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$\mathcal{T}$-CURVATURE TENSOR ON A SEMI-RIEMANNIAN MANIFOLD

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ABSTRACT. We introduce a new curvature tensor named as the $\mathcal{T}$-curvature tensor. We show that the quasi-conformal, conformal, conharmonic, concircular, pseudo-projective, projective, $\mathcal{M}$-projective, $\mathcal{W}_i$-curvature tensors ($i = 0, \ldots, 9$), $\mathcal{W}_j^*$-curvature tensors ($j = 0, 1$) are the particular cases of the $\mathcal{T}$-curvature tensor. Some properties for the $\mathcal{T}$-curvature tensor are given. We obtain the results for $\mathcal{T}$-conservative and $\mathcal{T}$-flat semi-Riemannian manifolds.

1. INTRODUCTION

The Weyl conformal curvature tensor [2] is the well known tensor, which is invariant of every conformal transformation. In particular, if a conformal transformation transforms a harmonic function into a harmonic function, then it is called a conharmonic transformation; and the conharmonic curvature tensor [3] is an invariant under conharmonic transformations. Next, it is well known that a semi-Riemannian manifold is locally projectively flat if and only if the projective curvature tensor [2] vanishes. Apart from conformal and projective curvature tensors, the concircular curvature tensor [8] is the next most important curvature tensor from the semi-Riemannian point of view, which is an invariant of concircular transformations. Later, Yano and Sawaki [10] generalized the conformal curvature tensor and concircular curvature tensor to the quasi-conformal curvature tensor. Note that conformal and concircular curvature tensors are curvature like tensors, while projective curvature tensor is not curvature like. Keeping these facts in mind, the first author introduced the $\mathcal{T}$-curvature tensor, which in particular cases reduces to several known curvature tensors and some new curvature tensors.

The object of this paper is to study $\mathcal{T}$-curvature tensors in semi-Riemannian manifolds. The paper is organized as follows. In section 2, the definition of a $\mathcal{T}$-curvature tensor is given. We give the properties and some identities of $\mathcal{T}$-curvature tensor. In section 3, we define $\mathcal{T}$-conservative semi-Riemannian manifolds and give necessary and sufficient conditions for semi-Riemannian manifolds to be

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$\mathcal{T}$-conservative. In section 4, we prove that every $\mathcal{T}$-flat semi-Riemannian manifold is Einstein. We also give the conditions for semi-Riemannian manifold to be $\mathcal{T}$-flat. In the last section, we give complete classifications of quasi-conformally flat and pseudo-projectively flat semi-Riemannian manifolds.

2. $\mathcal{T}$-CURVATURE TENSOR

Let (M, g) be an n-dimensional semi-Riemannian manifold and $\mathfrak{X}(M)$ the Lie algebra of vector fields in M. Throughout the paper we assume that $X, Y, Z, V, W \in \mathfrak{X}(M)$, unless specifically stated otherwise.

Definition 2.1. In an n-dimensional semi-Riemannian manifold (M, g), a $\mathcal{T}$-*curvature tensor* is a tensor of type $(1, 3)$, which is defined by

$$\begin{aligned} \mathcal{T}(X,Y)Z &= a_0 R(X,Y)Z \\ &\quad + a_1 S(Y,Z)X + a_2 S(X,Z)Y + a_3 S(X,Y)Z \\ &\quad + a_4 g(Y,Z)QX + a_5 g(X,Z)QY + a_6 g(X,Y)QZ \\ &\quad + a_7 r\left(g(Y,Z)X - g(X,Z)Y\right), \end{aligned} \tag{2.1}$$

where $a_0, \ldots, a_7$ are some smooth functions on M; and R, S, Q and r are the curvature tensor, the Ricci tensor, the Ricci operator of type $(1, 1)$ and the scalar curvature respectively.

In particular, the $\mathcal{T}$-curvature tensor is reduced to

(1) the *curvature tensor* R if

$$a_0 = 1, \quad a_1 = \cdots = a_7 = 0,$$

(2) the *quasiconformal curvature tensor* $\mathcal{C}_*$ [10] if

$$a_1 = -a_2 = a_4 = -a_5, \quad a_3 = a_6 = 0, \quad a_7 = -\frac{1}{n}\left(\frac{a_0}{n-1} + 2a_1\right),$$

(3) the *conformal curvature tensor* $\mathcal{C}$ [2, p. 90] if

$$a_0 = 1,\ a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{n-2},\ a_3 = a_6 = 0,\ a_7 = \frac{1}{(n-1)(n-2)},$$

(4) the *conharmonic curvature tensor* $\mathcal{L}$ [3] if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{n-2}, \quad a_3 = a_6 = 0, \quad a_7 = 0,$$

(5) the *concircular curvature tensor* $\mathcal{V}$ ([8], [9, p. 87]) if

$$a_0 = 1, \quad a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{n(n-1)},$$

(6) the *pseudo-projective curvature tensor* $\mathcal{P}_*$ [7] if

$$a_1 = -a_2, \quad a_3 = a_4 = a_5 = a_6 = 0, \quad a_7 = -\frac{1}{n}\left(\frac{a_0}{n-1} + a_1\right),$$

(7) the *projective curvature tensor* $\mathcal{P}$ [9, p. 84] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(8) the $\mathcal{M}$-*projective curvature tensor* [5] if

$$a_0 = 1, \quad a_1 = -a_2 = a_4 = -a_5 = -\frac{1}{2(n-1)}, \quad a_3 = a_6 = a_7 = 0,$$

(9) the $\mathcal{W}_0$-*curvature tensor* [5, Eq. (1.4)] if

$$a_0 = 1, \quad a_1 = -a_5 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(10) the $\mathcal{W}_0^*$-*curvature tensor* [5, Eq. (2.1)] if

$$a_0 = 1, \quad a_1 = -a_5 = \frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_6 = a_7 = 0,$$

(11) the $\mathcal{W}_1$-*curvature tensor* [5] if

$$a_0 = 1, \quad a_1 = -a_2 = \frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(12) the $\mathcal{W}_1^*$-*curvature tensor* [5] if

$$a_0 = 1, \quad a_1 = -a_2 = -\frac{1}{(n-1)}, \quad a_3 = a_4 = a_5 = a_6 = a_7 = 0,$$

(13) the $\mathcal{W}_2$-*curvature tensor* [4] if

$$a_0 = 1, \quad a_4 = -a_5 = -\frac{1}{(n-1)}, \quad a_1 = a_2 = a_3 = a_6 = a_7 = 0,$$

(14) the $\mathcal{W}_3$-*curvature tensor* [5] if

$$a_0 = 1, \quad a_2 = -a_4 = -\frac{1}{(n-1)}, \quad a_1 = a_3 = a_5 = a_6 = a_7 = 0,$$

(15) the $\mathcal{W}_4$-*curvature tensor* [5] if

$$a_0 = 1, \quad a_5 = -a_6 = \frac{1}{(n-1)}, \quad a_1 = a_2 = a_3 = a_4 = a_7 = 0,$$

(16) the $\mathcal{W}_5$-*curvature tensor* [6] if

$$a_0 = 1, \quad a_2 = -a_5 = -\frac{1}{(n-1)}, \quad a_1 = a_3 = a_4 = a_6 = a_7 = 0,$$

(17) the $\mathcal{W}_6$-*curvature tensor* [6] if

$$a_0 = 1, \quad a_1 = -a_6 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_4 = a_5 = a_7 = 0,$$

(18) the $\mathcal{W}_7$-*curvature tensor* [6] if

$$a_0 = 1, \quad a_1 = -a_4 = -\frac{1}{(n-1)}, \quad a_2 = a_3 = a_5 = a_6 = a_7 = 0,$$

(19) the $\mathcal{W}_8$-*curvature tensor* [6] if

$$a_0 = 1, \quad a_1 = -\,a_3 = -\,\frac{1}{(n-1)}, \quad a_2 = a_4 = a_5 = a_6 = a_7 = 0,$$

(20) the $\mathcal{W}_9$-*curvature tensor* [6] if

$$a_0 = 1, \quad a_3 = -\,a_4 = \frac{1}{(n-1)}, \quad a_1 = a_2 = a_5 = a_6 = a_7 = 0.$$

Even some new tensors may be retrieved from the definition of the $\mathcal{T}$-curvature tensor, which are not introduced so far. In [5], a (projective) curvature tensor is defined by

$$\mathcal{W}(X,Y)Z = R(X,Y)Z + \frac{1}{n-1}(g(X,Z)QY - S(Y,Z)X),$$

which we shall call $\mathcal{W}_0$-curvature tensor.

Denoting

$$\mathcal{T}\,(X,Y,Z,V) = g(\mathcal{T}\,(X,Y)\,Z,V),$$

we write the curvature tensor $\mathcal{T}$ in its $(0,4)$ form as follows.

$$\begin{aligned}\mathcal{T}\,(X,Y,Z,V) \;=\;& a_0\,R\,(X,Y,Z,V) + a_1\,S\,(Y,Z)\,g\,(X,V) + a_2\,S\,(X,Z)\,g\,(Y,V)\\ &+ a_3\,S(X,Y)g(Z,V) + a_4\,g\,(Y,Z)\,S\,(X,V)\\ &+ a_5\,g\,(X,Z)\,S\,(Y,V) + a_6\,S(Z,V)g(X,Y)\\ &+ a_7\,r\,(g\,(Y,Z)\,g\,(X,V) - g\,(X,Z)\,g\,(Y,V)).\end{aligned} \tag{2.2}$$

Lemma 2.1. *In a semi-Riemannian manifold* (M,g)*, the* $\mathcal{T}$*-curvature tensor satisfies*

$$\begin{aligned}&\mathcal{T}\,(X,Y,Z,V) + \mathcal{T}\,(Y,X,Z,V)\\ &\quad= (a_1+a_2)(S\,(Y,Z)\,g\,(X,V) + S\,(X,Z)\,g\,(Y,V))\\ &\qquad+ (a_4+a_5)(g\,(Y,Z)\,S\,(X,V) + g\,(X,Z)\,S\,(Y,V))\\ &\qquad+2a_3\,S(X,Y)g(Z,V) + 2a_6\,S(Z,V)g(X,Y),\end{aligned}$$

$$\begin{aligned}&\mathcal{T}\,(X,Y,Z,V) + \mathcal{T}\,(X,Y,V,Z)\\ &\quad= (a_1+a_5)(S\,(Y,Z)\,g\,(X,V) + S\,(Y,V)\,g\,(X,Z))\\ &\qquad+ (a_2+a_4)(S\,(X,V)\,g\,(Y,Z) + S\,(X,Z)\,g\,(Y,V))\\ &\qquad+ 2a_3\,S(X,Y)g(Z,V) + 2a_6\,S(Z,V)g(X,Y),\end{aligned}$$

$$\begin{aligned}&\mathcal{T}\,(X,Y,Z,V) + \mathcal{T}\,(Y,Z,X,V) + \mathcal{T}\,(Z,X,Y,V)\\ &\quad= (a_1+a_2+a_3)(S(X,Y)g(Z,V)\\ &\qquad\quad+ S(Y,Z)g(X,V) + S(X,Z)g(Y,V))\\ &\qquad\quad+ (a_4+a_5+a_6)(S(X,V)g(Y,Z)\\ &\qquad\quad+ S(Y,V)g(X,Z) + S(Z,V)g(X,Y)),\end{aligned}$$

$$\begin{aligned}
&\mathcal{T}(X,Y,Z,V)-\mathcal{T}(Z,V,X,Y)\\
&\quad= (a_1-a_4)(S(Y,Z)\,g(X,V)-S(X,V)\,g(Y,Z))\\
&\qquad+(a_3-a_6)(S(X,Y)\,g(Z,V)-S(Z,V)\,g(X,Y)),
\end{aligned}$$

Remark 2.1. A $(0,4)$-tensor T is known to be a curvature like tensor if it satisfies

$$\begin{gathered}
T(X,Y,Z,V)+T(Y,X,Z,V)=0,\\
T(X,Y,Z,V)+T(X,Y,V,Z)=0,\\
T(X,Y,Z,V)+T(Y,Z,X,V)+T(Z,X,Y,V)=0.
\end{gathered}$$

If T is a curvature like tensor, then it follows that $T(X,Y,Z,V)=T(Z,V,X,Y)$. In general, $\mathcal{T}$-curvature tensor is not a curvature like tensor. In particular, quasi-conformal, conformal, conharmonic, concircular and $\mathcal{M}$-projective curvature tensors are curvature like tensors. But, pseudo-projective, projective, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$ curvature tensors are not curvature like tensors.

In a semi-Riemannian manifold (M,g), let $\{e_i\}$, $i=1,\dots,n$, be a local orthonormal basis. Define

$$(\operatorname{div}\mathcal{T})(X,Y,Z)=\sum_{i=1}^{n}\varepsilon_i g((\nabla_{e_i}\mathcal{T})(X,Y)Z,e_i),$$

$$S_{\mathcal{T}}(X,Y)=\sum_{i=1}^{n}\varepsilon_i\mathcal{T}(e_i,X,Y,e_i),\quad r_{\mathcal{T}}=\sum_{i=1}^{n}\varepsilon_i S_{\mathcal{T}}(e_i,e_i),\quad \|S_{\mathcal{T}}\|^2=(S_{\mathcal{T}})_{hk}(S_{\mathcal{T}})^{hk},$$

where $\varepsilon_i=g(e_i,e_i)$. In particular, if $\mathcal{T}=R$, then $S_{\mathcal{T}}$ and $r_{\mathcal{T}}$ become the well known Ricci tensor S and the scalar curvature r, respectively.

Lemma 2.2. *In a semi-Riemannian manifold (M,g), we have the following identities*:

$$S_{\mathcal{T}}(X,Y)=(a_0+na_1+a_2+a_3+a_5+a_6)S(X,Y)+(a_4+(n-1)a_7)r\,g(X,Y),$$

$$r_{\mathcal{T}}=(a_0+na_1+a_2+a_3+na_4+a_5+a_6+n(n-1)a_7)r,$$

$$\begin{aligned}
\|S_{\mathcal{T}}\|^2 &= (a_0+na_1+a_2+a_3+a_5+a_6)^2\,\|S\|^2\\
&\quad+\big(2(a_0+na_1+a_2+a_3+a_5+a_6)(a_4+(n-1)a_7)\\
&\quad+n(a_4+(n-1)a_7)^2\big)\,r^2,
\end{aligned}$$

$$\begin{aligned}
(\operatorname{div}\mathcal{T})(X,Y,Z) &= (a_0+a_1)(\nabla_X S)(Y,Z)+(-a_0+a_2)(\nabla_Y S)(X,Z)\\
&\quad+a_3(\nabla_Z S)(X,Y)+\left(\frac{a_4}{2}+a_7\right)(\nabla_X r)g(Y,Z)\\
&\quad+\left(\frac{a_5}{2}-a_7\right)(\nabla_Y r)g(X,Z)+\frac{a_6}{2}(\nabla_Z r)g(X,Y),
\end{aligned}$$

$$
\begin{aligned}
(\mathrm{div}\mathcal{T})(X,Y,Z) &= \frac{1}{A}((a_0+a_1)(\nabla_X S_{\mathcal{T}})(Y,Z) + (-a_0+a_2)(\nabla_Y S_{\mathcal{T}})(X,Z) \\
&+ a_3(\nabla_Z S_{\mathcal{T}})(X,Y)) \\
&+ \left(\frac{2Ba_0+2Ba_1+Aa_4+2Aa_7}{2A(A+nB)}\right)(\nabla_X r_{\mathcal{T}})g(Y,Z) \\
&+ \left(\frac{-2Ba_0+2Ba_2+Aa_5-2Aa_7}{2A(A+nB)}\right)(\nabla_Y r_{\mathcal{T}})g(X,Z) \\
&+ \left(\frac{2Ba_3+Aa_6}{2A(A+nB)}\right)(\nabla_Z r_{\mathcal{T}})g(X,Y),
\end{aligned}
$$

where

$$A = a_0 + na_1 + a_2 + a_3 + a_5 + a_6, \quad B = a_4 + (n-1)a_7.$$

3. $\mathcal{T}$-CONSERVATIVE SEMI-RIEMANNIAN MANIFOLDS

We begin with the definition of $\mathcal{T}$-conservative semi-Riemannian manifold.

Definition 3.1. A semi-Riemannian manifold is called $\mathcal{T}$-*conservative* if $\mathrm{div}\,\mathcal{T} = 0$.

In particular, if $\mathcal{T}$ is equal to R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes conservative, quasi-conformally conservative, conformally conservative, conharmonically conservative, concircularly conservative, pseudo-projectively conservative, projectively conservative, $\mathcal{M}$-conservative, $\mathcal{W}_0$-conservative, $\mathcal{W}_0^*$-conservative, $\mathcal{W}_1$-conservative, $\mathcal{W}_1^*$-conservative, $\mathcal{W}_2$-conservative, $\mathcal{W}_3$-conservative, $\mathcal{W}_4$-conservative, $\mathcal{W}_5$-conservative, $\mathcal{W}_6$-conservative, $\mathcal{W}_7$-conservative, $\mathcal{W}_8$-conservative, $\mathcal{W}_9$-conservative, respectively.

Theorem 3.1. *Let M be an n-dimensional ($n > 2$) semi-Riemannian manifold. Then M is $\mathcal{T}$-conservative if and only if*

$$
\begin{aligned}
& -(a_0+a_1)g((\nabla_X Q)Y,Z) + (a_0-a_2)g((\nabla_Z Q)X,Y) - a_3 g((\nabla_Y Q)X,Z) \\
= & \left(\frac{a_4}{2}+a_7\right)g(Y,Z)Xr + \left(\frac{a_5}{2}-a_7\right)g(X,Y)Zr + \frac{a_6}{2}g(X,Z)Yr. \qquad (3.1)
\end{aligned}
$$

Remark 3.1. In [11], for conformally flat Riemannian manifold, it is proved that

$$g((\nabla_X Q)Y,Y) - g((\nabla_Y Q)X,Y) = \frac{1}{2(n-1)}(g(Y,Y)Xr - g(X,Y)Yr).$$

However, even for the weaker condition $\mathrm{div}\,\mathcal{C} = 0$, the above equation is true, which is given in the following Table.

From Theorem 3.1, we have the following

Theorem 3.2. *Let M be an n-dimensional ($n > 2$) $\mathcal{T}$-conservative semi-Riemannian manifold. Then*

$$
\begin{aligned}
& -(a_0+a_1)g((\nabla_X Q)Y,Y) + (a_0-a_2-a_3)g((\nabla_Y Q)X,Y) \\
= & \left(\frac{a_4}{2}+a_7\right)g(Y,Y)Xr + \left(\frac{a_5+a_6}{2}-a_7\right)g(X,Y)Yr. \qquad (3.2)
\end{aligned}
$$

Consequently, we have the following:

Condition	**Result**
$\operatorname{div} \mathcal{C}_* = 0$	$-(a_0 + a_1)(g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y)) = \frac{1}{n}\left(\frac{(n-4)a_1}{2} - \frac{a_0}{n-1}\right)(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{C} = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = \frac{1}{2(n-1)}(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{L} = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = \frac{1}{2(n-3)}(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{V} = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = \frac{1}{n(n-1)}(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{P}_* = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = \frac{a_0 + (n-1)a_1}{n(n-1)(a_0+a_1)}(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{M} = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = \frac{1}{2(2n-3)}(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{W}_0 = 0$	$-2(n-2)g((\nabla_X Q)Y, Y) + 2(n-1)g((\nabla_Y Q)X, Y) = g(X,Y)Yr$
$\operatorname{div} \mathcal{W}_0^* = 0$	$2ng((\nabla_X Q)Y, Y) - 2(n-1)g((\nabla_Y Q)X, Y) = g(X,Y)Yr$
$\operatorname{div} \mathcal{W}_2 = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = \frac{1}{2(n-1)}(g(Y,Y)Xr - g(X,Y)Yr)$
$\operatorname{div} \mathcal{W}_3 = 0$	$-2(n-1)g((\nabla_X Q)Y, Y) + 2ng((\nabla_Y Q)X, Y) = g(Y,Y)Xr$
$\operatorname{div} \mathcal{W}_5 = 0$	$-2(n-1)g((\nabla_X Q)Y, Y) + 2ng((\nabla_Y Q)X, Y) = g(X,Y)Yr$
$\operatorname{div} \mathcal{W}_6 = 0$	$-2(n-2)g((\nabla_X Q)Y, Y) + 2(n-1)g((\nabla_Y Q)X, Y) = g(X,Y)Yr$
$\operatorname{div} \mathcal{W}_7 = 0$	$-2(n-2)g((\nabla_X Q)Y, Y) + 2(n-1)g((\nabla_Y Q)X, Y) = g(Y,Y)Xr$
$\operatorname{div} \mathcal{W}_9 = 0$	$2(n-1)g((\nabla_X Q)Y, Y) - 2(n-2)g((\nabla_Y Q)X, Y) = g(Y,Y)Xr$
$\operatorname{div}\{\mathcal{P}, \mathcal{W}_1, \mathcal{W}_1^*, \mathcal{W}_4, \mathcal{W}_8\} = 0$	$g((\nabla_X Q)Y, Y) - g((\nabla_Y Q)X, Y) = 0$

Proof. Replacing Z by Y in (3.1), we get (3.2). □

4. $\mathcal{T}$-FLAT SEMI-RIEMANNIAN MANIFOLDS

In this section, we define and study $\mathcal{T}$-flat semi-Riemannian manifold.

Definition 4.1. A semi-Riemannian manifold is said to be $\mathcal{T}$-flat if $\mathcal{T} = 0$.

In particular, if $\mathcal{T}$ is equal to R, $\mathcal{C}_*$, $\mathcal{C}$, $\mathcal{L}$, $\mathcal{V}$, $\mathcal{P}_*$, $\mathcal{P}$, $\mathcal{M}$, $\mathcal{W}_0$, $\mathcal{W}_0^*$, $\mathcal{W}_1$, $\mathcal{W}_1^*$, $\mathcal{W}_2$, $\mathcal{W}_3$, $\mathcal{W}_4$, $\mathcal{W}_5$, $\mathcal{W}_6$, $\mathcal{W}_7$, $\mathcal{W}_8$, $\mathcal{W}_9$, then it becomes flat, quasi-conformally flat, conformally flat, conharmonically flat, concircularly flat, pseudo-projectively flat, projectively flat, $\mathcal{M}$-flat, $\mathcal{W}_0$-flat, $\mathcal{W}_0^*$-flat, $\mathcal{W}_1$-flat, $\mathcal{W}_1^*$-flat, $\mathcal{W}_2$-flat, $\mathcal{W}_3$-flat, $\mathcal{W}_4$-flat, $\mathcal{W}_5$-flat, $\mathcal{W}_6$-flat, $\mathcal{W}_7$-flat, $\mathcal{W}_8$-flat, $\mathcal{W}_9$-flat, respectively.

Theorem 4.1. *Let M be an n-dimensional $(n > 2)$ semi-Riemannian manifold. Let M be a $\mathcal{T}$-flat.*

(i) *If $a_0 + na_1 + a_2 + a_3 + a_5 + a_6 \neq 0$, then it is Einstein with*

$$S = C_1 g, \tag{4.1}$$

where

$$C_1 = -\frac{r(a_4 + (n-1)a_7)}{a_0 + na_1 + a_2 + a_3 + a_5 + a_6} g.$$

(ii) *If $a_0 - a_1 - a_3 - a_4 - na_5 - a_6 \neq 0$, then it is Einstein with*

$$S = C_2 g, \tag{4.2}$$

where

$$C_2 = \frac{r(a_2 - (n-1)a_7)}{a_0 - a_1 - a_3 - a_4 - na_5 - a_6} g.$$

Consequently we have the following:

Curvature tensor	$S =$
$\mathcal{C}_* = 0$	$\frac{r}{n} g$ *if* $a_0 + (n-2)a_1 \neq 0$
$\mathcal{P}_* = 0$	$\frac{r}{n} g$ if $a_0 + (n-1)a_1 \neq 0$ or $a_0 - a_1 \neq 0$
$\mathcal{W}_3 = 0$	$-\frac{r}{n-2} g$
$\{\mathcal{V}, \mathcal{P}, \mathcal{M}, \mathcal{W}_1^*, \mathcal{W}_2\} = \{0\}$	$\frac{r}{n} g$
$\{\mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_4, \ldots, \mathcal{W}_9\} = \{0\}$	0

Proof. Let M be an n-dimensional semi-Riemannian manifold. Consider $\mathcal{T} = 0$, then by (2.2), we have

$$\begin{aligned} -a_0 R(X,Y,Z,V) &= a_1 S(Y,Z) g(X,V) + a_2 S(X,Z) g(Y,V) \\ &\quad + a_3 S(X,Y) g(Z,V) + a_4 g(Y,Z) S(X,V) \\ &\quad + a_5 g(X,Z) S(Y,V) + a_6 S(Z,V) g(X,Y) \\ &\quad + a_7 r (g(Y,Z) g(X,V) - g(X,Z) g(Y,V)). \end{aligned} \tag{4.3}$$

Contracting (4.3) with respect to X and V, we get (4.1). This gives the statement (i).

Next, (4.3) can be rewritten as

$$\begin{aligned} a_0\, R\,(X,Y,V,Z) &= a_1\, S\,(Y,Z)\, g\,(X,V) + a_2\, S\,(X,Z)\, g\,(Y,V) \\ &+ a_3\, S(X,Y) g(Z,V) + a_4\, g\,(Y,Z)\, S\,(X,V) \\ &+ a_5\, g\,(X,Z)\, S\,(Y,V) + a_6\, S(Z,V) g(X,Y) \\ &+ a_7\, r\,(g\,(Y,Z)\, g\,(X,V) - g\,(X,Z)\, g\,(Y,V))\,. \end{aligned} \tag{4.4}$$

Contracting (4.4) with respect to X and Z, we get (4.2). This gives the statement (ii). □

Theorem 4.2. *Let M be an n-dimensional $(n > 2)$ semi-Riemannian manifold. Let M be $\mathcal{T}$-flat.*

(i) *If $a_0 + na_1 + a_2 + a_3 + a_5 + a_6 \neq 0$, then*

$$\begin{aligned} -a_0 R(X,Y)Z &= (a_1C_1 + a_4C_1 + a_7r) g(Y,Z)X \\ &+ (a_2C_1 + a_5C_1 - a_7r) g(X,Z)Y \\ &+ (a_3C_1 + a_6C_1) g(X,Y)Z. \end{aligned} \tag{4.5}$$

(ii) *If $a_0 - a_1 - a_3 - a_4 - na_5 - a_6 \neq 0$, then*

$$\begin{aligned} -a_0 R(X,Y)Z &= (a_1C_2 + a_4C_2 + a_7r) g(Y,Z)X \\ &+ (a_2C_2 + a_5C_2 - a_7r) g(X,Z)Y \\ &+ (a_3C_2 + a_6C_2) g(X,Y)Z. \end{aligned} \tag{4.6}$$

Consequently, we have the following:

Condition	$R(X,Y)Z =$
$\mathcal{C}_* = 0$	$\frac{r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y)$ *if* $a_0 + (n-2)a_1 \neq 0$
$\mathcal{P}_* = 0$ [7]	$\frac{r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y)$ *if* $a_0 + (n-1)a_1 \neq 0$ *or* $a_0 - a_1 \neq 0$
$\mathcal{W}_3 = 0$	$\frac{r}{(n-1)(n-2)}(g(Y,Z)X - g(X,Z)Y)$
$\{\mathcal{V}, \mathcal{P}, \mathcal{M}, \mathcal{W}_1^*, \mathcal{W}_2\} = \{0\}$	$\frac{r}{n(n-1)}(g(Y,Z)X - g(X,Z)Y)$
$\{\mathcal{W}_0^*, \mathcal{W}_1, \mathcal{W}_4, \ldots, \mathcal{W}_9\} = \{0\}$	0

Proof. Using (4.1) in (2.1) and taking $\mathcal{T} = 0$, we get (4.5). Next, using (4.2) in (2.1) and taking $\mathcal{T} = 0$, we get (4.6). □

Theorem 4.3. *Let M be an n-dimensional $(n > 2)$ semi-Riemannian manifold. If*

(i) $a_0 + na_1 + a_2 + a_3 + a_5 + a_6 \neq 0$, (4.5) *is true and* $a_0C_1 + B_1C_1 + (n-1)a_7r = 0$ *or*

(ii) $a_0-a_1-a_3-a_4-na_5-a_6 \neq 0$, (4.6) *holds and* $a_0C_2+B_1C_2+(n-1)a_7r=0$, *then* M *is* $\mathcal{T}$*-flat. In particular, for* $\mathcal{T} \in \{\mathcal{C}_*, \mathcal{V}, \mathcal{P}_*, \mathcal{P}, \mathcal{M}, \mathcal{W}_0^*, \mathcal{W}_1^*, \mathcal{W}_2, \mathcal{W}_4, \mathcal{W}_6, \mathcal{W}_8\}$, *if* (4.5) *or* (4.6) *is true, then* M *is* $\mathcal{T}$*-flat.*

Proof. Let M be an n-dimensional $(n > 2)$ semi-Riemannian manifold. Let (4.5) holds. Then contracting (4.5), we get

$$-a_0S = (B_1C_1 + (n-1)a_7r)\, g, \tag{4.7}$$

where

$$B_1 = na_1 + a_2 + a_3 + na_4 + a_5 + a_6.$$

Using (4.5) and (4.7) in (2.1), we get

$$\begin{aligned} -a_0\mathcal{T}(X,Y)Z &= (a_0C_1 + B_1C_1 + (n-1)a_7r) \times \\ &\quad ((a_1+a_4)g(Y,Z)X + (a_2+a_5)g(X,Z)Y + (a_3+a_6)g(X,Y)Z), \end{aligned}$$

which gives the statement (i). The proof of statement (ii) is similar to statement (i). This completes the proof. □

5. TWO CLASSIFICATIONS

In this section, we give complete classifications of quasi-conformally flat and pseudo-projectively flat semi-Riemannian manifolds.

Theorem 5.1. *Let* M *be a semi-Riemannian manifold of dimension* n *greater than* 2*. Then* M *is quasi-conformally flat if and only if one of the following statements is true*:

(i) $a_0 + (n-2)a_1 = 0$, $a_0 \neq 0 \neq a_1$ *and* M *is conformally flat.*
(ii) $a_0 + (n-2)a_1 \neq 0$, $a_0 \neq 0$, M *is of constant curvature.*
(iii) $a_0 + (n-2)a_1 \neq 0$, $a_0 = 0$ *and* M *is Einstein manifold.*

Proof. The quasi-conformal curvature tensor in M is given by

$$\begin{aligned} \mathcal{C}_*(X,Y,Z,V) &= a_0R(X,Y,Z,V) + a_1(S(Y,Z)\,g(X,V) - S(X,Z)\,g(Y,V) \\ &\quad + g(Y,Z)\,S(X,V) - g(X,Z)\,S(Y,V)) \\ &\quad - \frac{r}{n}\left(\frac{a_0}{n-1} + 2a_1\right)(g(Y,Z)\,g(X,V) - g(X,Z)\,g(Y,V)). \end{aligned} \tag{5.1}$$

This can be rewritten as

$$\mathcal{C}_*(X,Y,Z,V) = -(n-2)a_1\,\mathcal{C}(X,Y,Z,V) + (a_0 + (n-2)a_1)\mathcal{V}(X,Y,Z,V). \tag{5.2}$$

Using $\mathcal{C}_* = 0$ in (5.1) we get

$$\begin{aligned} 0 &= a_0R(X,Y,Z,V) + a_1(S(Y,Z)\,g(X,V) - S(X,Z)\,g(Y,V) \\ &\quad + g(Y,Z)\,S(X,V) - g(X,Z)\,S(Y,V)) \\ &\quad - \frac{r}{n}\left(\frac{a_0}{n-1} + 2a_1\right)(g(Y,Z)\,g(X,V) - g(X,Z)\,g(Y,V)), \end{aligned} \tag{5.3}$$

from which we obtain

$$(a_0 + (n-2)a_1)\left(S - \frac{r}{n}\,g\right) = 0. \tag{5.4}$$

Case 1. $a_0 + (n-2)a_1 = 0$ and $a_0 \neq 0 \neq a_1$. Then from (5.1) and (5.2), it follows that $(n-2)a_1\,\mathcal{C} = 0$, which gives $\mathcal{C} = 0$. This gives the statement (i).
Case 2. $a_0 + (n-2)a_1 \neq 0$ and $a_0 \neq 0$. Then from (5.4)

$$S\,(Y, Z) = \frac{r}{n}\,g(Y, Z). \tag{5.5}$$

Using (5.5) in (5.3), we get

$$a_0(R\,(X, Y, Z, V) - \frac{r}{n(n-1)}\,(g(Y, Z)g(X, V) - g(X, Z)g(Y, V))) = 0. \tag{5.6}$$

Since $a_0 \neq 0$, then by definition of concircular curvature tensor, $\mathcal{V} = 0$ and using (5.6) and (5.3), we get $\mathcal{C} = 0$. This gives the statement (ii).
Case 3. $a_0 + (n-2)a_1 \neq 0$ and $a_0 = 0$, we get (5.5). This gives the statement (iii).
Converse is true in all cases. □

Remark 5.1. In [1], the following three results are known:

(a) [1, Proposition 1.1] A quasi-conformally flat manifold is either conformally flat or Einstein.
(b) [1, Corollary 1.1] A quasi-conformally flat manifold is conformally flat if the constant $a_0 \neq 0$.
(c) [1, Corollary 1.2] A quasi-conformally flat manifold is Einstein if the constants $a_0 = 0$ and $a_1 \neq 0$.

However, the converses need not be true in these three results. But, in Theorem 5.1 we get a complete classification of quasi-conformally flat manifolds.

Theorem 5.2. *Let M be a semi-Riemannian manifold of dimension n greater than 2. Then M is pseudo-projectively flat if and only if one of the following statements is true:*

(i) $a_0 + (n-1)a_1 = 0$, $a_0 \neq 0 \neq a_1$ *and M is projectively flat.*
(ii) $a_0 + (n-1)a_1 \neq 0$, $a_0 \neq 0$, *M is of constant curvature.*
(iii) $a_0 + (n-1)a_1 \neq 0$, $a_0 = 0$ *and M is Einstein manifold.*

Proof. The pseudo-projective curvature tensor in M is given by

$$\begin{aligned}\mathcal{P}_*(X, Y, Z, V) &= a_0R(X, Y, Z, V) + a_1(S(Y, Z)g(X, V) - S(X, Z)g(Y, V)) \\ &\quad - \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)(g(Y, Z)g(X, V) - g(X, Z)g(Y, V)).\end{aligned} \tag{5.7}$$

This can be rewritten as

$$\mathcal{P}_*\,(X, Y, Z, V) = -(n-1)a_1\,\mathcal{P}\,(X, Y, Z, V) + (a_0 + (n-1)a_1)\mathcal{V}\,(X, Y, Z, V)\,. \tag{5.8}$$

Using $\mathcal{P}_* = 0$ in (5.7) we get

$$\begin{aligned} 0 &= a_0 R(X,Y,Z,V) + a_1(S(Y,Z)\,g(X,V) - S(X,Z)\,g(Y,V)) \\ &\quad - \frac{r}{n}\left(\frac{a_0}{n-1} + a_1\right)(g(Y,Z)\,g(X,V) - g(X,Z)\,g(Y,V)), \end{aligned} \tag{5.9}$$

from which we obtain

$$(a_0 + (n-1)a_1)\left(S - \frac{r}{n}g\right) = 0. \tag{5.10}$$

Case 1. $a_0 + (n-1)a_1 = 0$ and $a_0 \neq 0 \neq a_1$. Then from (5.8) and (5.9), it follows that $a_0\,\mathcal{P} = 0$, which gives $\mathcal{P} = 0$. This gives the statement (i).
Case 2. $a_0 + (n-1)a_1 \neq 0$ and $a_0 \neq 0$. Then from (5.10)

$$S(Y,Z) = \frac{r}{n}g(Y,Z). \tag{5.11}$$

Using (5.11) in (5.9), we get

$$a_0(R(X,Y,Z,V) - \frac{r}{n(n-1)}(g(Y,Z)g(X,V) - g(X,Z)g(Y,V))) = 0. \tag{5.12}$$

Since $a_0 \neq 0$, then by definition of concircular curvature tensor, $\mathcal{V} = 0$ and by using (5.12), (5.11) in the definition of projective curvature tensor, we get $\mathcal{P} = 0$. This gives the statement (ii).
Case 3. $a_0 + (n-1)a_1 \neq 0$ and $a_0 = 0$, we get (5.11). This gives the statement (iii).

Converse is true in all the three cases. □

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