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Tail sums of convergent series of independent random variables

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Let $X_1, X_2, \ldots$ be a sequence of independent random variables such that, for each $n \geqslant 1$, $EX_n = 0$ and $EX_n^2 = \sigma_n^2 < \infty$, and assume that $\sum_{n=1}^{\infty} \sigma_n^2 < \infty$: then $\sum_{n=1}^{N} X_n$ converges almost surely as $N \to \infty$. Let $S_n = \sum_{j=n}^{\infty} X_j$, $s_n^2 = \operatorname{Var} S_n = \sum_{j=n}^{\infty} \sigma_j^2$, and let $F_n(x)$ denote the distribution function of X_n. Loynes (2) observed that the sequence $\{S_n\}$ is a reversed martingale, and applied his central limit theorem to it: however, stronger results are obtainable, in precise duality with the classical theory of partial sums of independent random variables. These results describe the fluctuations of the sequence $\{S_n\}$, and hence the way in which $\sum_{n=1}^{N} X_n$ converges to its limit.

For example, as in the Lindeberg–Feller theorem,

$$S_n/s_n \xrightarrow{D} N(0,1) \quad \text{and} \quad \sigma_n/s_n \to 0 \quad \text{as} \quad n \to \infty \tag{1}$$

if and only if, for each $\epsilon > 0$,

$$s_n^{-2} \sum_{k=n}^{\infty} \int_{|x| \geqslant \epsilon s_n} x^2 \, dF_k(x) \to 0 \quad \text{as} \quad n \to \infty: \tag{2}$$

similarly, as in Kolmogorov's law, setting $\phi_n = [2 \log\log s_n^{-1}]^{\frac{1}{2}}$,

$$|X_n| = o(s_n/\phi_n) \quad \text{a.s.} \quad \text{as} \quad n \to \infty \tag{3}$$

implies that

$$\limsup_{n \to \infty} S_n/s_n \phi_n = 1 \quad \text{a.s.} \tag{4}$$

Counterparts of the Berry–Esseen theorems, and of the functional weak and strong laws, can also easily be derived.

All these theorems can be proved by suitably adapting the classical proofs. Alternatively, the weak laws can be deduced from the classical theorems for triangular arrays, by looking at the array with nth row $X_n, X_{n+1}, \ldots, X_{k(n)}$, where, because $\sum_{j=1}^{\infty} \sigma_j^2 < \infty$, the index $k(n)$ can be chosen so that $s_{k(n)}^2/s_n^2$ is small enough to make $S_{k(n)+1}$ negligible. When $\{X_n\}$ is a martingale difference sequence, this method is also applicable.

Neither of the above two means of proof really explains why there should exist a duality between the fluctuations of classical partial sums and the fluctuations of tails of convergent sums. Yet the fact that each classical theorem has its 'tail' counterpart strongly suggests that there is a reason for the duality, and, if so, a general method might be found for deducing 'tail' theorems directly from their classical analogues. The rest of this paper is concerned with providing such a method.

The basic idea is to obtain *functional* weak and strong laws for $\{X_n\}$ from the classical theorems, by finding suitable continuous mappings which transform random functions associated with classical partial sum processes into random functions associated with the tails of convergent series, and vice versa. The continuous mapping theorem can then be used to establish the desired duality. The method is analogous to a device used by Whitt (5) in another context, and so most details are omitted here. The approach uses stronger mathematical techniques than are needed to prove the results mentioned in (1)–(4), but the general method which emerges is rather powerful.

Let $D[0,\infty)$ be the space of all right continuous functions with left limits on $[0,\infty)$, and let T_3 be the subspace consisting of those functions x which satisfy

$$(A)\quad \limsup_{t\to\infty} t^{-1}|x(t)| = 0,$$

$$(B)\quad \int_1^\infty u^{-2}|x(u)|\,du < \infty,$$

$$(C)\quad \int_0^1 u^{-1}|x(u)|\,du < \infty.$$

Let m_3 be the metric on T_3 such that $m_3(x,y)$ is the infimum of those $\epsilon > 0$ for which there exists some continuous strictly increasing function λ: $[0,\infty) \to [0,\infty)$, with $\lambda(0) = 0$, such that

$$(A')\quad \sup_{t\geq 0} |x(t)-y(\lambda(t))|/(t \vee 1) < \epsilon,$$

$$(B')\quad \int_1^\infty u^{-2}|x(u)-y(\lambda(u))|\,du < \epsilon,$$

$$(C')\quad \int_0^1 u^{-1}|x(u)-y(\lambda(u))|\,du < \epsilon,$$

$$(D)\quad \sup_{t\neq s}\left|\log\frac{\lambda(t)-\lambda(s)}{t-s}\right| < \epsilon.$$

Let $T_2(T_1)$ and $m_2(m_1)$ be defined similarly, but omitting restrictions $C, C'(C, C', B, B')$. Let T_i^*, $i = 1,2,3$, denote the corresponding subspaces of $D^*[0,\infty)$, the space of all left continuous functions with right limits on $[0,\infty)$.

Consider the mappings g: $T_3 \to T_2^*$ and: $T_2^* \to T_1$ defined by

$$g(x)(0) = 0:\quad g(x)(s) = sx(s^{-1}) - \int_{s^{-1}}^\infty u^{-2}x(u)\,du,\quad 0 < s < \infty.$$

A straightforward calculation shows that both mappings are continuous, and that for $x \in T_3$ with $x(0) = 0$, $g(g(x)) = x$. Thus, if a sequence of random functions $\{Z_n\}$ converges weakly in (T_3, m_3) $[(T_2^*, m_2)]$ to a random function Z, the sequence of random functions $\{g(Z_n)\}$ converges weakly in (T_2^*, m_2) $[(T_1, m_1)]$ to $g(Z)$: similarly, if $\{Z_n\}$ has compact closure almost surely in (T_3, m_3) $[(T_2^*, m_2)]$ with set of limit points the compact set K, then $\{g(Z_n)\}$ has compact closure almost surely in (T_2^*, m_2) $[(T_1, m_1)]$ with set of limit points $g(K)$.

The relevance of this result in the present problem is that, for Z a random function in T_2^* (T_3), $g(Z)(s)$ is the stochastic integral

$$\int_{1/s}^{\infty} u^{-1}\, dZ(u),$$

defined by its integration by parts formula. Suppose that $\{X_n\}$ is defined as in the initial paragraph, and that $\sigma_n/s_n \to 0$ as $n\to\infty$: put $Z_n(t) = s_{n+1}^{-1} f(n)\, S_{k(n,t)}$, $t \geqslant 0$, where $f(n)$ is a normalization function, and where

$$k(n,t) = \max\,[r\colon r \text{ integral},\ s_r^2 \geqslant t s_{n+1}^2].$$

Then $$g(Z_n)(s) = s_{n+1} f(n) \sum_{j=1}^{k(n,s^{-1})-1} X_j/s_j^2,$$

and $$\sum_{j=1}^{k(n,s^{-1})-1} \operatorname{Var}(X_j/s_j^2) = \sum_{j=1}^{k(n,s^{-1})-1} \sigma_j^2/s_j^4 \sim s/s_{n+1}^2 \quad \text{as} \quad s\to\infty.$$

Hence $g(Z_n)$ is a random function derived from the partial sums of the sequence of independent random variables $Y_j = X_j/s_j^2$, and $\sum_{j=1}^{\infty} \operatorname{Var} Y_j = \infty$, so that the classical theorems may be applied to it.

For instance, taking $f(n) = 1$, $g(Z_n) \Rightarrow W$, the standard Wiener process, in (T_3, m_3), if and only if the random variables Y_j satisfy Lindeberg's condition: the extra conditions required for convergence in T_3 rather than in $D[0,1]$ are readily verified using arguments similar to those in Muller(3), p. 177. This can be used to deduce corresponding results for $Z_n = g(g(Z_n))$, and it can be shown, by choosing a specially simple set of Y_j, that $g(W)$ is also a standard Wiener process.

The Lindeberg condition for $\{Y_n\}$ does not at first sight appear to be the same as condition (2) for $\{X_n\}$, yet they are in fact equivalent. The two conditions can be expressed as

$$(L)\colon\ g_n^*(\epsilon) = s_{n+1}^2 \sum_{j=1}^{n} \int_{|z| \geqslant \epsilon s_j^2/s_{n+1}} s_j^{-4} z^2\, dF_j(z) \to 0 \quad \text{as} \quad n\to\infty,$$

for each fixed $\epsilon > 0$, and

$$(T)\colon\ g_n(\epsilon) = s_{n+1}^{-2} \sum_{j=n+1}^{\infty} \int_{|z| \geqslant \epsilon s_{n+1}} z^2\, dF_j(z) \to 0 \quad \text{as} \quad n\to\infty,$$

for each fixed $\epsilon > 0$. Define, for $s > t > 0$,

$$g_n^*(\epsilon, s, t) = s_{n+1}^2 \sum_{j=k(n,s)+1}^{k(n,t)} \int_{|z| \geqslant \epsilon s_j^2/s_{n+1}} s_j^{-4} z^2\, dF_j(z),$$

$$g_n(\epsilon, s, t) = s_{n+1}^{-2} \sum_{j=k(n,s)+1}^{k(n,t)} \int_{|z| \geqslant \epsilon s_{n+1}} z^2\, dF_j(z)\colon$$

then, since $k(n,s)+1 \leqslant j \leqslant k(n,t)$ implies $t \leqslant s_j^2 s_{n+1}^{-2} < s$,

$$g_n(\epsilon, s, t) \geqslant t^2 g_n^*(\epsilon/t, s, t) \geqslant t^2 s^{-2} g_n(\epsilon s/t, s, t),$$

so that $$(L')\colon\ g_n^*(\epsilon, s, t) \to 0 \quad \text{as} \quad n\to\infty$$

for each fixed $\epsilon > 0$ and $s > t > 0$ is equivalent to

$$(T'):\ g_n(\epsilon, s, t) \to 0 \quad \text{as} \quad n \to \infty$$

for each fixed $\epsilon > 0$ and $s > t > 0$. But

$$g_n(\epsilon, s, t) = s_{n+1}^{-2}[s_{k(n,s)+1}^2 g_{k(n,s)}(\epsilon s_{n+1}/s_{k(n,s)+1}) - s_{k(n,t)+1}^2 g_{k(n,t)}(\epsilon s_{n+1}/s_{k(n,t)+1})],$$

and, for any $u > 0$, $k(n, u) \to \infty$ and $s_{k(n,u)+1}^2/s_{n+1}^2 \to u$ as $n \to \infty$: hence (T) implies (T'). On the other hand, if (T') holds, fix $\epsilon > 0$ and $s = 1$, so that $k(n, s) = n+1$: then $g_n(\epsilon, 1, t) \to 0$ as $n \to \infty$ for all $0 < t < 1$, so that, by Chung(1), § 7·2, Lemma 1, there exists a sequence $t^{(n)} \to 0$ such that $g_n(\epsilon, 1, t^{(n)}) \to 0$ as $n \to \infty$. Hence

$$g_n(\epsilon) \leqslant g_n(\epsilon, 1, t^{(n)}) + t^{(n)} \to 0 \quad \text{as} \quad n \to \infty,$$

and so (T') implies (T). Thus (T') is equivalent to (T). A similar argument shows that (L') is equivalent to (L), and so the equivalence of (L) and (T) is proved.

If, now, $f(n)$ is taken to be $[2 \log \log s_n^{-1}]^{-\frac{1}{2}}$, invariance principles for the law of the iterated logarithm can be deduced for $\{X_n\}$ from those for $\{Y_n\}$. In this case, it is easy to prove directly that, if K is the set of all absolutely continuous real functions x on $[0, \infty)$ for which $\int_0^\infty [\dot{x}(t)]^2 dt \leqslant 1$, cf. Strassen(4), then $g(K) = K$.

Because the basic mapping theorem makes little restriction on $\{Z_n\}$ and Z, it can be used in much more general contexts than that chosen here.

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