Taking the average over t and once again utilizing (A2) we obtain

$$D'' = \int_0^1 E\epsilon''(t)^2\,dt = \int \sum_{\nu} |G_\nu(\lambda)|^2 \Phi_W(\lambda - \nu)\,d\lambda$$

$$= \int \boldsymbol{G}(\lambda)\boldsymbol{Q}''(\lambda)\boldsymbol{G}^*(\lambda)\,d\lambda \tag{A4}$$

where in the last step we have introduced the diagonal matrix

$$\boldsymbol{Q}''(\lambda) \triangleq \{Q''_{ij}(\lambda)\}$$

$$Q''_{ij}(\lambda) \triangleq \delta_{ij}\Phi_W(\lambda - i).$$

If we finally define

$$\boldsymbol{Q}(\lambda) \triangleq \boldsymbol{Q}'(\lambda) + \boldsymbol{Q}''(\lambda),$$

and add the expressions (A3) and (A4), we obtain the desired result (3.7).

REFERENCES

[1] T. Berger and D. W. Tufts, "Optimum pulse amplitude modulation: part I," *IEEE Trans. Inform. Theory*, vol. IT-13, pp. 196–208, Apr. 1967.
[2] T. Berger, *Rate Distortion Theory*. Englewood Cliffs, NJ: Prentice Hall, 1971, pp. 151–165.
[3] H. L. van Trees, *Detection, Estimation, and Modulation Theory: Part I*. New York: Wiley, 1968, ch. 6.
[4] T. Ericson, "Optimum amplitude modulation," Univ. Linköping, Linköping, Sweden, Rep. LiTH-ISY-R-0040. 1974.
[5] E. Wong, *Stochastic Processes in Information and Dynamical Systems*. New York: McGraw-Hill, 1971.
[6] W. A. Gardner and L. E. Franks, "Characterization of cyclostationary random signal processes," *IEEE Trans. Inform. Theory*, vol. IT-21, no. 1, pp. 4–14, Jan. 1975.
[7] W. R. Bennet, "Statistics of regenerative digital transmission," *Bell. Syst. Tech. J.*, vol. 37, pp. 1501–1542, Nov. 1958.

Tables of Sphere Packings and Spherical Codes

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Abstract—**The theta function of a sphere packing gives the number of centers at each distance from the origin. The theta functions of a number of important packings (A_n, D_n, E_n, the Leech lattice, and others) and tables of the first fifty or so of their coefficients are given in this paper.**

I. SUMMARY

THE MAIN RESULTS in this paper are formulas for the theta functions of the sphere packings A_2, given in (30), A_n (34), the face-centered cubic lattice D_3 (40), the body-centered cubic lattice $D_3^\perp$ (41), the hexagonal close-packing (42), D_n (40), $D_n^\perp$ (41), E_6 (43), E_7 (44), E_8 (45), K_{12} (49), Λ_{16} (51), and the Leech lattice Λ_{24} (54). Tables I–X give the first fifty or so coefficients of these theta functions and are far more extensive than any tables hitherto published. (Some earlier tables were given in [4] and [44]; see also [40] and [41].)

Spherical codes are defined in Section II, and one of the motivations for constructing these tables is that they supply excellent examples of spherical codes. The maximum inner product of any one of these codes is given by (12) and (13). Sections III–IX deal with the general properties of sphere packings and the associated spherical codes, while Section X gives formulas for the number of centers inside or on a large spherical shell. The connections with number theory are sketched in Section XI, and then Sections XII–XX give the most important packings in greater detail (including their generator matrices, densities and kissing numbers).

A sequel to this paper will discuss the encoding and decoding of these spherical codes and the nearest neighbor regions associated with the codewords.

II. SPHERICAL CODES

Just as a binary error-correcting code [36] is a subset of the vertices of an n-dimensional cube, so a spherical code [16], [21] is a subset of the points of an n-dimensional sphere. More precisely, let Ω_n denote the unit sphere in $\boldsymbol{R}^n$,

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i.e. the points $x=(x_1,\cdots,x_n)\in \boldsymbol{R}^n$ with[1]

$$\|x\|:=x_1^2+\cdots+x_n^2=1.$$

A *spherical code C of dimension n, size M and maximum dot product s* is a set of M points of Ω_n with the property that

$$x\cdot y\leq s \qquad \text{for all } x, y\in C,\ x\neq y, \tag{1}$$

where

$$x\cdot y:=x_1y_1+\cdots+x_ny_n.$$

The problem of finding the largest spherical code with a given dot product has a long history, especially in the case $n=3$ [13], [23]. Spherical codes have been extensively studied in connection with the design of signals for the Gaussian channel [3]–[6], [8], [31], [50], [55], [60], [63]. They also have applications to the design of quantizers and samplers [18], [19], [39], [58], in numerical quadrature [22], [56], tomography [49], and above all to diophantine equations (see Section XI). The theory of these codes borrows heavily from group representations, modular forms, and harmonic analysis [24], [27], [29], [35], [45], [46]. Spherical codes may be efficiently constructed from sphere packings, as we now demonstrate.

III. Sphere Packings

Stated informally, a sphere packing in $\boldsymbol{R}^n$ is an arrangement of infinitely many nonoverlapping spheres, all of the same size. More precisely, a *sphere packing* Λ *in* $\boldsymbol{R}^n$ *of radius* ρ consists of an infinite sequence of points $x^{(1)}, x^{(2)},\cdots$ in $\boldsymbol{R}^n$ (the centers of the spheres) such that

$$\operatorname{dist}(x^{(i)},x^{(j)})^2=\|x^{(i)}-x^{(j)}\|\geq 4\rho^2 \tag{2}$$

for all $i\neq j$. Then if spheres of radius ρ are drawn around the centers $x^{(1)}, x^{(2)},\cdots$ the spheres will not overlap. Λ is called a *lattice* packing if the centers $x^{(i)}$ form a group under componentwise addition. The literature on sphere packings is extensive, and the subject is intimately related to the theory of error-correcting codes. For more information the reader is referred to [1], [15], [34], [47], [48], [51]–[54].

IV. The Parameters of a Sphere Packing

The *dimension d* of a lattice packing Λ is the maximum number of linearly independent centers in Λ. Since $\Lambda\subseteq \boldsymbol{R}^n$, $d\leq n$. Let $x^{(1)}=(x_1^{(1)},\cdots,x_n^{(1)}),\cdots,x^{(d)}=(x_1^{(d)},\cdots,x_n^{(d)})\in\Lambda$ be linearly independent centers which span the lattice. The $d\times n$ matrix

$$M:=\left(x_j^{(i)}\right),\qquad 1\leq i\leq d,\ 1\leq j\leq n,$$

is a *generator matrix* for Λ, and Λ consists of all integer combinations of the rows of M. The *determinant* of Λ is

$$\det\Lambda:=(\det MM^T)^{1/2}. \tag{3}$$

When the lattice has the same dimension as the space in which it lies, i.e., when $d=n$, as is usually the case for the packings we shall consider, M is a square matrix and

$$\det\Lambda=|\det M|. \tag{4}$$

The *density* Δ of any (lattice or nonlattice) sphere packing is, loosely speaking, the fraction of the space $\boldsymbol{R}^n$ that is covered by the spheres. For a lattice packing Λ of radius ρ and dimension $d=n$, the density is given by the formula

$$\Delta=\frac{V_n\rho^n}{\det\Lambda}, \tag{5}$$

where

$$V_n:=\frac{\pi^{n/2}}{\Gamma((n/2)+1)}$$

is the volume of the unit sphere Ω_n. The density of a nonlattice packing must be defined in a more complicated way—see [47]. The main sphere packing problem is to determine the sphere packings in $\boldsymbol{R}^n$ with the greatest density. References [34], [52], and [54] give the most recent results.

The *kissing number* $\tau(x)$ of the sphere centered at x is the number of neighboring spheres, i.e., the number of spheres which just kiss the sphere centered at x (using a term borrowed from billiards). The maximum value of $\tau(x)$ for $x\in\Lambda$ is denoted by $\tau_{\max}$. For a lattice packing $\tau(x)=\tau_{\max}=\tau$, independently of the choice of x. A second important problem is to find the packings in $\boldsymbol{R}^n$ with the greatest value of $\tau_{\max}$ [2], [34], [43], [52].

V. An Example: The Lattice D_4

The lattice packing D_4 (see Section XVI) is a four-dimensional lattice in $\boldsymbol{R}^4$ spanned by the vectors

$$x^{(1)}=\frac{1}{\sqrt{2}}(2,0,0,0),$$

$$x^{(2)}=\frac{1}{\sqrt{2}}(1,1,0,0),$$

$$x^{(3)}=\frac{1}{\sqrt{2}}(1,0,1,0),$$

$$x^{(4)}=\frac{1}{\sqrt{2}}(1,0,0,1).$$

Thus

$$M=\frac{1}{\sqrt{2}}\begin{pmatrix}2&0&0&0\\1&1&0&0\\1&0&1&0\\1&0&0&1\end{pmatrix}$$

is a generator matrix, and $\det D_4=|\det M|=1/2$. The lattice points closest to the origin are the 24 points

$$\frac{1}{\sqrt{2}}(\pm1,\pm1,0,0),\ \frac{1}{\sqrt{2}}(\pm1,0,\pm1,0),\ \frac{1}{\sqrt{2}}(\pm1,0,0,\pm1),$$
$$\frac{1}{\sqrt{2}}(0,\pm1,\pm1,0),\ \frac{1}{\sqrt{2}}(0,\pm1,0,\pm1),\ \frac{1}{\sqrt{2}}(0,0,\pm1,\pm1), \tag{6}$$

[1]Note that this definition of $\|x\|$ is the square of the usual one.

so we may take the radius of the spheres to be $\rho = 1/2$. The kissing number is $\tau = 24$, and the density is

$$\Delta = \frac{\pi^2}{16} = 0.61686\cdots.$$

VI. THE THETA FUNCTION OF A LATTICE

Many properties of a sphere packing can be obtained from its theta function, which is analogous to the weight enumerator of a code [36, ch. 5] in that it gives the number of centers at each distance from the origin. There is one important difference, however: a weight enumerator is a polynomial while a theta function is an infinite sum.

Definition: The *theta function* of a sphere packing Λ is

$$\Theta_\Lambda(z) := \sum_{x \in \Lambda} q^{\|x\|}, \tag{7}$$

where $q = e^{\pi i z}$. If Λ is a lattice packing, as it usually is in this paper, the theta function is a holomorphic function of z for $Im(z) > 0$ (see [24, p. 71]).

If N_m denotes the number of centers $x \in \Lambda$ with $\|x\| = m$, i.e., at a squared distance of m from the origin, then (7) can be rewritten as

$$\Theta_\Lambda(z) = \sum_{m=0}^{\infty} N_m q^m,$$

where m runs through all the values of $\|x\|$ for $x \in \Lambda$. The first two terms are $\Theta_\Lambda(z) = 1 + \tau q^{4\rho^2} + \cdots$, where ρ is the radius of Λ and $\tau = \tau(0)$ is the kissing number of the sphere at the origin. For example the theta function of D_4 begins $\Theta_{D_4}(z) = 1 + 24q + 24q^2 + 96q^3 + \cdots$ (see Table V).

If Λ is a lattice packing in $\boldsymbol{R}^n$ of dimension n, the *dual lattice* $\Lambda^\perp$ is defined to be $\Lambda^\perp := \{x \in \boldsymbol{R}^n | x \cdot y \in \boldsymbol{Z}$ for all $y \in \Lambda\}$. A generator matrix for $\Lambda^\perp$ is $(M^{-1})^{\text{tr}}$, and its determinant and theta function are

$$\det \Lambda^\perp = (\det \Lambda)^{-1},$$

$$\Theta_{\Lambda^\perp}(z) = (\det \Lambda)\left(\frac{i}{z}\right)^{n/2} \Theta_\Lambda\left(-\frac{1}{z}\right). \tag{8}$$

VII. RESCALING

It is frequently necessary to rescale a sphere packing, replacing Λ by $\Lambda' = c\Lambda = \{cx\colon\ x \in \Lambda\}$ for some appropriate constant $c \in \boldsymbol{R}$. The parameters of Λ' and Λ are related as follows:

$$\rho' = c\rho,$$
$$\dim \Lambda' = \dim \Lambda,$$
$$M' = cM,$$
$$\det \Lambda' = c^{\dim \Lambda} \cdot \det \Lambda,$$
$$(\Lambda')^\perp = \frac{1}{c} \Lambda^\perp,$$
$$\Delta' = \Delta,$$
$$\tau'(cx) = \tau(x), \qquad \tau' = \tau,$$
$$\Theta_{\Lambda'}(z) = \Theta_\Lambda(c^2 z),$$

or

$$\Theta_{\Lambda'}(z) = \Theta_\Lambda(z)|_{\text{replace } q \text{ by } q^{c^2}}.$$

VIII. JACOBI THETA FUNCTIONS

The theta functions of many packings can be specified concisely in terms of the classical Jacobi theta functions θ_2, θ_3, and θ_4, which are defined as follows:[2]

$$\begin{aligned}
\theta_2(z) &:= 2 \sum_{m=0}^{\infty} q^{(m+(1/2))^2} \\
&= 2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots \\
&= 2q^{1/4}(1 + q^2 + q^6 + q^{12} + q^{20} + \cdots), \\
\theta_3(z) &:= 1 + 2 \sum_{m=1}^{\infty} q^{m^2} \\
&= 1 + 2q + 2q^4 + 2q^9 + \cdots \\
&= 1 + 2q(1 + q^3 + q^8 + q^{15} + \cdots), \\
\theta_4(z) &:= 1 + 2 \sum_{m=1}^{\infty} (-q)^{m^2} \\
&= 1 - 2q + 2q^4 - 2q^9 + \cdots \\
&= 1 - 2q(1 - q^3 + q^8 - q^{15} + \cdots).
\end{aligned}$$

It is important to notice that θ_3 is itself the theta function of the one-dimensional lattice of integer points, $\boldsymbol{Z}$, in $\boldsymbol{R}^1$:

$$\theta_3(z) = \Theta_{\boldsymbol{Z}}(z), \tag{9}$$

and that

$$\theta_2(z) = \Theta_{\boldsymbol{Z}+(1/2)}(z). \tag{10}$$

Furthermore,

$$\theta_4(z) = \theta_3(z+1).$$

We also point out that these theta functions can be written as infinite products:

$$\theta_2(z) = 2q^{1/4} \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m})^2,$$

$$\theta_3(z) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2,$$

$$\theta_4(z) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 - q^{2m-1})^2.$$

There are many other useful identities relating these functions—see [45], [46], [52, eqs. (14)–(23)], [53, eqs. (9)–(19)], [57], [62].

Example: The theta function of D_4 is (see Section XVI)

$$\Theta_{D_4}(z) = \frac{1}{2}\left(\theta_3\left(\frac{z}{2}\right)^4 + \theta_4\left(\frac{z}{2}\right)^4\right).$$

[2] In the older literature these functions are denoted by $\theta_2(0|z)$, $\theta_3(0|z)$, and $\theta_4(0|z)$, but except in Section XV we simply omit the first argument (see also [46, sec. 7.5], [62, ch. XXI]).

IX. CONSTRUCTION OF SPHERICAL CODES FROM SPHERE PACKINGS

The construction is very simple. Suppose there are N_m centers x with $\|x\| = m$ in a sphere packing Λ. Then these points, rescaled by dividing them by $\sqrt{m}$, form a spherical code of dimension n and size N_m. In other words we take a *shell* of points around the origin as the spherical code. The theta function of Λ is thus a generating function for the sizes of these codes.

To find the maximum dot product s of one of these codes we may either rescale the points and use (1), or work directly with the centers of Λ and replace (1) by

$$s = \max_{x \neq y} \cos \measuredangle (x\mathbf{0}y) = \max_{x \neq y} \frac{x \cdot y}{m} \tag{11}$$

taken over distinct centers x, y, in the shell. It is easy to find s. Suppose $2r$ is the smallest distance between any two points in the code. Then

$$s = 1 - \frac{2r^2}{m} \tag{12}$$

—see Fig. 1. Certainly r cannot be less than ρ, the radius of the spheres in Λ, so

$$s \leq 1 - \frac{2\rho^2}{m}, \tag{13}$$

and in the majority of cases (13) holds with equality.

For example consider the lattice D_4. The first shell consists of the 24 points given in (6), and corresponds to $m = 1$. The points $(1/\sqrt{2})(1,1,0,0)$ and $(1/\sqrt{2})(1,0,1,0)$ are at a distance $2r = 1 = 2\rho$, and for this spherical code

$$s = 1 - \frac{2r^2}{m} = \frac{1}{2}.$$

Similarly (13) holds with equality for most of the shells. On the other hand (13) can be improved for the shells corresponding to $m = 2, 4, 8, 16, \cdots$. These spherical codes also contain just 24 points, which after rescaling are either (6) again or

$$(\pm 1,0,0,0), \cdots, (0,0,0,\pm 1),\ \frac{1}{2}(\pm 1, \pm 1, \pm 1, \pm 1). \tag{14}$$

For these codes s is still $1/2$, but r is $\sqrt{m}/2$.

X. CODES WITH BOUNDED ENERGY

A spherical code represents a set of signals for the Gaussian channel in which each signal has the same energy [50]. A signal set with *bounded* energy may be obtained by taking all the centers of a packing Λ that are within or on a large sphere of radius ρ_0. The total number of such centers is

$$S(\rho_0) := \sum_{m \leq \rho_0^2} N_m. \tag{15}$$

The values of $S(\rho_0)$ are not included in the Tables for two

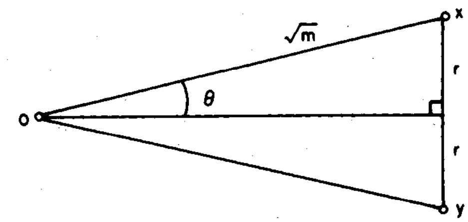

Fig. 1 The maximum dot product of the spherical code consisting of all points in Λ at distance $\sqrt{m}$ from $\mathbf{0}$ is $\cos 2\theta = 1 - 2(r/\sqrt{m})^2$, where $2r$ is the smallest distance between two such points.

reasons: 1) they are easily found by summing N_m; and 2) there is an excellent approximation to $S(\rho_0)$ given by

$$S(\rho_0) \approx \Delta \cdot \left(\frac{\rho_0}{\rho}\right)^n, \tag{16}$$

or

$$S(\rho_0) \approx \frac{V_n \rho_0^n}{\det \Lambda} \tag{17}$$

if Λ is a lattice packing. Also if Λ corresponds to an integral quadratic form (see Section XI) then a theorem of Val'fiš [61] states that

$$S(\rho_0) = \frac{V_n \rho_0^n}{\det \Lambda} + O(\rho_0^{n-1}), \tag{18}$$

and so the error in the approximations (16) or (17) is of smaller order than the main term. Furthermore (18) gives a crude estimate for the order of magnitude of N_m. Since

$$N_m = S(\sqrt{m}) - S(\sqrt{m-1}),$$

(18) implies

$$c_1 m^{(n/2)-1} < N_m < c_2 m^{(n/2)-1} \tag{19}$$

for some positive constants c_1, c_2.

XI. CONNECTIONS WITH NUMBER THEORY

A very old problem asks for the number of ways of expressing an integer m as the sum of four squares, or in other words for the number of quadruples of integers (u_1, u_2, u_3, u_4) such that

$$u_1^2 + u_2^2 + u_3^2 + u_4^2 = m. \tag{20}$$

For example when m is 2 there are 24 solutions, given (ignoring the factor $\sqrt{2}$) by (6): we agree to count $(1,1,0,0)$, $(1,0,1,0)$, etc., as different solutions. A moment's thought shows that the general answer is given by the coefficient of q^m in the expansion of

$$(1 + 2q + 2q^4 + 2q^9 + \cdots)^4 = \theta_3(z)^4$$

in powers of q. However, this power series is also the theta function of the lattice $\boldsymbol{Z}^4$ of integer points in $\boldsymbol{R}^4$:

$$\theta_3(z)^4 = \Theta_{Z^4}(z).$$

In other words *the coefficient of* q^m *in this theta function gives the number of solutions to* (20). Call this number $r_4(m)$. There is in fact a simple formula for this number, due to Jacobi:

$$r_4(m) = \begin{cases} 8 \sum_{d|m} d, & \text{if } m \text{ is odd,} \\ 24 \sum_{d|m,\, d \text{ odd}} d, & \text{if } m \text{ is even.} \end{cases} \quad (21)$$

(See [28, thm. 386], [45, sec. 83], [46, eq. (7.4.23)].)

To generalize this, suppose Λ is a lattice of dimension n in $\boldsymbol{R}^n$ with the property that $\|\boldsymbol{x}\|$ is always an integer for $\boldsymbol{x} \in \Lambda$. The *integral quadratic form* associated with Λ is

$$f(u_1, \cdots, u_n) := \sum_{i,j,k=1}^{n} u_i M_{ij} M_{kj} u_k,$$

or in vector notation

$$f(\boldsymbol{u}) = \boldsymbol{u} M M^{\text{tr}} \boldsymbol{u}^{\text{tr}}, \qquad \text{for } \boldsymbol{u} \in \boldsymbol{Z}^n. \quad (22)$$

As $\boldsymbol{u}$ runs through $\boldsymbol{Z}^n$, $\boldsymbol{x} = \boldsymbol{u}M$ runs through Λ, and

$$f(\boldsymbol{u}) = \boldsymbol{x}\boldsymbol{x}^{\text{tr}} = \|\boldsymbol{x}\|.$$

Thus the theta function of Λ can be rewritten as

$$\Theta_\Lambda(z) = \sum_{\boldsymbol{u} \in \boldsymbol{Z}^n} q^{\boldsymbol{u}MM^{\text{tr}}\boldsymbol{u}^{\text{tr}}} = \sum_{m=0}^{\infty} N_m q^m, \quad (23)$$

and the coefficient N_m is equal to the number of solutions to the diophantine equation

$$\boldsymbol{u} M M^{\text{tr}} \boldsymbol{u}^{\text{tr}} = m. \quad (24)$$

In the above example M is the 4×4 identity matrix I_4 and the quadratic form is $u_1^2 + u_2^2 + u_3^2 + u_4^2$.

This link with number theory makes it possible to apply the vast literature on diophantine equations and modular forms to the study of lattices [17], [20], [24]–[28], [30], [37], [45], [46], [59], [61].

XII. Description of Some Important Lattices

In the following sections we describe some of the most important lattices and give tables of their theta functions. In view of the construction in Section IX these are also tables of spherical codes. The maximum dot product of any of these codes is given by (12) and (13). For lattices in higher dimensions see [33], [34], [48], [51], [52].

Our notation is that

d	dimension of Λ,
M	generator matrix,
ρ	radius,
det	determinant,
τ	kissing number,
Δ	density,
$\Theta_\Lambda(z)$	theta function,
N_m	number of centers $\boldsymbol{x} \in \Lambda$ with $\|\boldsymbol{x}\| = m$.

XIII. The Cubic Lattice $\boldsymbol{Z}^n$ in $\boldsymbol{R}^n$

The simplest lattice is $\boldsymbol{Z}^n$, which consists of all points in $\boldsymbol{R}^n$ with integer coordinates. For this lattice $d = n$, $M = I_n$,

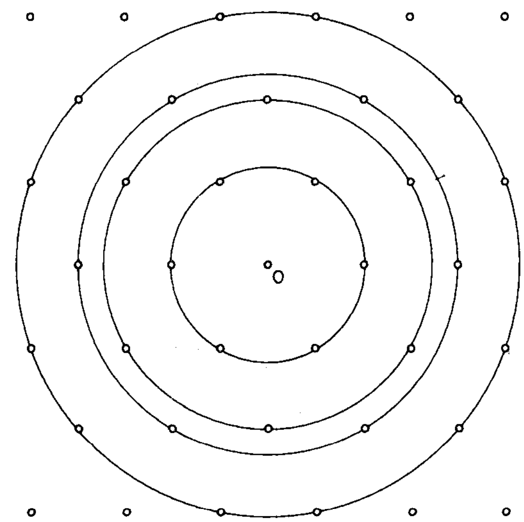

Fig. 2 The hexagonal lattice A_2 in the plane. The first five shells around the origin contain 1, 6, 6, 6, and 12 points, respectively.

$\rho = 1/2$, det $= 1$, $\tau = 2n$, $\Delta = V_n/2^n$, and

$$\Theta_{\boldsymbol{Z}^n}(z) = \theta_3(z)^n = \sum_{m=0}^{\infty} r_n(m) q^m. \quad (25)$$

The coefficient $r_n(m)$ is the number of ways of representing m as a sum of n squares. There are explicit formulas for small even values of m. For example

$$r_2(m) = 4|\{d|n, d \equiv 1 \pmod 4\}| - 4|\{d|n, d \equiv 3 \pmod 4\}|; \quad (26)$$

$r_4(m)$ is given by (21);

$$r_6(m) = 16 \sum_{d|m} \chi(d') d^2 - 4 \sum_{d|m} \chi(d) d^2, \quad (27)$$

where $dd' = m$ and $\chi(d)$ is 1, -1, or 0 according as d is of the form $4k+1$, $4k-1$, or $2k$; and

$$r_8(m) = 16(-1)^m \sum_{d|m} (-1)^d d^3. \quad (28)$$

Much more is known about $r_n(m)$—see the references given at the end of Section XI.

XIV. The Two-Dimensional Lattice A_2

This familiar lattice is shown in Fig. 2, and is spanned by the vectors (1,0) and $(-1/2, \sqrt{3}/2)$. Thus $d = 2$,

$$M = \begin{pmatrix} 1 & 0 \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{pmatrix},$$

$\rho = 1/2$, det $= \sqrt{3}/2$, $\tau = 6$, and the density is

$$\Delta = \frac{\pi}{2\sqrt{3}} = 0.9069 \cdots,$$

which is the highest attainable in $\boldsymbol{R}^2$ (see [47, p. 11]). The associated quadratic form is

$$u^2 - uv + v^2 \tag{29}$$

and the theta function is

$$\begin{aligned}\Theta_{A_2}(z) &= \sum_{u,v=-\infty}^{\infty} q^{u^2-uv+v^2} \\ &= \sum_{u,v=-\infty}^{\infty} q^{(u-(1/2)v)^2+(3/4)v^2} \\ &= \sum_{\substack{u,v=-\infty \\ v \text{ even}}}^{\infty} q^{(u-(1/2)v)^2+(3/4)v^2} \\ &\quad + \sum_{\substack{u,v=-\infty \\ v \text{ odd}}}^{\infty} q^{(u-(1/2)v)^2+(3/4)v^2} \\ &= \sum_{r,s=-\infty}^{\infty} q^{r^2+3s^2} + \sum_{r,s=-\infty}^{\infty} q^{(r+(1/2))^2+3(s+(1/2))^2} \\ &= \theta_3(z)\theta_3(3z) + \theta_2(z)\theta_2(3z) \\ &= \phi_0(z) \quad \text{(say).}\end{aligned} \tag{30}$$

If we write

$$\Theta_{A_2}(z) = \sum_{m=0}^{\infty} N_m q^m,$$

then N_m is the number of times (29) represents m. It follows from the standard theory,[3] writing

$$N'(m) = \frac{1}{6} N_m,$$

that $N'(m)$ is *multiplicative*, i.e., satisfies

$$N'(rs) = N'(r)N'(s), \quad \text{whenever } r \text{ and } s \text{ are relatively prime} \tag{31}$$

(see [28]). Therefore it is sufficient to calculate $N'(m)$ when $m = p^a$ is a power of a prime. These values are

$$N'(3^a) = 1, \qquad \text{for all } a,$$

$$N'(p^a) = a+1, \qquad \text{if } p \equiv 1 \pmod 3,$$

$$N'(p^a) = \begin{cases} 0, & \text{if } p \equiv 2 \pmod 3, \quad a \text{ odd}, \\ 1, & \text{if} \equiv 2 \pmod 3, \quad a \text{ even}. \end{cases}$$

The theta function begins

$$\Theta_{A_2}(z) = 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \cdots$$

(see Fig. 2), and further values are given in Table I.

XV. The n-Dimensional Lattice A_n

A_2 may be generalized to higher dimensions as follows. We describe A_n as an n-dimensional lattice in $\boldsymbol{R}^{n+1}$. Thus for $n \geq 2$ let

$$A_n := \left\{ \boldsymbol{x} \in \boldsymbol{Z}^{n+1} \,\middle|\, \sum_{j=1}^{n+1} x_j = 0 \right\}. \tag{32}$$

[3]Since (21) has class number 1.

TABLE I
THE HEXAGONAL LATTICE A_2 IN $\boldsymbol{R}^2$

m	$\frac{1}{6}N_m$	m	$\frac{1}{6}N_m$	m	$\frac{1}{6}N_m$
0	1/6	64	1	147	3
1	1	67	2	148	2
3	1	73	2	151	2
4	1	75	1	156	2
7	2	76	2	157	2
9	1	79	2	163	2
12	1	81	1	169	3
13	2	84	2	171	2
16	1	91	4	172	2
19	2	93	2	175	2
21	2	97	2	181	2
25	1	100	1	183	2
27	1	103	2	189	2
28	2	108	1	192	1
31	2	109	2	193	2
36	1	111	2	196	3
37	2	112	2	199	2
39	2	117	2	201	2
43	2	121	1	208	2
48	1	124	2	211	2
49	3	127	2	217	4
52	2	129	2	219	2
57	2	133	4	223	2
61	2	139	2	225	1
63	2	144	1	228	2

This definition of A_2 agrees (apart from a rotation) with the definition in the previous section if the old version is rescaled by multiplying it by $c = \sqrt{2}$. Then A_n has $d = n$,

$$M = \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & \cdots & 0 \\ & \cdots & & \cdots & \\ 1 & 0 & 0 & & -1 \end{bmatrix},$$

an $n \times (n+1)$ matrix, $\rho = 1/\sqrt{2}$, $\det = \sqrt{n+1}$, and $\tau = 2n(n-1)$. We may regard A_n as a lattice in $\boldsymbol{R}^n$ by restricting our attention to the hyperplane

$$\{\boldsymbol{x} \in \boldsymbol{R}^{n+1} | x_1 + \cdots + x_{n+1} = 0\}.$$

Looked at in this way, A_n has density

$$\Delta = \frac{V_n}{2^{n/2}\sqrt{n+1}}.$$

The theta function of A_n is most simply expressed in terms of the more general Jacobi theta function [62, ch. XXI]

$$\theta_3(\xi | z) := \sum_{m=-\infty}^{\infty} e^{2mi\xi + \pi i z m^2}. \tag{33}$$

$\theta_2(z)$, $\theta_3(z)$, and $\theta_4(z)$ may be expressed in terms of this function by

$$\theta_2(z) = e^{\pi i z/4}\theta_3\left(\frac{\pi z}{2}\middle| z\right),$$

$$\theta_3(z) = \theta_3(0|z),$$

$$\theta_4(z) = \theta_3\left(\frac{\pi}{2}\middle| z\right).$$

Then the theta function of A_{n-1} is

$$\Theta_{A_{n-1}}(z) = \frac{1}{n\theta_3(nz)} \sum_{k=0}^{n-1} \theta_3\left(\frac{k\pi}{n}\middle| z\right)^n. \tag{34}$$

To prove this, observe that A_{n-1} is a subgroup of the lattice

$$L_n := \left\{ x \in Z^n \,\middle|\, \sum_{j=1}^{n} \left| x_j \equiv 0 \pmod{n} \right. \right\},$$

and in fact we may write

$$L_n = \bigcup_{m=-\infty}^{\infty} \{(m, m, \cdots, m) + A_{n-1}\}.$$

Therefore

$$\Theta_{L_n}(z) = \sum_{m=-\infty}^{\infty} q^{nm^2}\Theta_{A_{n-1}}(z)$$
$$= \theta_3(nz)\Theta_{A_{n-1}}(z). \tag{35}$$

To find the theta function of L_n we weight each point x of Z^n according to the value of Σx_j. Let

$$\chi(x) := e^{2\pi i \Sigma_{j=1}^{n} x_j / n}, \qquad \text{for } x \in R^n, \tag{36}$$

and

$$\Theta_{\chi^k, Z^n}(z) := \sum_{x \in Z^n} \chi^k(x) q^{x \cdot x} \tag{37}$$

for $k = 0, 1, \cdots, n-1$. Then

$$\Theta_{\chi^k, Z^n}(z) = \left(\sum_{x \in Z} \chi^k(x) q^{x^2} \right)^n$$
$$= \theta_3\left(\frac{k\pi}{n} \middle| z \right)^n \tag{38}$$

from (33). Finally the sum of

$$\Theta_{\chi^k, Z^n}(z)$$

over $k = 0, 1, \cdots, n-1$ picks out those $x \in Z^n$ with $\Sigma x_j \equiv 0 \pmod{n}$:

$$\Theta_{L_n}(z) = \frac{1}{n} \sum_{k=0}^{n-1} \Theta_{\chi^k, Z^n}(z), \tag{39}$$

and (34) follows from (35), (38), (39).

XVI. The n-Dimensional Lattice D_n

The lattice packing D_n in R^n is obtained by applying Construction A to the code consisting of all binary vectors of even weight,[4] and then rescaling by multiplying all the centers by $1/\sqrt{2}$. Alternatively, color the points of Z^n red and blue with a checkerboard coloring, take the red points, and multiply by $1/\sqrt{2}$. For this lattice $d = n$,

$$M = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ & \cdots & & \cdots & \\ 1 & 0 & 0 & \cdots & 1 \end{bmatrix},$$

$$\rho = \frac{1}{2},$$

$$\det = 2^{-(n-2)/2},$$

$$\tau = 2n(n-1),$$

[4]Construction A: if C is a binary code of length n, the set of centers $c + 2x$ $(c \in C, x \in Z^n)$ forms a sphere packing in R^n. Most of the properties of this packing can be obtained directly from the code C—see [52] for details.

and

$$\Delta = V_n 2^{-(n+2)/2}.$$

For $n = 3$, 4, and 5 D_n is the densest possible lattice packing (as well as the densest known packing, although for $n = 3$ and 5 there are equally dense nonlattice packings [34]). Also

$$\Theta_{D_n}(z) = \frac{1}{2}\left\{ \theta_3\left(\frac{z}{2}\right)^n + \theta_4\left(\frac{z}{2}\right)^n \right\}$$
$$= \sum_{m=0}^{\infty} r_n(2m) q^m, \tag{40}$$

using the notation of (25). This begins

$$\Theta_{D_n}(z) = 1 + 2n(n-1)q + 2nq^2 + 2n(n-1)q^4 + \cdots.$$

The *dual lattice* $D_n^{\perp}$ $(n \geq 3)$ can of course be obtained by finding the dual of D_n, or more simply by applying Construction A to the repetition code $\{(0,0,\cdots,0), (1,1,\cdots,1)\}$. For this lattice $d = n$,

$$M = \begin{bmatrix} 2 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 2 & \cdots & 0 & 0 \\ & \cdots & & \cdots & & \\ 0 & 0 & 0 & \cdots & 2 & 0 \\ 1 & 1 & 1 & \cdots & 1 & 1 \end{bmatrix},$$

$$\rho = \begin{cases} \sqrt{3}/2, & \text{if } n = 3, \\ 1, & \text{if } n \geq 4, \end{cases}$$

$$\det = 2^{n-1},$$

$$\tau = \begin{cases} 8, & \text{if } n = 3, \\ 24, & \text{if } n = 4, \\ 2n, & \text{if } n \geq 5, \end{cases}$$

$$\Delta = \begin{cases} \dfrac{\pi\sqrt{3}}{8} = 0.680175\cdots, & \text{if } n = 3, \\ \dfrac{V_n}{2^{n-1}}, & \text{if } n \geq 4, \end{cases}$$

and

$$\Theta_{D_n^{\perp}}(z) = \theta_3(4z)^n + \theta_2(4z)^n. \tag{41}$$

The case $n = 3$. References for these three-dimensional packings are [1, ch. V], [14, ch. IV] and [15, ch. 9]. D_3 is the familiar *face-centered cubic lattice* (as seen in a square *or* triangular pyramid of billiard balls), and has density $\Delta = \pi/3\sqrt{2} = 0.740480\cdots$. There is a complicated formula for the coefficients $r_3(2m)$ (see [17, vol. II, chapter VII]). The dual lattice $D_3^{\perp}$ is the *body-centered cubic lattice*. The coefficients of the theta functions are given in Tables II and III.

There is a further packing in R^3 that has not yet been mentioned. This is the *hexagonal close-packing*, which is a nonlattice packing with the same density and kissing number as D_3. It may be defined to be the union of the lattice L spanned by $(\sqrt{3},0,0)$, $(\sqrt{3}/2,3/2,0)$, and $(0,0,2\sqrt{2})$, and

TABLE II
THE FACE-CENTERED CUBIC LATTICE D_3 IN $\boldsymbol{R}^3$ (THE TABLE GIVES $\frac{1}{6}N_m$ FOR $m=10r+s$)

r/s	0	1	2	3	4	5	6	7	8	9
0	1/6	2	1	4	2	4	4/3	8	1	6
1	4	4	4	12	0	8	2	8	5	12
2	4	8	4	8	4/3	14	4	16	8	4
3	0	16	1	16	8	8	6	20	4	8
4	4	8	8	20	4	20	0	16	4	18
5	5	8	12	12	16/3	24	0	16	12	12
6	8	20	0	24	2	8	8	28	8	16
7	8	8	5	32	4	20	12	16	0	16
8	4	18	16	20	8	24	4	24	4	16
9	12	24	8	24	0	8	4/3	40	9	20

TABLE III
THE BODY-CENTERED CUBIC LATTICE $D_3^{\perp}$ IN $\boldsymbol{R}^3$

m	N_m	m	N_m	m	N_m
0	1	67	24	136	48
3	8	68	48	139	72
4	6	72	36	140	48
8	12	75	56	144	30
11	24	76	24	147	56
12	8	80	24	148	24
16	6	83	72	152	72
19	24	84	48	155	96
20	24	88	24	160	24
24	24	91	48	163	24
27	32	96	24	164	96
32	12	99	72	168	48
35	48	100	30	171	120
36	30	104	72	172	24
40	24	107	72	176	24
43	24	108	32	179	120
44	24	115	48	180	72
48	8	116	72	184	48
51	48	120	48	187	48
52	24	123	48	192	8
56	48	128	12	195	96
59	72	131	120	196	54
64	6	132	48	200	84

the coset

$$L'=\left(\frac{\sqrt{3}}{2},\frac{1}{2},\sqrt{2}\right)+L.$$

By following the same kind of algebra used to derive (30), one can show that the theta functions of L and L' are respectively

$$\theta_3(8z)\phi_0(3z)$$

and

$$\frac{1}{2}\theta_2(8z)\{\phi_0(z)-\phi_0(3z)\}.$$

Therefore the theta function of the hexagonal close-packing is

$$\phi_0(3z)\left\{\theta_3(8z)-\frac{1}{2}\theta_2(8z)\right\}+\frac{1}{2}\phi_0(z)\theta_2(8z)$$
$$=1+12q^3+6q^6+2q^8+\cdots \quad (42)$$

—see Table IV.

The case $n=4$ (see also Sec. V). D_4 is a self-dual lattice: $D_4=D_4^{\perp}$. The coefficients $N_m=r_4(2m)$ of the theta function are given by the second formula in (21), and the first 50 terms are given in Table V. Furthermore $(24)^{-1}N_m$ is multiplicative. It is worth mentioning that N_m is the number of integral quaternions of norm m [15, p. 25].

TABLE IV
THE HEXAGONAL CLOSE-PACKING: A NONLATTICE PACKING IN $\boldsymbol{R}^3$ (THE TABLE GIVES $\frac{1}{6}N_m$ FOR $m=10r+s$)

r/s	0	1	2	3	4	5	6	7	8	9
0	1/6	0	0	2	0	0	1	0	1/3	3
1	0	2	1	0	0	2	0	2	1	1
2	2	4	1	0	0	2	0	2	0	4
3	2	2	1/3	2	1	4	1	2	0	4
4	0	2	0	1	4	2	2	4	1	2
5	0	4	0	4	3	2	2	4	0	2
6	0	2	0	6	0	4	2	3	2	4
7	2	8	1/3	0	0	6	0	0	4	2
8	2	7	1	2	4	2	0	2	0	6
9	0	2	4	12	2	4	0	2	0	8

TABLE V
THE LATTICE D_4 IN $\boldsymbol{R}^4$

m	$(24)^{-1}N_m$	m	$(24)^{-1}N_m$
1	1	26	14
2	1	27	40
3	4	28	8
4	1	29	30
5	6	30	24
6	4	31	32
7	8	32	1
8	1	33	48
9	13	34	18
10	6	35	48
11	12	36	13
12	4	37	38
13	14	38	20
14	8	39	56
15	24	40	6
16	1	41	42
17	18	42	32
18	13	43	44
19	20	44	12
20	6	45	78
21	32	46	24
22	12	47	48
23	24	48	4
24	4	49	57
25	31	50	31

XVII. THE LATTICES E_6, E_7, E_8

The lattice packings E_6 in $\boldsymbol{R}^6$, E_7 in $\boldsymbol{R}^7$, and E_8 in $\boldsymbol{R}^8$ are the densest possible lattice packings in these dimensions (and also the densest known packings there). E_6 is most easily constructed from the ternary code {000,111,222}. The complex version of Construction A then produces a complex lattice[5] in $\boldsymbol{C}^3$ with generator matrix

$$\frac{1}{\sqrt{3}}\begin{pmatrix} i\sqrt{3} & 0 & 0 \\ 0 & i\sqrt{3} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

—see [54, sec. 5.8.2]. By regarding this as a real lattice in $\boldsymbol{R}^6$ we obtain E_6, for which $d=6$,

$$M=\frac{1}{\sqrt{3}}\begin{bmatrix} 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & 0 & \sqrt{3} & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ \frac{3}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 & 0 \\ 0 & \frac{3}{2} & 0 & 0 & \frac{\sqrt{3}}{2} & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix},$$

[5]Strictly speaking, this is a $\boldsymbol{Z}[e^{2\pi i/3}]$-module in $\boldsymbol{C}^3$.

$\rho = 1/2$, $\det = \sqrt{3}/2^3$, $\tau = 72$, $\Delta = \pi^3/48\sqrt{3} = 0.372948\cdots$ and

$$\Theta_{E_6}(z) = \phi_0(z)^3 + \frac{1}{4}\left\{\phi_0\left(\frac{z}{3}\right) - \phi_0(z)\right\}^3$$
$$= 1 + 72q + 270q^2 + \cdots. \quad (43)$$

E_7 may be obtained by applying Construction A to the little Hamming code of length seven containing eight codewords. For this lattice $d = 7$,

$$M = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix},$$

$\rho = 1/2$, $\det = 1/8$, $\tau = 126$, $\Delta = \pi^3/105 = 0.295298\cdots$, and the theta function is, from [52, thm. 6],

$$\Theta_{E_7}(z) = \theta_3(z)^7 + 7\theta_3(z)^3\theta_2(z)^4$$
$$= 1 + 126q + 756q^2 + \cdots. \quad (44)$$

E_8 is obtained similarly from the extended Hamming code of length eight. For this lattice $d = 8$,

$$M = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix},$$

$\rho = 1/2$, $\det = 1/16$, $\tau = 240$, $\Delta = \pi^4/384 = 0.253670\cdots$, and the theta function is (see for example [52, eqs. (34), (47), (48)])

$$\Theta_{E_8}(z) = \theta_3(z)^8 + 14\theta_3(z)^4\theta_2(z)^4 + \theta_2(z)^8$$
$$= \frac{1}{2}\left\{\theta_2\left(\frac{z}{2}\right)^8 + \theta_3\left(\frac{z}{2}\right)^8 + \theta_4\left(\frac{z}{2}\right)^8\right\}$$
$$= 1 + \sum_{m=1}^{\infty} N_m q^m, \quad (45)$$

where

$$N_m = 240\sigma_3(m) \quad (46)$$

and

$$\sigma_r(m) = \sum_{d \mid m} d^r. \quad (47)$$

The coefficients lie in the range

$$240m^3 < N_m < 240\zeta(3)m^3 \approx 288.5m^3. \quad (48)$$

Furthermore N_m is equal to the number of integral Cayley numbers of norm m [15, ch. 2]. The first fifty terms of these three theta functions are given in Table VI.

TABLE VI
THE LATTICES E_6, E_7, AND E_8 IN $\boldsymbol{R}^6$, $\boldsymbol{R}^7$, AND $\boldsymbol{R}^8$

m	$N_m(E_6)$	$N_m(E_7)$	$(240)^{-1}\cdot N_m(E_8)$
1	72	126	1
2	270	756	9
3	720	2072	28
4	936	4158	73
5	2160	7560	126
6	2214	11592	252
7	3600	16704	344
8	4590	24948	585
9	6552	31878	757
10	5184	39816	1134
11	10800	55944	1332
12	9360	66584	2044
13	12240	76104	2198
14	13500	99792	3096
15	17712	116928	3528
16	14760	133182	4681
17	25920	160272	4914
18	19710	177660	6813
19	26064	205128	6860
20	28080	249480	9198
21	36000	265104	9632
22	25920	281736	11988
23	47520	350784	12168
24	37638	382536	16380
25	43272	390726	15751
26	45900	470232	19782
27	59040	505568	20440
28	46800	532800	25112
29	75600	615384	24390
30	51840	640080	31752
31	69264	701568	29792
32	73710	799092	37449
33	88560	809424	37296
34	62208	853776	44226
35	108000	1006992	43344
36	85176	1051974	55261
37	98640	1031688	50654
38	97740	1195992	61740
39	122400	1286208	61544
40	88128	1313928	73710
41	151200	1469664	68922
42	110700	1474704	86688
43	133200	1547784	79508
44	140400	1797768	97236
45	157680	1776600	95382
46	114048	1809360	109512
47	198720	2104704	103824
48	147600	2130968	131068
49	176472	2123982	117993
50	162270	2382156	141759

XVIII. THE 12-DIMENSIONAL LATTICE K_{12}

This is the densest packing known in $\boldsymbol{R}^{12}$ and like E_6 is most easily constructed from a complex lattice. The starting point is the three-dimensional self-dual code g_6 of length six over GF(4) with generator matrix

$$\begin{bmatrix} 1 & \omega & \omega & 1 & 0 & 0 \\ \omega & 1 & \omega & 0 & 1 & 0 \\ \omega & \omega & 1 & 0 & 0 & 1 \end{bmatrix},$$

where $\omega \in \mathrm{GF}(4)$ is a primitive cube root of unity. By applying the complex version of Construction A to this code [53], [54], we obtain a lattice in $\boldsymbol{C}^6$ with generator matrix

$$\frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 1 & \omega & \omega & 1 & 0 & 0 \\ \omega & 1 & \omega & 0 & 1 & 0 \\ \omega & \omega & 1 & 0 & 0 & 1 \end{bmatrix},$$

where now $\omega = e^{2\pi i/3}$. Then K_{12} is obtained by regarding this as a real lattice in $\boldsymbol{R}^{12}$. For this lattice $d = 12$, M is

$$\frac{1}{4}\left[\begin{array}{ccc|ccc|ccc|ccc}
4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
\hline
2 & -1 & -1 & 2 & 0 & 0 & 0 & \sqrt{3} & \sqrt{3} & 0 & 0 & 0\\
-1 & 2 & -1 & 0 & 2 & 0 & \sqrt{3} & 0 & \sqrt{3} & 0 & 0 & 0\\
-1 & -1 & 2 & 0 & 0 & 2 & \sqrt{3} & \sqrt{3} & 0 & 0 & 0 & 0\\
\hline
2 & 0 & 0 & 0 & 0 & 0 & -2\sqrt{3} & 0 & 0 & 0 & 0 & 0\\
0 & 2 & 0 & 0 & 0 & 0 & 0 & -2\sqrt{3} & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -2\sqrt{3} & 0 & 0 & 0\\
\hline
1 & 1 & 1 & 1 & 0 & 0 & \sqrt{3} & \sqrt{3} & \sqrt{3} & -\sqrt{3} & 0 & 0\\
1 & 1 & 1 & 0 & 1 & 0 & \sqrt{3} & -\sqrt{3} & \sqrt{3} & 0 & -\sqrt{3} & 0\\
1 & 1 & 1 & 0 & 0 & 1 & \sqrt{3} & \sqrt{3} & -\sqrt{3} & 0 & 0 & -\sqrt{3}
\end{array}\right]$$

Fig. 3 Generator matrix for the 12-dimensional lattice packing K_{12}.

TABLE VII
K_{12}, THE BEST PACKING KNOWN IN $\boldsymbol{R}^{12}$

m	N_m	m	N_m
0	1	19	48009024
1	0	20	64049832
2	756	21	70709184
3	4032	22	102958128
4	20412	23	124782336
5	60480	24	142254252
6	139860	25	189423360
7	326592	26	237588120
8	652428	27	248250240
9	1020096	28	344391264
10	2000376	29	397510848
11	3132864	30	433936440
12	4445532	31	554879808
13	7185024	32	671393772
14	10747296	33	677557440
15	13148352	34	908374824
16	21003948	35	1018507392
17	27506304	36	1079894844
18	33724404		

shown in Fig. 3, $\rho = 1/2$, $\det = 3^3/2^{12}$, $\tau = 756$, and

$$\Delta = \frac{\pi^6}{19440} = 0.0494542\cdots.$$

Furthermore from the weight enumerator of g_6 we can write down immediately that

$$\begin{aligned}\Theta_{K_{12}}(z) &= \phi_0(2z)^6 + 45\phi_0(2z)^2\phi_1(2z)^4 + 18\phi_1(2z)^6\\ &= 1 + 756q^2 + 4032q^3 + \cdots, \qquad (49)\end{aligned}$$

where $\phi_0(z)$ is defined in (30) and

$$\begin{aligned}\phi_1(z) &= \theta_2(z)\theta_3(3z) + \theta_2(3z)\theta_3(z)\\ &= \frac{1}{2}\theta_2\left(\frac{z}{4}\right)\theta_2\left(\frac{3z}{4}\right)\\ &= 2q^{1/4}(1 + q^{1/2} + 2q^{3/2} + q^2 + 2q^3 + \cdots). \qquad (50)\end{aligned}$$

The first 36 terms of the theta function are given in Table VII.

XIX. The 16-Dimensional Lattice Λ_{16}

Only a brief description is given here of Λ_{16}, the densest known lattice in $\boldsymbol{R}^{16}$, which is obtained by applying Construction B to the first-order Reed–Muller code of length 16 (see [34, sec. 3.4], [52, ex. 9]). For this lattice $d = 16$, M is shown in Fig. 4, $\rho = 2^{-1/2}$, $\det = 2^{-4}$, $\tau = 4320$, and

$$\Delta = \frac{\pi^8}{8!2^4} = 0.0147082\cdots.$$

$$\frac{1}{2}\left[\begin{array}{cccccccccccccccc}
4 &&&&&&&&&&&&&&&\\
2 & 2 &&&&&&&&&&&&&&\\
2 & 0 & 2 &&&&&&&&&&&&&\\
2 & 0 & 0 & 2 &&&&&&&&&&&&\\
2 & 0 & 0 & 0 & 2 &&&&&&&&&&&\\
2 & 0 & 0 & 0 & 0 & 2 &&&&&&&&&&\\
2 & 0 & 0 & 0 & 0 & 0 & 2 &&&&&&&&&\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 2 &&&&&&&&\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 &&&&&&&\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 &&&&&&\\
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 &&&&&\\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 &&&&\\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 &&&\\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 &&\\
0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 &\\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}\right] \quad \bigcirc$$

Fig. 4 Generator matrix for the 16-dimensional lattice packing Λ_{16}. The last six rows are a generator matrix for the first-order Reed–Muller code of length 16.

TABLE VIII
Λ_{16}, THE BEST PACKING KNOWN IN $\boldsymbol{R}^{16}$

m	N_m	m	N_m
0	1	16	8593797600
1	0	17	11585617920
2	4320	18	19590534240
3	61440	19	25239859200
4	522720	20	40979580480
5	2211840	21	50877235200
6	8960640	22	79783021440
7	23224320	23	96134307840
8	67154400	24	146902369920
9	135168000	25	172337725440
10	319809600	26	256900127040
11	550195200	27	295487692800
12	1147643520	28	431969276160
13	1771683840	29	487058227200
14	3371915520	30	699846624000
15	4826603520		

Again the theta function can be written down from the weight enumerator of the code ([52, thm. 17]) and is

$$\begin{aligned}\Theta_{\Lambda_{16}}(z) &= \frac{1}{2}\left\{\theta_2(z)^{16} + \theta_3(z)^{16} + \theta_4(z)^{16} + 30\theta_2(z)^8\theta_3(z)^8\right\}\\ &= 1 + 4320q^2 + 61440q^3 + \cdots. \qquad (51)\end{aligned}$$

The first 30 terms are given in Table VIII.

At this point it is worth mentioning an interesting unsolved problem. The Nordstrom–Robinson code [36, ch. 2, sec. 8], [42] is a union of the first-order Reed–Muller code above and seven of its translates; it is optimal in the sense that it contains the greatest number of codewords of any binary code of length 16 and minimum distance 6. It is possible that there exists an analogous nonlattice packing, perhaps consisting of a union of Λ_{16} and 15 translates. The theta function can be determined exactly and the first ten terms are given in Table IX. Such a packing would have an extremely high kissing number and density (see [52, open prob. 7]).

XX. The 24-Dimensional Leech Lattice Λ_{24}

A great deal has already been written about this important lattice, the densest known packing in $\boldsymbol{R}^{24}$ [2], [4],

TABLE IX
THE HYPOTHETICAL "NORDSTROM–ROBINSON" NONLATTICE PACKING IN $\boldsymbol{R}^{16}$

m	N_m
0	1
1/2	0
1	0
3/2	7680
2	4320
5/2	276480
3	61440
7/2	2903040
4	522720
9/2	16896000

[9]–[12], [32], [34], [43], [52], [54], [55]. The centers are the points

$$\frac{1}{2\sqrt{2}}(\mathbf{0}+2\boldsymbol{c}+4\boldsymbol{x}) \tag{52}$$

and

$$\frac{1}{2\sqrt{2}}(\mathbf{1}+2\boldsymbol{c}+4\boldsymbol{y}), \tag{53}$$

where $\mathbf{0}=(0,0,\cdots,0)$, $\mathbf{1}=(1,1,\cdots,1)$, $\boldsymbol{c}$ is any codeword in the binary extended Golay code of length 24, and $\boldsymbol{x}\in\boldsymbol{Z}^{24}$ and $\boldsymbol{y}\in\boldsymbol{Z}^{24}$ satisfy

$$\sum_{i=1}^{24} x_i \equiv 0 \pmod 2,$$

$$\sum_{i=1}^{24} y_i \equiv 1 \pmod 2.$$

For this lattice $d=24$, M is shown in Fig. 5, $\rho=1$, $\det=1$, $\tau=196560$,

$$\Delta=\frac{\pi^{12}}{12!}=0.00192957\cdots,$$

and

$$\Theta_{\Lambda_{24}}(z)=\Theta_{E_8}(z^2)^3-720q^2\prod_{m=1}^{\infty}(1-q^{2m})^{24}$$
$$=1+196560q^4+16773120q^6+\cdots. \tag{54}$$

The first 40 terms are given in Table X, together with their prime factors.

There is an explicit formula for N_m, the coefficient of q^m:

$$N_m=\frac{65520}{691}\left(\sigma_{11}\left(\frac{m}{2}\right)-\tau\left(\frac{m}{2}\right)\right), \tag{55}$$

where $\sigma_{11}(n)$ is defined by (47) and $\tau(n)$ is a Ramanujan number defined by

$$q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n. \tag{56}$$

Furthermore

$$N_m=\frac{65520}{691}\sigma_{11}\left(\frac{m}{2}\right)+O(m^6), \tag{57}$$

the second term on the right being of much smaller order than the first.

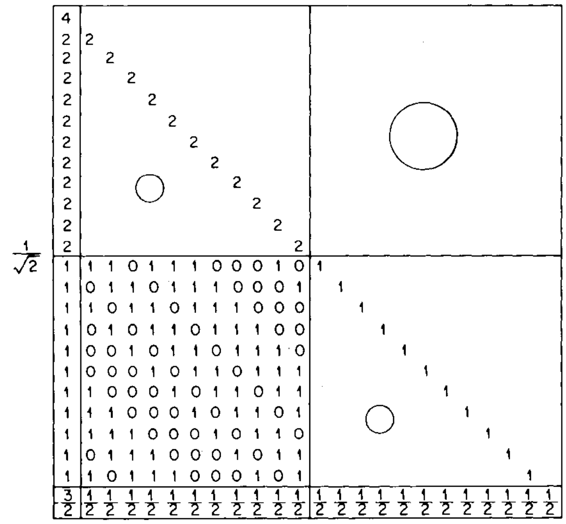

Fig. 5 Generator matrix for the 24-dimensional Leech lattice. The 11×11 circulant matrix in the bottom left corner comes from the generator matrix of the Golay code (see [36, fig. 2.13]).

TABLE X
THE LEECH LATTICE IN $\boldsymbol{R}^{24}$

m	N_m	Prime Factors of N_m
0	1	1
2	0	0
4	196560	$2^4\cdot3^3\cdot5\cdot7\cdot13$
6	16773120	$2^{12}\cdot3^2\cdot5\cdot7\cdot13$
8	398034000	$2^4\cdot3^7\cdot5^3\cdot7\cdot13$
10	4629381120	$2^{14}\cdot3^3\cdot5\cdot7\cdot13\cdot23$
12	34417656000	$2^6\cdot3^3\cdot5^3\cdot7\cdot13\cdot17\cdot103$
14	187489935360	$2^{13}\cdot3^7\cdot5\cdot7\cdot13\cdot23$
16	814879774800	$2^4\cdot3^3\cdot5^2\cdot7\cdot13\cdot17^2\cdot19\cdot151$
18	2975551488000	$2^{15}\cdot3^2\cdot5^3\cdot7\cdot13\cdot887$
20	9486551299680	$2^5\cdot3^7\cdot5\cdot7\cdot13\cdot23\cdot12953$
22	27052945920000	$2^{12}\cdot3^3\cdot5^4\cdot7\cdot11\cdot13\cdot17\cdot23$
24	70486236999360	$2^6\cdot3^2\cdot5\cdot7^2\cdot13^2\cdot59\cdot50093$
26	169931095326720	$2^{14}\cdot3^7\cdot5\cdot7^2\cdot13\cdot1489$
28	384163586352000	$2^7\cdot3^3\cdot5^3\cdot7\cdot13\cdot23\cdot83\cdot5119$
30	820166620815360	$2^{13}\cdot3^3\cdot5\cdot7\cdot13\cdot17\cdot19\cdot23\cdot1097$
32	1668890090322000	$2^4\cdot3^8\cdot5^3\cdot7\cdot13\cdot751\cdot1861$
34	3249631112232960	$2^{15}\cdot3^4\cdot5\cdot7\cdot13\cdot23\cdot116993$
36	6096882661243920	$2^4\cdot3^3\cdot5\cdot7^2\cdot13\cdot17\cdot260654803$
38	11045500816896000	$2^{12}\cdot3^8\cdot5^3\cdot7\cdot13\cdot23\cdot1571$
40	19428439855275360	$2^5\cdot3^3\cdot5\cdot7^2\cdot13\cdot23\cdot1747\cdot175709$

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REFERENCES

[1] W. W. R. Ball and H. S. M. Coxeter, *Mathematical Recreations and Essays*, 12th ed. Toronto: Univ. Toronto Press, 1974.
[2] E. Bannai and N. J. A. Sloane, "Uniqueness of certain spherical codes," *Canad. J. Math.*, 1981, to be published.
[3] E. Biglieri and M. Elia, "Cyclic-group codes for the Gaussian channel," *IEEE Trans. Inform. Theory*, vol. IT-22, pp. 624–629, 1976.
[4] I. F. Blake, "The Leech lattice as a code for the Gaussian channel," *Inform. Contr.* vol. 19, pp. 66–74, 1971.
[5] I. F. Blake, "Distance properties of group codes for the Gaussian channel," *SIAM J. Appl. Math.*, vol. 23, pp. 312–324, 1972.
[6] I. F. Blake and R. C. Mullin, *The Mathematical Theory of Coding*. New York: Academic, 1975.

[7] W. S. Brown, *Altran User's Manual*, 4th ed. Murray Hill, NJ: Bell Laboratories, 1977.
[8] R. de Buda, "The upper error bound of a new near-optimal code," *IEEE Trans. Inform. Theory*, vol. IT-21, pp. 441–445, 1975.
[9] J. H. Conway, "A perfect group of order 8, 315, 553, 613, 086, 720, 000, and the sporadic simple groups," *Proc. Nat. Acad. Sci.*, vol. 61, pp. 398–400, 1968.
[10] J. H. Conway, "A group of order 8, 315, 553, 613, 086, 720, 000," *Bull. London Math. S.*, vol. 1, pp. 79–88, 1969.
[11] J. H. Conway, "A characterization of Leech's lattice," *Invent. Math.*, vol. 7, pp. 137–142, 1969.
[12] J. H. Conway, "Three lectures on exceptional groups," in *Finite Simple Groups*, M. B. Powell and G. Higman, Eds. London: Academic, 1971, pp. 215–247.
[13] H. S. M. Coxeter, "The problem of packing a number of equal nonoverlapping circles on a sphere," *Trans. New York Acad. Sci.*, series II, vol. 24, no. 3, pp. 320–331, Jan. 1962.
[14] H. S. M. Coxeter, *Regular Polytopes*, 2nd ed. New York: Macmillan, 1963.
[15] H. S. M. Coxeter, *Twelve Geometric Essays*. Carbondale, IL: Southern Illinois Univ. Press, 1968.
[16] P. Delsarte, J. M. Goethals, and J. J. Seidel, "Spherical codes and designs," *Geometriae Dedicata*, vol. 6, pp. 363–388, 1977.
[17] L. E. Dickson, *History of the Theory of Numbers*. New York: Chelsea, 1966.
[18] A. Gersho, "Asymptotically optimal block quantization," *IEEE Trans. Inform. Theory*, vol. IT-25, pp. 373–380, 1979.
[19] A. Gersho, "Principles of quantization," *IEEE Trans. Circuit. Syst.*, vol. CAS-25, pp. 427–436, July 1978.
[20] J. W. L. Glaisher, "On the numbers of representations of a number as a sum of $2r$ squares, where $2r$ does not exceed eighteen," *Proc. London Math. Soc.*, series 2, vol. 5, pp. 479–490, 1907.
[21] J. M. Goethals and J. J. Seidel, "Spherical designs," in *Relations Between Combinatorics and Other Parts of Mathematics, Proc. Sympos. Pure Math. XXXIV*, D. K. Ray-Chaudhuri, Ed. Providence RI: Amer. Math. Soc., 1979, pp. 255–272.
[22] J. M. Goethals and J. J. Seidel, "Cubature formulae, polytopes and spherical designs," in *Proc. Coxeter Symposium*, Toronto, May 1979, to be published.
[23] M. Goldberg, "An improved packing of 33 equal circles on a sphere," *Elemente der Mathematik*, vol. 22, pp. 110–111, 1967.
[24] R. C. Gunning, *Lectures on Modular Forms*. Princeton: Princeton Univ. Press, 1962.
[25] G. H. Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five or seven," *Proc. Nat. Acad. Sci.*, vol. 4, pp. 189–193, 1918, (also *Collected Papers*, vol. I, pp. 340–344).
[26] G. H. Hardy, "On the representation of a number as the sum of any number of squares, and in particular of five," *Trans. Amer. Math. Soc.*, vol. 21, pp. 255–284, 1920, (also *Collected Papers*, vol. I, pp. 345–374).
[27] G. H. Hardy, *Ramanujan*. Bronx, NY: Chelsea, 1959.
[28] G. H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, 3rd Ed. Oxford: Clarendon, 1954.
[29] G. A. Kabatiansky and V. I. Levenshtein, "Bounds for packings on a sphere and in space," *Prob. Inform. Transmiss.*, vol. 14, no. 1, pp. 1–17, 1978.
[30] M. I. Knopp, *Modular Functions in Analytic Number Theory*. Chicago: Markham, 1970.
[31] H. J. Landau, "How does a porcupine separate its quills?," *IEEE Trans. Inform. Theory.*, vol. IT-17, pp. 157–161, 1971.
[32] J. Leech, "Notes on sphere packings," *Canad. J. Math.*, vol. 19, pp. 251–267, 1967.
[33] J. Leech and N. J. A. Sloane, "New sphere packings in more than thirty-two dimensions," in *Proc. Second Chapel Hill Conference on Combinatorial Mathematics and Its Applications*, Chapel Hill, NC: Univ. of North Carolina, 1970, pp. 345–355.
[34] J. Leech and N. J. A. Sloane, "Sphere packings and error-correcting codes," *Canad. J. Math.*, vol. 23, pp. 718–745, 1971.
[35] S. P. Lloyd, "Hamming association schemes and codes on spheres," *SIAM J. Math. Anal.*, vol. 11, pp. 488–505, May 1980.
[36] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam; North Holland, and New York: Elsevier–North Holland, 1977.
[37] A. V. Malyšev, "On the representation of integers by positive quadratic forms with four or more variables I, *Amer. Math. Soc. Translations*, series 2, vol. 46, pp. 17–47, 1965.
[38] Mathlab Group, MIT, *MACSYMA Reference Manual, Version* 9. Cambridge MA: MIT, 1977.
[39] R. M. Mersereau, "The processing of hexagonally sampled two-dimensional signals," *Proc. IEEE*, vol. 67, pp. 930–949, June 1979.
[40] W. C. Mitchell, "The number of lattice points in a k-dimensional hypersphere," *Math. Comp.*, vol. 20, pp. 300–310, 1966.
[41] M. Newman, "A table of the coefficients of the power of $\eta(\tau)$," *Proc. Kon. Ned. Akad. Wetensch., Series A*, vol. 59, pp. 204–216, 1956.
[42] A. W. Nordstrom and J. P. Robinson, "An optimum nonlinear code," *Inform. Contr.*, vol. 11, pp. 613–616, 1967.
[43] A. M. Odlyzko and N. J. A. Sloane, "New bounds on the number of unit spheres that can touch a unit sphere in n dimensions," *J. Comb. Theory*, vol. 26A, pp. 210–214, 1979.
[44] G. Pry and I. Prigogine, "Rayons et nombres de coordination de quelques réseaux simples," *Acad. Roy. Belgique, Bull. Cl. Sci.* vol. 28, pp. 866–873, 1942.
[45] H. Rademacher, *Topics in Analytic Number Theory*. New York: Springer, 1973.
[46] R. A. Rankin, *Modular Forms and Functions*. Cambridge: The University Press, 1977.
[47] C. A. Rogers, *Packing and Covering*. Cambridge: The University Press, 1964.
[48] A. Schrijver, Ed., *Packing and Covering in Combinatorics*, Math. Centre Tracts 106. Amsterdam: Mathematisch Centrum, 1979.
[49] L. A. Shepp, "Computerized tomography and nuclear magnetic resonance," *J. Comp. Assist. Tomograph.*, vol. 4, no. 1, pp. 94–107, Feb. 1980.
[50] D. Slepian, "Group codes for the Gaussian channel," *Bell Syst. Tech. J.*, vol. 47, pp. 575–602, 1968.
[51] N. J. A. Sloane, "Sphere packings constructed from BCH and Justesen codes," *Mathematika*, vol. 19, pp. 183–190, 1972.
[52] N. J. A. Sloane, "Binary codes, lattices, and sphere packings," in *Combinatorial Surveys*, P. J. Cameron, Ed. London and New York: Academic, 1977, pp. 117–164.
[53] N. J. A. Sloane, "Codes over GF(4) and complex lattices," *J. Algebra*, vol. 52, pp. 168–181, 1978.
[54] N. J. A. Sloane, "Self-dual codes and lattices," in *Relations Between Combinatorics and Other Parts of Mathematics, Proc. Symp. Pure Math. XXXIV*, D. K. Ray-Chaudhuri, Ed. Providence, RI: Amer. Math. Soc., 1979, pp. 273–308.
[55] N. J. A. Sloane, "A note on the Leech lattice as a code for the Gaussian channel," *Inform. Contr.*, vol. 46, pp. 270–272, 1980.
[56] S. L. Sobolev, "Cubature formulas on the sphere invariant under finite groups of rotations," *Soviet Math. Doklady*, vol. 3, pp. 1307–1310, 1962.
[57] J. Tannery and J. Molk, *Éléments de la théorie des Fonctions Elliptiques*. 2nd Ed. Bronx, NY: Chelsea, 1972.
[58] Y. Tashiro, "On methods for generating uniform random points on the surface of a sphere," *Ann. Inst. Statist. Math.*, vol. 29, pt. A, pp. 295–300, 1977.
[59] A. Z. Val'fiš, "On the representation of numbers by sums of squares —asymptotic formulas," *Amer. Math. Soc. Translations*, series 2, vol. 3, pp. 163–248, 1956.
[60] B. L. van der Waerden, "Pollenkörner, Punktverteilungen auf der Kugel und Informationstheorie," *Die Naturwissenschaften*, no. 7, pp. 189–192, 1961.
[61] A. Walfisz (= A. Z. Val'fiš), "Uber Gittergunkte in mehrdimensionalen Ellipsoiden," *Math. Zeit.*, vol. 19, pp. 300–307, 1924.
[62] E. T. Whittaker and G. N. Watson, *A Course of Modern Analysis*, 4th Ed. Cambridge: The University Press, 1963.
[63] A. D. Wyner, "On coding and information theory," *SIAM Review*, vol. 11, pp. 317–346, 1969.