

## TABLES FOR COMPUTING BIVARIATE NORMAL PROBABILITIES

By DONALD B. OWEN

*Sandia Corporation*

**1. Introduction.** Various tables have been published for obtaining probabilities over rectangles for correlated bivariate normal variables. Some of these tables give the probabilities as functions of three parameters (see [1], [2], and [3]). Others tabulate related two-parameter families from which these probabilities may be computed (see [3], [4], [5], [6], and [7]). The tables given here are of the latter type. They have been computed for use with a special two-dimensional interpolation scheme, which is described in Section 4. These new tabulations reduce considerably the amount of interpolation work required over that needed with previous tables. The function tabulated also eliminates an arctangent function from the formula for the bivariate normal over a region outside of a rectangle as compared with the formula for Nicholson's tabulation in [5]. Section 3 contains a derivation of the formulas given in Section 2 for using a two-parameter table to compute probabilities over rectangles. The tables given below should prove very useful, since examples where bivariate normal integrals over polygons are needed to solve practical problems abound in the literature. For example, see [6], [8], [9], and [10].

The usefulness of the  $T(h, a)$  function tabulated below was also recognized by Professor Harry A. Bender, University of Rhode Island, who submitted, after this paper was received by the editor, a somewhat shorter tabulation than given here. An abstract of Professor Bender's paper appears in [15].

For  $h$  and  $a > 0$ ,  $T(h, a)$ , the function tabulated, gives the volume of an uncorrelated bivariate normal distribution with zero means and unit variances over the area between  $y = ax$  and  $y = 0$  and to the right of  $x = h$ , i.e., the area shaded in Fig. 1.

Cadwell in [11] gives a method for obtaining the volume of a bivariate normal over any polygon. In Fig. 2, if  $AB$  is a side of any polygon, then the volume over the shaded area for an uncorrelated bivariate normal with zero means and unit variances is given by

$$T(h, a_2) - T(h, a_1)$$

for  $a_2 > a_1$ , where  $h$  is the length of the perpendicular from the origin to the line through  $AB$  and  $a_1 h$  is the distance from the foot of the perpendicular,  $C$ , to  $B$  and  $a_2 h$  is  $CA$ . If  $C$  lies between  $A$  and  $B$ , then the  $T$ -functions are added instead of subtracted. By composition of volumes like this, it is possible to obtain the volume over the area outside of any polygon. Section 2 includes some useful formulas for doing this.

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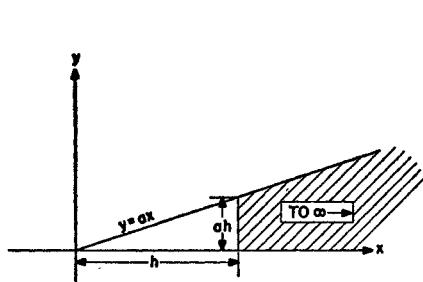


FIG. 1

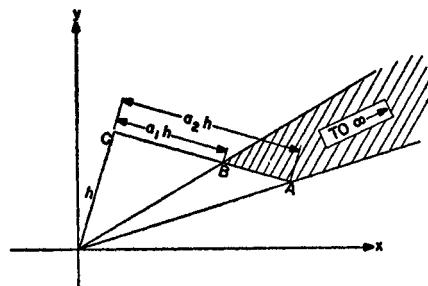


FIG. 2

FIG. 1. The area over which  $T(h, a)$  gives the volume of a standardized bivariate normal with correlation zero.

FIG. 2. A typical area for computing the bivariate normal over a polygon.

**2. Summary of formulas.** The fundamental formula for finding volumes over rectangles is

$$(2.1) \quad B(h, k; \rho) = \frac{1}{2}G(h) + \frac{1}{2}G(k) - T(h, a_h) - T(k, a_k) - \begin{cases} 0, \\ \frac{1}{2}, \end{cases}$$

where the upper choice is made if  $hk > 0$  or if  $hk = 0$  but  $h + k \geq 0$ , and the lower choice is made otherwise, where

$$(2.2) \quad a_h = \frac{k}{h\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}}, \quad a_k = \frac{h}{k\sqrt{1-\rho^2}} - \frac{\rho}{\sqrt{1-\rho^2}},$$

and where  $B(h, k; \rho)$  is the volume of a bivariate normal with zero means and unit variances and correlation  $\rho$  over the lower left-hand quadrant of the  $xy$ -plane when divided at  $x = h$  and  $y = k$ ,  $G(h)$  is the univariate normal with zero mean and unit variance integral from minus infinity to  $h$ , and  $T(h, a)$  is the function tabulated below.

The  $T$ -function is tabulated only for  $0 < a \leq 1$ , and  $\infty$ , but it is possible to obtain values for  $1 < a < \infty$  by use of the following formula:

$$(2.3) \quad T(h, a) = \frac{1}{2}G(h) + \frac{1}{2}G(ah) - G(h)G(ah) - T\left(ah, \frac{1}{a}\right).$$

Values for negative  $a$  or  $h$  may be obtained by using

$$(2.4) \quad T(h, -a) = -T(h, a)$$

and

$$(2.5) \quad T(-h, a) = T(h, a).$$

Note that (2.3) requires  $a$  to be positive and hence when  $a$  is negative, first apply (2.4) and then (2.3).

Other useful formulas are:

$$T(h, 0) = 0,$$

$$T(0, a) = \frac{1}{2\pi} \arctan a,$$

$$T(h, 1) = \frac{1}{2}G(h)[1 - G(h)],$$

and

$$T(h, \infty) = \begin{cases} \frac{1}{2}[1 - G(h)] & \text{if } h \geq 0, \\ \frac{1}{2}G(h) & \text{if } h \leq 0. \end{cases}$$

For finding volumes of the general correlated bivariate normal over polygons, the first step is to make a rotation and stretching of the axes to reduce the function under the integral to the form of the  $T$ -function. A transformation that will do this is

$$\begin{aligned} u &= \frac{1}{\sqrt{2 + 2\rho}} \left[ \frac{x - \mu_x}{\sigma_x} + \frac{y - \mu_y}{\sigma_y} \right], \\ v &= \frac{-1}{\sqrt{2 - 2\rho}} \left[ \frac{x - \mu_x}{\sigma_x} - \frac{y - \mu_y}{\sigma_y} \right], \end{aligned}$$

for  $\rho^2 < 1$ , where  $\mu_x, \mu_y$  are the means of the  $X$  and  $Y$  variables and  $\sigma_x, \sigma_y$  are the standard deviations of the  $X$  and  $Y$  variables, respectively. This will take the original polygon into another polygon in the  $uv$  plane. The vertices of the new polygon should be computed and a graph drawn. For each side of the polygon the volume over a region like that shown in Fig. 2 may be computed with the aid of these formulas:

$$\begin{aligned} h &= \frac{|h_1 k_2 - h_2 k_1|}{\sqrt{(h_2 - h_1)^2 + (k_2 - k_1)^2}}, \\ a_1 &= \frac{|h_1(h_2 - h_1) + k_1(k_2 - k_1)|}{|h_1 k_2 - h_2 k_1|}, \\ a_2 &= \frac{|h_2(h_2 - h_1) + k_2(k_2 - k_1)|}{|h_1 k_2 - h_2 k_1|}, \end{aligned}$$

where the vertical bars indicate absolute value and where  $(h_1, k_1)$  and  $(h_2, k_2)$  are the coordinates of two adjacent vertices on the polygon. With the aid of the graph, these volumes are then easily combined to give the volume over the outside (or inside) of the polygon.

**3. Derivation of the relationship between the bivariate normal and the tabulated function.**

Let

$$(3.1) \quad B(h, k; \rho) = \frac{1}{2\pi \sqrt{1 - \rho^2}} \int_{-\infty}^h \int_{-\infty}^k \exp [ -\frac{1}{2}(x^2 - 2\rho xy + y^2)/(1 - \rho^2) ] dx dy,$$

$$(3.2) \quad G(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x \exp (-\frac{1}{2}t^2) dt,$$

and

$$(3.3) \quad T(h, a) = \frac{1}{2\pi} \int_0^a \frac{\exp [ -\frac{1}{2}h^2(1 + x^2) ]}{1 + x^2} dx.$$

It is also convenient to have a second form of (3.3), which is the function in Tables A, B and C. It may be obtained by differentiating with respect to  $h$  and then reintegrating. The result is

$$(3.4) \quad T(h, a) = \frac{-1}{2\pi} \int_0^h \int_0^{ax} \exp [ -\frac{1}{2}(x^2 + y^2) ] dy dx + \frac{\arctan a}{2\pi}.$$

The  $T$ -function is related to the  $V$ -function tabulated by Nicholson in [5] as follows:

$$T(h, a) = \frac{1}{2\pi} \arctan a - V(h, ah).$$

If (3.4) is integrated by parts,

$$(3.5) \quad T(h, a) = \begin{cases} \frac{1}{2}G(h) + \frac{1}{2}G(ah) - G(h)G(ah) - T(ah, 1/a) & \text{if } a \geq 0, \\ \frac{1}{2}G(h) + \frac{1}{2}G(ah) - G(h)G(ah) - T(ah, 1/a) - \frac{1}{2} & \text{if } a < 0. \end{cases}$$

It will be shown that (3.1) can be expressed as a function of expressions like (3.2) and (3.3). If (3.1) is differentiated with respect to  $\rho$ , then integration with respect to  $x$  and  $y$  can be effected. Integrating that result with respect to  $\rho$  yields

$$(3.6) \quad \begin{aligned} & B(h, k; \rho) \\ &= \frac{1}{2\pi} \int_0^\rho (1 - z^2)^{-1/2} \exp [ -\frac{1}{2}(h^2 - 2hkz + k^2)/(1 - z^2) ] dz + G(h)G(k). \end{aligned}$$

From this  $B(0, 0; \rho) = 1/(2\pi) \arcsin \rho + \frac{1}{4}$ , a well-known result (see [12], [13], and [14]). Now (3.6) may be rewritten as

$$\begin{aligned} & B(h, k; \rho) \\ &= \frac{1}{2\pi} \int_0^\rho (1 - z^2)^{-1/2} \frac{h(h - kz)}{h^2 - 2hkz + k^2} \exp [ -\frac{1}{2}(h^2 - 2hkz + k^2)/(1 - z^2) ] dz \\ &+ \frac{1}{2\pi} \int_0^\rho (1 - z^2)^{-1/2} \frac{k(k - hz)}{h^2 - 2hkz + k^2} \exp [ -\frac{1}{2}(h^2 - 2hkz + k^2)/(1 - z^2) ] dz \\ &+ G(h)G(k). \end{aligned}$$

In the integrals above, making the substitutions

$$u = \frac{k - hz}{h \sqrt{1 - z^2}} \quad \text{and} \quad v = \frac{h - kz}{k \sqrt{1 - z^2}},$$

respectively, produces

$$(3.7) \quad \begin{aligned} B(h, k; \rho) &= T\left(h, \frac{k}{h}\right) + T\left(k, \frac{h}{k}\right) - T\left(h, \frac{k - \rho h}{h \sqrt{1 - \rho^2}}\right) \\ &\quad - T\left(k, \frac{h - \rho k}{k \sqrt{1 - \rho^2}}\right) + G(h)G(k). \end{aligned}$$

Applying (3.5) to (3.7), gives

$$(3.8) \quad B(h, k; \rho) = \begin{cases} \frac{1}{2}G(h) - T\left(h, \frac{k - \rho h}{h \sqrt{1 - \rho^2}}\right) + \frac{1}{2}G(k) - T\left(k, \frac{h - \rho k}{k \sqrt{1 - \rho^2}}\right), \\ \qquad \qquad \qquad \text{if } hk > 0 \text{ or if } hk = 0, h \text{ or } k \geq 0 \\ \frac{1}{2}G(h) - T\left(h, \frac{k - \rho h}{h \sqrt{1 - \rho^2}}\right) + \frac{1}{2}G(k) \\ \qquad \qquad \qquad - T\left(k, \frac{h - \rho k}{k \sqrt{1 - \rho^2}}\right) - \frac{1}{2}, \text{ if } hk < 0 \text{ or if } hk = 0, h \text{ or } k < 0 \end{cases}$$

which expresses the bivariate normal in terms of the  $G$ - and  $T$ -functions in a compact form.

A series expression for  $T(h, a)$  may be obtained by expanding the numerator of the integrand of (3.3) in the usual exponential series, dividing by the denominator, and integrating term by term. Rearrangement of the terms of this series gives

$$(3.9) \quad T(h, a) = \frac{\arctan a}{2\pi} - \frac{1}{2\pi} \sum_{j=0}^{\infty} c_j a^{2j+1},$$

where

$$c_j = (-1)^j \frac{1}{2j+1} \left[ 1 - \exp(-\frac{1}{2}h^2) \sum_{i=0}^j \frac{h^{2i}}{2^i i!} \right],$$

which converges rapidly for small values of  $a$  and  $h$ .

The values of  $T(h, a)$  given in Tables A, B, and C were computed using the series (3.9). They were checked by using Gauss' seven-point integration formula on (3.3). The tables were also checked by taking differences. These checks show that at the points of tabulation the table is accurate to as many places as given, i.e., to six decimal places.

**4. Interpolation in the tables.** Table A has a coarse interval in the parameter  $a$  and an interval fine enough for ordinary linear interpolation in the parameter  $h$ . Table B has intervals in parameter  $a$  fine enough for ordinary linear interpolation and has parameter  $h$  at a coarse interval. Ordinary linear interpolation

TABLE A OF  $T(h, 0) = 0$ Note that  $T(h, 0) = 0$ 

$\Delta$	$\Delta^2$	$.35$	$.50$	$.75$	$1.00$	$.25$	$.50$	$.75$	$1.00$
		$\Delta$		$h$		$h^2$		$h^3$	
0.00	.038990	.073792	.102116	.125000	.0.30	.037210	.0.07287	.0.097186	.118048
0.01	.9886	.768	.110	.125992	.0.31	.121	.0.066	.0.096811	.117592
0.02	.982	.776	.392	.968	.0.32	.005	.0.069828	.487	.123
0.03	.972	.756	.363	.928	.0.33	.036882	.580	.122	.116641
0.04	.958	.788	.321	.873	.0.34	.756	.233	.095748	.116
0.05	.910	.692	.267	.801	.0.35	.627	.0.068812	.0.094271	.115639
0.06	.918	.649	.202	.714	.0.36	.495	.542	.569	.11587
0.07	.892	.597	.124	.611	.0.37	.359	.0.068828	.0.094271	.11587
0.08	.862	.538	.035*	.492	.0.38	.220	.265*	.157	.044
0.09	.829	.470	.101314	.357	.0.39	.078	.0.067983	.0.093736	.113489
0.10	.791	.395-	.821	.207	.0.40	.035933	.694	.306	.112922
0.11	.750-	.312	.697	.011	.0.41	.785-	.399	.0.092668	.344
0.12	.704	.222	.561	.12860	.0.42	.634	.0.098	.1.20	.111755*
0.13	.655-	.122	.113	.663	.0.43	.479	.0.066791	.0.093965-	.155*
0.14	.602	.016	.253	.450*	.0.44	.322	.479	.501	.110545-
0.15	.515-	.072802	.082	.223	.0.45	.162	.161	.028	.109924
0.16	.484-	.780	.100900	.125980	.0.46	.034999	.0.065337	.0.090448	.293
0.17	.449	.651	.706	.722	.0.47	.834	.508	.0.060	.108652
0.18	.350*	.514	.501	.449	.0.48	.685*	.173	.0.08564	.001
0.19	.278	.369	.285-	.162	.0.49	.194.	.0.061834	.0.060	.107344
0.20	.202	.217	.057	.128859	.0.50	.320	.489	.0.088519	.106671
0.21	.122	.058	.099618	.542	.0.51	.111	.139	.0.087506	.105993
0.22	.038	.071891	.569	.210	.0.52	.0.033965-	.0.063784	.0.085973	.104669
0.23	.037951	.717	.308	.120864	.0.53	.783	.424	.435-	.103965+
0.24	.860	.535*	.037	.503	.0.54	.599	.059		
0.25	.766	.347	.098755-	.129	.0.55	.113	.0.062690	.0.085889	.193
0.26	.668	.151	.462	.119710	.0.56	.224	.316	.337	.102473
0.27	.566	.070916	.158	.337	.0.57	.033	.0.061938	.0.084779	.101715*
0.28	.461	.738	.097644	.11921	.0.58	.0.032810	.555*	.215-	.100268
0.29	.352	.521	.520	.492	.0.59	.645-	.168	.0.083645-	
0.30	.210	.297	.186	.0.18	.0.60	.147	.0.060778	.069	.095519

TABLE A OF  $T(h,a)$ 

$\sqrt{a}$	.25	.50	.75	1.00	$\sqrt{a}$	.25	.50	.75	1.00
0.60	.032117	.060778	.083669	.099519	0.90	.055791	.017700	.064013	.075091
0.61	217	383	651	.098764	0.91	.0554	.063317	.071451	.071451
0.62	016	.059811	.082187	.081901	0.92	.0551	.062861	.071451	.071451
0.63	.031812	581	175+	.081234	0.93	.0548	.06775-	.072572	.072572
0.64	636	175+	.080712	.096560	0.94	.054818	.061349	.071734	.071734
0.65	429	.058655+	.079504	.110	.095681	0.95	.0602	.070898	.070898
0.66	219	352	.079504	.091896	0.96	.0601	.06920	.071451	.071451
0.67	908	.057936	.078693	.106	0.97	.053886	.05954	.069230	.069230
0.68	.030794	516	278	.093312	0.98	.053886	.05890	.068398	.068398
0.69	581	093	.077658	.099512	0.99	.053886	.05392	.067569	.067569
0.70	365-	.056667	.035+	.091709	1.00	1.08	.0602	.06504	.06504
0.71	117	239	.076108	.096901	1.01	.0601	.06920	.071451	.071451
0.72	.02928	.055077	.075777	.089	1.02	.022931	.041677	.055386	.055386
0.73	707	373	113	.082714	1.03	.0601	.064276	.063159	.063159
0.74	485+	.051937	.071506	.088455+	1.04	.0548	.051729	.051729	.051729
0.75	262	198	.073866	.087634	1.05	.021978	.039336	.052770	.052770
0.76	938	056	223	.086809	1.06	.021978	.039336	.052770	.052770
0.77	.028812	.053613	.072577	.089282	1.07	.038922	.051714	.062626	.062626
0.78	585-	167	.071928	.152	1.08	.038922	.051714	.059174	.059174
0.79	357	.052120	278	.081320	1.09	.038922	.051714	.059174	.059174
0.80	128	270	.070625-	.083186	1.10	.030	.056830	.056830	.056830
0.81	.02798	.051819	.06970	.082651	1.11	.020724	.012	.057839	.057839
0.82	667	367	313	.081814	1.12	.0559	.037559	.048311	.051
0.83	435-	.050912	.068555-	.080975+	1.13	.0325-	.0107	.052668	.052668
0.84	202	457	.057995-	.07995-	1.14	.091	.036657	.277	.055189
0.85	.026968	000-	.06333	.079296	1.15	.019057	.209	.047616	.051714
0.86	734	.019542	.064671	.078145+	1.16	.025-	.035162	.053945	.053945
0.87	499	083	007	.077614	1.17	.393	.317	.06393	.180
0.88	264	.018622	.065313	.076773	1.18	.162	.034874	.052771	.052771
0.89	027	161	.061678	.075932	1.19	.01831	.432	.051664	.152
0.90	.025791	.017700	013	.091	1.20	.702	.033993	.041537	.056911

TABLE A OF T(h, a)

$\sqrt{a}$	$h$	.25	.50	.75	1.00	.25	.50	.75	1.00
1.20	.018702	.033993	.015537	.050914	1.50	.012372	.022006	.028029	.031172
1.21	1.73	.556	.015925+	.0169	1.51	.021654	.027553	.030514	.030514
1.22	246	120	.016930	.017	1.52	.011997	.305+	.082	.063
1.23	.019	.032687	.042712	.018696	1.53	.812	.020939	.026616	.029519
1.24	.017794	.256	.011967	.112	1.54	.628	.617	.155+	.028982
1.25	.569	.031828	.015155-	.016527	1.55	1.46	.025700	.451	
1.26	.345+	.402	.016922	.016527	1.56	.266	.019913	.219	.027927
1.27	123	.030978	.015925	.015815-	1.57	.088	.611	.024804	.110
1.28	.016902	.556	.039748	.167	1.58	.010911	.283	.364	.026899
1.29	682	138	.011969	.109	1.59	.736	.018958	.023929	.395+
1.30	.163	.029721	.038590	.013715+	1.60	.563	.637	.500-	.025898
1.31	.215-	.303	.018	.027	1.61	.391	.319	.075+	.408
1.32	028	.028897	.037150+	.012345+	1.62	.231	.005-	.022656	.021924
1.33	.015513	.488	.036886	.011670	1.63	.053	.017694	.212	.147
1.34	599	083	.036886	.000	1.64	.009837	.387	.021833	.023976
1.35	.387	.027680	.035773	.010337	1.65	.723	.083	.130	.512
1.36	.176	.281	.021680	.019680	1.66	.560	.016083	.032	.055-
1.37	.011966	.026884	.031678	.030	1.67	.399	.486	.020638	.022604
1.38	757	1.90	.038386	.137	1.68	.210	.193	.250+	.159
1.39	.550+	.099	.033601	.037749	1.69	.082	.015903	.019868	.021211
1.40	.345-	.025711	.070	.118	1.70	.008927	.617	.190	.290
1.41	110	.326	.032543	.036493	1.71	.773	.334	.117	.020862
1.42	.013938	.021694	.022	.035875+	1.72	.621	.055+	.018750-	.146
1.43	737	.566	.031505+	.261	1.73	.470	.014780	.388	.033
1.44	537	190	.036994	.031659	1.74	.322	.508	.030	.019627
1.45	.339	.023818	.187	.061	1.75	175+	.017678	.227	
1.46	.112	.449	.029985+	.033170	1.76	.030	.013974	.331	.018833
1.47	.012947	.081	.028855+	.012885	1.77	.007887	.713	.016989	.446
1.48	754	.022721	.028997	.308	1.78	.745+	.454	.651	.064
1.49	562	362	.031736	.511	1.79	.605+	.200	.319	.017689
1.50	.372	.006	.029	.172	1.80	.467	.012949	.015992	.320

TABLE A OF  $\mathcal{N}(h, a)$ 

$h^a$	.25	.50	.75	1.00	$h^a$	.25	.50	.75	1.00
.007167	.012949	.015992	.017320	.017320	.002655*	.004334	.005079	.005305-	.005305-
1.80	331	701	669	669	.016956	2.32	2.32	.001825*	.034
1.81	197	457	352	352	.015956	2.34	2.34	.001825*	.034
1.82	664	216	039	039	.015956	2.36	2.36	.001825*	.034
1.83	006933	011978	011731	011731	.015901	2.38	2.38	.001825*	.034
1.84						169	169	.001825*	.034
1.85	804	744	428	561	.013836	2.40	2.40	.001967	206
1.86	676	514	130	227	.013836	2.42	2.42	.001967	211
1.87	550*	286	547*	561	.013836	2.44	2.44	.001967	217
1.88	426	062	547*	575*	.013836	2.46	2.46	.001967	227
1.89	304	010811	263	258	.013836	2.48	2.48	.001967	231
1.90	183	624	012983	013916	.013916	2.50	2.50	.001967	235
1.91	064	410	708	639	.013916	2.52	2.52	.001967	239
1.92	005927	199	137	338	.013916	2.54	2.54	.001967	243
1.93	831	006991	171	012752	.013916	2.56	2.56	.001967	247
1.94	717	786	011909	011909	.013916	2.58	2.58	.001967	251
1.95	605*	585*	652	667	.013916	2.60	2.60	.001967	255
1.96	495*	387	399	187	.013916	2.62	2.62	.001967	259
1.97	386	192	150*	011911	.013916	2.64	2.64	.001967	263
1.98	278	000	00965*	011911	.013916	2.66	2.66	.001967	267
1.99	172	008611	665*	665*	.013916	2.68	2.68	.001967	271
2.00	068	624*	129	116	.013916	2.70	2.70	.001967	275
2.02	004865*	263	009710	010611	.013916	2.72	2.72	.001967	279
2.04	667	007912	527	009656	.013916	2.74	2.74	.001967	283
2.06	176	573	099	009656	.013916	2.76	2.76	.001967	287
2.08	006929	245*	008687	008687	.013916	2.78	2.78	.001967	291
2.10	112	006929	291	008773	.013916	2.80	2.80	.001967	295
2.12	003939	624	007909	007909	.013916	2.82	2.82	.001967	299
2.14	330	330	542	007958	.013916	2.84	2.84	.001967	303
2.16	610	046	183	571*	.013916	2.86	2.86	.001967	307
2.18	153	005772	006819	207	.013916	2.88	2.88	.001967	311
2.20	303	509	523	006855*	.013916	2.90	2.90	.001967	315
2.22	157	255*	209	517	.013916	2.92	2.92	.001967	319
2.24	017	011	191	005909	.013916	2.94	2.94	.001967	323
2.26	002861	001777	620	005884	.013916	2.96	2.96	.001967	327
2.28	751	551	344	588	.013916	2.98	2.98	.001967	331
2.30	625*	334	079	305*	.013916	3.00	3.00	.001967	335

TABLE B OF  $T(h, \alpha)$

TABLE B OF  $\mathbf{T}(b, a)$ 

$b$	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00	
.30	.046387	.044918	.040787	.034728	.020761	.014578	.009599	.005929	.003434	.001866	.000951	.000455-		
.31	.047843	.046326	.042571	.021381	.01503	.01503	.006092	.002692	.001424	.000642	.000319	.000165+		
.32	.049291	.047725-	.043319	.029407	.029407	.029407	.010142	.004253	.001425	.000615+	.000319	.000165+		
.33	.050750	.049115+	.045573	.037916	.030235-	.026601	.018339	.004066	.001111	.000423	.0002006	.0001019	.0000486	
.34	.052161	.050497	.045810	.038963	.031868	.024138	.017050+	.011175+	.007011	.003434	.001619	.000642	.000250	
.35	.053593	.051871	.041871	.040001	.031868	.024138	.017050+	.011175+	.007011	.003434	.001619	.000642	.000250	
.36	.054997	.053254+	.048232	.041031	.032673	.024138	.017050+	.011175+	.007011	.003434	.001619	.000642	.000250	
.37	.056401	.055921	.049501	.042052	.033470	.025539	.014361	.009266	.004433	.002286	.001034	.000455+	.000165+	
.38	.057797	.05937	.050711	.043052	.034259	.025533	.014361	.009266	.004433	.002286	.001034	.000455+	.000165+	
.39	.059183	.057275-	.051922	.044040	.035040	.026098	.018210	.009140	.004394	.002140	.001033	.000455+	.000165+	
.40	.060559	.068602	.053103	.040563	.03103	.021207	.012140	.00584+	.002140	.001033	.000455+	.000165+	.0000486	
.41	.061927	.059921	.054254+	.046048	.036576	.021207	.012140	.00584+	.002140	.001033	.000455+	.000165+	.0000486	
.42	.063284	.061230	.055498	.047025-	.031332	.021207	.012140	.00584+	.002140	.001033	.000455+	.000165+	.0000486	
.43	.064633	.062559	.056621	.048950-	.038079	.022886	.017500+	.013918	.009318	.00403	.00171	.000642	.000250	
.44	.065971	.063818	.057775-	.048950-	.035957	.022886	.017500+	.013918	.009318	.00403	.00171	.000642	.000250	
.45	.067299	.065098	.058918	.049898	.039547	.029336	.019898	.012140	.00584+	.002140	.001033	.000455+	.000165+	
.46	.068618	.066368	.060032	.050837	.040268	.03849	.027207	.017500+	.013918	.009318	.00403	.00171	.000642	
.47	.069966	.067638	.061176	.051767	.048079	.030352+	.021043	.014361	.009266	.004433	.002286	.001034	.000455+	
.48	.071225+	.068817	.062290	.052687	.041684	.03854	.027207	.017500+	.013918	.009318	.00403	.00171	.000642	
.49	.072513	.070117	.063395-	.053597	.042379	.031345-	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.50	.073792	.071347	.064499	.054498	.043052-	.032304	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.51	.075060	.072556	.065273	.055389	.04742	.032304	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.52	.076318	.073774+	.066647	.056270	.044410	.032304	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.53	.077566	.074974	.067710	.057141	.045069	.033232	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.54	.078803	.076153	.068764	.058003	.04720	.036232	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.55	.080039	.077311	.069807	.059854	.046361	.031131	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.56	.081247	.078509	.070841	.059697	.046361	.031131	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.57	.082453	.079667	.071861	.060530	.047618	.031131	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.58	.083619	.080834	.072876	.061353	.048233	.031131	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.59	.084835+	.082191	.073879	.062105+	.04939	.031131	.02379	.014361	.009266	.004433	.002286	.001034	.000455+	
.60	.086010	.083078	.074871	.064081	.049436	.036246	.031131	.02379	.014361	.009266	.004433	.002286	.001034	
.61	.087176	.084194	.075841	.063762	.050024	.04646	.025081	.016048	.006048	.00383	.00184	.000642	.000250	
.62	.088330	.085306	.076846	.064546	.0604	.037040	.0326	.0188	.007917	.00418	.00217	.001034	.000455+	
.63	.089475-	.086396	.077788	.065320	.051177+	.0426	.0566	.0324	.017500+	.00452	.00227	.001034	.000455+	
.64	.090609	.0871482	.078739	.066034	.056034	.0738	.0804+	.0800	.0456	.0181	.00917	.00409	.00171	
.65	.091733	.088557	.079681	.0839	.052291	.038177	.026028	.026028	.015124	.0112	.005642	.002286	.000455+	

TABLE B OF T(h, a)

$\Delta h$	0.00	0.25	0.50	0.75	1.00	1.25	1.50	1.75	2.00	2.25	2.50	2.75	3.00
.65	.091733	.088557	.079681	.066939	.052291	.038177	.026228	.016585	.009885	.005516	.002884	.001115	.000651
.66	.092817	.089622	.080612	.067584	.053373	.041129	.029125	.017099	.009479	.005016	.002097	.000870	.000353
.67	.093950*	.090677	.081534	.063320	.050913	.039550-	.027162	.016969	.009381	.005010	.002124	.000892	.000355
.68	.095033	.092122	.082455*	.069046	.054421	.042122	.029492	.018021	.009710	.005212	.002321	.000949	.000356
.69	.096127	.092756	.083317	.0762	.051135-	.037063	.027091	.017063	.0127	.007028	.003288	.001333	.000658
.70	.097200	.093781	.084239	.070169	.051167	.035135-	.026161	.018989	.0127	.007363	.003677	.001437	.000659
.71	.098263	.094796	.085120	.075120	.056116	.040166	.028192	.018126	.0127	.007777	.003952	.001431	.000660
.72	.099316	.095800	.086179*	.082454	.070535-	.056116	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.73	.100360	.096795-	.086854	.07854	.073205-	.057166	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.74	.101393	.097780	.087707	.073205-	.057166	.041121	.028191	.018126	.0127	.007770	.003950	.001431	.000664
.75	.102416	.098755-	.088519	.0866	.057165-	.05115-	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.76	.103430	.099720	.089382	.0871518	.071518	.05115-	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.77	.104434	.100675*	.090206	.075161	.058282	.041121	.028191	.018126	.0127	.007770	.003950	.001431	.000665
.78	.105428	.101621	.091020	.0791	.0728	.05115-	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.79	.106433	.102557	.0824	.0761420	.059167	.0666	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.80	.107388	.103484	.092620	.077036	.0598	.041121	.028191	.018126	.0127	.007770	.003950	.001431	.000668
.81	.108354	.104101	.093106	.09143	.060022	.041305*	.029002	.018126	.0127	.007770	.003950	.001431	.000669
.82	.109310	.105309	.091182	.078242	.0538	.041121	.028191	.018126	.0127	.007770	.003950	.001431	.000669
.83	.110257	.106208	.0950	.07832	.0832	.0721	.05115-	.041121	.028191	.018126	.0127	.007770	.003950
.84	.111192*	.107097	.09708	.079411	.061218	.0970	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.85	.112124	.108813	.0977	.096158	.0988	.0642	.041121	.028191	.018126	.0127	.007770	.003950	.001431
.86	.113043	.108818	.097198	.080553	.0801109	.0409	.0683	.041121	.028191	.018126	.0127	.007770	.003950
.87	.114054	.109710	.09653	.09653	.0958	.0782	.0210	.030068	.018126	.0127	.007770	.003950	.001431
.88	.114955	.110563	.09653	.09653	.0958	.0782	.0210	.030068	.018126	.0127	.007770	.003950	.001431
.89	.115747	.111107	.099367	.082199	.063118	.045132	.030068	.018126	.0127	.007770	.003950	.001431	.000672
.90	.116631	.112243	.100073	.731	.507	.348	.185-	.717	.929	.969	.978	.986	.993
.91	.117506	.113069	.770	.083256	.859	.559	.298-	.877	.974	.986	.993	.999	.999
.92	.118372	.114087	.103158	.0811281	.544	.967	.514	.927	.995-	.995-	.995-	.995-	.995-
.93	.119230	.114696	.102138	.0811281	.544	.967	.514	.927	.995-	.995-	.995-	.995-	.995-
.94	.120079	.115497	.810	.783	.877	.046363	.617	.011015-	.006001	.0714	.482	.673	.673
.95	.920	.116290	.103174	.081276	.0655203	.355-	.717	.019021	.034	.0036	.076	.683	.673
.96	.121752	.117074	.101129	.762	.523	.512	.811	.066	.052	.015-	.078	.683	.674
.97	.122577	.11850-	.777	.086241	.837	.721	.908	.109	.059	.021	.080	.684	.674
.98	.123392	.118617	.105116	.713	.066115-	.902	.999	.150-	.086	.027	.082	.485-	.674
.99	.124601	.119377	.106047	.087177	.116	.017075-	.031087	.189	.101	.032	.084	.485+	.674
1.00	.125000	.120129	.671	.742	.244	.348	.227	.116	.038	.086	.485+	.674	.674
$\infty$	.250000	.206647	.154269	.113311	.079328	.052825-	.033404	.020030	.375+	.112	.105-	.485+	.674

TABLE C OF T(h, a)

may be used throughout Table C. Tables A and B were designed for interpolation as follows: To interpolate for a value  $T(h_2, a_2)$ , say,  $a_1$  and  $a_3$  should be picked closest to  $a_2$  from Table A so that  $a_1 \leq a_2 < a_3$ , and  $h_1$  and  $h_3$  should be picked closest to  $h_2$  from Table B so that  $h_1 \leq h_2 < h_3$ . Then the interpolated value of  $T(h_2, a_2)$  is obtained from

$$T(h_2, a_2) = \sum_{i=1}^3 \sum_{j=1}^3 w_{ij} T(h_i, a_j),$$

where the weights  $w_{ij}$  are given by

$$w_{ij} = \begin{pmatrix} -(1-b)(1-c) & 1-c & -b(1-c) \\ (1-b) & 0 & b \\ -(1-b)c & c & -bc \end{pmatrix},$$

where

$$b = \frac{a_2 - a_1}{a_3 - a_1} \quad \text{and} \quad c = \frac{h_2 - h_1}{h_3 - h_1}.$$

The weights were obtained by considering the result of ordinary linear interpolation where nearby values of the function are subtracted before interpolating, say,  $T(h_2, a_1) - T(h_1, a_1)$  and  $T(h_2, a_3) - T(h_1, a_3)$ . These numbers are interpolated with respect to  $a$  to obtain  $T(h_2, a_2) - T(h_1, a_2)$ , and then  $T(h_1, a_2)$  is added. This process may also be followed with  $(h_3, a_1)$ ,  $(h_3, a_3)$ , and  $(h_3, a_2)$ . If the two estimates of  $T(h_2, a_2)$  are then combined as in linear interpolation with respect to  $h$ , i.e.,  $(1-c)$  times the first estimate plus  $c$  times the second, the above weights  $w_{ij}$  follow. The interpolation on the differences could also have been first with respect to  $h$  to obtain the two estimates and then with respect to  $a$  between these two. The same weights  $w_{ij}$  are obtained by doing this.

This method of interpolation has resulted in approximately a 90 per cent reduction over the size of a table needed for linear interpolation. Quadratic interpolation using Bessel's formula would give comparable results to the new method with approximately an additional 80 per cent reduction in the number of entries, but the additional work involved more than outweighs that reduction in the number of entries, even though the table is used only a few times. The procedure given here may be termed a compromise between linear and quadratic interpolation.

**EXAMPLE.** Find  $T(.15, .625)$ . From the tables, the following entries are extracted:

$h$	$a$		
	.50	.625	.75
0	.073792	.088903	.102416
.15	.072902		.101082
.25	.071347	.085848	.098755

The weights to be applied are

$h$	$a$		
	.50	.625	.75
0	-.2	.4	-.2
.15	.5		.5
.25	-.3	.6	-.3

The result is  $T(.15, .625) = .0877898$ . Calculation of this number from the series gives .0877919. The result of the interpolation therefore provides a difference of two in the sixth place. Further calculations show that this difference could be reduced to five in the seventh place if the linear interpolations for  $T(0, .625)$  and  $T(.25, .625)$  were eliminated and the exact values for these points were used. The value for  $T(0, .625)$  was rounded up during the linear interpolation with respect to  $a$  in Table B, since second differences in the  $a$  direction for all  $h$  are negative. A similar working rule for rounding when interpolating in the  $h$  direction is to round up the interpolated value when  $0 < h < .9$  and to round down for  $h \geq .9$  in Table A. The value obtained from the above interpolation scheme should be rounded up for  $0 < h < 1.50$  and rounded down for  $h \geq 1.50$ , for all values of  $a > 0$ .

Empirical examination of the errors in interpolation by this scheme shows that the maximum error that would occur anywhere in Tables A and B is seven in the sixth decimal place, and that this could be reduced to six in the sixth decimal place if the linear interpolations in Tables A and B were eliminated and the exact values were used. Linear interpolation in Table C gives errors less than four in the sixth decimal place. Table D gives the maximum error in the sixth decimal place, which will be committed when using the above

TABLE D

$h$	$a$			
	0.00-0.25	0.25-0.50	0.50-0.75	0.75-1.00
0.00-0.25	+0.9	+2.1	+2.8	+3.3
0.25-0.50	+1.3	+3.1	+4.4	+5.3
0.50-0.75	+1.7	+4.4	+6.2	+7.1
0.75-1.00	+2.0	+4.9	+6.5	+6.4
1.00-1.25	+1.8	+4.2	+4.6	+3.1
1.25-1.50	+1.3	+2.5	+1.5	-1.3
1.50-1.75	+0.5	+0.5	-1.7	-3.9
1.75-2.00	-0.3	-1.6	-3.6	-4.7
2.00-2.25	-0.8	-2.6	-4.0	-3.9
2.25-2.50	-1.0	-2.8	-3.4	-2.6
2.50-2.75	-1.0	-2.4	-2.4	-1.4
2.75-3.00	-0.8	-1.7	-1.4	-0.7

interpolation scheme over the ranges of  $h$  and  $a$  indicated. The sign preceding the entry is the sign of the exact value of  $T(h, a)$  minus the interpolated value for that difference which is the largest in absolute value. These are empirical results obtained on the digital computer by using the interpolation process and the exact value for fifteen points systematically spaced in each block. A number in Tables A, B, and C whose last nonzero digit is five is followed by a plus or minus sign, respectively, to indicate that the number should be rounded up or down when dropping the digit with the five. Any entry having the first three digits the same as those of the entry directly above it has had these digits dropped.

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