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T-FRACTIONS FROM A DIFFERENT POINT OF VIEW

HAAKON WAADELAND

The continued fractions of the form

$$(1)\qquad 1 + d_0 z + \frac{z}{1 + d_1 z} + \cdots + \frac{z}{1 + d_n z} + \cdots$$

were introduced by W. J. Thron in 1948 [5], and are referred to as T-fractions.

Between the class of T-fractions and the class of formal power series $1 + c_1 z + c_2 z^2 + \cdots$ there is a one-to-one correspondence. The formal approximation is however slow, and none of the approximants are in the Padé table. But on the other hand, the T-fractions have several interesting properties, such as a Stieltjes integral representation (under certain conditions on the parameters d_n) and simple convergence properties associated with properties of the sequence (d_n) [2], [3], [5].

The purpose of this paper is to present some convergence theorems of a different type, namely theorems for the T-fraction expansions of certain functions.

Let f_0 be holomorphic in some neighborhood D_0 of the origin and normalized by $f_0(0) = 1$. Let furthermore (f_n) be the sequence of functions defined by

$$(2)\qquad f_n(z) = 1 + (f_n{}'(0) - 1)z + \frac{z}{f_{n+1}(z)}, \; z \neq 0, f_{n+1}(0) = 1.$$

Then every f_n is analytic in some neighborhood D_n of the origin, and with $d_n = f_n{}'(0) - 1$, the identity

$$(3)\quad f_0(z) = 1 + d_0 z + \frac{z}{1 + d_1 z} + \cdots + \frac{z}{1 + d_{n-1} z} + \frac{z}{f_n(z)}$$

holds in a neighborhood of the origin, and the continued fraction (1) is the T-fraction expansion of f_0.

The theorems to be presented are of the form:

Boundedness property of $f_0 \Rightarrow$ convergence property of the T-fraction expansion of f_0. The first result of this type was proved in 1964 [6]:

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Theorem 1. *There exists an $R_0 \geqq 1$ and to each $R > R_0$ a $K(R) > 0$, such that the following holds:*

If f_0 is holomorphic in $|z| < R$, normalized by $f_0(0) = 1$ and satisfying the boundedness condition

$$|f_0(z) - 1| < K(R) \text{ in } |z| < R,$$

then the functions f_n, defined by (2), *are all holomorphic in $|z| < R$, and $f_n(z) \to 1$ as $n \to \infty$ uniformly on $|z| < R$. Furthermore, the T-fraction is "limitärperiodisch" with $d_n \to -1$ as $n \to \infty$, and it converges to f_0 uniformly on any compact subset of $|z| < 1$.*

The main tool of the proof is Schwarz' lemma. It is easy to prove (in particular if $R > 2$ and $K(R) \leqq R/2 - 1$) not only that the boundedness condition is inherited from f_0 to all the succeeding functions f_n, but also that a contraction takes place, from which follow the stated limit properties of f_n and d_n. The convergence of the T-expansion is proved by using a well-known theorem on "limitär-periodisch" continued fractions [4, p. 93].

The convergence to the "right" function can not be extended beyond the unit circle, as may be seen from the T-expansion of $f_0(z) = 1$

$$1 - z + \frac{z}{1 - z} + \cdots + \frac{z}{1 - z} + \cdots,$$

which converges to 1 for $|z| < 1$ and to $-z$ for $|z| > 1$ (and diverges on $|z| = 1$, except for $z = -1$, where it converges to 1) [5, p. 208]. This example reflects a "normal" behavior of T-expansions for the functions in question, i.e., convergence to the "right" function in $|z| < 1$ and to a "wrong" function for $|z| > 1$ [7].

A slight modification of the T-approximation takes care of this difficulty to a certain extent. The nth approximant $A_n(z)/B_n(z)$ of f_0 is obtained from the identity (3) by replacing $f_n(z)$ by 0. But since $f_n(z) \to 1$ for the functions in question it seemed to be a better idea to replace $f_n(z)$ by 1. (A similar idea is considered by T. E. Phipps. See, e.g., [8].) The corresponding approximants $\tilde{A}_n(z)/\tilde{B}_n(z)$ are called modified T-approximants. In the example mentioned above the sequence of modified T-approximants converges to the right function $f_0(z) = 1$ in the whole plane. In the paper [7] it is proved that the domain of convergence actually is increased for the functions we are dealing with, but it was not until 1971 it was proved that the convergence could be extended to the whole disk $|z| < R$. Hovstad proved in [1] a result on the rate at which $f_n(z) \to 1$. From this result followed first that $R_0 = 1$ in theorem 1 and next a theorem on modified T-fractions:

THEOREM 2. *To each* $R > 1$ *there exists a* $K(R) > 0$, *such that the following holds*:

If f_0 *is holomorphic in* $|z| < R$, *normalized by* $f_0(0) = 1$ *and satisfying the boundedness condition*

$$|f_0(z) - 1| < K(R) \quad \text{in } |z| < R,$$

then the sequence of modified T-approximants converges to f_0 *uniformly on any compact subset of the disk* $|z| < R$. (Example: For $R > 2$ any number $< R/2 - 1$ will do as a $K(R)$.)

FINAL REMARK. The modified T-approximation is here presented as an *example* of a summability method in the theory of continued fractions. The "usefulness" of the method is (for the functions we have studied here) at most at "power series level", and the non-modified T-approximation is even less useful. But the theorems have a wider scope. If we require the original f_0-conditions to hold for some f_n rather than for f_0, the conclusions of the theorems still hold (with obvious minor modifications), and if f_0 has poles in the unit disk, the power series expansion is useful only in some $|z| < r$, where $r < 1$, whereas the T-expansion is useful in $|z| < 1$ and the modified T-expansion in some $|z| < R$, where $R > 1$.

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MATEMATISK INSTITUTT, UNIVERSITETET I TRONDHEIM NORGES LAERERHOGSKOLE, 7000 TRONDHEIM, NORWAY