. Applications to uniform and Lipschitz homeomorphisms

We first recall the Gorelik principle. We use the version proved in [6]; see [12] for an earlier version. Both are based on the original idea of Gorelik [8]. If $V : X \to Y$ is uniformly continuous we denote the modulus of continuity by $\omega(V, t) = \sup\{\|Vx - Vy\|; \ \|x - y\| \le t\}$.

Proposition 5.1. *(The Gorelik Principle) Let X and Y be two Banach spaces and let U be a homeomorphism of X onto Y whose inverse is uniformly continuous. Suppose $b > 0$ and $d > \omega(U^{-1}, b)$ and let X_0 be a closed subspace of X of finite codimension. Then there is a compact subset K of Y so that*

$$bB_Y \subset K + U(2dB_{X_0}).$$

Let us recall at this point that if $V : X \to Y$ is uniformly continuous it is Lipschitz at large distances. More precisely if $0 < t < \infty$ then we have

$$\|Vx - Vy\| \le t^{-1}\omega(V, t)(t + \|x - y\|).$$

Let us define the asymptotic Lipschitz constant of V by

$$\|V\|_u = \lim_{t \to \infty} t^{-1}\omega(V, t).$$

If X and Y are uniformly homeomorphic Banach spaces we define

$$d_u(X, Y) = \inf\{\|U\|_u\|U^{-1}\|_u : \ U : X \to Y \text{ is a uniform homeomorhism}\}.$$

Its clear that $d_u(X, Y) \ge 1$ and that $d_u(X, Z) \le d_u(X, Y)d_u(Y, Z)$ if X, Y and Z are all uniformly homeomorphic.

Lemma 5.2. *Suppose X, Y are uniformly homeomorphic. Then if $M^2 > d_u(X, Y)$ and $\eta > 0$ there is a uniform homeomorphism $V : X \to Y$ so that we have*

$$(5.6) \qquad \begin{aligned} \|Vx_1 - Vx_2\| &\le M\max(\|x_1 - x_2\|, \eta) & x_1, x_2 \in X \\ \|V^{-1}y_1 - V^{-1}y_2\| &\le M\max(\|y_1 - y_2\|, \eta) & y_1, y_2 \in Y. \end{aligned}$$

Proof. Let $U : X \to Y$ be a uniform homeomorphism with $\|U\|_u\|U^{-1}\|_u = L^2 < M^2$. For any $s > 0$ define

$$V_s(x) = L\|U\|_u^{-1}s^{-1}U(sx)$$

so that

$$V_s^{-1}(y) = s^{-1}U^{-1}(L^{-1}\|U\|_u sy).$$

Thus $\omega(V_s, t) = L\|U\|_u^{-1}s^{-1}\omega(U, st)$ and $\omega(V_s^{-1}, t) = s^{-1}\omega(U^{-1}, L^{-1}\|U\|_u st)$. Hence for any $\delta > 0$ we have

$$\|V_s x_1 - V_s x_2\| \le L\|U\|_u^{-1}\delta^{-1}s^{-1}\omega(U, s\delta)(\delta + \|x_1 - x_2\|)$$

and

$$\|V_s^{-1}y_1 - V_s^{-1}y_2\| \le \delta^{-1}s^{-1}\omega(U^{-1}, L^{-1}\|U\|_u s\delta)(\|y_1 - y_2\| + \delta).$$

It is clear that if we take s large enough and δ small enough, we obtain the lemma. □

We now state our main result of this section:

Theorem 5.3. *Suppose X and Y are separable Banach spaces which are uniformly homeomorphic. Then for any $\lambda > d_u(X, Y)$ and any $\epsilon > 0$ there is a λ-equivalent norm $|\cdot|$ on Y with*

$$\theta_{Y,|\cdot|}(\tau) \geq \theta_X(\tau/C) - \epsilon, \qquad \forall 0 \leq \tau \leq 1,$$

where $C = 32\lambda^2$.

Proof. Let $M = \lambda^{1/2}$. We assume that $U : X \to Y$ is a uniform homeomorphism of X onto Y which satisfies:

(5.7)
$$\|Ux_1 - Ux_2\| \leq M \max(\|x_1 - x_2\|, 1) \qquad x_1, x_2 \in X$$
$$\|U^{-1}y_1 - U^{-1}y_2\| \leq M \max(\|y_1 - y_2\|, 1) \qquad y_1, y_2 \in Y.$$

This is possible by Lemma 5.2. Recall that $C = 32\lambda^2 = 32M^4$.

We first define a decreasing sequence of dual norms $\{|\cdot|_k\}_{k=1}^{\infty}$ on Y^* by

$$|y^*|_k = \sup \left\{ \frac{|y^*(Ux_1 - Ux_2)|}{\|x_1 - x_2\|}; \quad x_1, x_2 \in X, \ \|x_1 - x_2\| \geq 2^k \right\}.$$

It is clear that we have $M^{-1}\|y^*\| \leq |y^*|_k \leq M\|y^*\|$ for every $k \geq 1$.

We will prove the following Claim concerning the norms $|\cdot|_k$:

Claim: suppose $0 < \tau < 1$ and $y^*, y_n^* \in Y^*$ are such that $\|y^*\| \leq M$, (y_n^*) is weak*-null and $\|y_n^*\| \geq M^{-1}\tau$ for all n. Then if $2^k C\tau^{-1}\theta_X(C^{-1}\tau) > 4M$,

(5.8)
$$\liminf_{n \to \infty} |y^* + y_n^*|_k \geq 2|y^*|_{k+1} - |y^*|_k + \theta_X(\frac{\tau}{C}).$$

To prove the claim we first define $\sigma = C\tau^{-1}\theta_X(\tau/C)$ so that $0 < \sigma \leq 1$. Let us also for convenience of exposition write $\beta = \theta_X(\tau/C)$. Now for any $\epsilon > 0$ we can choose $x, x' \in X$ so that $\|x - x'\| \geq 2^{k+1}$ and $y^*(Ux - Ux') \geq (1 - \epsilon)\|x - x'\||y^*|_{k+1}$. As usual in such arguments we can suppose by using translations that $x' = -x$ and $Ux' = -Ux$. Next we apply Proposition 2.6 and the separability of X. We deduce the existence of a finite codimensional subspace X_0 of X so that

(5.9)
$$\|x + z\| \geq \|x\| \geq 2^k \qquad \forall z \in X_0$$

and

(5.10)
$$\|x + z\| \leq (1 + 2\beta)\|x\| \qquad \forall z \in \sigma\|x\|B_{X_0}.$$

Now, by assumption we have $\omega(U^{-1}, b) < 2Mb$ if $b \geq 1$. Hence we apply the Gorelik principle, Proposition 5.1 for $b = \frac{\sigma\|x\|}{4M} > 1$ by choice of k, and

$d = \frac{\sigma\|x\|}{2}$. We deduce that there is a compact subset K of Y so that:

$$(5.11) \qquad \frac{\sigma\|x\|}{4M} B_Y \subset K + U(\sigma\|x\| B_{X_0}).$$

It follows from (5.11) that there is a sequence $z_n \in \sigma\|x\| B_{X_0}$ so that

$$\liminf_{n\to\infty} y_n^*(-Uz_n) \geq \frac{\sigma\tau}{4M^2}\|x\| = \frac{C\beta}{4M^2}\|x\|.$$

Now $\|x + z_n\| = \|z_n - x'\|$ so that by (5.9) and (5.10) we have

$$y^*(Ux + Uz_n) = y^*(Uz_n - Ux') \leq (1 + 2\beta)|y^*|_k\|x\|$$

But

$$y^*(Ux) = \frac{1}{2}y^*(Ux - Ux') \geq (1 - \epsilon)|y^*|_{k+1}\|x\|.$$

Hence we deduce

$$y^*(Uz_n) \leq [(1 + 2\beta)|y^*|_k - (1 - \epsilon)|y^*|_{k+1}]\,\|x\|.$$

Combining these estimates gives:

$$\liminf_{n\to\infty}(y^* + y_n^*)(Ux - Uz_n) \geq (2 - 2\epsilon)|y^*|_{k+1}\|x\| - |y^*|_k\|x\| + \frac{C\beta}{4M^2}\|x\| - 2\beta|y^*|_k\|x\|.$$

The left-hand side is estimated by $(1 + 2\beta)\liminf|y^* + y_n^*|_k\|x^*\|$. Hence since $\epsilon > 0$ is arbitrary and $\|y^*\| \leq M$,

$$(5.12) \qquad \begin{aligned} \liminf_{n\to\infty}|y^* + y_n^*|_k &\geq (1 - 2\beta)(2|y^*|_{k+1} - |y^*|_k + \frac{C - 8M^3}{4M^2}\beta) \\ &\geq 2|y^*|_{k+1} - |y^*|_k + \frac{C - 8M^3 - 16M^4}{4M^2}\beta. \end{aligned}$$

Our choice of C gives

$$\frac{C - 8M^3 - 16M^4}{4M^2} \geq 1$$

and so the Claim (5.8) is proved.

The completion of the argument from the claim is easy. If $\theta_X(1/C) \leq \epsilon$ the original norm will suffice. Otherwise choose τ_0 so that $\theta_X(\tau_0/C) = \epsilon$.

Pick $k_0 \in \mathbb{N}$ so that $\frac{C}{\tau_0}\theta_X(\frac{\tau_0}{C})2^{k_0} > 4M$. Pick an integer N so that

$$\frac{2M^2}{N} < \epsilon.$$

Now let

$$|y^*| = \frac{1}{N}\sum_{k=k_0+1}^{k=k_0+N}|y^*|_k$$

which clearly defines a dual norm on Y^* with $M^{-1}\|y^*\| \leq |y^*| \leq M\|y^*\|$.

Assume $|y^*| = 1$ and that (y_n^*) is weak* null with $|y_n^*| \geq \tau$. Then $\|y^*\| \leq M$ and $\|y_n^*\| \geq \tau/M$. Hence for each $k_0 + 1 \leq k \leq k_0 + N$ we have (5.8). Summing gives

$$\liminf |y^* + y_n^*| \geq |y^*| - \frac{2}{N}|y^*|_{k_0+1} + \theta_X(\tau/C).$$

Now

$$\frac{2}{N}|y^*|_{k_0+1} \leq \frac{2M}{N}\|y^*\| \leq \frac{2M^2}{N} < \epsilon$$

so that the theorem is proved. $\square$

Before turning to applications for this result, let us notice that in the case of Lipschitz equivalence the same techniques can be applied to give a rather stronger result. A special case of the theorem below is shown in [6].

Theorem 5.4. *Suppose X and Y are separable Banach spaces which are Lipschitz isomorphic. Then there is an equivalent norm on Y so that θ_Y C-dominates θ_X for some $C > 0$.*

Proof. We only provide a sketch. In this case one can use the norm on Y^* defined by

$$|y^*| = \sup\left\{\frac{|y^*(Ux_1) - y^*(Ux_2)|}{\|x_1 - x_2\|} \qquad x_1 \neq x_2\right\}.$$

The calculations are similar but rather simpler. We leave the details to the reader. $\square$

Our main application of Theorem 5.3 is the fact that the convex Szlenk index is (up to equivalence) invariant under uniform homeomorphisms.

Theorem 5.5. *Suppose X and Y are uniformly homeomorphic. Then:*
(i) $Sz(X) \leq \omega_0$ if and only if $Sz(Y) \leq \omega_0$.
(ii) There exists a constant C so that if $0 \leq \tau \leq 1$

$$Cz(X, C\tau) \leq Cz(Y, \tau) \leq Cz(X, \tau/C).$$

Proof. We note first that the relevant functionals are separable determined ([17]). Hence we may assume that X and Y are separable. The result is then an almost immediate deduction from Theorem 5.3. In fact it is immediately clear that the statement of the Theorem yields that for the original norm on Y,

$$Cz(Y, \tau) \leq (\theta_X(\frac{\tau}{C\lambda}))^{-1} + 1.$$

If we then consider all 2-equivalent norms on X we deduce an estimate of the form

$$Cz(Y, \tau) \leq \psi_X(\tau/C)^{-1} + 1$$

for a different constant C. However by Theorem 4.7 this implies an estimate

$$Cz(Y, \tau) \leq Cz(X, \tau/C).$$

This estimate and its converse establish both (i) and (ii). $\qquad\square$

Theorem 5.6. *(i) Let X be a Banach space which is uniformly homeomorphic to a subspace of c_0. Then X has summable Szlenk index (and in particular, X^* is separable).*
(ii) If X is uniformly isomorphic to c_0 then X^* is linearly isomorphic to ℓ_1.

Proof. Since any subspace of c_0 has summable Szlenk index then X must also have summable Szlenk index by Theorem 5.5 and Theorem 4.10. For the second part note that X must be a $\mathcal{L}_\infty-$space [10] and so (i) yields that X^* is isomorphic to ℓ_1 [20]. $\qquad\square$

*Remarks.*1) It is shown in [1] that the Bourgain-Delbaen preduals of l^1 have a finite Szlenk index, although they do not contain c_0. On the other hand, it is shown in [9] that these spaces contain hereditarily l^p for some $p \in (1, \infty)$. It follows now from Theorem 5.6 that no infinite-dimensional subspace of a Bourgain-Delbaen space is uniformly homeomorphic to a subspace of c_0.
  2)As remarked above, there are unfortunately examples of Banach spaces with summable Szlenk index which do not embed into c_0 as shown in [16]. We conjecture however that any predual of ℓ_1 with summable Szlenk index is isomorphic to c_0.
  A Banach space which is Lipschitz isomorphic to c_0 is (in the separable case) linearly isomorphic to c_0 ([6]). We do not know whether a space which is uniformly homeomorphic to c_0 is isomorphic to c_0. Curiously however, we can show that if $d_u(X, c_0)$ is small enough then X is isomorphic to c_0. This result is very similar to the quantitative Lipschitz result in [6]. Note that the following statement is true for any equivalent renorming of c_0, although the function f and ϵ_0 depend upon the given norm.

Theorem 5.7. *There is $\epsilon_0 > 0$ and a function $f : (0, \epsilon_0] \to (0, \infty)$ with $\lim_{\epsilon \to 0} f(\epsilon) = 0$ so that if $d_u(X, c_0) < 1 + \epsilon$ then $d(X, c_0) < 1 + f(\epsilon)$.*

Proof. We use the ideas of [14]: in particular we use the the Gromov-Hausdorff distance for two Banach spaces $d_{GH}(X, Y)$, for which we refer to [14]. From Theorem 5.9 of [14] we have the fact that if X_n is a sequence of Banach spaces converging to c_0 in d_{GH} then the Banach-Mazur distance $d(X_n, c_0) \to 1$. It therefore suffices to show that $d_u(X_n, c_0) \to 1$ implies $d_{GH}(X, c_0) \to 0$.
  To achieve this suppose that $d_u(X, Y) < 1 + \epsilon$. Then for any $\eta > 0$ we use Lemma 5.2 to produce a uniform homeomorphism $V : X \to Y$ such that $V(0) = 0$ and

$$(5.13) \qquad \begin{aligned} \|Vx_1 - Vx_2\| &\leq (1 + \epsilon)^{1/2} \max(\|x_1 - x_2\|, \eta) & x_1, x_2 &\in X \\ \|V^{-1}y_1 - V^{-1}y_2\| &\leq (1 + \epsilon)^{1/2} \max(\|y_1 - y_2\|, \eta) & y_1, y_2 &\in Y. \end{aligned}$$

Now define $\Phi : B_X \to B_Y$ by $\Phi(x) = (1+\epsilon)^{-1/2} V x$ and $\Psi : B_Y \to B_X$ by $\Psi(y) = (1+\epsilon)^{-1/2} V^{-1} y$.

Then if $x \in B_X$, $y \in B_Y$,

$$\left| \|x - \Psi(y)\| - \|x - V^{-1}(y)\| \right| \leq \frac{1}{2}\epsilon$$

and

$$\left| \|\Phi(x) - y\| - \|Vx - y\| \right| \leq \frac{1}{2}\epsilon.$$

Now

$$\|Vx - y\| \leq (1+\epsilon)^{1/2}(\|x - V^{-1}y\| + \eta)$$

and a similar reverse inequality gives

$$\left| \|Vx - y\| - \|x - V^{-1}y\| \right| \leq \frac{1}{2}\epsilon(\max \|x - V^{-1}y\|, \|Vx - y\|) + 2\eta.$$

Since $\|x - V^{-1}y\|, \|Vx - y\| \leq (1+\epsilon)^{1/2} + 1 < 3$ we obtain

$$\left| \|x - \Psi(y)\| - \|\Phi(x) - y\| \right| < \frac{5}{2}\epsilon + 2\eta.$$

This implies ([14]) that $d_{GH}(X, Y) \leq \frac{5}{4}\epsilon + \eta$. The result then follows. $\qquad\square$

We now give one more application of our main result on the invariance of the convex Szlenk index:

Theorem 5.8. *Suppose $2 < p < \infty$ and X is a quotient (resp. subspace) of ℓ_p. If Y is uniformly homeomorphic to X then Y is linearly isomorphic to a quotient (resp. subspace) of ℓ_p.*

Remark. In fact the methods of [12] give the subspace version rather easily so the quotient case is the more interesting.

Proof. In both cases we have $\mathrm{Cz}(X, \epsilon) \leq C\epsilon^{-q}$ where $\frac{1}{p} + \frac{1}{q} = 1$. We therefore deduce by Theorem 5.5 that $\mathrm{Cz}(Y, \epsilon) \leq C\epsilon^{-q}$.

Now we use the "standard ultraproduct technique" (cf.[12] p. 438 or [3]). The spaces X and Y are super-reflexive and hence we can find Lipschitz-isomorphic separable spaces $X_1 \supset X, Y_1 \supset Y$ so that X_1 is one-complemented in an ultraproduct $X_{\mathcal{U}}$ and Y_1 is one-complemented in $Y_{\mathcal{U}}$. Then Y_1 embeds complementably into X_1 and X_1 embeds complementably in Y_1. Since X and Y are complemented in their ultraproducts this leads to the fact that Y is a quotient (respectively a subspace) of L_p.

In the case when X is a subspace of ℓ_p we can complete the argument very simply. Since $p > 2$, if Y does not embed into ℓ_p then ℓ_2 embeds complementably in Y [13] and so $\mathrm{Cz}(X, \epsilon) \geq c\epsilon^{-2}$ for some $c > 0$ which yields a contradiction.

Now consider the case when X is a quotient of ℓ_p so that Y is a quotient of $L_p[0,1]$. We have $\mathrm{Cz}(Y,\epsilon) \leq C\epsilon^{-q}$. It follows from Theorem 4.4 and Proposition 2.8 that $\psi_Y(\tau) \leq C_1\epsilon^p$ for a suitable constant C_1. We will argue that this implies Y is of type p-Banach-Saks in the sense of [11], i.e. there is a constant $\lambda > 0$ so that every normalized weakly null basic sequence $(w_n)_{n=1}^\infty$ has a subsequence (v_n) satisfying $\|\sum_{i=1}^n v_i\| \leq \lambda n^{1/p}$.

To do this we note that by the definition of ψ_Y there is, for each $k \in \mathbb{N}$ an equivalent norm $|\cdot|_k$ on Y so that $\|y\| \leq |y|_k \leq 2\|y\|$ for $y \in Y$ and

$$\limsup |y + y_n|_k^p \leq |y|_k^p(1 + 2C_1 k^{-1})$$

if (y_n) is weakly null and $\sup |y_n|_k \leq k^{-1/p}|y|_k$.

Now if (w_n) is a normalized weakly null basic sequence in $(Y, \|\cdot\|)$, we pass to a subsequence (v_n) such that if $1 \leq k \leq 2^n$ is such that $|v_n|_k \leq k^{-1/p}|\sum_{i=1}^{n-1} v_i|_k$ then

$$|\sum_{i=1}^n v_i|_k \leq |\sum_{i=1}^{n-1} v_i|_k(1 + 3C_1 k^{-1})^{1/p}.$$

For fixed n, let $m = [\log_2 n]$. Let r be the greatest integer with $1 \leq r \leq n$ so that $|v_r|_n > n^{-1/p}|\sum_{i=1}^{r-1} v_i|_n$. Provided $1 \geq 2n^{-1/p}(\log_2 n + 1)$, we have $r > m$. Hence for $r \leq j \leq n-1$ we have

$$|\sum_{i=1}^{j+1} v_i|_n \leq (1 + 3C_1 n^{-1})^{1/p}|\sum_{i=1}^{j} v_i|_n$$

and so

$$\|\sum_{i=1}^n v_i\| \leq |\sum_{i=1}^n v_i|_n$$

$$\leq (1 + 3C_1 n^{-1})^{n/p}|\sum_{i=1}^{r} v_i|_n$$

(5.14)
$$\leq e^{3C_1/p}(|\sum_{i=1}^{r-1} v_i|_n + |v_r|_n)$$

$$\leq e^{3C_1/p}(n^{1/p} + 1)|v_r|_n$$

$$\leq 4e^{3C_1/p}n^{1/p}.$$

We can now combine Theorems III.1 and III.2 (or Remark III.3) of [11] to deduce that Y is isomorphic to a quotient of ℓ_p. $\qquad\square$

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