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T-Spherical Fuzzy Power Muirhead Mean Operator Based on Novel Operational Laws and Their Application in Multi-Attribute Group Decision Making

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ABSTRACT T-spherical fuzzy set (T-SPFS) is a generalization of several fuzzy concepts such as fuzzy set (FS), intuitionistic FS, picture FS, Pythagorean FS, and q-rung orthopair FS. T-SPFS is a more powerful mathematical tool to handle uncertain, inconsistent, and vague information than the above-defined sets. In this paper, some limitations in the operational laws for SPF numbers (SPFNs) are discussed and some novel operational laws for SPFNs are proposed. Furthermore, two new aggregation operators for aggregating SPF information are proposed and are applied to multiple-attribute group decision-making (MAGDM). To take the advantages of Muirhead mean (MM) operator and power average operator, the SPF power MM (SPFPMM) operator, weighted SPFPMM operator, SPF power dual MM (SPFPDMM) operator, weighted SPFPDMM operator are introduced and their anticipated properties are discussed. The main advantage of these developed aggregation operators is that they take the relationship among fused data and the interrelationship among aggregated values, thereby getting more information in the process of MAGDM. Moreover, a novel approach to MAGDM based on the developed aggregation operators is established. Finally, a numerical example is given to show the effectiveness of the developed approach and comparison with the existing approaches is also given.

INDEX TERMS T-Spherical fuzzy set, novel operational laws, MAGDM, aggregation operator, power average operator, MM operator, power Murihead mean operator.

I. INTRODUCTION

There are a large number of multiple attribute decision-making (MADM) or MAGDM problems in decision making, and the attributes used are usually ambiguous and can easily be represented by fuzzy information. Since the initiation of FS by Zadeh [1], FS has gained a significant concentration from the researchers all over the world and they studied its theoretical as well as practical aspects. Several extensions of FS has been developed such as interval-valued FS (IVFS) [2], which can be explained by the truth-membership degree (TMD) by some closed interval the unit [0, 1], intuitionistic FS (IFS) [3], which can be explained by the TMD and falsity-membership degree (FMD). Clearly IFS can define fuzziness and uncertainty more utterly than that of FS. Although, FS and IFS work very well in many circumstances but still these have many issues, where we oppose opinions that contained many kinds of answers, such as yes, abstain, no and refusal. To deal with such type of situation, Coung and Kreinovich [4] developed the approach of picture fuzzy sets (PFSs), as a generalization of FSs and IFSs. PFS consist of four membership degree namely, positive membership (PM), abstain membership (AM), negative-membership (NM) and refusal-membership (RM). Certainly, utilizing PFS to express the uncertain information are more realistic and perfect than FS and IFS. After the initiation of PFS, some authors applied this notion into the picture fuzzy

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clustering (PFC) [5]–[7]. Recently, studies have been originated on MADM with PF information [8]. Wei *et al.* [9], [10] proposed cross entropy for PFS and proposed operational laws for PFNs and applied these to deal with MADM problem. Wang *et al.* [11] advanced some new operational rules for PFNs and proposed geometric aggregation operators based on these operational laws and applied these to MADM problems. Ashraf *et al.* [12] proposed some aggregation operator for PFNs and applied these to MAGDM problem.

Some other generalizations of IFS, which gain much more attention from the scholars, are PGFS [13]–[15], q-rung orthopair fuzzy sets (QROFS) [16]. But these two extensions have same limitations as IFS have. To deal with such situations, recently, Mahmood *et al.* [17] developed the concepts of T-spherical fuzzy set (T-SPFS), T-spherical fuzzy number (T-SPFN) and defined some relations, operational laws, aggregation operators and discussed there applications in medical diagnosis and pattern recognition. Some similarity measures for T-SPFS are defined by Ullah *et al.* [18] and discussed its application in pattern recognition. However, the defined operational laws for SPFNs have some limitations which will discuss in Section 2.2.

Due to the enhanced complexity in real decision making problems, we have to examine the following questions, when modifying the best alternative. (1) In some situations, the values of the attributes provided by the decision makers may be too low or too high, have a negative impact on the final ranking results. The power average (PA) operator initially developed by Yager [19] is a handy aggregation operator that permits the evaluated values to mutually supported and enhanced. Therefore, we may use the PA operator to diminish such awful impact by designating distinct weights produced by the support measure. (2) In some practical decision making the values of attribute are dependent. Therefore, the interrelationship among the values of the attributes should be examined. The Bonferroni mean (BM) operator [20], [21], Heronian mean (HM) operator [22], Murihead mean (MM) operator [23] can attain this function. However, some advantages of MM operator over BM and HM are discussed by Liu and Li [24], Liu and You [25]. Some existing aggregation operators such as, BM and Maclaurin symmetric mean (MSM) operator [26] are special cases of MM operator. Moreover, MM operator consists of the parameter vector, which enlarged the flexibility in the aggregation process. Recently, Li *et al.* [27] developed the concept of power Murihead mean operator under PGF environment and apply them to MADM. From the existing literature, PA operator and MM operator was not combined to deal with T-spherical fuzzy environment.

Therefore, the main aim of this article is to propose some novel operational laws for T-SPFNs, combine PA operator with MM operator, and extend the idea to T-spherical fuzzy environment, and develop some new aggregation operators such as T-spherical fuzzy power Muirhead mean operator, weighted T-spherical fuzzy power Muirhead mean operator, T-spherical fuzzy power dual Muirhead mean operator, weighted T-spherical fuzzy power dual Muirhead mean operator and discussed some special cases of the developed aggregation operator and apply them to MAGDM to achieve the two requirements discussed above.

To do so, the rest of the article is organized as follows. In Section 2, some basic definitions, about spherical fuzzy sets, T- spherical fuzzy sets, Muirhead Mean operator, PA operator are given. In Section 3 we extend Muirhead mean operator to T- spherical fuzzy environment. In Section 4, based on these proposed aggregation operators a novel method to MAGDM is developed. In Section 5, a numerical example is illustrated to show the effectiveness and practicality of the developed approach and a comparison with some existing methods are given.

II. PRELIMINARIES

A. SPHERICAL FUZZY SETS AND THEIR OPERATIONS

Definition 1 [17]: Let Υ be a universe of discourse set. A SPFS Θ is defined and mathematically denoted as:

$$\Theta = \{\langle a, \Xi(a), \Psi(a), Z(a)\rangle \text{ for all } a \in \Upsilon\} \tag{1}$$

where $\Xi(a), \Psi(a), Z(a) \in [0, 1]$ are respectively, representing the membership degree (MD), abstinence degree (AD) and non-membership degree (NMD) such that $0 \le (\Xi(a))^2 + (\Psi(a))^2 + (Z(a))^2 \le 1$, and refusal degree (RD) is expressed by $\Gamma = \sqrt{1 - (\Xi(a))^2 + (\Psi(a))^2 + (Z(a))^2}$. For computational simplicity, we shall denote a spherical fuzzy number (SPFN) by the triplet $\Theta = (\Xi, \Psi, Z)$.

The operational laws for SPFS were defined by Mahmood *et al.* [17] and are given below:

Definition 2 [17]: Let Θ_1 and Θ_2 be any two SPFSs. Then

(1) $\Theta_1 \subseteq \Theta_2$ iff $\Xi_1(a) \le \Xi_2(a), \Psi_1(a) \le \Psi_2(a), Z_1(a) \ge Z_2(a)$ for all $a \in \Upsilon$.

(2) $\Theta_1 = \Theta_2$ iff $\Theta_1 \subseteq \Theta_2$ and $\Theta_2 \subseteq \Theta_1$.

(3) $\Theta_1 \cup \Theta_2 = \{\langle a, \max(\Xi_1(a), \Xi_2(a)), \min(\Psi_1(a), \Psi_2(a)), \min(Z_1(a), Z_2(a))\rangle \text{ for all } a \in \Upsilon\}$.

(4) $\Theta_1 \cap \Theta_2 = \{\langle a, \min(\Xi_1(a), \Xi_2(a)), \min(\Psi_1(a), \Psi_2(a)), \max(Z_1(a), Z_2(a))\rangle \text{ for all } a \in \Upsilon\}$. For comparison of SPFSs Θ_1 and Θ_2 Mahmood *et al.* [17] defined the score function, accuracy function and comparison rules which are described as follows:

$$\widetilde{SF}(\Theta_1) = \Xi_1^2(a) - Z_1^2(a), \quad \widetilde{SF} \in [-1, 1]. \tag{2}$$

$$\widetilde{AF}(\Theta_1) = \Xi_1^2(a) + \Psi_1^2(a) + Z_1^2(a); \quad \widetilde{AF} \in [0, 1]. \tag{3}$$

Comparison rules for comparing two SPFSs.

i. If $\widetilde{SF}(\Theta_1) > \widetilde{SF}(\Theta_2)$, then Θ_1 is greater to Θ_2 and is denoted by $\Theta_1 > \Theta_2$;
ii. If $\widetilde{SF}(\Theta_1) = \widetilde{SF}(\Theta_2)$ and $\widetilde{AF}(\Theta_1) > \widetilde{AF}(\Theta_2)$, then Θ_1 is greater to Θ_2 and is denoted by $\Theta_1 > \Theta_2$;
iii. If $\widetilde{SF}(\Theta_1) = \widetilde{SF}(\Theta_2)$ and $\widetilde{AF}(\Theta_1) = \widetilde{AF}(\Theta_2)$, then Θ_1 is equal to Θ_2 and is denoted by $\Theta_1 = \Theta_2$.

B. SHORTCOMINGS IN THE OPERATIONS OF SPFSs AND T-SPFSs

The defined partial order for SPFSs by Mahmood *et al.* [17], has some limitations in some situations while comparing two SPFNs. For example let $\Theta_1 = \langle 0.4, 0.2, 0.2\rangle$ and $\Theta_2 = \langle 0.5, 0.4, 0.1\rangle$ be two SPFNs. Then by the partial order defined by Mahmood *et al.* [17] $\Theta_1 \subseteq \Theta_2$. It means that the score value of $\Theta_1 < \Theta_2$. But when we calculate the score values of Θ_1 and Θ_2 using Equation (2), we can have $\widetilde{SF}(\Theta_1) = 0.12$ and $\widetilde{SF}(\Theta_2) = 0.09$. This shows that $\Theta_1 > \Theta_2$. But from the inclusion relation we have $\Theta_1 \subseteq \Theta_2$. Hence the defined inclusion relation fails in this situation. So there is need to improve the inclusion relation defined by Mahmood *et al.* [17].

Further, we discuss some limitations in the operations of SPFNs and T-SPFNs defined by Mahmood *et al.* [17].

Definition 3 [17]: Let $\Theta_1 = \langle \Xi_1, \Psi_1, Z_1\rangle$ and $\Theta_2 = \langle \Xi_2, \Psi_2, Z_2\rangle$ be any two T-SPFNs $m > 0$. Then the operational laws defined by Mahmood *et al.* [17] are as follows:

$$(1)\ \Theta_1 \otimes \Theta_2 = \left\langle (\Xi_1+\Psi_1)(\Xi_2+\Psi_2) - \Psi_1\Psi_2, \Psi_1\Psi_2, \sqrt[m]{1-(1-Z_1^m)(1-Z_2^m)}\right\rangle; \quad (4)$$

$$(2)\ \Theta_1 \oplus \Theta_2 = \left\langle \sqrt[m]{1-(1-\Xi_1^m)(1-\Xi_2^m)}, \Psi_1\Psi_2, (Z_1+\Psi_1)(Z_2+\Psi_2) - \Psi_1\Psi_2\right\rangle; \quad (5)$$

$$(3)\ \alpha\Theta_1 = \left\langle \sqrt[m]{1-(1-\Xi_1^m)^\alpha}, \Psi_1^\alpha, (\Xi_1+\Psi_1)^\alpha - \Psi_1^\alpha\right\rangle; \quad (6)$$

$$(4)\ \Theta_1^\alpha = \left\langle (\Xi_1+\Psi_1)^\alpha - \Psi_1^\alpha, \Psi_1^\alpha, \sqrt[m]{1-(1-Z_1^m)^\alpha}\right\rangle. \quad (7)$$

When $m = 2$, the above operational laws degenerate into the operational laws for SPFNs defined by Mahmood *et al.* [17]. In the above operational rules, there exist some limitations in the multiplication operation which is discussed below.

Let $\Theta_1 = \langle 1, 0, 0\rangle$ and $\Theta_2 = \langle 0.7, 0.5, 0.5\rangle$ be two T-SPFNs or SPFNs and $m = 2$. When we multiply these two T-SPFNs or SPFNs, then we can get $\Theta = \langle 1.2, 0, 0.5\rangle$ and we add the squares of the three functions of the obtained SPFN the value is equal to 1.69, and the result violates the condition for SPFN. Hence Θ is not SPFN. Similarly, for $m = 1, 2, \ldots, n$ the values of the obtained T-SPFNs exceeds from the previous one. Similar limitations exist in the sum operation defined in Equation (5). For example, let $\Theta_1 = \langle 0, 0, 1\rangle$ and $\Theta_2 = \langle 0.5, 0.5, 0.7\rangle$ be two T-SPFNs or SPFNs and $m = 2$. When we add these two T-SPFNs or SPFNs, then we can get $\Theta = \langle 0.5, 0, 1.2\rangle$ and we add the squares of the three functions of the obtained SPFN the value is equal to 1.69, and the result violates the condition for SPFN. Hence Θ is not SPFN. Similarly, for $m = 1, 2, \ldots, n$ the values of the obtained T-SPFNs exceeds from the previous one.

Since SPFSs and T-SPFSs are the generalization of PFSs, PGFSs, q-ROFSs. Motivated by the operational rules defined for PGFNs and q-ROFNs, we propose some novel operational laws for SPFNs and T-SPFNs.

C. NOVEL OPERATIONAL LAWS FOR SPFSs AND T-SPFSs

In this subpart, some novel operational laws for SPFNs and for T-SPFNs are described:

Definition 4: Let $\Theta_1 = \langle \Xi_1, \Psi_1, Z_1\rangle$ and $\Theta_2 = \langle \Xi_2, \Psi_2, Z_2\rangle$ be any two T-SPFNs and $m > 0$. Then the operational laws for T-SPFNs and SPFNs are described as follows:

$$(1)\ \Theta_1 \oplus \Theta_2 = \left\langle \left(\Xi_1^m + \Xi_2^m - \Xi_1^m\Xi_2^m\right)^{\frac{1}{m}}, \Psi_1\Psi_2, Z_1Z_2\right\rangle; \quad (8)$$

$$(2)\ \Theta_1 \otimes \Theta_2 = \left\langle \Xi_1\Xi_2, \left(\Psi_1^m + \Psi_2^m - \Psi_1^m\Psi_2^m\right)^{\frac{1}{m}}, \left(Z_1^m + Z_2^m - Z_1^mZ_2^m\right)^{\frac{1}{m}}\right\rangle; \quad (9)$$

$$(3)\ \alpha\Theta_1 = \left\langle \left(1-\left(1-\Xi_1^m\right)^\alpha\right)^{\frac{1}{m}}, \Psi_1^\alpha, Z_1^\alpha\right\rangle; \quad \alpha > 0 \quad (10)$$

$$(4)\ \Theta_1^\alpha = \left\langle \Xi_1^\alpha, \left(1-\left(1-\Psi_1^m\right)^\alpha\right)^{\frac{1}{m}}, \left(1-\left(1-Z_1^m\right)^\alpha\right)^{\frac{1}{m}}\right\rangle; \quad \alpha > 0 \quad (11)$$

$$(5)\ \Theta_1^c = \langle Z_1, \Psi_1, \Xi_1\rangle \quad (12)$$

If $m = 2$, Then, Equations (8-12) becomes the operational laws for SPFNs.

If we take the above example and use Equations (8) and Equation (9), then we get $\Theta = \langle 0.25, 0, 0.49\rangle$ and $\Theta = \langle 0.49, 0.25, 0.25\rangle$ respectively and are SPFNs or T-SPFNs.

Theorem 1: Assume that $\Theta = \langle \Xi, \Psi, Z\rangle, \Theta_1 = \langle \Xi_1, \Psi_1, Z_1\rangle$ and $\Theta_2 = \langle \Xi_2, \Psi_2, Z_2\rangle$ are three T-SPFNs and $k, k_1, k_2 > 0$, then

$$(1)\ \Theta_1 \oplus \Theta_2 = \Theta_2 \oplus \Theta_1 \quad (13)$$

$$(2)\ \Theta_1 \otimes \Theta_2 = \Theta_2 \otimes \Theta_1 \quad (14)$$

$$(3)\ k(\Theta_1 \oplus \Theta_2) = k\Theta_1 \oplus k\Theta_2 \quad (15)$$

$$(4)\ k_1\Theta \oplus k_2\Theta = (k_1+k_2)\Theta \quad (16)$$

$$(5)\ \Theta^{k_1} \otimes \Theta^{k_2} = \Theta^{k_1+k_2} \quad (17)$$

$$(6)\ \Theta_1^k \otimes \Theta_2^k = (\Theta_1 \otimes \Theta_2)^k \quad (18)$$

Proof: We prove Equation (13), Equation (15) and Equation (17). The proof of other Equation is similar to these Equations.

(1) From Equation (8), we have

$$\begin{aligned}\Theta_1 \oplus \Theta_2 &= \left\langle \left(\Xi_1^m + \Xi_2^m - \Xi_1^m\Xi_2^m\right)^{\frac{1}{m}}, \Psi_1\Psi_2, Z_1Z_2\right\rangle \\ &= \left\langle \left(\Xi_2^m + \Xi_1^m - \Xi_2^m\Xi_1^m\right)^{\frac{1}{m}}, \Psi_2\Psi_1, Z_2Z_1\right\rangle \\ &= \Theta_2 \oplus \Theta_1.\end{aligned}$$

(3) From Equation (10), for the left hand side of Equation (15), we can have

$$\begin{aligned}&k(\Theta_1 \oplus \Theta_2) \\ &= k\left\langle \left(\Xi_1^m + \Xi_2^m - \Xi_1^m\Xi_2^m\right)^{\frac{1}{m}}, \Psi_1\Psi_2, Z_1Z_2\right\rangle \\ &= \left\langle \left(1-\left(1-\left(\Xi_1^m+\Xi_2^m-\Xi_1^m\Xi_2^m\right)\right)^k\right)^{\frac{1}{m}}, (\Psi_1\Psi_2)^k, (Z_1Z_2)^k\right\rangle\end{aligned}$$

Furthermore, we have

$$k\Theta_1 = \left\langle \left(1-\left(1-\Xi_1^m\right)^k\right)^{\frac{1}{m}}, \Psi_1^k, Z_1^k \right\rangle,$$

and

$$k\Theta_2 = \left\langle \left(1-\left(1-\Xi_2^m\right)^k\right)^{\frac{1}{m}}, \Psi_2^k, Z_2^k \right\rangle$$

$$\begin{aligned} k\Theta_1 \oplus k\Theta_2 &= \left\langle \left(\left(\left(1-\left(1-\Xi_1^m\right)^k\right)^{\frac{1}{m}}\right)^m + \left(\left(1-\left(1-\Xi_2^m\right)^k\right)^{\frac{1}{m}}\right)^m \right.\right. \\ &\quad \left. - \left(\left(1-\left(1-\Xi_1^m\right)^k\right)^{\frac{1}{m}}\right)^m \left(\left(1-\left(1-\Xi_2^m\right)^k\right)^{\frac{1}{m}}\right)^m\right)^{\frac{1}{m}}, \\ &\quad \left. (\Psi_1\Psi_2)^k, (Z_1Z_2)^k \right\rangle \\ &= \left\langle \left(\left(1-\left(1-\Xi_1^m\right)^k\right) + \left(1-\left(1-\Xi_2^m\right)^k\right) \right.\right. \\ &\quad \left. - \left(1-\left(1-\Xi_1^m\right)^k\right)\left(1-\left(1-\Xi_2^m\right)^k\right)\right)^{\frac{1}{m}}, \\ &\quad \left. (\Psi_1\Psi_2)^k, (Z_1Z_2)^k \right\rangle \\ &= \left\langle \left(1-\left(1-\left(\Xi_1^m+\Xi_2^m-\Xi_1^m\Xi_2^m\right)\right)^k\right)^{\frac{1}{m}}, \right. \\ &\quad \left. (\Psi_1\Psi_2)^k, (Z_1Z_2)^k \right\rangle \end{aligned}$$

Thus, we can have, $k(\Theta_1 \oplus \Theta_2) = k\Theta_1 \oplus k\Theta_2$.

(5) From Equation (11), we have

$$\Theta^{k_1} = \left\langle \Xi^{k_1}, \left(1-\left(1-\Psi^m\right)^{k_1}\right)^{\frac{1}{m}}, \left(1-\left(1-Z^m\right)^{k_1}\right)^{\frac{1}{m}} \right\rangle,$$

and

$$\Theta^{k_2} = \left\langle \Xi^{k_2}, \left(1-\left(1-\Psi^m\right)^{k_2}\right)^{\frac{1}{m}}, \left(1-\left(1-Z^m\right)^{k_2}\right)^{\frac{1}{m}} \right\rangle,$$

Then, from Equation (9), we have the equation can be derived, as shown at the top of next page

From the right side of Equation (17), we have

$$\begin{aligned} \Theta^{k_1+k_2} &= \left\langle \Xi^{k_1+k_2}, \left(1-\left(1-\Psi^m\right)^{k_1+k_2}\right)^{\frac{1}{m}}, \right. \\ &\quad \left. \left(1-\left(1-Z^m\right)^{k_1+k_2}\right)^{\frac{1}{m}} \right\rangle \\ &= \left\langle \Xi^{k_1}\Xi^{k_2}, \left(1-\left(1-\Psi^m\right)^{k_1}\left(1-\Psi^m\right)^{k_2}\right)^{\frac{1}{m}}, \right. \\ &\quad \left. \left(1-\left(1-Z^m\right)^{k_1}\left(1-Z^m\right)^{k_2}\right)^{\frac{1}{m}} \right\rangle. \end{aligned}$$

Hence we can have, $\Theta^{k_1} \otimes \Theta^{k_2} = \Theta^{k_1+k_2}$.

Definition 5: Assume that $\Theta_1 = \langle \Xi_1, \Psi_1, Z_1 \rangle$ and $\Theta_2 = \langle \Xi_2, \Psi_2, Z_2 \rangle$ are two T-SPFNs. Then the normalized Hamming distance between Θ_1 and Θ_2 is defined as follows:

$$\overline{\overline{D}}(\Theta_1, \Theta_2) = \frac{1}{3}\left(\left|\Xi_1^m - \Xi_2^m\right| + \left|\Psi_1^m - \Psi_2^m\right| + \left|Z_1^m - Z_2^m\right|\right) \tag{19}$$

D. POWER AVERAGE OPERATOR

The PA operator was first introduced by Yager [19] for classical number. The dominant edge of PA operator is its capacity to diminish the inadequate effect of unreasonably high and low arguments on the inconclusive results.

Definition 6 [19]: Let $R_z(z = 1, 2, \ldots, a)$ be a group of classical numbers. Then the PA operator is represented as follows:

$$PA(R_1, R_2, \ldots, R_a) = \sum_{z=1}^{a} \left(\frac{(1+T(R_z))}{\sum_{x=1}^{a}(1+T(R_x))} R_z \right) \tag{20}$$

where $T(R_z) = \sum_{x=1}^{a} Supp(R_z, R_x)$ and $Supp(R_z, R_x)$ are the support degree for R_z and R_x. The support degree must satisfy the following axioms:

(1) $Supp(R_z, R_x) \in [0, 1]$;

(2) $Supp(R_z, R_x) = Supp(R_x, R_z)$;

(3) if $\overline{\overline{D}}(R_z, R_x) < \overline{\overline{D}}(R_l, R_m)$, then $Supp(R_z, R_x) > Supp(R_l, R_m)$, where $\overline{\overline{D}}(R_z, R_x)$ is the distance measure among R_z and R_x.

E. MUIRHEAD MEAN OPERATOR

The MM operator was first introduced by Murihead [23] for classical numbers. MM operator has the advantage of considering the interrelationship among all aggregated arguments.

Definition 7: Let $R_z(z = 1, 2, \ldots, a)$ be a group of classical numbers and $Q = (q_1, q_2, \ldots, q_a) \in \widetilde{R}^a$ be a vector of parameters. Then, the MM operator is explained as

$$MM^Q(R_1, R_2, \ldots, R_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{z=1}^{a} R_{\theta(z)}^{q_z} \right)^{\frac{1}{\sum_{z=1}^{a} q_z}} \tag{21}$$

where, S_a is the group of permutation of $(1, 2, \ldots, a)$ and $\theta(z)$ is any permutation of $(1, 2, \ldots, a)$.

Now we can give some special cases with respect to the parameter vector Q of MM operator. Which are shown as follows:

(1) If $Q = (1, 0, 0, \ldots, 0)$, then MM operator degenerate to the following form:

$$MM^{(1,0,\ldots,0)}(R_1, R_2, \ldots, R_a) = \frac{1}{a}\sum_{z=1}^{a} R_z \tag{22}$$

That is the MM operator degenerate into arithmetic averaging operator.

(2) If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)$, then MM operator degenerate to the following form:

$$MM^{\left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)}(R_1, R_2, \ldots, R_a) = \frac{1}{a}\prod_{z=1}^{a} R_a \tag{23}$$

That is the MM operator degenerate into geometric averaging operator.

$$\Theta^{k_1}\otimes\Theta^{k_2}=\Bigg\langle \Xi^{k_1}\Xi^{k_1},\left(\left(\left(1-(1-\Psi^m)^{k_1}\right)^{\frac{1}{m}}\right)^m+\left(\left(1-(1-\Psi^m)^{k_2}\right)^{\frac{1}{m}}\right)^m-\left(\left(1-(1-\Psi^m)^{k_1}\right)^{\frac{1}{m}}\right)^m\left(\left(1-(1-\Psi^m)^{k_2}\right)^{\frac{1}{m}}\right)^m\right),\left(\left(\left(1-(1-Z^m)^{k_1}\right)^{\frac{1}{m}}\right)^m\left(\left(1-(1-Z^m)^{k_2}\right)^{\frac{1}{m}}\right)^m-\left(\left(1-(1-Z^m)^{k_1}\right)^{\frac{1}{m}}\right)^m\left(\left(1-(1-Z^m)^{k_2}\right)^{\frac{1}{m}}\right)^m\right)\Bigg\rangle$$

$$=\left\langle \Xi^{k_1}\Xi^{k_1},\left(\left(1-(1-\Psi^m)^{k_1}\right)+\left(1-(1-\Psi^m)^{k_2}\right)-\left(1-(1-\Psi^m)^{k_1}\right)\left(1-(1-\Psi^m)^{k_2}\right)\right)^{\frac{1}{m}},\left(\left(1-(1-Z^m)^{k_1}\right)+\left(1-(1-Z^m)^{k_2}\right)-\left(1-(1-Z^m)^{k_1}\right)\left(1-(1-Z^m)^{k_2}\right)\right)^{\frac{1}{m}}\right\rangle$$

$$=\left\langle \Xi^{k_1}\Xi^{k_2},\left(1-(1-\Psi^m)^{k_1}(1-\Psi^m)^{k_2}\right)^{\frac{1}{m}},\left(1-(1-Z^m)^{k_1}(1-Z^m)^{k_2}\right)^{\frac{1}{m}}\right\rangle$$

(3) If $Q=(1,1,0,\ldots,0)$, then MM operator degenerate to the following form:

$$MM^{(1,1,0,\ldots,0)}(R_1,R_2,\ldots,R_a)=\left(\frac{1}{a(a+1)}\sum_{\substack{z,x=1\\z\neq x}}^{a}R_zR_x\right)^{\frac{1}{2}}\tag{24}$$

That is the MM operator degenerates into BM operator.

(4) If $Q=\left(\overbrace{1,1,\ldots,1}^{c},\overbrace{0,\ldots,0}^{a-c}\right)$, then MM operator degenerates to the following form:

$$MM^{\left(\overbrace{1,1,\ldots,1}^{d},\overbrace{0,\ldots,0}^{a-d}\right)}(R_1,R_2,\ldots,R_a)=\left(\frac{\sum_{1\le x_1<x_2<\ldots<x_d\le a}\prod_{y=1}^{d}R_{z_y}}{C_a^d}\right)^{\frac{1}{d}}\tag{25}$$

That is the MM operator degenerate into MSM operator.

III. POWER MUIRHEAD MEAN OPERATOR FOR T-SPFSs

In this part, we first give the definitions of PMM operator and propose the concept of PDMM operator. Then, we extend both the aggregation operators to SVN environment.

Definition 8 [27]: Let $R_z(z=1,2,\ldots,a)$ be a group of classical numbers and $Q=(q_1,q_2,\ldots,q_a)\in\widetilde{R}^a$ be a vector of parameters. Then, the PMM operator is defined as

$$PMM^Q(R_1,R_2,\ldots,R_a)=\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(\frac{a\left(1+T\left(R_{\theta(z)}\right)\right)}{\sum_{x=1}^{a}(1+T(R_x))}R_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}\tag{26}$$

where $T(R_z)=\sum_{\substack{x=1\\x\neq z}}^{a}Supp(R_z,R_x)$ and $Supp(R_z,R_x)$ is the support degree for R_z and R_z, satisfying the above conditions.

Definition 9: Let $R_z(z=1,2,\ldots,a)$ be a group of classical numbers and $Q=(q_1,q_2,\ldots,q_a)\in\widetilde{R}^a$ be a vector of parameters. Then, the PDMM operator is given as

$$PDMM^Q(R_1,R_2,\ldots,R_a)=\frac{1}{\sum_{z=1}^{a}q_z}\left(\sum_{\theta\in S_a}\prod_{z=1}^{a}q_zR_{\theta(z)}^{\frac{a\left(1+T\left(R_{\theta(z)}\right)\right)}{\sum_{x=1}^{a}(1+T(R_x))}}\right)^{\frac{1}{a!}}\tag{27}$$

where $T(R_z)=\sum_{\substack{x=1\\x\neq z}}^{a}Supp(R_z,R_x)$ and $Supp(R_z,R_x)$ is the support degree for R_z and R_z, satisfying the above conditions.

A. THE T-SPFNPMM OPERATOR

Definition 10: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs and $Q=(q_1,q_2,\ldots,q_a)\in\widetilde{R}^a$ be a vector of parameters. If

$$T-SPFPMM^Q(\Theta_1,\Theta_2,\ldots,\Theta_a)=\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(\frac{a\left(1+T\left(\Theta_{\theta(z)}\right)\right)}{\sum_{x=1}^{a}(1+T(\Theta_x))}\Theta_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}\tag{28}$$

Then, we call $SPFPMM^Q$ the T-spherical fuzzy power Murihead mean operator, where S_a is the group of all permutation, $\theta(z)$ is any permutation of $(1,2,\ldots,a)$ and $T(\Theta_x)=\sum_{\substack{x=1\\x\neq z}}^{a}Supp(\Theta_z,\Theta_x)$, $Supp(\Theta_z,\Theta_x)$ is the support degree for Θ_z and Θ_x, satisfying the following axioms:

(1) $Supp(\Theta_z,\Theta_x)\in[0,1]$;

(2) $Supp(\Theta_z,\Theta_x)=Supp(\Theta_x,\Theta_z)$;

(3) If $\overline{\overline{D}}(\Theta_z, \Theta_x) < \overline{\overline{D}}(\Theta_u, \Theta_v)$, then $Supp(\Theta_z, \Theta_x) > Supp(\Theta_u, \Theta_v)$, where $\overline{\overline{D}}(\Theta_z, \Theta_x)$ is distance among Θ_z and Θ_x.

To write Equation (28) in a simple form, we can specify it as

$$\Gamma_z = \frac{(1 + T(\Theta_z))}{\sum_{x=1}^{a} (1 + T(\Theta_x))} \tag{29}$$

For suitability, we can call $(\Gamma_1, \Gamma_2, \ldots, \Gamma_a)^T$ the power weight vector (PMV), such that $\Gamma_z \in [0, 1]$ and $\sum_{Z=1}^{A} \Gamma_z = 1$. From the use of Equation (29), Equation (28) can be expressed as

$$T - SPFPMM^Q(\Theta_1, \Theta_2, \ldots, \Theta_a) = \left(\frac{1}{a!} \sum_{\theta \in S_a} \prod_{z=1}^{a} (a\Gamma_z \Theta_{\theta(z)})^{q_z} \right)^{\frac{1}{\sum_{z=1}^{a} q_z}} \tag{30}$$

Based on the operational rules given in Definition 4 for T-SPFNs, and Definition 10, we can have the following result.

Theorem 2: Let $\Theta_z (z = 1, 2, \ldots, a)$ be a group of T-SPFNNs and $Q = (q_1, q_2, \ldots, q_a) \in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (16) is still a SPFNN and (31), as shown at the top of the next page

Proof: According to operational laws for T-SPFNs, we have

$$a\Gamma_z \Theta_{\theta(z)} = \left\langle \sqrt[m]{1 - \left(1 - \Xi^m_{\theta(z)}\right)^{a\Gamma_z}}, \Psi^{ma\Gamma_z}_{\theta(z)}, Z^{ma\Gamma_z}_{\theta(z)} \right\rangle,$$

Therefore,

$$(a\Gamma_z \Theta_{\theta(z)})^{q_z} = \left\langle \left(\sqrt[m]{1 - \left(1 - \Xi^m_{\theta(z)}\right)^{a\Gamma_z}} \right)^{q_z}, \sqrt[m]{1 - \left(1 - \Psi^{ma\Gamma_z}_{\theta(z)}\right)^{q_z}}, \sqrt[m]{1 - \left(1 - Z^{ma\Gamma_z}_{\theta(z)}\right)^{q_z}} \right\rangle,$$

So,

$$\prod_{z=1}^{a} (a\Gamma_z \Theta_{\theta(z)})^{q_z} = \left\langle \prod_{z=1}^{a} \left(\sqrt[m]{1 - \left(1 - \Xi^m_{\theta(z)}\right)^{a\Gamma_z}} \right)^{q_z}, \sqrt[m]{1 - \prod_{z=1}^{a} \left(1 - \Psi^{ma\Gamma_z}_{\theta(z)}\right)^{q_z}}, \sqrt[m]{1 - \prod_{z=1}^{a} \left(1 - Z^{ma\Gamma_z}_{\theta(z)}\right)^{q_z}} \right\rangle,$$

and

$$\sum_{\theta \in S_a} \prod_{z=1}^{a} (a\Gamma_z \Theta_{\theta(z)})^{q_z}$$

$$= \left\langle \sqrt[m]{1 - \prod_{\theta \in S_a} \left(\prod_{z=1}^{a} \left(1 - \left(1 - \Xi^m_{\theta(z)}\right)^{a\Gamma_z}\right)^{q_z} \right)}, \prod_{\theta \in S_a} \left(\sqrt[m]{1 - \prod_{z=1}^{a} \left(1 - \Psi^{ma\Gamma_z}_{\theta(z)}\right)^{q_z}} \right), \prod_{\theta \in S_a} \left(\sqrt[m]{1 - \prod_{z=1}^{a} \left(1 - Z^{ma\Gamma_z}_{\theta(z)}\right)^{q_z}} \right) \right\rangle,$$

Furthermore,

$$\frac{1}{a!} \sum_{\theta \in S_a} \prod_{z=1}^{a} (a\Gamma_z \Theta_{\theta(z)})^{q_z}$$

$$= \left\langle \sqrt[m]{1 - \left(\prod_{\theta \in S_a} \left(1 - \prod_{z=1}^{a} \left(1 - \left(1 - \Xi^m_{\theta(z)}\right)^{a\Gamma_z}\right)^{q_z} \right) \right)^{\frac{1}{a!}}}, \left(\prod_{\theta \in S_a} \left(\sqrt[m]{1 - \prod_{z=1}^{a} \left(1 - \Psi^{ma\Theta_z}_{\theta(z)}\right)^{q_z}} \right) \right)^{\frac{1}{a!}}, \left(\prod_{\theta \in S_a} \left(\sqrt[m]{1 - \prod_{z=1}^{a} \left(1 - Z^{ma\Theta_z}_{\theta(z)}\right)^{q_z}} \right) \right)^{\frac{1}{a!}} \right\rangle$$

Hence, the equation can be derived, as shown at the top of next page.

This is the required proof of Theorem 2.

In the above equations, we calculate the PWV Γ, we first have to calculate the support degree $Supp(\Theta_z, \Theta_x)$. According to the Equation (19), we can get $Supp(\Theta_z, \Theta_x)$ utilizing

$$Supp(\Theta_z, \Theta_x) = 1 - \overline{\overline{D}}(\Theta_z, \Theta_x) \tag{32}$$

Therefore, we use the equation

$$T(\Theta_z) = \sum_{\substack{z=1 \\ z \neq x}}^{a} Supp(\Theta_z, \Theta_x) \tag{33}$$

To determine, $T(\Theta_z)$ $(z = 1, 2 \ldots, a)$. Then according to Equation (29) we can get the PWV.

Theorem 3 (Idempotency): Let $\Theta_z (z = 1, 2, \ldots, a)$ be a group of T-SPFNs, and $\Theta_z = \Theta$ for all $z = 1, 2, \ldots, a$. Then

$$T - SPFPMM^Q(\Theta_1, \Theta_2, \ldots, \Theta_a) = \Theta \tag{34}$$

Proof: As $\Theta_z = \Theta$ for all $z = 1, 2, \ldots, a$, we have $Supp(\Theta_z, \Theta_x) = 1$ for all $z, x = 1, 2, \ldots, a$. Therefore, we can get $\Gamma_z = \frac{1}{a}$ for all z. Moreover, the equation can be derived, as shown at the top of next page, Which is the require proof of Theorem 3.

Theorem 4 (Boundedness): Let $\Theta_z (z = 1, 2, \ldots, a)$ be a group of T-SPFNs, $\overset{-}{\Theta} = \min(\Theta_1, \Theta_2, \ldots, \Theta_a) = (\Xi^-, \Psi^+, Z^+)$ and $\overset{+}{\Theta} = \max(\Theta_1, \Theta_2, \ldots, \Theta_a) = (\Xi^+, \Psi^-, Z^-)$. Then

$$u \leq T - SPFPMM^Q(\Theta_1, \Theta_2, \ldots, \Theta_a) \leq v \tag{35}$$

$$T-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)=\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.$$
$$\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},$$
$$\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle. \quad (31)$$

$$\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}$$
$$=\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.$$
$$\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle.$$

$$T-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)$$
$$=\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.$$
$$\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle.$$

where, u, as shown at the top of next page, and v, as shown at the top of next page,

Proof: Since

$$a\Gamma_z\Theta_{\theta(z)}=\left\langle\sqrt[m]{1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}},\Psi_{\theta(z)}^{ma\Gamma_z},Z_{\theta(z)}^{ma\Gamma_z}\right\rangle$$
$$\geq\left\langle\sqrt[m]{1-\left(1-\overset{m}{\underset{\theta(z)}{\Xi}}\right)^{a\Gamma_z}},\overset{+}{\Psi}{}_{\theta(z)}^{ma\Gamma_z},\overset{+}{Z}{}_{\theta(z)}^{ma\Gamma_z}\right\rangle,$$

Therefore, $\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}$, as shown at the top the page 9, So, $\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}$, as shown at the top the page 9, and $\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}$, as shown at the top the page 9, Furthermore, $\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}$, as shown at the top the page 9,

Hence, $\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}$, as shown at the top the page 10.

This implies that $u\leq T-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)$.

In a similar way we can also show that $T-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)\leq v$.. So $u\leq T-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)\leq v$.

T-SPFPMM operator does not have the property of monotonicity.

$$
\begin{aligned}
&T-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)\\
&\quad= T-SPFPMM^{Q}\left(\Theta,\Theta,\ldots,\Theta\right)\\
&\quad=\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi^m\right)^{a\frac{1}{a}}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.\\
&\qquad\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Psi^{m\left(a\frac{1}{a}\right)}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-Z^{m\left(a\frac{1}{a}\right)}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle\\
&\quad=\left\langle\sqrt[m]{\left(1-\left(\left(1-\left(1-\left(1-\Xi^m\right)\right)^{\sum_{z=1}^{a}q_z}\right)^{a!}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\right.\\
&\qquad\left.\sqrt[m]{1-\left(1-\left(1-\left(1-\Psi^m\right)^{\sum_{z=1}^{a}q_z}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\sqrt[m]{1-\left(1-\left(1-\left(1-Z^m\right)^{\sum_{z=1}^{a}q_z}\right)^{a!\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle\\
&\quad=\left\langle\sqrt[m]{\Xi^m},\sqrt[m]{\Psi^m},\sqrt[m]{Z^m}\right\rangle=\langle\Xi,\Psi,Z\rangle=\Theta.
\end{aligned}
$$

$$
\begin{aligned}
u=&\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.\\
&\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{+}{\Psi}{}_{\theta(z)}^{ma\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{-}{Z}{}_{\theta(z)}^{ma\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
v=&\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{+}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.\\
&\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{-}{\Psi}{}_{\theta(z)}^{ma\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{-}{Z}{}_{\theta(z)}^{ma\Theta_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle,
\end{aligned}
$$

$$
\begin{aligned}
\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}&=\left\langle\left(\sqrt[m]{1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}}\right)^{q_z},\sqrt[m]{1-\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}},\sqrt[m]{1-\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right\rangle\\
&\geq\left\langle\left(\sqrt[m]{1-\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}}\right)^{q_z},\sqrt[m]{1-\left(1-\overset{+}{\Psi}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}},\sqrt[m]{1-\left(1-\overset{+}{Z}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right\rangle,
\end{aligned}
$$

$$\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z} = \left\langle \prod_{z=1}^{a}\left(\sqrt[m]{1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}}\right)^{q_z}, \sqrt[m]{1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}, \sqrt[m]{1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right\rangle$$
$$\geq \left\langle \prod_{z=1}^{a}\left(\sqrt[m]{1-\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}}\right)^{q_z}, \sqrt[m]{1-\prod_{z=1}^{a}\left(1-\overset{+}{\Psi}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}, \sqrt[m]{1-\prod_{z=1}^{a}\left(1-\overset{-}{Z}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right\rangle,$$

$$\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z} = \left\langle \sqrt[m]{1-\prod_{\theta\in S_a}\left(\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)}, \prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right),\right.$$
$$\left.\prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right)\right\rangle \geq \left\langle \sqrt[m]{1-\prod_{\theta\in S_a}\left(\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)},\right.$$
$$\left.\prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-\overset{+}{\Psi}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right), \prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-\overset{+}{Z}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}}\right)\right\rangle,$$

$$\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z} = \left\langle \sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}},\right.$$
$$\left.\left(\prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Theta_z}\right)^{q_z}}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Theta_z}\right)^{q_z}}\right)\right)^{\frac{1}{a!}}\right\rangle$$
$$\geq \left\langle \sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}},\right.$$
$$\left.\left(\prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-\overset{+}{\Psi}{}_{\theta(z)}^{ma\Theta_z}\right)^{q_z}}\right)\right)^{\frac{1}{a!}}, \left(\prod_{\theta\in S_a}\left(\sqrt[m]{1-\prod_{z=1}^{a}\left(1-\overset{+}{Z}{}_{\theta(z)}^{ma\Theta_z}\right)^{q_z}}\right)\right)^{\frac{1}{a!}}\right\rangle,$$

One of the leading advantage of T-SPFPMM is its capacity to represent the interrelationship among T-SPFNs. Furthermore, T-SPFPMM operator is more flexible in aggregation process due to parameter vector. Now we will discuss some special cases of T-SPFPMM operators by assigning different values to the parameter vector.

Case 1: If $Q = (1, 0, \ldots, 0)$, then T-SPFPMM operators degenerate into the following form:

$$T-SPFPMM^{(1,0,\ldots,0)}\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right) = \left(\sum_{z=1}^{a}\frac{\left(1+T\left(\Theta_z\right)\right)}{\sum_{x=1}^{a}\left(1+T\left(\Theta_x\right)\right)}\Theta_z\right) \quad (36)$$

This is the SPF power averaging operator.

Case 2: If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)$, then T-SPFPMM operators degenerate into the following form:

$$T-SPFPMM^{\left(\frac{1}{a},\frac{1}{a},\ldots,\frac{1}{a}\right)}\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right) = \prod_{z=1}^{a}\Theta_z^{\frac{\left(1+T\left(\Theta_{\theta(z)}\right)\right)}{\sum_{x=1}^{a}\left(1+T\left(\Theta_x\right)\right)}} \quad (37)$$

This is T-SPF power geometric operator.

Case 3: If $Q = (1, 1, \ldots, 0)$, then T-SPFPMM operators degenerate into the following form (38), as shown at the top of the next page:

This is the T-SPF power Bonferroni mean operator $(p = q = 1)$.

$$\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(a\Gamma_z\Theta_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}$$
$$=\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.$$
$$\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\ \sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle$$
$$\geq\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\ \sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{+}{\Psi}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\right.$$
$$\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{+}{Z}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle.$$

$$T-SPFPMM^{(1,1,0,\ldots,0)}=(\Theta_1,\Theta_2,\ldots,\Theta_a)=\left\langle\left(1-\left(\prod_{\substack{z,x=1\\z\neq x}}^{a}\left(1-\left(1-\left(1-\Xi_z^{m}\right)^{\Gamma_z}\right)\left(1-\left(1-\Xi_x^{m}\right)^{\Gamma_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2m}},\right.$$
$$1-\left(1-\left(\prod_{\substack{z,x=1\\z\neq x}}^{a}\left(1-\left(1-\Psi_z^{m\Gamma_z}\right)\left(1-\Psi_x^{m\Gamma_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2m}},$$
$$\left.1-\left(1-\left(\prod_{\substack{z,x=1\\z\neq x}}^{a}\left(1-\left(1-Z_z^{m\Gamma_z}\right)\left(1-Z_x^{m\Gamma_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2m}}\right\rangle. \tag{38}$$

Case 4: If $Q=\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)$, then T-SPFPMM operators degenerate into the following form (39), as shown at the top of the next page: This is the T-SPF power Maclaurin symmetric mean operator.

IV. WEIGHTED T-SPHERICAL FUZZY POWER MURIHEAD MEAN (WSPFPMM) OPERATOR

The T-SPFPMM operator does not consider the weight of the aggregated T-SPFNs. In this subpart, we develop the weighted T-spherical fuzzy power Murihead mean (WT-SPFPMM) operator, which has the capacity of taking the weights of T-SPFNs.

Definition 11: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. If

$$WT-SPFPMM^{Q}(\Theta_1,\Theta_2,\ldots,\Theta_a)$$
$$=\left(\frac{1}{a!}\sum_{\theta\in S_a}\prod_{z=1}^{a}\left(\frac{a\Upsilon_{\vartheta(z)}\Gamma_{\theta(z)}}{\sum_{x=1}^{a}\Upsilon_x\Gamma_x}\Theta_{\theta(z)}\right)^{q_z}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}} \tag{40}$$

$$T-SPFPMM^{\left(\overbrace{1,1,..,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)}=(\Gamma_1,\Gamma_2,\ldots,\Gamma_a)$$
$$=\left\langle\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-\Xi_{z_x}^{m}\right)^{\Theta_{z_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{mk}},\right.$$
$$1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\Psi_{z_x}^{m\Theta_{z_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{mk}},$$
$$\left.1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-Z_{z_x}^{\Theta_{z_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{mk}}\right\rangle. \tag{39}$$

Then, we call $T-WSPFPMM^{Q}$ the weighted T-spherical fuzzy power Murihead mean operator, where $\Upsilon=(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_a)^T$ is the weight vector of Θ_z ($z=1,2,\ldots,a$) such that $\Upsilon_z\in[0,1]$, $\sum_{z=1}^{a}\Upsilon_z=1$, S_a is the group of all permutation, $\theta(z)$ is any permutation of $(1,2,\ldots,a)$ and Θ_z is PVW satisfying $\Gamma_z=(1+T(\Theta_z))/\sum_{z=1}^{a}(1+T(\Theta_z))$, $T(\Theta_x)=\sum_{x=1,x\ne z}^{a}Supp(\Theta_z,\Theta_x)$, $Supp(\Theta_z,\Theta_x)$ is the support degree for Θ_z and Θ_x, satisfying the following axioms:

(1) $Supp(\Theta_z,\Theta_x)\in[0,1]$;

(2) $Supp(\Theta_z,\Theta_x)=Supp(\Theta_x,\Theta_z)$;

(3) If $\overline{\overline{D}}(\Theta_z,\Theta_x)<\overline{\overline{D}}(\Theta_u,\Theta_v)$, then $Supp(\Theta_z,\Theta_x)>Supp(\Theta_u,\Theta_v)$, where $\overline{\overline{D}}(\Theta_z,\Theta_x)$ is distance among Θ_z and Θ_x.

From Definition 11, we have the following Theorem 5.

Theorem 4: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (40) is still a T-SPFN and (41), as shown at the top of the next page.

Proof: Proof of Theorem 5 is same as Theorem 2.

A. THE T-SPHERICAL FUZZY POWER DUAL MURIHEAD MEAN (SPFPDMM) OPERATOR

In this subpart, we develop the T-SPFPDMM operator and discuss some related properties.

Definition 12: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. If

$$T-SPFPDMM^{Q}(\Theta_1,\Theta_2,\ldots,\Theta_a)=\frac{1}{\sum_{z=1}^{a}q_z}\left(\prod_{\theta\in S_a}\sum_{z=1}^{a}\left(q_z\Theta_{\theta(z)}^{\frac{a\left(1+T\left(\Theta_{\theta(z)}\right)\right)}{\sum_{x=1}^{a}(1+T(\Theta_x))}}\right)\right)^{\frac{1}{a!}} \tag{42}$$

Then, we call $T-SPFPDMM^{Q}$ the T-spherical fuzzy power dual Murihead mean operator, where S_a is the group of all permutation, $\theta(z)$ is any permutation of $(1,2,\ldots,a)$ and $T(\Theta_x)=\sum_{x=1,x\ne z}^{a}Supp(\Theta_z,\Theta_x)$, $Supp(\Theta_z,\Theta_x)$ is the support degree for Θ_z and Θ_x, satisfying the following axioms:

(1) $Supp(\Theta_z,\Theta_x)\in[0,1]$;

(2) $Supp(\Theta_z,\Theta_x)=Supp(\Theta_x,\Theta_z)$;

(3) If $\overline{\overline{D}}(\Theta_z,\Theta_x)<\overline{\overline{D}}(\Theta_u,\Theta_v)$, then $Supp(\Theta_z,\Theta_x)>Supp(\Theta_u,\Theta_v)$, where $\overline{\overline{D}}(\Theta_z,\Theta_x)$ is distance among Θ_z and Θ_x.

To write Equation (42) in a simple form, we can specify it as

$$\Gamma_z=\frac{(1+T(\Theta_z))}{\sum_{x=1}^{a}(1+T(\Theta_x))} \tag{43}$$

For suitability, we can call $(\Gamma_1,\Gamma_2,\ldots,\Gamma_a)^T$ the power weight vector (PMV), such that $\Gamma_z\in[0,1]$ and $\sum_{z=1}^{a}\Gamma_z=1$. From the use of Equation (43), Equation (42) can be expressed as

$$T-SPFPDMM^{Q}(\Theta_1,\Theta_2,\ldots,\Theta_a)=\frac{1}{\sum_{z=1}^{a}q_z}\left(\prod_{\theta\in S_a}\sum_{z=1}^{a}\left(q_z\Theta_{\theta(z)}^{a\Gamma_{\theta(z)}}\right)\right)^{\frac{1}{a!}} \tag{44}$$

Theorem 6: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (42) is still a T-SPFN and (45), as shown at the top of the next page.

Proof: Same as Theorem 1.

Theorem 7 (Idempotency): Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs, and $\Theta_z=\Theta$ for all $z=1,2,\ldots,a$. Then

$$T-SPFPDMM^{Q}(\Theta_1,\Theta_2,\ldots,\Theta_a)=\Theta. \tag{46}$$

$$
\begin{aligned}
&WT-SPFPMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)\\
&=\left\langle\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Xi_{\theta(z)}^{m}\right)^{\frac{a\Gamma_{\theta(z)}\Upsilon_{\vartheta(z)}}{\sum_{x=1}^{a}\Gamma_x\Upsilon_x}}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.\\
&\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Psi_{\theta(z)}^{m\frac{a\Gamma_{\theta(z)}\Upsilon_{\vartheta(z)}}{\sum_{x=1}^{a}\Gamma_x\Upsilon_x}}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\\
&\left.\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-Z_{\theta(z)}^{m\frac{a\Gamma_{\theta(z)}\Upsilon_{\vartheta(z)}}{\sum_{x=1}^{a}\Gamma_x\Upsilon_x}}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}}\right\rangle. \qquad (41)
\end{aligned}
$$

$$
\begin{aligned}
&T-SPFPDMM^{Q}\left(\Theta_1,\Theta_2,\ldots,\Theta_a\right)\\
&=\left\langle\sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Xi_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\right.\\
&\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Psi_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\\
&\left.\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-Z_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}\right\rangle \qquad (45)
\end{aligned}
$$

Theorem 8 (Boundedness): Let $\Theta_z (z = 1, 2, \ldots, a)$ be a group of T-SPFNs, $\overset{-}{\Theta} = \min\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right) = \left(\Xi^-, \Psi^+, Z^+\right)$ and $\overset{+}{\Theta} = \max\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right) = \left(\Xi^+, \Psi^-, Z^-\right)$. Then

$$u \leq SPFPDMM^{Q}\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right) \leq v \qquad (47)$$

where u, as shown at the top of next page, and v, as shown at the top of next page,

Now we will discuss some special cases of T-SPFPDMM operator with respect to the parameter vector Q.

Case 1: If $Q = (1, 0, \ldots, 0)$, then T-SPFPDMM operators degenerate into the following form:

$$
\begin{aligned}
&T-SPFPMM^{(1,0,\ldots,0)}\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right)\\
&=\left(\prod_{z=1}^{a}\Theta_z^{\frac{(1+T(\Theta_z))}{\sum_{x=1}^{a}(1+T(\Theta_x))}}\right) \qquad (48)
\end{aligned}
$$

This is the T-SPF power geometric averaging operator.

Case 2: If $Q = \left(\frac{1}{a}, \frac{1}{a}, \ldots, \frac{1}{a}\right)$, then T-SPFPMM operators degenerate into the following form:

$$
\begin{aligned}
&T-SPFPMM^{\left(\frac{1}{a},\frac{1}{a},\ldots,\frac{1}{a}\right)}\left(\Theta_1, \Theta_2, \ldots, \Theta_a\right)\\
&=\sum_{z=1}^{a}\frac{\left(1+T\left(\Theta_z\right)\right)}{\sum_{x=1}^{a}\left(1+T\left(\Theta_x\right)\right)}\Theta_z \qquad (49)
\end{aligned}
$$

This is T-SPF power arithmetic averaging operator.

$$u=\left\langle \sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{-}{\Xi}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{m}{\underset{\theta(z)}{\Psi}}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.$$
$$\left.\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{Z}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}\right\rangle.$$

$$v=\left\langle \sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\overset{+}{\Xi}{}_{\theta(z)}^{ma\Gamma_z}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{\Psi}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},\right.$$
$$\left.\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\overset{-}{Z}{}_{\theta(z)}^{m}\right)^{a\Gamma_z}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}\right\rangle.$$

$$\begin{aligned}T-SPFPDMM^{(1,1,0,\ldots,0)}&=(\Theta_1,\Theta_2,\ldots,\Theta_a)\\&=\left\langle 1-\left(1-\left(\prod_{\substack{z,x=1\\z\neq x}}^{a}\left(1-\left(1-\Xi_z^{m\Gamma_z}\right)\left(1-\Xi_x^{m\Gamma_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2m}},\right.\\&\quad\left(1-\left(\prod_{\substack{z,x=1\\z\neq x}}^{a}\left(1-\left(1-\left(1-\Psi_z^{m}\right)^{\Theta_z}\right)\left(1-\left(1-\Psi_x^{m}\right)^{\Gamma_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2m}}\\&\quad\left.\left(1-\left(\prod_{\substack{z,x=1\\z\neq x}}^{a}\left(1-\left(1-\left(1-Z_z^{m}\right)^{\Gamma_z}\right)\left(1-\left(1-Z_x^{m}\right)^{\Gamma_x}\right)\right)\right)^{\frac{1}{a^2-a}}\right)^{\frac{1}{2m}}\right\rangle.\end{aligned}\tag{50}$$

Case 3: If $Q=(1,1,0,\ldots,0)$, then T-SPFPDMM operators degenerate into the following form (50), as shown at the top of this page:
This is the T-SFP power geometric Bonferroni mean operator $(p=q=1)$.

Case 4: If $Q=\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)$, then T-SPFPDMM operators degenerate into the following form (51), as shown at the top of the next page.
This is the T-SPF power Dual Maclaurin symmetric mean operator.

B. WEIGHTED T-SPHERICAL FUZZY POWER DUAL MURIHEAD MEAN (WSPFPDMM) OPERATOR

The T-SPFPDMM operator does not consider the weight of the aggregated T-SPFNs. In this subpart, we develop the weighted spherical fuzzy power dual Murihead mean (WSPF-PDMM) operator, which has the capacity of taking the weights of T-SPFNs.

Definition 13: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. If

$$WT-SPFPDMM^{Q}(\Theta_1,\Theta_2,\ldots,\Theta_a)$$

$$T-SPFPDMM^{\left(\overbrace{1,1,\ldots,1}^{i},\overbrace{0,0,\ldots,0}^{z-i}\right)}(\Theta_1,\Theta_2,\ldots,\Theta_a)$$
$$=\left\langle\left(1-\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\Xi_{z_x}^{m\Gamma_{z_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{k}}\right)^{\frac{1}{m}},\right.$$
$$\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-\Psi_{z_x}^{m}\right)^{\Gamma_{z_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{mk}},$$
$$\left.\left(1-\prod_{1\le y_1<y_2<\ldots<y_i\le a}\left(1-\prod_{x=1}^{i}\left(1-\left(1-Z_{z_x}^{m}\right)^{\Gamma_{z_x}}\right)\right)^{\frac{1}{C_a^i}}\right)^{\frac{1}{mk}}\right\rangle. \quad (51)$$

$$=\frac{1}{\sum_{z=1}^{a}q_z}\left(\prod_{\theta\in S_a}\sum_{z=1}^{a}\left(q_z\Theta_{\theta(z)}^{\frac{a\Upsilon_{\vartheta(z)}\Gamma_{\theta(z)}}{\sum_{x=1}^{a}\Upsilon_x\Gamma_x}}\right)\right)^{\frac{1}{a!}} \quad (52)$$

Then, we call $WT-SPFPDMM^{Q}$ the weighted T-spherical fuzzy power dual Murihead mean operator, where $\Upsilon=(\Upsilon_1,\Upsilon_2,\ldots,\Upsilon_a)^T$ is the weight vector of Θ_z $(z=1,2,\ldots,a)$ such that $\Upsilon_z\in[0,1]$, $\sum_{z=1}^{a}\Upsilon_z=1$, S_a is the group of all permutation, $\theta(z)$ is any permutation of $(1,2,\ldots,a)$ and Θ_z is PVW satisfying

$$\Gamma_z=(1+T(\Theta_z))/\sum_{z=1}^{a}(1+T(\Theta_z)),\quad \sum_{z=1}^{a}\Gamma_z=1,$$
$$T(\Theta_x)=\sum_{x=1,x\neq z}^{a}Supp(\Theta_z,\Theta_x),$$

$Supp(\Theta_z,\Theta_x)$ is the support degree for Θ_z and Θ_x, satisfying the following axioms:

(1) $Supp(\Theta_z,\Theta_x)\in[0,1]$;

(2) $Supp(\Theta_z,\Theta_x)=Supp(\Theta_x,\Theta_z)$;

(3) If $\overline{\overline{D}}(\Theta_z,\Theta_x)<\overline{\overline{D}}(\Theta_u,\Theta_v)$, then $Supp(\Theta_z,\Theta_x)>Supp(\Theta_u,\Theta_v)$, where $\overline{\overline{D}}(\Theta_z,\Theta_x)$ is distance among Θ_z and Θ_x.

From Definition 13, we have the following Theorem 9.

Theorem 8: Let $\Theta_z(z=1,2,\ldots,a)$ be a group of T-SPFNs and $Q=(q_1,q_2,\ldots,q_a)\in R^a$ be a vector of parameters. Then, the aggregated value obtained by using Equation (52) is still a T-SPFN and (53), as shown at the top of the next page.

Proof: Proof of Theorem 9 is same as Theorem 2.

V. THE MAGDM METHOD BASED ON WT-SPFPMM AND WT-SPFPDMM OPERATORS

In this part, an innovative method to MAGDM with T-SPFNs is introduced, in which the weights of the expert's and attributes are known. Let the set of alternatives and attributes be respectively, expressed as $\hbar=\{\hbar_1,\hbar_2,\ldots,\hbar_a\}$, $\lambda=\{\lambda_1,\lambda_2,\ldots,\lambda_b\}$ and the set of experts is represented by $\mathbb{Z}=\{\mathbb{Z}_1,\mathbb{Z}_2,\ldots,\mathbb{Z}_c\}$. Suppose that the assessment value for the alternative $\hbar_g$ given by the expert Z_k about the attribute λ_h is expressed by the form $\Theta_{gh}^k=\left\langle\Xi_{gh}^k,\Psi_{gh}^k,Z_{gh}^k\right\rangle$. The weight vector of the attributes $\{\lambda_1,\lambda_2,\ldots,\lambda_b\}$ is denoted by $\varpi=(\varpi_1,\varpi_2,\ldots,\varpi_b)^T$ such that $\varpi_h\in[0,1]$, $\sum_{h=1}^{b}\varpi_h=1$. $\Lambda=(\Lambda_1,\Lambda_2,\ldots,\Lambda_c)^T$ represent the weight vector of the expert such that $\Lambda_k\in[0,1]$, $\sum_{k=1}^{c}\Lambda_k=1$. Then the aim of this MAGDM problem is to rank the alternatives. To do the following steps are to followed.

Step 1: Standardize the decision matrix. Generally, there are two types of attributes, one is of cost type and the other is of benefit type. We need to convert the cost type of attributes into benefit types of attributes by utilizing the following formula:

$$\Theta_{gh}^k=\left\langle\Xi_{gh}^k,\Psi_{gh}^k,Z_{gh}^k\right\rangle=\begin{cases}\left\langle\Xi_{gh}^k,\Psi_{gh}^k,Z_{gh}^k\right\rangle & \text{for benefit attribute }\Theta_h\\ \left\langle Z_{gh}^k,\Psi_{gh}^k,\Xi_{gh}^k\right\rangle & \text{for cost attribute }\Theta_h\end{cases} \quad (54)$$

Hence the decision matrix $M=\left[\Theta_{gh}^k\right]_{a\times b}$ can be transformed to matrix $N=\left[\delta_{gh}^k\right]_{a\times b}$.

Step 2: Determine the supports

$$Supp\left(\delta_{gh}^k,\delta_{gl}^k\right)\quad(1,2,\ldots,a;\; h,l=1,2,\ldots,b,k=1,2,\ldots,c)$$

by

$$Supp\left(\delta_{gh}^k,\delta_{gl}^k\right)=1-\overline{\overline{D}}\left(\delta_{gh}^k,\delta_{gl}^k\right) \quad (55)$$

$$WT-PFPDMM^{Q}(\Theta_1,\Theta_2,\ldots,\Theta_a)$$
$$=\left\langle \sqrt[m]{1-\left(1-\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\Xi_{\theta(z)}^{m\frac{a\Gamma_{\theta(z)}\Upsilon_{\vartheta(z)}}{\sum_{x=1}^{a}\Gamma_x\Upsilon_x}}\right)^{q_z}\right)^{\frac{1}{a!}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}},\right.$$
$$\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-\Psi_{\theta(z)}^{m}\right)^{\frac{a\Gamma_{\theta(z)}\Upsilon_{\vartheta(z)}}{\sum_{x=1}^{a}\Gamma_x\Upsilon_x}}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}},$$
$$\left.\left(\sqrt[m]{1-\left(\prod_{\theta\in S_a}\left(1-\prod_{z=1}^{a}\left(1-\left(1-Z_{\theta(z)}^{m}\right)^{\frac{a\Gamma_{\theta(z)}\Upsilon_{\vartheta(z)}}{\sum_{x=1}^{a}\Gamma_x\Upsilon_x}}\right)^{q_z}\right)\right)^{\frac{1}{a!}}}\right)^{\frac{1}{\sum_{z=1}^{a}q_z}}\right\rangle \tag{53}$$

where $\overline{\overline{D}}\left(\delta_{gh}^{k},\delta_{gl}^{k}\right)$ is the distance measure among two SVNNs δ_{gh}^{k} and δ_{gl}^{k} defined in Definition (5).

Step 3: Determine $T\left(\delta_{gh}^{k}\right)$ by

$$T\left(\delta_{gh}^{k}\right)=\sum_{\substack{l=1\\l\neq h}}^{b}Supp\left(\delta_{gh}^{k},\delta_{gl}^{k}\right)\quad(1,2,\ldots,a;\quad h,l=1,2,\ldots,b,\quad k=1,2,\ldots,c) \tag{56}$$

Step 4: Determine

$$\Phi_{gh}^{k}=\frac{b\varpi_h\left(1+T\left(\delta_{gh}^{k}\right)\right)}{\sum_{d=1}^{b}\varpi_d\left(1+T\left(\delta_{gd}^{k}\right)\right)}\quad(g=1,2,\ldots,a;\quad h,d=1,2,\ldots,b,k=1,2,\ldots,c). \tag{57}$$

Step 5: Utilize the WT-SPFPMM or WT-SPFPDMM operators

$$\delta_g^k=\left\langle\Xi_g^k,\Psi_g^k,Z_g^k\right\rangle=WT-SPFPMM^{Q}\left(\delta_{g1}^{k},\delta_{g2}^{k},\ldots,\delta_{gb}^{k}\right) \tag{58}$$

or

$$=WT-SPFPDMM^{Q}\left(\delta_{g1}^{k},\delta_{g2}^{k},\ldots,\delta_{gb}^{k}\right) \tag{59}$$

To calculate the overall T-SPFNs $\delta_g^k(g=1,2,\ldots,a;$ $k=1,2,\ldots,c)$.

Step 6: Determine the supports $Supp\left(\delta_g^k,\delta_g^m\right)$ $(g=1,2,\ldots,a;m,k=1,2,\ldots,c)$ by

$$Supp\left(\delta_g^k,\delta_g^m\right)=1-\overline{\overline{D}}\left(\delta_g^k,\delta_g^m\right) \tag{60}$$

where, $\overline{\overline{D}}\left(\delta_g^k,\delta_g^m\right)$ is the distance measure among two T-SPFNs δ_g^k and δ_g^m defined in Definition (5).

Step 7: Determine $T\left(\delta_g^k\right)$ by

$$T\left(\delta_g^k\right)=\sum_{\substack{m=1\\m\neq g}}^{b}Supp\left(\delta_g^k,\delta_g^m\right)\quad(g=1,2,\ldots,a;\quad h,m,k=1,2,\ldots,c) \tag{61}$$

Step 8: Determine

$$K_g^k=\frac{c\Lambda_k\left(1+T\left(\delta_g^k\right)\right)}{\sum_{k=1}^{c}\Lambda_c\left(1+T\left(\delta_g^k\right)\right)}\quad(g=1,2,\ldots,a;\quad h,k=1,2,\ldots,c). \tag{62}$$

Step 9: Utilize the WT-SPFPMM or WT-SPPDMM operators

$$\delta_g=\left\langle\Xi_g,\Psi_g,Z_g\right\rangle=WT-SPFPMM^{Q}\left(\delta_g^1,\delta_g^2,\ldots,\delta_g^c\right) \tag{63}$$

or

$$=WT-SPFPDMM^{Q}\left(\delta_g^1,\delta_g^2,\ldots,\delta_g^c\right) \tag{64}$$

Step 10: Using Definition 2, Equation (2), to calculate the score values of the overall T-SPFNs $\delta_g(g=1,2,\ldots,a)$.

Step 11: Rank all the alternatives and the select the best one.

Step 12: End.

TABLE 1. Air quality data from station $\mathbb{Z}_1$.

	$ƛ_1$	$ƛ_2$	$ƛ_3$
$\hbar_1$	$\langle 0.265,0.350,0.385\rangle$	$\langle 0.330,0.390,0.280\rangle$	$\langle 0.245,0.275,0.480\rangle$
$\hbar_2$	$\langle 0.345,0.245,0.410\rangle$	$\langle 0.430,0.290,0.280\rangle$	$\langle 0.245,0.375,0.380\rangle$
$\hbar_3$	$\langle 0.365,0.300,0.335\rangle$	$\langle 0.480,0.315,0.205\rangle$	$\langle 0.340,0.370,0.290\rangle$
$\hbar_4$	$\langle 0.430,0.300,0.270\rangle$	$\langle 0.460,0.245,0.295\rangle$	$\langle 0.310,0.520,0.170\rangle$

TABLE 2. Air quality data from station $\mathbb{Z}_2$.

	$ƛ_1$	$ƛ_2$	$ƛ_3$
$\hbar_1$	$\langle 0.125,0.470,0.405\rangle$	$\langle 0.220,0.420,0.36\rangle$	$\langle 0.345,0.490,0.165\rangle$
$\hbar_2$	$\langle 0.335,0.335,0.330\rangle$	$\langle 0.300,0.370,0.330\rangle$	$\langle 0.205,0.630,0.165\rangle$
$\hbar_3$	$\langle 0.250,0.445,0.305\rangle$	$\langle 0.310,0.585,0.105\rangle$	$\langle 0.240,0.580.220\rangle$
$\hbar_4$	$\langle 0.365,0.365,0.270\rangle$	$\langle 0.355,0.320,0.325\rangle$	$\langle 0.325,0.485,0.190\rangle$

TABLE 3. Air quality data from station $\mathbb{Z}_3$.

	$ƛ_1$	$ƛ_2$	$ƛ_3$
$\hbar_1$	$\langle 0.260,0.425,0.315\rangle$	$\langle 0.220,0.450,0.330\rangle$	$\langle 0.255,0.500,0.245\rangle$
$\hbar_2$	$\langle 0.270,0.370,0.360\rangle$	$\langle 0.320,0.215,0.465\rangle$	$\langle 0.135,0.575,0.290\rangle$
$\hbar_3$	$\langle 0.510,0.220,0.290\rangle$	$\langle 0.450,0.370,0.180\rangle$	$\langle 0.490,0.350,0.160\rangle$
$\hbar_4$	$\langle 0.390,0.340,0.270\rangle$	$\langle 0.305,0.475,0.220\rangle$	$\langle 0.465,0.485,0.050\rangle$

VI. ILLUSTRATIVE EXAMPLE

In this section, a numerical example is given to show the effectiveness and practicality of the proposed aggregation operators and decision making approach is initiated. The following examples are adapted from Ashraf *et al.* [12].

Example 1: Let $\{\hbar_1, \hbar_2, \hbar_3, \hbar_4\}$ be the set of four alternatives, respectively, showing the air quality in Guangzhou city for November 2006, November 2007, November 2008 and of November 2009. The following three attributes are taken under consideration, which are $SO_2(ƛ_1)$, $NO_2(ƛ_2)$ and $PM_{10}(ƛ_3)$. The weight vector of the attributes is $\varpi = (0.314, 0.355, 0.331)^T$. Let us assume that there are three decision makers, that three air quality monitoring stations represented by $\{Z_1, Z_2, Z_3\}$ and the weight vector of these monitoring stations is $\Lambda = (0.40, 0.20, 0.40)^T$. The evaluation values of the three air quality monitoring stations under the three attributes are provided in the form of T-SPFNs, which are given in Table 1,2 and 3.

The evaluation steps by utilizing WT-SPFPMM operator or WTSPFPMM are as follows.

Step 1: Since all the attributes are of the same type, so there is no need to normalize it.

Step 2: Determine the supports

$$Supp\left(\Theta^k_{gh}, \Theta^k_{gl}\right) \quad (1, 2, 3, 4; h, l = 1, 2, 3, k = 1, 2, 3, h \neq l)$$

using Equation (55). We shall denote $Supp\left(\Theta^k_{gh}, \Theta^k_{gl}\right)$ by $Supp^k_{gh,gl}$, which are given below, (assume $m = 2$):

$$\begin{aligned}
Supp^1_{11,12} &= Supp^1_{21,11} = 0.9540,\\
Supp^1_{11,13} &= Supp^1_{31,11} = 0.9536,\\
Supp^1_{12,13} &= Supp^1_{13,12} = 0.9076;\\
Supp^1_{21,22} &= Supp^1_{22,21} = 0.9401,\\
Supp^1_{21,23} &= Supp^1_{23,21} = 0.9456,\\
Supp^1_{22,23} &= Supp^1_{23,22} = 0.91753;\\
Supp^1_{31,32} &= Supp^1_{32,31} = 0.9411,\\
Supp^1_{31,33} &= Supp^1_{33,31} = 0.9691,\\
Supp^1_{32,33} &= Supp^1_{33,32} = 0.9352;\\
Supp^1_{41,42} &= Supp^1_{42,41} = 0.9764,\\
Supp^1_{41,43} &= Supp^1_{43,41} = 0.8956,\\
Supp^1_{42,43} &= Supp^1_{43,42} = 0.8720;\\
Supp^2_{11,12} &= Supp^2_{21,11} = 0.9628,\\
Supp^2_{11,13} &= Supp^2_{31,11} = 0.9135,\\
Supp^2_{12,13} &= Supp^2_{13,12} = 0.9211;\\
Supp^2_{21,22} &= Supp^2_{22,21} = 0.9844,\\
Supp^2_{21,23} &= Supp^2_{23,21} = 0.85448,\\
Supp^2_{22,23} &= Supp^2_{23,22} = 0.8701;\\
Supp^2_{31,32} &= Supp^2_{32,31} = 0.9134,\\
Supp^2_{31,33} &= Supp^2_{33,31} = 0.9374,\\
Supp^2_{32,33} &= Supp^2_{33,32} = 0.9728;\\
Supp^2_{41,42} &= Supp^2_{42,41} = 0.9764,\\
Supp^2_{41,43} &= Supp^2_{43,41} = 0.9445,\\
Supp^2_{42,43} &= Supp^2_{43,42} = 0.9258;\\
Supp^3_{11,12} &= Supp^3_{21,11} = 0.9831,\\
Supp^3_{11,13} &= Supp^3_{31,11} = 0.9630,\\
Supp^3_{12,13} &= Supp^3_{13,12} = 0.9623;\\
Supp^3_{21,22} &= Supp^3_{22,21} = 0.9311,\\
Supp^3_{21,23} &= Supp^3_{23,21} = 0.9020,\\
Supp^3_{22,23} &= Supp^3_{23,22} = 0.8331;\\
Supp^3_{31,32} &= Supp^3_{32,31} = 0.9341,\\
Supp^3_{31,33} &= Supp^3_{33,31} = 0.9491,\\
Supp^3_{32,33} &= Supp^3_{33,32} = 0.9804;
\end{aligned}$$

$$Supp^3_{41,42} = Supp^3_{42,41} = 0.9355,$$
$$Supp^3_{41,43} = Supp^3_{43,41} = 0.9153,$$
$$Supp^3_{42,43} = Supp^3_{43,42} = 0.9404.$$

Step 3: Determine

$$T\left(\Theta^k_{gh}\right) \quad (g = 1, 2, 3, 4, h = 1, 2, 3, k = 1, 2, 3),$$

using formula (56) for simplicity we shall denote $T\left(\Theta^k_{gh}\right)$ by T^k_{gh}, which is given below:

$$T^1_{11} = 1.9080, \quad T^1_{12} = 1.8615, \quad T^1_{31} = 1.8611,$$
$$T^1_{21} = 1.8857, \quad T^1_{22} = 1.8576, \quad T^1_{23} = 1.8631;$$
$$T^1_{31} = 1.9103, \quad T^1_{32} = 1.8763, \quad T^1_{33} = 1.9043,$$
$$T^1_{41} = 1.8720, \quad T^1_{42} = 1.8484, \quad T^1_{43} = 1.7676;$$
$$T^2_{11} = 1.8763, \quad T^2_{12} = 1.8839, \quad T^2_{31} = 1.8346,$$
$$T^2_{21} = 1.8389, \quad T^2_{22} = 1.8545, \quad T^2_{23} = 1.7246;$$
$$T^2_{31} = 1.8508, \quad T^2_{32} = 1.886, \quad T^2_{33} = 1.9101,$$
$$T^2_{41} = 1.9209, \quad T^2_{42} = 1.9022, \quad T^2_{43} = 1.8703;$$
$$T^3_{11} = 1.9460, \quad T^3_{12} = 1.9454, \quad T^3_{31} = 1.9253,$$
$$T^3_{21} = 1.8331, \quad T^3_{22} = 1.7642, \quad T^3_{23} = 1.7351;$$
$$T^3_{31} = 1.8832, \quad T^3_{32} = 1.9145, \quad T^3_{33} = 1.9295,$$
$$T^3_{41} = 1.8508, \quad T^3_{42} = 1.8759, \quad T^3_{43} = 1.8557;$$

Step 4: Determine Φ^k_{gh} by utilizing formula (57)

$$\Phi^1_{11} = 0.9524, \quad \Phi^1_{12} = 1.05969, \quad \Phi^1_{13} = 0.9879,$$
$$\Phi^1_{21} = 0.9477, \quad \Phi^1_{22} = 1.0611, \quad \Phi^1_{23} = 0.9912,$$
$$\Phi^1_{31} = 0.9466, \quad \Phi^1_{32} = 1.0577, \quad \Phi^1_{33} = 0.9958,$$
$$\Phi^1_{41} = 0.9563, \quad \Phi^1_{42} = 1.0723, \quad \Phi^1_{43} = 0.9714;$$
$$\Phi^2_{11} = 0.9457, \quad \Phi^2_{12} = 1.07194, \quad \Phi^2_{13} = 0.9824,$$
$$\Phi^2_{21} = 0.9528, \quad \Phi^2_{22} = 1.0831, \quad \Phi^2_{23} = 0.9640,$$
$$\Phi^2_{31} = 0.9315, \quad \Phi^2_{32} = 1.0662, \quad \Phi^2_{33} = 1.0024,$$
$$\Phi^2_{41} = 0.9496, \quad \Phi^2_{42} = 1.0667, \quad \Phi^2_{43} = 0.9837;$$
$$\Phi^3_{11} = 0.9443, \quad \Phi^3_{12} = 1.0674, \quad \Phi^3_{13} = 0.9884,$$
$$\Phi^3_{21} = 0.9613, \quad \Phi^3_{22} = 1.0604, \quad \Phi^3_{23} = 0.9783,$$
$$\Phi^3_{31} = 0.9334, \quad \Phi^3_{32} = 1.0668, \quad \Phi^3_{33} = 0.9998,$$
$$\Phi^3_{41} = 0.9385, \quad \Phi^3_{42} = 1.0704, \quad \Phi^3_{43} = 0.9911.$$

Step 5: Utilize the WT-SPFPMM operators defined in Equation (58), we have (assume $m = 2$). The collective decision matrix is given in Table 4.

Step 6: Determine the supports

$$Supp\left(\Theta^k_g, \Theta^m_g\right) \quad (g = 1, 2, \ldots, 4; m, k = 1, 2, 3)$$

using Equation (60), we have

$$S_{11} = 0.9405, \quad S_{12} = 0.9399, \quad S_{13} = 0.9876,$$
$$S_{21} = 0.9261, \quad S_{22} = 0.9479, \quad S_{23} = 0.9565;$$
$$S_{31} = 0.9023, \quad S_{31} = 0.9597, \quad S_{31} = 0.8798,$$

TABLE 4. Collective decision matrix *U* WT-SPFPMM operator.

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
Θ_1	$\langle 0.2776, 0.3418, 0.3969\rangle$	$\langle 0.2115, 0.4636, 0.3308\rangle$	$\langle 0.2442, 0.4606, 0.2997\rangle$
Θ_2	$\langle 0.3309, 0.3092, 0.3663\rangle$	$\langle 0.2739, 0.4797, 0.2856\rangle$	$\langle 0.2266, 0.4298, 0.3784\rangle$
Θ_3	$\langle 0.3901, 0.3307, 0.2878\rangle$	$\langle 0.2647, 0.5419, 0.2370\rangle$	$\langle 0.4825, 0.3178, 0.2255\rangle$
Θ_4	$\langle 0.3939, 0.3864, 0.2499\rangle$	$\langle 0.3477, 0.4014, 0.2665\rangle$	$\langle 0.3809, 0.4386, 0.2075\rangle$

$$S_{41} = 0.9818, \quad S_{42} = 0.9758, \quad S_{43} = 0.9722.$$

Step 7: Determine the $T\left(\Theta^k_g\right)$ $(g = 1, 2, 3, 4; k = 1, 2, 3)$ using Equation (61), we have

$$T_{11} = 1.8804, \quad T_{12} = 1.9281, \quad T_{13} = 1.9274,$$
$$T_{21} = 1.8741, \quad T_{22} = 1.8826, \quad T_{23} = 1.9044,$$
$$T_{31} = 1.8620, \quad T_{32} = 1.7821, \quad T_{33} = 1.8395,$$
$$T_{41} = 1.9576, \quad T_{42} = 1.9540, \quad T_{43} = 1.9480.$$

Step 8: Determine K^k_g utilizing formula (62), we have

$$K_{11} = 1.1883, \quad K_{12} = 0.6040, \quad K_{13} = 1.2077,$$
$$K_{21} = 1.1943, \quad K_{22} = 0.5989, \quad K_{23} = 1.2069;$$
$$K_{31} = 1.2106, \quad K_{32} = 0.5884, \quad K_{33} = 1.201,$$
$$K_{41} = 1.2018, \quad K_{42} = 0.6002, \quad K_{43} = 1.1980.$$

Step 9: Utilize the WT-SPFPMM given in Equation (63), to get the overall T-SPFN. (Assume $m = 2$), we have

$$\Theta_1 = \langle 0.2370, 0.6382, 0.5545\rangle,$$
$$\Theta_2 = \langle 0.2672, 0.6341, 0.5550\rangle,$$
$$\Theta_3 = \langle 0.3569, 0.6401, 0.4819\rangle,$$
$$\Theta_4 = \langle 0.3643, 0.6169, 0.4846\rangle.$$

Step 10: Utilizing Equation (2), to get the score values of the T-SPFNs.

$$\widetilde{SC}(\Theta_1) = -0.2513, \quad \widetilde{SC}(\Theta_2) = -0.2366,$$
$$\widetilde{SC}(\Theta_3) = -0.1049, \quad \widetilde{SC}(\Theta_4) = -0.1021.$$

Step 11: Utilizing the comparison rules defined for T-SPFNs in Definition (2), and select the best one.

$$\Theta_4 > \Theta_3 > \Theta_2 > \Theta_1.$$

Hence Θ_4 is the best alternative, while Θ_1 is the worst one.

In a similar way, we utilize WT-SPFPDMM operator.

Steps 1-4 are same

Step 5: Utilize the WT-SPFPDMM operators defined in Equation (59), we have (assume $m = 2$). The collective decision matrix is given in Table 5.

TABLE 5. Collective decision matrix U utilizing WT-SPPDMM operator.

	$\mathbb{Z}_1$	$\mathbb{Z}_2$	$\mathbb{Z}_3$
Θ_1	$\langle 0.2814,0.3346,0.3726\rangle$	$\langle 0.2486,0.4589,0.2885\rangle$	$\langle 0.2487,0.4570,0.2940\rangle$
Θ_2	$\langle 0.5217,0.2985,0.3520\rangle$	$\langle 0.2869,0.4274,0.2616\rangle$	$\langle 0.2525,0.3578,0.3644\rangle$
Θ_3	$\langle 0.5732,0.3268,0.2710\rangle$	$\langle 0.2679,0.5316,0.1917\rangle$	$\langle 0.4865,0.3050,0.2028\rangle$
Θ_4	$\langle 0.5701,0.3369,0.2382\rangle$	$\langle 0.3499,0.3840,0.2553\rangle$	$\langle 0.3978,0.4273,0.1437\rangle$

Step 6: Determine the supports

$$Supp\left(\Theta_g^k, \Theta_g^m\right) \quad (g = 1, 2, \ldots, 4; m, k = 1, 2, 3)$$

using Equation (60), we have

$$S_{11} = 0.9428, \quad S_{12} = 0.9445, \quad S_{13} = 0.9983,$$
$$S_{21} = 0.8871, \quad S_{22} = 0.9146, \quad S_{23} = 0.9542;$$
$$S_{31} = 0.8436, \quad S_{31} = 0.9540, \quad S_{31} = 0.8804,$$
$$S_{41} = 0.9184, \quad S_{42} = 0.9093, \quad S_{43} = 0.9615.$$

Step 7: Determine the $T\left(\Theta_g^k\right)$ $(g = 1, 2, 3, 4; k = 1, 2, 3)$ using Equation (61), we have

$$T_{11} = 1.8873, \quad T_{12} = 1.9411, \quad T_{13} = 1.9428,$$
$$T_{21} = 1.8017, \quad T_{22} = 1.8412, \quad T_{23} = 1.8688,$$
$$T_{31} = 1.7976, \quad T_{32} = 1.724, \quad T_{33} = 1.8344,$$
$$T_{41} = 1.828, \quad T_{42} = 1.880, \quad T_{43} = 1.8709.$$

Step 8: Determine K_g^k utilizing formula (62), we have

$$\mathrm{K}_{11} = 1.1865, \quad \mathrm{K}_{12} = 0.6043, \quad \mathrm{K}_{13} = 1.2093,$$
$$\mathrm{K}_{21} = 1.1853, \quad \mathrm{K}_{22} = 0.6010, \quad \mathrm{K}_{23} = 1.2137;$$
$$\mathrm{K}_{31} = 1.2000, \quad \mathrm{K}_{32} = 0.5842, \quad \mathrm{K}_{33} = 1.2158,$$
$$\mathrm{K}_{41} = 1.1883, \quad \mathrm{K}_{42} = 0.6051, \quad \mathrm{K}_{43} = 1.2065.$$

Step 9: Utilize the WT-SPFPDMM given in Equation (64), to get the overall T-SPFN. (Assume $m = 2$), we have

$$\Theta_1 = \langle 0.3043, 0.4035, 0.3084\rangle,$$
$$\Theta_2 = \langle 0.4020, 0.3498, 0.3142\rangle,$$
$$\Theta_3 = \langle 0.4668, 0.3685, 0.2133\rangle,$$
$$\Theta_4 = \langle 0.4700, 0.3719, 0.2012\rangle.$$

Step 10: Utilizing Equation (2), to get the score values of the T-SPFNs.

$$\widetilde{SC}(\Theta_1) = -0.00253, \quad \widetilde{SC}(\Theta_2) = 0.0629,$$
$$\widetilde{SC}(\Theta_3) = 0.1724, \quad \widetilde{SC}(\Theta_4) = 0.1804.$$

Step 11: Utilizing the comparison rules defined for T-SPFNs in Definition (2), and select the best one.

$$\Theta_4 > \Theta_3 > \Theta_2 > \Theta_1.$$

Hence Θ_4 is the best alternative, while Θ_1 is the worst one.

TABLE 6. Effect of the parameter on the decision making results utilizing WT-SPFPMM.

Parameter	Score values	Ranking order
(1,0,0)	$\widetilde{SC}(\Theta_1) = -0.1481, \widetilde{SC}(\Theta_2) = -0.1286,$ $\widetilde{SC}(\Theta_3) = 0.0446, \widetilde{SC}(\Theta_4) = 0.0409.$	$\Theta_3 > \Theta_4 > \Theta_2 > \Theta_1.$
(1,1,0)	$\widetilde{SC}(\Theta_1) = -0.2137, \widetilde{SC}(\Theta_2) = -0.2064,$ $\widetilde{SC}(\Theta_3) = -0.0437, \widetilde{SC}(\Theta_4) = -0.0516.$	$\Theta_3 > \Theta_4 > \Theta_2 > \Theta_1.$
(1,1,1)	$\widetilde{SC}(\Theta_1) = -0.2513, \widetilde{SC}(\Theta_2) = -0.2366,$ $\widetilde{SC}(\Theta_3) = -0.1049, \widetilde{SC}(\Theta_4) = -0.1021.$	$\Theta_4 > \Theta_3 > \Theta_2 > \Theta_1.$
(2,0,0)	$\widetilde{SC}(\Theta_1) = -0.1321, \widetilde{SC}(\Theta_2) = -0.1016,$ $\widetilde{SC}(\Theta_3) = 0.0741, \widetilde{SC}(\Theta_4) = 0.0612.$	$\Theta_3 > \Theta_4 > \Theta_2 > \Theta_1.$
(5,0,0)	$\widetilde{SC}(\Theta_1) = -0.0913, \widetilde{SC}(\Theta_2) = -0.0329,$ $\widetilde{SC}(\Theta_3) = 0.1253, \widetilde{SC}(\Theta_4) = 0.1085.$	$\Theta_3 > \Theta_4 > \Theta_2 > \Theta_1.$
(10,0,0)	$\widetilde{SC}(\Theta_1) = -0.0435, \widetilde{SC}(\Theta_2) = 0.0337,$ $\widetilde{SC}(\Theta_3) = 0.1627, \widetilde{SC}(\Theta_4) = 0.1470.$	$\Theta_3 > \Theta_4 > \Theta_2 > \Theta_1.$

A. EFFECT OF THE DIFFERENT PARAMETER VALUE Q

In this subpart, different values to the parameter vector Q utilizing WT-SPFPMM and WT-SPFPDMM operators are given. The score values and ranking order are shown in Table 6 and Table 7. From Table 6 and Table 7, we can see that, when the value of the parameter vector is $(1, 0, 0)$, then the best alternative is Θ_3, while the worst one remains the same. In simple words, when we do not consider the inter-relationship among the input arguments the best alternative is Θ_3. When the value of the parameter vector is $(1, 1, 0)$, then from Table 6 the best alternative is Θ_3, while in Table 7 the best alternative is Θ_4. Similarly, for other values the score values vary.

B. EFFECT OF m ON THE DECISION RESULT UTILIZING PROPOSE AGGREGATION OPERATOR

In this subpart, we take different values for m and the score values and ranking order are shown in Table 8 and Table 9.

From Table 8 we see that the sore values are different for different values of m. The best alternative remains the same but the worst alternative changed, when the values of m are odd using WT-SPFPMM operator. From Table 9, we can see that when the values of $m \leq 4$, utilizing WT-SPFPDMM operator, the ranking order are same as obtained above. values of $m > 4$, utilizing WT-SPFPDMM operator, the ranking order change and the best alternative is Θ_3, while the worst one remain the same.

C. COMPARISON AND DISCUSSION

To epitomize the usefulness and advantages of the developed method, a comparative analysis is managed. We utilize some existing methods to solve the same example and examine the result. In this subsection, we compare our developed approach with that developed by Wei [10] based on picture fuzzy weighted averaging (PFWA) operator, Mahmood *et al.* [17] based on T-SPFGWA operator,

TABLE 7. Effect of the parameter on the decision making results DPMMM.

Parameter	Score Values	Ranking order
(1,0,0)	$\widetilde{SC}(\Theta_1)=-0.05657, \widetilde{SC}(\Theta_2)=-0.0252,$ $\widetilde{SC}(\Theta_3)=0.1401, \widetilde{SC}(\Theta_4)=0.1296.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$
(1,1,0)	$\widetilde{SC}(\Theta_1)=-0.02811, \widetilde{SC}(\Theta_2)=0.0357,$ $\widetilde{SC}(\Theta_3)=0.1598, \widetilde{SC}(\Theta_4)=0.1620.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
(1,1,1)	$\widetilde{SC}(\Theta_1)=-0.00253, \widetilde{SC}(\Theta_2)=0.0629,$ $\widetilde{SC}(\Theta_3)=0.1724, \widetilde{SC}(\Theta_4)=0.1804.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
(2,0,0)	$\widetilde{SC}(\Theta_1)=-0.0729, \widetilde{SC}(\Theta_2)=-0.0459,$ $\widetilde{SC}(\Theta_3)=0.1277, \widetilde{SC}(\Theta_4)=0.1178.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$
(5,0,0)	$\widetilde{SC}(\Theta_1)=-0.11691, \widetilde{SC}(\Theta_2)=-0.0942,$ $\widetilde{SC}(\Theta_3)=0.1006, \widetilde{SC}(\Theta_4)=0.0917.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$
(10,0,0)	$\widetilde{SC}(\Theta_1)=-0.1567, \widetilde{SC}(\Theta_2)=-0.1403,$ $\widetilde{SC}(\Theta_3)=0.0743, \widetilde{SC}(\Theta_4)=0.0605.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$

TABLE 8. Score values and ranking result for different values of *m* using WT-SPFPMM (For fix *Q*(1, 1, 1)).

Parameter	Score Value	Ranking order
$m=2$	$\widetilde{SC}(\Theta_1)=-0.2513, \widetilde{SC}(\Theta_2)=-0.2366,$ $\widetilde{SC}(\Theta_3)=-0.1049, \widetilde{SC}(\Theta_4)=-0.1021.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
$m=4$	$\widetilde{SC}(\Theta_1)=-0.2928, \widetilde{SC}(\Theta_2)=-0.2922,$ $\widetilde{SC}(\Theta_3)=-0.2183, \widetilde{SC}(\Theta_4)=-0.2176.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
$m=7$	$\widetilde{SC}(\Theta_1)=-0.3043, \widetilde{SC}(\Theta_2)=-0.3055,$ $\widetilde{SC}(\Theta_3)=-0.2489, \widetilde{SC}(\Theta_4)=-0.2435.$	$\Theta_4>\Theta_3>\Theta_1>\Theta_2.$
$m=10$	$\widetilde{SC}(\Theta_1)=-0.3108, \widetilde{SC}(\Theta_2)=-0.3116,$ $\widetilde{SC}(\Theta_3)=-0.2583, \widetilde{SC}(\Theta_4)=-0.2493.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
$m=12$	$\widetilde{SC}(\Theta_1)=-0.3141, \widetilde{SC}(\Theta_2)=-0.3145,$ $\widetilde{SC}(\Theta_3)=-0.2620, \widetilde{SC}(\Theta_4)=-0.2516.$	$\Theta_4>\Theta_3>\Theta_1>\Theta_2.$
$m=15$	$\widetilde{SC}(\Theta_1)=-0.3177, \widetilde{SC}(\Theta_2)=-0.3179,$ $\widetilde{SC}(\Theta_3)=-0.2660, \widetilde{SC}(\Theta_4)=-0.2539.$	$\Theta_4>\Theta_3>\Theta_1>\Theta_2.$

TABLE 9. Score values and ranking result for different values of *m* using WT-SPFPDMM (For fix *Q*(1, 1, 1)).

Parameter	Score Value	Ranking order
$m=2$	$\widetilde{SC}(\Theta_1)=-0.2513, \widetilde{SC}(\Theta_2)=-0.2366,$ $\widetilde{SC}(\Theta_3)=-0.1049, \widetilde{SC}(\Theta_4)=-0.1021.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
$m=4$	$\widetilde{SC}(\Theta_1)=0.00665, \widetilde{SC}(\Theta_2)=0.0734,$ $\widetilde{SC}(\Theta_3)=0.1072, \widetilde{SC}(\Theta_4)=0.1084.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
$m=7$	$\widetilde{SC}(\Theta_1)=0.0018, \widetilde{SC}(\Theta_2)=0.0667,$ $\widetilde{SC}(\Theta_3)=0.0870, \widetilde{SC}(\Theta_4)=0.0820.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$
$m=10$	$\widetilde{SC}(\Theta_1)=0.00030, \widetilde{SC}(\Theta_2)=0.065,$ $\widetilde{SC}(\Theta_3)=0.0852, \widetilde{SC}(\Theta_4)=0.0783.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$
$m=12$	$\widetilde{SC}(\Theta_1)=0.0000021, \widetilde{SC}(\Theta_2)=0.0646,$ $\widetilde{SC}(\Theta_3)=0.0850, \widetilde{SC}(\Theta_4)=0.0777.$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$
$m=15$	$\widetilde{SC}(\Theta_1)=0.000005, \widetilde{SC}(\Theta_2)=0.0647,$ $\widetilde{SC}(\Theta_3)=0.0851, \widetilde{SC}(\Theta_4)=0.0777$	$\Theta_3>\Theta_4>\Theta_2>\Theta_1.$

TABLE 10. Comparison with other aggregation.

Aggregation operator	Score values	Ranking Order
PFWA [10]	$\widetilde{SC}(\Theta_1)=-0.4170, \widetilde{SC}(\Theta_2)=-0.3648,$ $\widetilde{SC}(\Theta_3)=-0.3619, \widetilde{SC}(\Theta_4)=-0.4070.$	$\Theta_3>\Theta_2>\Theta_4>\Theta_1.$
SGWA [17]	$\widetilde{SC}(\Theta_1)=-0.4246, \widetilde{SC}(\Theta_2)=-0.4169,$ $\widetilde{SC}(\Theta_3)=-0.2310, \widetilde{SC}(\Theta_4)=-0.2432.$	$\Theta_3>\Theta_2>\Theta_4>\Theta_1.$
PFBM extended from [21]	$\widetilde{SC}(\Theta_1)=-0.2513, \widetilde{SC}(\Theta_2)=-0.2366,$ $\widetilde{SC}(\Theta_3)=-0.1049, \widetilde{SC}(\Theta_4)=-0.1021.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
WT-SPFPMM operator in this article	$\widetilde{SC}(\Theta_1)=-0.00253, \widetilde{SC}(\Theta_2)=0.0629,$ $\widetilde{SC}(\Theta_3)=0.1724, \widetilde{SC}(\Theta_4)=0.1804.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$
WT-SPFPDMM operator in this article	$\widetilde{SC}(\Theta_1)=-0.2513, \widetilde{SC}(\Theta_2)=-0.2366,$ $\widetilde{SC}(\Theta_3)=-0.1049, \widetilde{SC}(\Theta_4)=-0.1021.$	$\Theta_4>\Theta_3>\Theta_2>\Theta_1.$

picture fuzzy Bonferroni mean (PFBM) operator extended from Xu *et al.* [21]. The ranking score values and ranking orders obtained by these methods are shown in Table 10.

The methods developed by Wei [10], Mahmood *et al.* [17] are based on basic weighted averaging and weighted geometric operators for PFNs and T-SPFNs. Both methods cannot consider the interrelationships among PFNs and T-SPFNs. In addition, both methods cannot diminish the effect of awkward data. Our developed method is based on WT-SPFPMM operator and WT-SPFPDMM operator, which can consider the interrelationship among input arguments and also eliminate the effect of awkward data at the same time. Thus our developed method is more judicious and practical in MADM and MAGDM problems.

The Xu *et al.* [21] is based on Bonferroni mean, for IFS, we extend it for picture fuzzy sets, and solve the same example. The Xu *et al.* [21] method can consider the interrelationship between any two arguments and cannot remove the influence of the awkward data. The main advantage of the proposed aggregation operator is that, these aggregation operators are special cases of it.

VII. CONCLUSION

In this article, some limitations in the operational laws for SPFNs and T-SPFNs are found out, and some novel operational laws for SPFNs and T-SPFNs are defined. Then based on these operational laws, some new aggregation operators are defined such as SPF power Murihead mean (SPFPMM) operator, weighted SPFPMM operator, SPF power dual MM operator, weighted SPFPDMM operator and discussed its desired properties. The developed aggregation operator take full advantage MM operator and PA operator at the same time. In simple words the developed aggregation operator can consider the interrelationship among input arguments by MM operator and eliminate the effect of awkward data by PA operator at the same time. Furthermore, based on these aggregation operators, we developed a novel MAGDM with T-SPF information. Finally, we give a numerical example to show the effectiveness and advantages of the proposed aggregation operators.

In future, we shall extend the proposed aggregation operators to different environment, such as IFS [3], PGFS [13], and so on.

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Authors' photographs and biographies not available at the time of publication.