The writer wishes to express his indebtedness to Professor C. W. Nelson of the University of California, Berkeley, who has checked the numerical tables.

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Tables for the Rapid and Accurate Numerical Evaluation of Certain Infinite Integrals Involving Bessel Functions

Introduction. In a recent paper [1] the author has formulated a method based on Euler's transformation of slowly convergent alternating series for the numerical evaluation of integrals of the form $\int_a^\infty f(x)dx$, where a is a constant and where $f(x)$ oscillates about zero in such a way that the integral over each half-cycle is smaller in absolute magnitude than (and opposite in sign to) that over the preceding half-cycle. The author has had occasion to make much use of this method for the evaluation of integrals of the type

$$\int_0^\infty J_0(x)g(x)dx, \tag{1}$$

and

$$\int_0^\infty J_1(x)h(x)dx, \tag{2}$$

where $g(x)$, $h(x)$ are well-behaved continuous functions which tend to a finite constant value or zero as x tends to infinity; here $J_0(x)$, $J_1(x)$ are Bessel Functions (of the first kind) of orders zero and one, respectively. The present paper gives tables useful in the evaluation of (1), (2).

The author is grateful to Yigal Accad who performed most of the numerical calculations for Tables 1 and 2.

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Description of the Method. The method employed [1] in evaluating integrals of the type (1) involved performing the integration over each of the first twenty half-cycles, i.e., evaluating the integrals

$$\int_{x_{i-1}}^{x_i} J_0(x)g(x)dx \quad i = 1, 2, \cdots, 20 \tag{3}$$

where x_0 is zero and x_i is the ith zero of $J_0(x)$. The first twenty terms of a slowly convergent alternating series for (1) were thus obtained. Euler's transformation (Bromwich [2], p. 62) was then applied to this series in order to obtain a rapidly convergent series for the numerical value of (1). Integrals of the type (2) were treated similarly.

In order to obtain high accuracy in the separate integrations (3) the Gauss quadrature formula (NBS AMS 37 [3]) was used for sixteen points of subdivision of each interval. Tables of abscissae and coefficients are given in [3] for $n = 2(1)16$, 15D, and in Davis and Rabinowitz [7] for $n = 2, 4, 8, 16, 20, 24, 32, 40, 48$, 20D. Values are also available in [7] for $n = 64$, 80 and 96. For a given value of n, Gauss' formula provides an approximation equivalent to replacing the integrand by a polynomial of degree $2n - 1$.

Description and Use of the Tables. Table 1 gives values of x and the corresponding values of $J_0(x)$. The values of x were obtained by computation from values given in [3] for the interval $(-1, 1)$ by means of the formula

$$x_i = x_i^1 \frac{q - p}{2} + \frac{q + p}{2}$$

where p, q are the lower and upper limits of integration, and in our case are zeros of $J_0(x)$ (the first value of p is zero), and the x_i^1 (given in [3]) refer to the interval $(-1, 1)$. The zeros of $J_0(x)$ are also given.

The values of $J_0(x)$ were obtained by interpolation from the Harvard University tables [4]. In view of the high accuracy needed interpolation was effected by means of the first four terms of Taylor's theorem

$$\begin{aligned} J_0(x + h) &\doteqdot J_0(x) + hJ_0'(x) + \frac{h^2}{2!} J_0''(x) + \frac{h^3}{3!} J_0'''(x) \\ &= J_0(x) - hJ_1(x) + \tfrac{1}{2}h^2 \left(\frac{J_1(x)}{x} - J_0(x) \right) \\ &\qquad + \tfrac{1}{6}h^3 \left(J_1(x) + \frac{J_0(x)}{x} - \frac{2J_1(x)}{x^2} \right), \end{aligned}$$

these terms being sufficient for 10 place accuracy with the maximum value of h required. The tables are believed to be accurate to within a few units in the tenth place of decimals.

Table 2 was similarly prepared for $J_1(x)$. Here the interpolation for $J_1(x)$

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was effected from [4] by means of the Taylor's theorem expansion

$$J_1(x+h) \doteqdot J_1(x) + h\left(J_0(x) - \frac{J_1(x)}{x}\right) + \tfrac{1}{2}h^2\left(\frac{2J_1(x)}{x^2} - \frac{J_0(x)}{x} - J_1(x)\right) + \tfrac{1}{6}h^3\left(\frac{3J_0(x)}{x^2} - \frac{6J_1(x)}{x^3} + \frac{2J_1(x)}{x} - J_0(x)\right).$$

In Tables 1 and 2 the zeros of $J_0(x)$, $J_1(x)$ were taken from tables in BAASMTC [5].

Table 3 gives the appropriate integration coefficients (these are quoted from

TABLE 3. *Gauss Integration Coefficients*

.02715 24594
.06225 35239
.09515 85117
.12462 89713
.14959 59888
.16915 65194
.18260 34150
.18945 06105
.18945 06105
.18260 34150
.16915 65194
.14959 59888
.12462 89713
.09515 85117
.06225 35239
.02715 24594

[3] for the interval $(-1, 1)$). After using them the result must be multiplied by half the length of the interval in each case.

Method of Checking. The tables were checked as follows:

Table 1 was checked by using it to calculate

$$\int_{x_{i-1}}^{x_i} xJ_0(x)dx$$

over each half-cycle and comparing the result with the known value of the integral

$$[xJ_1(x)]_{x_{i-1}}^{x_i} = x_iJ_1(x_i) - x_{i-1}J_1(x_{i-1}),$$

the values of $J_1(x_i)$ being obtained from [5]. In each case agreement was obtained to within the accuracy warranted by the number of significant figures in the table.

Table 2 was checked by using it to calculate

$$\int_{y_{i-1}}^{y_i} J_1(x)dx = [-J_0(x)]_{y_{i-1}}^{y_i} = J_0(y_{i-1}) - J_0(y_i)$$

over each half-cycle, where the y_i are the zeros of $J_1(x)$. The results were calculated to ten places of decimals and in each case were accurate to within two units in the last place.

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TABLE 1

	x	$J_0(x)$		x	$J_0(x)$
	.00000 00000			5.52007 81103	
1	.01274 44512	.99995 93952	1	5.53668 49893	.00564 19942
2	.06664 37005	.99888 99624	2	5.60691 93443	.02928 21569
3	.16156 67593	.99348 46849	3	5.73061 04883	.06978 62151
4	.29410 48650	.97849 22054	4	5.90331 65740	.12301 12685
5	.45947 04869	.94791 40344	5	6.11879 91260	.18206 24818
6	.65168 75524	.89661 10444	6	6.36927 09191	.23760 14372
7	.86380 90708	.82197 96824	7	6.64567 95559	.27944 55482
8	1.08816 85230	.72517 29892	8	6.93803 50453	.29921 03673
9	1.31665 70348	.61136 06080	9	7.23577 09779	.29294 41222
10	1.54101 64869	.48883 00670	10	7.52812 64673	.26243 80011
11	1.75313 80053	.36721 18144	11	7.80453 51041	.21449 26872
12	1.94535 50709	.25549 12057	12	8.05500 68972	.15853 58154
13	2.11072 06927	.16052 05519	13	8.27048 94492	.10381 91971
14	2.24325 87984	.08645 02192	14	8.44319 55349	.05742 97048
15	2.33818 18573	.03506 03860	15	8.56688 66789	.02366 25625
16	2.39208 11065	.00663 36641	16	8.63712 10339	.00451 20991
	2.40482 55577			8.65372 79129	
1	2.42133 49398	−.00854 11306	1	8.67035 68206	−.00450 94160
2	2.49115 69581	−.04397 83230	2	8.74068 43400	−.02345 71287
3	2.61412 19276	−.10342 59686	3	8.86453 95579	−.05612 15388
4	2.78581 40782	−.17919 07027	4	9.03747 47347	−.09944 20369
5	3.00003 15602	−.26006 26555	5	9.25324 31200	−.14809 16553
6	3.24903 28661	−.33251 74168	6	9.50404 71589	−.19457 96817
7	3.52381 87438	−.38328 09269	7	9.78082 24462	−.23045 12070
8	3.81445 78522	−.40269 93971	8	10.07356 57393	−.24845 57197
9	4.11044 58158	−.38757 11113	9	10.37169 66127	−.24484 61857
10	4.40108 49242	−.34203 66196	10	10.66443 99058	−.22066 46191
11	4.67587 08019	−.27600 29383	11	10.94121 51931	−.18130 82267
12	4.92487 21078	−.20186 70491	12	11.19201 92320	−.13461 51679
13	5.13908 95898	−.13110 13628	13	11.40778 76173	−.08848 00142
14	5.31078 17404	−.07207 25099	14	11.58072 27941	−.04908 31439
15	5.43374 67099	−.02957 09059	15	11.70457 80120	−.02026 32192
16	5.50356 87282	−.00562 57234	16	11.77490 55314	−.00386 81029
	5.52007 81103			11.79153 44391	

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TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	11.79153 44391			18.07106 39679	
1	11.80817 17029	.00386 45915	1	18.08770 75348	.00312 28948
2	11.87853 45618	.02012 43811	2	18.15809 70515	.01627 88282
3	12.00245 20168	.04823 70070	3	18.28206 14541	.03908 92889
4	12.17547 40932	.08568 63326	4	18.45514 90820	.06960 69216
5	12.39135 09018	.12799 26729	5	18.67110 76781	.10428 43725
6	12.64228 09694	.16873 86824	6	18.92213 28135	.13794 52734
7	12.91919 53359	.20055 59550	7	19.19915 20923	.16454 42527
8	13.21208 57321	.21699 70655	8	19.49215 34535	.17868 44763
9	13.51036 64157	.21458 33409	9	19.79054 71443	.17733 10142
10	13.80325 68119	.19401 42035	10	20.08354 85055	.16087 71698
11	14.08017 11784	.15987 41093	11	20.36056 77843	.13297 88950
12	14.33110 12460	.11900 00109	12	20.61159 29197	.09924 99767
13	14.54697 80546	.07837 94720	13	20.82755 15158	.06551 96366
14	14.72000 01310	.04355 03186	14	21.00063 91437	.03646 96948
15	14.84391 75860	.01799 93958	15	21.12460 35463	.01509 18172
16	14.91428 04449	.00343 81248	16	21.19499 30630	.00288 47612
	14.93091 77086			21.21163 66299	
1	14.94755 90159	−.00343 51301	1	21.22828 15866	−.00288 27393
2	15.01793 89756	−.01789 91458	2	21.29867 69809	−.01503 12702
3	15.14188 65473	−.04294 96191	3	21.42265 17345	−.03611 16663
4	15.31495 06747	−.07640 67698	4	21.59575 38153	−.06434 88934
5	15.53087 99495	−.11433 61880	5	21.81173 04440	−.09648 81035
6	15.78187 10027	−.15103 88400	6	22.06277 65401	−.12775 47593
7	16.05885 26700	−.17990 39278	7	22.33981 89501	−.15254 57750
8	16.35181 42497	−.19507 68842	8	22.63284 47770	−.16583 05446
9	16.65016 74269	−.19331 88429	9	22.93126 33838	−.16474 72215
10	16.94312 90066	−.17514 01558	10	23.22428 92107	−.14961 08067
11	17.22011 06739	−.14458 58966	11	23.50133 16207	−.12378 03876
12	17.47110 17271	−.10779 31239	12	23.75237 77168	−.09245 99613
13	17.68703 10019	−.07109 30831	13	23.96835 43455	−.06107 91462
14	17.86009 51293	−.03954 30642	14	24.14145 64263	−.03401 63997
15	17.98404 27010	−.01635 51985	15	24.26543 11799	−.01408 19757
16	18.05442 26607	−.00312 53477	16	24.33582 65742	−.00269 23112
	18.07106 39679			24.35247 15308	

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TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	24.35247 15308			30.63460 64684	
1	24.36911 74027	.00269 06195	1	30.65125 34327	.00239 91633
2	24.43951 66677	.01403 25020	2	30.72165 73180	.01251 61523
3	24.56349 82381	.03372 47328	3	30.84564 70254	.03009 59796
4	24.73660 98370	.06012 63461	4	31.01876 99856	.05369 50891
5	24.95259 83413	.09021 33604	5	31.23477 26651	.08063 46686
6	25.20365 82413	.11953 24224	6	31.48584 90422	.10694 79499
7	25.48071 58846	.14283 84354	7	31.76292 48688	.12793 92673
8	25.77375 78237	.15540 20499	8	32.05598 60401	.13934 93292
9	26.07219 28391	.15450 96015	9	32.35444 06419	.13870 48969
10	26.36523 47782	.14042 06959	10	32.64750 18132	.12619 35228
11	26.64229 24215	.11625 87219	11	32.92457 76398	.10458 43845
12	26.89335 23215	.08689 57273	12	33.17565 40169	.07823 96934
13	27.10934 08258	.05743 36567	13	33.39165 66964	.05175 15312
14	27.28245 24247	.03199 94405	14	33.56477 96566	.02885 08462
15	27.40643 39951	.01325 09005	15	33.68876 93640	.01195 21353
16	27.47683 32601	.00253 38391	16	33.75917 32493	.00228 60352
	27.49347 91320			33.77582 02136	
1	27.51012 56384	−.00253 24019	1	33.79246 75193	−.00228 49556
2	27.58052 75870	−.01320 95154	2	33.86287 28479	−.01192 16140
3	27.70451 38837	−.03175 59526	3	33.98686 50970	−.02867 17157
4	27.87763 20816	−.05663 87732	4	34.15999 16062	−.05116 72201
5	28.09362 88193	−.08502 21095	5	34.37599 87138	−.07686 29839
6	28.34469 82897	−.11271 68644	6	34.62708 02379	−.10198 26451
7	28.62176 64945	−.13477 53040	7	34.90416 17446	−.12204 77212
8	28.91481 96043	−.14672 13774	8	35.19722 89237	−.13298 71140
9	29.21326 59961	−.14596 96665	9	35.49568 96437	−.13242 66679
10	29.50631 91059	−.13273 89002	10	35.78875 68228	−.12052 95168
11	29.78338 73107	−.10995 96619	11	36.06583 83295	−.09992 72078
12	30.03445 67811	−.08222 81150	12	36.31691 98536	−.07478 03172
13	30.25045 35188	−.05437 12708	13	36.53292 69612	−.04947 71799
14	30.42357 17167	−.03030 32039	14	36.70605 34704	−.02758 90427
15	30.54755 80134	−.01255 14239	15	36.83004 57195	−.01143 12041
16	30.61795 99620	−.00240 04013	16	36.90045 10481	−.00218 65932
	30.63460 64684			36.91709 83537	

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TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	36.91709 83537			43.19979 17132	
1	36.93374 59205	.00218 56417	1	43.21643 96470	.00202 05537
2	37.00415 23534	.01140 44585	2	43.28684 76319	.01054 45035
3	37.12814 65473	.02743 21950	3	43.41084 45592	.02536 98152
4	37.30127 57720	.04896 56861	4	43.58397 76004	.04529 96158
5	37.51728 62677	.07357 53320	5	43.79999 28578	.06809 50035
6	37.76837 17300	.09765 02107	6	44.05108 38550	.09041 97790
7	38.04545 75826	.11690 15020	7	44.32817 58158	.10830 19105
8	38.33852 93584	.12742 34635	8	44.62125 40521	.11811 38815
9	38.63699 47599	.12693 02235	9	44.91972 60329	.11772 07481
10	38.93006 65358	.11556 53707	10	45.21280 42692	.10723 69760
11	39.20715 23884	.09584 13189	11	45.48989 62300	.08897 80881
12	39.45823 78507	.07174 25318	12	45.74098 72272	.06663 43209
13	39.67424 83464	.04747 84574	13	45.95700 24846	.04411 43990
14	39.84737 75711	.02647 94775	14	46.13013 55258	.02461 06003
15	39.97137 17650	.01097 29247	15	46.25413 24531	.01020 06291
16	40.04177 81979	.00209 90896	16	46.32454 04380	.00195 15847
	40.05842 57646			46.34118 83717	
1	40.07507 35355	−.00209 82432	1	46.35783 64373	−.00195 08992
2	40.14548 08322	−.01094 92382	2	46.42824 49795	−.01018 15645
3	40.26947 65473	−.02634 06534	3	46.55224 28884	−.02449 89606
4	40.44260 78959	−.04702 58304	4	46.72537 73001	−.04375 05072
5	40.65862 10416	−.07067 63342	5	46.94139 42674	−.06577 72703
6	40.90970 95842	−.09382 67282	6	47.19248 72523	−.08735 88503
7	41.18679 88362	−.11235 56730	7	47.46958 14065	−.10465 74117
8	41.47987 42074	−.12250 42137	8	47.76266 19629	−.11416 40209
9	41.77834 32704	−.12206 57451	9	48.06113 63063	−.11380 89156
10	42.07141 86416	−.11116 79231	10	48.35421 68627	−.10369 54948
11	42.34850 78936	−.09221 87488	11	48.63131 10169	−.08605 66385
12	42.59959 64362	−.06904 71411	12	48.88240 40018	−.06445 79246
13	42.81560 95819	−.04570 38541	13	49.09842 09691	−.04267 99914
14	42.98874 09305	−.02549 38175	14	49.27155 53808	−.02381 32307
15	43.11273 66456	−.01056 56711	15	49.39555 32897	−.00987 09780
16	43.18314 39423	−.00202 13118	16	49.46596 18319	−.00188 86072
	43.19979 17132			49.48260 98974	

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TABLE 1—*Continued*

	x	$J_0(x)$		x	$J_0(x)$
	49.48260 98974			55.76551 07550	
1	49.49925 80712	.00188 79842	1	55.78215 90943	.00177 84922
2	49.56966 70711	.00985 36907	2	55.85256 87941	.00928 29778
3	49.69366 57858	.02371 20292	3	55.97656 87414	.02234 17954
4	49.86680 13228	.04235 01575	4	56.14970 59995	.03991 06704
5	50.08281 96942	.06368 11286	5	56.36572 65182	.06002 74072
6	50.33391 43111	.08458 90824	6	56.61682 36311	.07975 79441
7	50.61101 02663	.10135 76521	7	56.89392 23408	.09559 78634
8	50.90409 27276	.11058 55944	8	57.18700 77154	.10433 45986
9	51.20256 90110	.11026 27587	9	57.48548 69658	.10406 33501
10	51.49565 14723	.10048 31520	10	57.77857 23404	.09486 31094
11	51.77274 74275	.08340 52240	11	58.05567 10501	.07876 32823
12	52.02384 20444	.06248 17090	12	58.30676 81630	.05901 96906
13	52.23986 04158	.04137 69630	13	58.52278 86817	.03909 30391
14	52.41299 59528	.02308 86479	14	58.69592 59398	.02181 80708
15	52.53699 46675	.00957 13465	15	58.81992 58871	.00904 57744
16	52.60740 36674	.00183 13567	16	58.89033 55869	.00173 09189
	52.62405 18411			58.90698 39261	
1	52.64070 01048	−.00183 07874	1	58.92363 23295	−.00173 04364
2	52.71110 94851	−.00955 55769	2	58.99404 23003	−.00903 24503
3	52.83510 88697	−.02299 63548	3	59.11804 27250	−.02174 01207
4	53.00824 53420	−.04107 61891	4	59.29118 06495	−.03883 90402
5	53.22426 48803	−.06177 33832	5	59.50720 19998	−.05842 15607
6	53.47536 08536	−.08206 70662	6	59.75830 00793	−.07763 33587
7	53.75245 83057	−.09835 15046	7	60.03539 98556	−.09306 32526
8	54.04554 23502	−.10732 38205	8	60.32848 63585	−.10158 19737
9	54.34402 02460	−.10702 86465	9	60.62696 67579	−.10133 15847
10	54.63710 42905	−.09755 19386	10	60.92005 32608	−.09238 50042
11	54.91420 17426	−.08098 46590	11	61.19715 30371	−.07671 51960
12	55.16529 77159	−.06067 67598	12	61.44825 11166	−.05749 13542
13	55.38131 72542	−.04018 64126	13	61.66427 24669	−.03808 43044
14	55.55445 37265	−.02242 64133	14	61.83741 03914	−.02125 66871
15	55.67845 31111	−.00929 74390	15	61.96141 08161	−.00881 34966
16	55.74886 24914	−.00177 90152	16	62.03182 07869	−.00168 65234
	55.76551 07550			62.04846 91902	

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TABLE 2

	x	$J_1(x)$		x	$J_1(x)$
	.00000 00000			7.01558 66698	
1	.02030 62503	.01015 26018	1	7.03232 19654	.00501 63154
2	.10618 61074	.05301 82575	2	7.10309 94235	.02606 88611
3	.25743 08620	.12765 21143	3	7.22774 70376	.06226 85746
4	.46860 91943	.22793 16731	4	7.40178 86125	.11009 66693
5	.73209 29378	.34206 47543	5	7.61893 74325	.16354 80168
6	1.03836 01743	.45228 19201	6	7.87134 60560	.21430 84782
7	1.37634 19817	.53758 52259	7	8.14989 20857	.25311 94883
8	1.73382 29846	.57949 53889	8	8.44450 82807	.27216 07618
9	2.09788 29856	.56851 14490	9	8.74454 65243	.26752 21198
10	2.45536 39885	.50780 82279	10	9.03916 27193	.24053 22625
11	2.79334 57959	.41190 88809	11	9.31770 87490	.19721 25700
12	3.09961 30324	.30107 17052	12	9.57011 73725	.14615 17084
13	3.36309 67759	.19458 39510	13	9.78726 61925	.09591 29504
14	3.57427 51082	.10623 42640	14	9.96130 77674	.05314 08898
15	3.72551 98628	.04329 54722	15	10.08595 53815	.02191 91616
16	3.81139 97199	.00819 97562	16	10.15673 28396	.00418 21325
	3.83170 59702			10.17346 81351	
1	3.84857 90494	−.00678 05723	1	10.19016 28485	−.00416 51430
2	3.91993 92273	−.03509 12559	2	10.26076 86758	−.02167 90724
3	4.04561 30809	−.08323 73938	3	10.38511 40273	−.05192 08365
4	4.22108 75619	−.14584 38197	4	10.55873 35623	−.09212 85148
5	4.44002 41954	−.21441 93722	5	10.77535 58100	−.13743 55137
6	4.69451 09307	−.27792 36119	6	11.02715 23585	−.18092 68681
7	4.97535 02648	−.32471 41295	7	11.30502 29316	−.21471 99134
8	5.27239 20719	−.34554 21188	8	11.59892 47008	−.23197 38712
9	5.57490 05681	−.33641 08609	9	11.89823 53706	−.22906 10599
10	5.87194 23752	−.29986 78712	10	12.19213 71398	−.20682 24276
11	6.15278 17093	−.24400 22677	11	12.47000 77129	−.17021 73009
12	6.40726 84446	−.17965 91930	12	12.72180 42614	−.12656 04959
13	6.62620 50781	−.11727 67942	13	12.93842 65091	−.08328 24257
14	6.80167 95591	−.06471 02564	14	13.11204 60441	−.04624 08298
15	6.92735 34127	−.02661 42650	15	13.23639 13956	−.01910 14741
16	6.99871 35906	−.00506 97404	16	13.30699 72229	−.00364 75651
	7.01558 66698			13.32369 19363	

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TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	13.32369 19363			19.61585 85105	
1	13.34036 92371	.00363 92005	1	19.63252 14378	.00299 89710
2	13.41090 14225	.01895 70437	2	19.70299 28337	.01563 52051
3	13.53511 70816	.04546 56026	3	19.82710 14354	.03755 37294
4	13.70855 55312	.08082 85751	4	20.00039 04040	.06689 70210
5	13.92495 18418	.12085 59397	5	20.21660 02095	.10026 96775
6	14.17648 57662	.15950 88072	6	20.46791 73449	.13270 31954
7	14.45406 65200	.18981 34949	7	20.74525 88608	.15837 97464
8	14.74766 17494	.20562 58827	8	21.03860 10500	.17208 93345
9	15.04666 02378	.20358 27840	9	21.33734 18412	.17088 33480
10	15.34025 54672	.18427 60601	10	21.63068 40304	.15511 21291
11	15.61783 62210	.15200 47172	11	21.90802 55463	.12827 75447
12	15.86937 01454	.11324 27689	12	22.15934 26817	.09578 30326
13	16.08576 64560	.07464 19546	13	22.37555 24872	.06325 40801
14	16.25920 49056	.04149 70116	14	22.54884 14558	.03521 86914
15	16.38342 05647	.01715 74777	15	22.67295 00575	.01457 70470
16	16.45395 27501	.00327 80206	16	22.74342 14534	.00278 66763
	16.47063 00509			22.76008 43806	
1	16.48729 82913	−.00327 29251	1	22.77674 39258	−.00278 40720
2	16.55779 21583	−.01705 76459	2	22.84720 10186	−.01451 84126
3	16.68194 03340	−.04094 59048	3	22.97128 44308	−.03488 63684
4	16.85528 45588	−.07288 03022	4	23.14453 82281	−.06218 23121
5	17.07156 33065	−.10912 92847	5	23.36070 41509	−.09327 05856
6	17.32296 05786	−.14426 61835	6	23.61197 02781	−.12354 19341
7	17.60039 05295	−.17197 26270	7	23.88925 55038	−.14757 67318
8	17.89382 62558	−.18662 79783	8	24.18253 81552	−.16049 77977
9	18.19266 23056	−.18509 40367	9	24.48121 83130	−.15951 75470
10	18.48609 80319	−.16781 59539	10	24.77450 09644	−.14492 09337
11	18.76352 79828	−.13863 52301	11	25.05178 61901	−.11994 54048
12	19.01492 52549	−.10341 91552	12	25.30305 23173	−.08962 51276
13	19.23120 40026	−.06824 25400	13	25.51921 82401	−.05922 29479
14	19.40454 82274	−.03797 23479	14	25.69247 20374	−.03298 98365
15	19.52869 64031	−.01570 98187	15	25.81655 54496	−.01365 90977
16	19.59919 02701	−.00300 24768	16	25.88701 25424	−.00261 16864
	19.61585 85105			25.90367 20876	

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TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	25.90367 20876			32.18967 99110	
1	25.92032 93474	.00260 96493	1	32.20633 43684	.00234 09996
2	25.99077 67744	.01361 13779	2	32.27676 99432	.01221 33904
3	26.11484 31639	.03271 75400	3	32.40081 54597	.02937 08010
4	26.28807 31930	.05834 28100	4	32.57401 63444	.05240 83042
5	26.50420 94607	.08755 98843	5	32.79011 62490	.07871 53604
6	26.75544 11174	.11605 08696	6	33.04130 56382	.10442 22839
7	27.03268 83032	.13872 26066	7	33.31850 61796	.12494 39101
8	27.32593 07199	.15097 45446	8	33.61169 92609	.13611 63937
9	27.62456 99027	.15015 74510	9	33.91028 82003	.13551 63765
10	27.91781 23194	.13650 89575	10	34.20348 12816	.12331 84545
11	28.19505 95052	.11305 33742	11	34.48068 18230	.10222 15009
12	28.44629 11619	.08452 19696	12	34.73187 12122	.07648 52251
13	28.66242 74296	.05587 69690	13	34.94797 11168	.05059 84188
14	28.83565 74587	.03113 74830	14	35.12117 20015	.02821 12428
15	28.95972 38482	.01289 55290	15	35.24521 75180	.01168 81140
16	29.03017 12752	.00246 60528	16	35.31565 30928	.00223 56390
	29.04682 85349			35.33230 75501	
1	29.06348 41780	−.00246 43999	1	35.34896 11121	−.00223 44653
2	29.13392 47675	−.01285 56731	2	35.41939 29002	−.01165 87150
3	29.25797 91152	−.03090 89818	3	35.54343 17477	−.02804 16792
4	29.43119 23308	−.05513 72476	4	35.71662 33207	−.05004 84236
5	29.64730 76205	−.08278 50383	5	35.93271 16074	−.07519 26901
6	29.89851 48930	−.10977 68854	6	36.18388 74921	−.09978 23087
7	30.17573 51694	−.13129 35471	7	36.46107 31306	−.11943 51446
8	30.46894 91244	−.14296 90393	8	36.75425 04492	−.13016 38763
9	30.76755 93216	−.14227 43786	9	37.05282 33358	−.12963 88064
10	31.06077 32766	−.12941 16541	10	37.34600 06544	−.11801 27490
11	31.33799 35530	−.10722 87815	11	37.62318 62929	−.09785 65260
12	31.58920 08255	−.08020 27911	12	37.87436 21776	−.07324 13135
13	31.80531 61152	−.05304 14604	13	38.09045 04643	−.04846 48482
14	31.97852 93308	−.02956 61656	14	38.26364 20373	−.02702 71631
15	32.10258 36785	−.01224 73498	15	38.38768 08848	−.01119 91603
16	32.17302 42680	−.00234 23778	16	38.45811 26729	−.00214 22896
	32.18967 99110			38.47476 62348	

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TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	38.47476 62348			44.75931 89977	
1	38.49141 91040	.00214 12735	1	44.77597 08809	.00198 52730
2	38.56184 79626	.01117 33773	2	44.84639 55686	.01036 06792
3	38.68588 16508	.02687 81611	3	44.97042 19116	.02492 87731
4	38.85906 60202	.04798 12388	4	45.14359 60250	.04451 51717
5	39.07514 53189	.07210 44757	5	45.35966 25275	.06692 15430
6	39.32631 07563	.09571 08701	6	45.61081 30909	.08887 03803
7	39.60348 48656	.11459 65682	7	45.88797 07859	.10645 75916
8	39.89664 99899	.12493 01313	8	46.18111 85490	.11611 55971
9	40.19521 04577	.12446 56086	9	46.47966 13360	.11574 22770
10	40.48837 55820	.11333 82116	10	46.77280 90991	.10544 63168
11	40.76554 96913	.09400 72264	11	47.04996 67941	.08750 13044
12	41.01671 51287	.07037 82521	12	47.30111 73575	.06553 43925
13	41.23279 44274	.04658 04345	13	47.51718 38600	.04338 95839
14	41.40597 87968	.02598 07709	14	47.69035 79734	.02420 77320
15	41.53001 24850	.01076 68903	15	47.81438 43164	.01003 40865
16	41.60044 13436	.00205 97434	16	47.88480 90041	.00191 97690
	41.61709 42128			47.90146 08872	
1	41.63374 65353	−.00205 88518	1	47.91811 24121	−.00191 90607
2	41.70417 30810	−.01074 40331	2	47.98853 55848	−.01001 56479
3	41.82820 26960	−.02584 84372	3	48.11255 92598	−.02410 07718
4	42.00138 13780	−.04615 08277	4	48.28572 96479	−.04304 20054
5	42.21745 35808	−.06936 81412	5	48.50179 15025	−.06471 68956
6	42.46861 07699	−.09210 05548	6	48.75293 66631	−.08595 79773
7	42.74577 57769	−.11030 23920	7	49.03008 83960	−.10298 88602
8	43.03893 12737	−.12028 11784	8	49.32322 98530	−.11235 50074
9	43.33748 19369	−.11986 63996	9	49.62176 62178	−.11201 66719
10	43.63063 74337	−.10917 88005	10	49.91490 76748	−.10207 23609
11	43.90780 24407	−.09057 93974	11	50.19205 94077	−.08471 72272
12	44.15895 96298	−.06782 68662	12	50.44320 45683	−.06345 97884
13	44.37503 18326	−.04490 01454	13	50.65926 64229	−.04202 19605
14	44.54821 05146	−.02504 72789	14	50.83243 68110	−.02344 73534
15	44.67224 01296	−.01038 11283	15	50.95646 04860	−.00971 96893
16	44.74266 66753	−.00198 60641	16	51.02688 36587	−.00185 97015
	44.75931 89977			51.04353 51836	

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TABLE 2—*Continued*

	x	$J_1(x)$		x	$J_1(x)$
	51.04353 51836			57.32752 54379	
1	51.06018 64126	.00185 90619	1	57.34417 62108	.00175 42236
2	51.13060 83339	.00970 29509	2	57.41459 62035	.00915 64627
3	51.25462 98048	.02335 01746	3	57.53861 42778	.02203 79680
4	51.42779 71154	.04170 60842	4	57.71177 68459	.03936 95712
5	51.64385 51303	.06271 67691	5	57.92782 89436	.05921 66524
6	51.89499 58277	.08331 44272	6	58.17896 27631	.07868 54349
7	52.17214 26353	.09983 86147	7	58.45610 19805	.09431 85737
8	52.46527 88827	.10893 77477	8	58.74923 01998	.10294 55141
9	52.76380 99421	.10862 92546	9	59.04775 30834	.10268 50402
10	53.05694 61895	.09900 29593	10	59.34088 13027	.09361 30005
11	53.33409 29971	.08218 31360	11	59.61802 05201	.07773 02633
12	53.58523 36945	.06157 05817	12	59.86915 43396	.05824 89261
13	53.80129 17094	.04077 60577	13	60.08520 64373	.03858 43703
14	53.97445 90200	.02275 44296	14	60.25836 90054	.02153 50060
15	54.09848 04909	.00943 31183	15	60.38238 70797	.00892 86590
16	54.16890 24122	.00180 49432	16	60.45280 70724	.00170 85351
	54.18555 36411			60.46945 78453	
1	54.20220 46228	−.00180 43618	1	60.48610 84404	−.00170 80464
2	54.27262 54982	−.00941 78326	2	60.55652 76810	−.00891 57171
3	54.39664 51273	−.02266 56242	3	60.68054 44308	−.02145 97360
4	54.56980 98663	−.04048 73357	4	60.85370 51497	−.03833 95708
5	54.78586 46726	−.06089 13712	5	61.06975 49401	−.05767 29573
6	55.03700 16405	−.08090 07468	6	61.32088 60776	−.07664 27230
7	55.31414 43323	−.09696 09091	7	61.59802 23352	−.09188 11714
8	55.60727 62264	−.10581 45509	8	61.89114 74241	−.10029 79275
9	55.90580 28526	−.10553 17575	9	62.18966 71197	−.10005 69840
10	56.19893 47467	−.09619 48735	10	62.48279 22086	−.09122 85052
11	56.47607 74385	−.07986 37153	11	62.75992 84662	−.07575 91800
12	56.72721 44064	−.05984 06981	12	63.01105 96037	−.05677 78096
13	56.94326 92127	−.03963 48225	13	63.22710 93941	−.03761 32667
14	57.11643 39517	−.02211 95390	14	63.40027 01130	−.02099 45050
15	57.24045 35808	−.00917 04951	15	63.52428 68628	−.00870 50043
16	57.31087 44562	−.00175 47553	16	63.59470 61034	−.00166 57858
	57.32752 54379			63.61135 66985	

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Example of the Use of the Tables. The example

$$\int_0^\infty J_0(x)dx = 1$$

given in [1] is repeated here partly since an error in Watson ([6], p. 752) is revealed, and partly as an extension of the tabulation of

$$\int_{x_n}^{x_{n+1}} J_0(x)dx$$

to a larger value of n than that obtainable from Watson ([6], p. 752).

Applying the integration coefficients directly to the values of $J_0(x)$ in Table 1 we get

n	$\pm\int_{x_{n-1}}^{x_n} J_0(x)dx$		Δ^2		Δ^4		Δ^6		Δ^8	
1	1.47030 0043									
2	.80145 4213									
3	.59932 2516									
4	.49904 9621									
5	.43653 5113									
6	.39282 2560									
7	.36005 6836									
8	.33432 0981									
9	.31341 5072									
10	.29599 6007									
11	.28119 1319									
		−12784 903								
12	.26840 6416		15987 49							
		−11186 154		−30760 6						
13	.25722 0262		12911 43		76952					
		− 9895 011		−23065 4		−23112				
14	.24732 5251		10604 89		53840		7965			
		− 8834 522		−17681 4		−15147		−3058		
15	.23849 0729		8836 75		38693		4907		1263	
		− 7950 847		−13812 1		−10240		−1795		−489
16	.23053 9882		7455 54		28453		3112		774	
		− 7205 293		−10966 8		− 7128		−1021		
17	.22333 4589		6358 86		21325		2091			
		− 6569 407		− 8834 3		− 5037				
18	.21676 5182		5475 43		16288					
		− 6021 864		− 7205 5						
19	.21074 3318		4754 88							
		− 5546 376								
20	.20519 6942									

Here x_0 denotes 0.

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Now, choosing the first advancing row of differences we have (by Euler's transformation) approximately

$$\int_0^\infty J_0(x)dx = 1.47030\ 0043 - 0.80145\ 4213 + .59932\ 2516 - .49904\ 9621 + .43653\ 5113 - .39282\ 2560 + .36005\ 6836 - .33432\ 0981 + .31341\ 5072 - .29599\ 6007 + (1/2)(0.28119\ 1319) + (1/4)(0.01278\ 4903) + (1/8)(0.00159\ 8749) + (1/16)(0.00030\ 7606) + (1/32)(0.00007\ 6952) + (1/64)(0.00002\ 3112) + (1/128)(0.00000\ 7965) + (1/256)(0.00000\ 3058) + (1/512)(0.00000\ 1263) + (1/1024)(0.00000\ 0489) = 0.99999\ 9992.$$

It should be noted that the first figure in the table

$$1.47030\ 00434 = \int_0^{x_1} J_0(x)dx \tag{4}$$

disagrees with the figure for this integral quoted in [1], where it is taken from Watson ([6], page 752). This integral was therefore calculated independently by interpolation,

$$\int_0^{x+h} J_0(t)dt = \int_0^x J_0(t)dt + hJ_0(x) - \frac{h^2}{2!}J_1(x) + \frac{h^3}{3!}\left(\frac{J_1(x)}{x} - J_0(x)\right),$$

(Taylor's theorem) using tables of $J_0(x)$, $J_1(x)$ [4] and a table [3] of $\int_0^x J_0(t)dt$. For this x was taken to be 2.4 and h to be 0.00482 55577, so that $x + h = 2.40482\ 55577$ is the first zero of $J_0(x)$. The result obtained agreed with (4) to ten decimal places, and this confirmation reveals an error in Watson [6] where the value

$$\frac{1}{2}\int_0^{x_1} J_0(x)dx = 0.73522\ 08$$

is quoted. This value should be corrected to 0.73515 00.

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