Tail-transported temporal correlations in the dynamics of a gravitating system

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The existence of retarded correlations over arbitrarily large time spans in the dynamics of a gravitating system, namely, the influence of the past evolution of a material system on its present gravitational internal dynamics, is investigated. This "hereditary" influence can be thought of as transported by the gravitational waves emitted by the system in the past and subsequently scattered off the curvature of spacetime back onto the system ("backscattered waves" or "tails"). The method used here applies to weakly self-gravitating slowly varying sources. It is a combination of a multipolar post-Minkowskian expansion for the metric in the weak-field region outside the system, and of a post-Newtonian-type expansion for the metric in the near zone. The two expansions are then "matched" in the weak-field-near-zone overlap region. The lowest-order nonlinear piece in the near-zone metric which depends on the full past history of the source ("hereditary" term) is determined. This term arises at the fourth post-Newtonian (PN) level. The arising of this "hereditary" term signifies the breakdown of one of the fundamental tenets of the post-Newtonian approximation schemes. Indeed, at the 4PN level it becomes impossible to express the near-zone metric as a functional of the instantaneous state of the material source. This means also that there is a fundamental breakdown of the concept of near zone versus the concept of wave zone. The direct dynamical influence of the above-determined hereditary, or tail, term on the evolution of the material system is then studied. This term is found to modify the Burke-Thorne gravitational radiation quadrupole damping force. This modification, although quite small in absolute magnitude, is rather large relative to the usual damping force, being $\sim(v/c)^3$ smaller ($1\frac{1}{2}$PN relative level). It could be important in the dynamics of inspiralling binaries. Finally, it is shown that the hereditary term is predominantly sensitive to the recent past evolution of the system and only negligibly dependent on its very remote past history.

I. INTRODUCTION

The assumption of localizability, in space and time, of physical systems is one of the most successful (and fundamental) assumptions of physics. This assumption means that one can make accurate predictions on the evolution of a physical system as if the system were isolated from other systems located far away in space, and as if its present state were uncorrelated with its states at epochs remote in time. Progress has been often slow and scarce in the fields where this assumption cannot be made (for instance, in some branches of cosmology). General relativity entails an essential spacetime nonlocality of the gravitational interaction. At the origin of this nonlocality is the combination of two remarkable features of Einstein's equations: their "hyperbolicity" (i.e., physically, the presence of propagation effects at a finite velocity) and their (infinite) nonlinearity (i.e., physically, the fact that gravity generates gravity, and influences its propagation). In this work we shall investigate a consequence of this general-relativistic spacetime nonlocality: the influence of the remote-past behavior of an (isolated) gravitationally interacting system on its present local-in-space gravitational field, and thereby on its present dynamical evolution.

The existence of such gravitationally induced dynamical correlations over very long time spans has been known, in principle, since the work of Choquet-Bruhat.[1] She investigated the existence and propagation of the solutions of Einstein's field equations as determined by an initial-value problem (Cauchy problem). She proved that, given some initial data on a spacelike hypersurface S (that we shall think of as lying in the remote past), the gravitational field at some event P in spacetime depends on the values of the data *on* and *within* the intersection of S with the past (curved) light cone having its vertex at P. This result can be interpreted by saying that the gravitational field propagates with all velocities smaller than or equal to the local "velocity of light" (by which we mean the maximum velocity of propagation of interactions). A second way of picturing this phenomenon is to say that the gravitational waves propagate with the local "velocity of light" but that they undergo a continuous backscattering off the curvature of spacetime, thereby developing the so-called wave tails. Thus, on the whole, they seem to propagate with all velocities smaller than or equal to the "velocity of light." Note that this phenomenon arises already for linear propagation equations, e.g., Maxwell's equations for the propagation of electromagnetic waves, in a curved background. However, we are here interested in the gravitational case, which is essentially a *nonlinear* phenomenon. Indeed, the gravitational field backscatters off the curvature of spacetime which is nothing more, in Einstein's theory, than the gravitational field itself. Thus one has a third way of viewing the propagation of the gravitational field within the light cone: the nonlinear beating of two linearized gravitational fields emitted by a material source acts as an effective nonlocal stress-energy distribution which, in turn, generates a third contribution to the field (this is the classical equivalent of a Feynman graph for a three-graviton vertex). In this picture, the nonlocality of the effective stress-energy distribution, together with the finite velocity of the linearized waves, imply that some pieces of the gravitational field, here and now, have been generated by the material source at arbitrarily remote epochs in the past. In other words, some pieces of the field irreducibly depend on the whole *past history* of the source. In the following, we shall qualify these pieces as "hereditary."

The phenomenon of propagation within (instead of on) the light cone has many aspects and has been studied by many authors. For instance we can quote some mathematical investigations of the existence and construction of solutions,[1-6] general physical investigations of the nonlinear structure of the relativistic gravitational field,[7-17] the influence of backscattering on outgoing (electromagnetic or gravitational) waves and the formation of "wave tails" in simple curved spacetimes,[18-25] and the effect of backscattering on the phenomenon of electromagnetic radiation damping.[26-28] The present investigation will use, from the mathematical point of view, some techniques and approximation methods which, up to now, have been mainly used to study the structure of the outgoing gravitational radiation.[11-17] However, from the physical point of view, our aim is closer to the one of the works[26-28] studying the modification of the electromagnetic radiation damping due to the propagation of the electromagnetic field within the light cone.

Another aspect of the present investigation is its bearing on the problem of determining the range of validity of the post-Newtonian approximation methods. These methods have been developed by many authors (notably Einstein, Fock, and Chandrasekhar) and they have reached a high degree of formal sophistication.[29-38] However they have also run into a number of difficulties which cast serious doubts on their validity. These difficulties are of two sorts.

First, there are fundamental problems in the sense that the mathematical meaning of these methods is unclear. Indeed, their basic assumption is that a solution of Einstein's theory can be approximated by, essentially, a power series in the inverse velocity of light, c^{-1}, whose first significant terms give back Newton's theory. However the precise definition and the mathematical status of this series have only been investigated quite recently[39,36] and no firm result has yet been obtained.[40]

The second difficulty encountered by the post-Newtonian methods is that, at some level of approximation, they lead to divergent integrals or ambiguities.[34,37,41,42] Several authors have discussed some of the causes of these divergences and possible cures for them,[33,42-45] but the situation is still confused: for example, Kerlick[34] and Futamase[37] find logarithmically divergent integrals at the "third post-Newtonian" (3PN) order, i.e., at order $1/c^8$ in g_{00}, while Anderson *et al.*[42] point out a breakdown of the post-Newtonian expansion at fourth post-Newtonian (4PN) order ($1/c^{10}$ in g_{00}). Concerning the cause of these divergences, the most recent discussions[43,42,36,37] place the blame on the traditional assumption that the post-Newtonian expansion proceeds along simple powers of $1/c$, and point out that it is necessary to introduce, at some level of approximation, *logarithms* of the expansion parameter. It has been recently shown[16] that all higher post-Newtonian orders of approximation can be expanded along the asymptotic sequence $(\ln c)^p/c^n$, with $p,n \in \mathbb{N}$ (and $2p \leq n-2$).

In this paper we present what we think is a deeper understanding of the cause of the breakdown of any post-Newtonian approximation method. Indeed, the basic tenet of the whole post-Newtonian approach is to model somehow Einstein's theory onto Newton's theory. In particular, the instantaneous character of the Newtonian gravitational interaction is taken over into the post-Newtonian approach. The fact that the Einsteinian gravitational interaction is not instantaneous poses no insurmountable problems at the level of the *lowest* post-Newtonian approximations for the following reason. The retardation due to a direct propagation between the source point $\mathbf{x}'$ and the field point $\mathbf{x}$ is essentially of order $|\mathbf{x}-\mathbf{x}'|/c$ which is small if the field point is well within the near zone of the source (see Sec. II). Therefore, it seems possible, by using suitable retardation expansions, to express the Einsteinian gravitational field, when considered in the near zone of the source, in terms of the *instantaneous* state of the material source. This is precisely what all the post-Newtonian approximation schemes do. (These approximation schemes are thus valid only in the near zone of the source.) However the preceding argument neglects the important fact that, as we recalled above, the propagation of the full Einsteinian gravitational field proceeds not only along the (curved) light cone ("direct propagation with the local velocity of light"), but also inside the light cone ("scattered propagation" or "tail"). This means that even for a field point $\mathbf{x}$ well within the near zone, the retardation (or time delay) $t-t'$, due to the propagation between $\mathbf{x}$ and some source point $\mathbf{x}'$, can be much larger than $|\mathbf{x}-\mathbf{x}'|/c$, and, in fact, arbitrarily large. We consider that the occurrence of such correlations over arbitrarily large time spans signals the fundamental breakdown of the concept of near zone. This conclusion differs from the one of Anderson *et al.*[42] Indeed, the latter authors have found a $\ln c$ at the 4PN order, and concluded from the link of the $\ln c$ with divergent integrals arising in the usual post-Newtonian schemes that "the PN divergences signify the breakdown of the PN power-series assumptions, rather than a breakdown of the near and wave zones" (see also the discussion of Futamase[37]). Our work shows that it is not sufficient to enlarge the set of gauge functions to the set $c^{-p}(\ln c)^q$ but that one must also allow for a noninstantaneous character of the expansion coefficients. On the other hand, coming back to a more technical level, our work confirms the main result of Anderson *et al.*[42] Indeed we recover (at the end of Sec. VI) the $\ln c$ term found by the latter authors from our result (1.1). (See also the Appendix for the discussion of other $\ln c$ terms, also pointed out by Anderson *et al.*) Note that if all post-Newtonian schemes must break down at the 4PN level, they may, evidently, break down before that level if the algorithm used generates a divergent integral of lower order. This is the case, for instance, in the Ehlers algorithm,[33] where, as shown in Ref. 34, divergent integrals arise at the 3PN level.

We shall determine below that the level of approximation at which the near-zone field can no longer be expressed in terms of the instantaneous state of the source is the fourth post-Newtonian (4PN) level, which corresponds to $1/c^{10}$ in g_{00}. We shall compute the lowest-order "hereditary" contribution to the near-zone field. It is given by[46]

$$\delta g_{00}(t,\mathbf{x})\,|^{\text{hereditary}} = -\frac{8}{5c^{10}} x^i x^j I(t) \times \int_{-\infty}^{t} dt' \ln\left[\frac{t-t'}{2P}\right] \times \frac{d^7 I_{ij}(t')}{dt'^7}\,, \tag{1.1}$$

where $I(t)$ and $I_{ij}(t)$ are, respectively, the Newtonian mass and quadrupole moment of the source at time t, and where P is a characteristic time scale of the source. The contribution (1.1) gives rise to a modification of the usual[47,48] gravitational radiation *damping force* (see the discussion in Sec. VII). This can be thought of as the radiation damping associated with the gravitational wave tail generated by the backscattering of the linearized quadrupole wave off the monopole curvature created by the total mass of the system. We shall also investigate the link of the breakdown of the post-Newtonian approach (at 4PN) caused by the near-zone term (1.1) with the weaker breakdown due to the appearance of logarithms of the expansion parameter. As far as we know these results are new and no attempt has been made before to derive them within the full framework of general relativity. However, special mention should be made of the work of Rudolph[28] (who first pointed out the link between the breakdown of the near-zone expansions and the propagation within the light cone), of Dixon[49] (who got explicit past-dependent contributions in the near-zone approximation of a simple scalar-field model) and of Anderson and co-workers[42–44] (who by their studies of scalar-field models, and of the matching between the wave-zone and near-zone fields, have greatly clarified the occurrence of the post-Newtonian breakdown).

The presentation of the plan of the paper is relegated to the end of the next section, in which we present our method.

II. METHOD

In this paper we shall restrict our attention to material systems which are both *weakly self-gravitating* and *slowly moving*. We thus have at our disposal two small dimensionless parameters. First, we have a field-weakness parameter

$$\gamma = \frac{Gm}{c^2 r_0} \ll 1\,, \tag{2.1}$$

where m is a characteristic mass and r_0 a characteristic size of our system (we shall assume that r_0 is strictly greater than the radius of a sphere in which the system can be completely enclosed). Second, we have a slowness parameter

$$\beta = \frac{r_0}{cP} \ll 1\,, \tag{2.2}$$

where P denotes a characteristic time scale for the evolution of the system [for instance, $(2\pi)^{-1}$ times the principal period if the motion of the system is quasiperiodic]. The slowness parameter is often introduced as $\beta = v/c$, where $v = r_0/P$ is a characteristic bulk velocity of the system. However, it will be more useful to think of β as the ratio r_0/λ, where $\lambda = cP$ is a characteristic (reduced) wavelength of the gravitational radiation field emitted by the system, so that $\beta \ll 1$ means that the system is well within its near zone.

A priori the two small parameters γ and β are independent. Indeed, we can conceive of weakly self-gravitating fast-moving sources, or slow-moving strongly self-gravitating ones. Often one assumes from the start the relation (linked with the virial theorem) $\gamma \sim \beta^2$, which is appropriate to the description of gravitationally bound systems. However, here we shall often use independently the two assumptions (2.1) and (2.2).

First we shall consider the limiting process[50] $\gamma \to 0$, or equivalently $G \to 0$ with c fixed, and only afterwards shall we consider the second limit[50] $\beta \to 0$ ($c^{-1} \to 0$). The first limit $G \to 0$ is physically a weak-field limit and corresponds to the so-called *post-Minkowskian* approximation methods.[7–10,38] Since we assumed that the field was weak in the source, we expect that these methods will yield good approximations to the true gravitational field all over space and time. Here we shall find it convenient to use a particular type of post-Minkowskian method, which is valid only *outside* the source: the *multipolar-post-Minkowskian* (MPM) method. This method was pioneered by Bonnor and co-workers[11–13] and Thorne,[15] and was further developed in a previous paper,[16] hereafter referred to as paper I (see also Ref. 51).

Let $D_e = \{(\mathbf{x},t)\,|\, r \geq r_0\}$ be an "exterior" or "outer" domain around the source. The MPM method assumes first that, in D_e, the "gothic" metric $\mathfrak{g}^{\alpha\beta} = \sqrt{g}\, g^{\alpha\beta}$ (see Ref. 46 for our notation and conventions) can be expanded in an asymptotic expansion in powers of γ (or equivalently in powers of G):

$$\mathfrak{g}^{\alpha\beta} = f^{\alpha\beta} + G h_1^{\alpha\beta}(\mathbf{x},t,c^{-1}) + G^2 h_2^{\alpha\beta}(\mathbf{x},t,c^{-1}) + \cdots + G^n h_n^{\alpha\beta}(\mathbf{x},t,c^{-1}) + \cdots\,, \tag{2.3}$$

where $f^{\alpha\beta}$ is the Minkowskian flat metric $f^{\alpha\beta} = \mathrm{diag}(-1,1,1,1)$. Second, it is assumed that each term of the series, $h_n^{\alpha\beta}(\mathbf{x},t,c^{-1})$, admits, in D_e, a multipolar expansion associated with the O(3) group of rotations of the spatial coordinates (which leaves invariant $r=|\mathbf{x}|=[(x^1)^2+(x^2)^2+(x^3)^2]^{1/2}$), i.e.,

$$h_n^{\alpha\beta}(\mathbf{x},t,c^{-1})=\sum_{l\geq 0}\hat{n}_L(\theta,\phi)h_{nL}^{\alpha\beta}(r,t,c^{-1}) , \tag{2.4}$$

where $L\equiv i_1i_2\cdots i_l$ denotes a multispatial index of order l, and $\hat{n}_L$ denotes the symmetric-trace-free (STF) part of $n_L\equiv n_{i_1}n_{i_2}\cdots n_{i_l}$, with $n_i=n^i=x^i/r$ being the unit coordinate direction vector from the origin (located in the source) towards the exterior field point $\mathbf{x}$. The expansion (2.4) is equivalent to the usual expansion in (scalar) spherical harmonics $Y_l^m(\theta,\phi)$ (see, e.g., Appendix A of paper I). In paper I we assumed, in order to avoid possible convergence problems, that the multipole expansion (2.4) contained only a finite number of terms. However, physical intuition and past experience in many fields of mathematical physics lead us to expect that the requirement of convergence of such multipole expansions, considered outside the source ($r>r_0$ with r_0 sufficiently greater than the source radius), should cause only very weak restrictions on the spatial and temporal structure of the source. In this paper we shall take formally the limit of the results of paper I for an infinite number of "multipoles." Finally we require, as in paper I, that the multipolar-post-Minkowskian (MPM) metrics [Eq. (2.3) with (2.4)] were, in the remote past, at once stationary and spatially asymptotically Minkowskian, in the sense that before some fixed instant $-T$, i.e., when $t\leq -T$, one had

$$(\partial/\partial t)\mathbf{g}^{\alpha\beta}(\mathbf{x},t)=0 \quad\text{and}\quad \lim_{r\to+\infty}\mathbf{g}^{\alpha\beta}(\mathbf{x},t)=f^{\alpha\beta} .$$

We view the latter assumptions as a way to express the physical fact that we are considering an isolated system which is the sole generator of the gravitational field ("no external fields" and "no incoming radiation"). The parameter T plays only the formal role of a "cutoff" parameter. We expect (and shall check to some extent) that it is possible to take the limit $-T\to-\infty$ under only weak conditions on the behavior of the source in the remote past.

Inserting the nonlinearity expansion (2.3) into the vacuum Einstein equations (appropriate to the exterior domain D_e), we get a hierarchy of inhomogeneous flat-space wave equations for the h_n's. Then the use of the multipole expansion (2.4) enables us to solve the latter equations quite explicitly, so that the MPM method just sketched allows us to have a detailed control of the structure of the external gravitational field everywhere outside the source. However, this knowledge is only structural, and disconnected from the actual source. We, therefore, need to complete the MPM method with a different, source-rooted, method.

We shall employ for this purpose a method of the *post-Newtonian type.* Let $D_i=\{(\mathbf{x},t)\,|\,r<\kappa r_0\}$, where κ is some constant >1, be an "inner domain" which contains the source. We assume for the metric in the domain D_i an asymptotic expansion when $c^{-1}\to 0$ (with G fixed) of the form

$$g_{\alpha\beta}=f_{\alpha\beta}+\sum_n\delta_n(c^{-1})_n g_{\alpha\beta}(\mathbf{x},t,G) , \tag{2.5}$$

where the δ_n's, for $n\in\mathbb{N}$, constitute an ordered set of gauge functions (see, e.g., Ref. 52) such that

$$\lim_{\epsilon\to 0}[\delta_{n+1}(\epsilon)/\delta_n(\epsilon)]=0 .$$

It turns out (see paper I) that one must use the set of gauge functions $\epsilon^p(\ln\epsilon)^q$ with $p,q\in\mathbb{N}$ such that $p\geq 2(q+1)$.

Inserting the expansion (2.5) into the Einstein equations with matter, and taking into account all explicit powers of c^{-1} (especially in $\partial/\partial x^0=c^{-1}\partial/\partial t$) gives a hierarchy of Poisson equations for the ${}_n g$'s.

Now the point is to notice that the post-Minkowskian external metric (2.3) should, when submitted to a further near-zone expansion, coincide in the overlap region $D_i\cap D_e$ ($r_0<r<\kappa r_0$) with the post-Newtonian inner metric (2.5). (Actually things are more complicated than that because the coordinate system used in D_i may differ from the one used in D_e, see Sec. VI.) This requirement is a variant of the method of "matched asymptotic expansions."[52] (However, it should be said that this type of method has not yet been fully clarified, neither from the point of view of its mathematical justification, nor even from the point of view of its formal structure.) The matching of the two metrics gives us a knowledge of the gravitational field everywhere (both outside and inside the system) and thereby a complete solution, in principle, of our problem.

The organization of this paper is as follows. In Sec. III we summarize from paper I the construction of the general MPM external metric satisfying, in D_e, Eqs. (2.3) and (2.4), the vacuum Einstein equations and the boundary conditions in the past. This external metric depends functionally on an infinite set of parametrizing functions, which we shall call the "algorithmic multipole moments." In Sec. IV we investigate the structure of the latter external metric and especially its functional dependence on the algorithmic moments. In Sec. V we reexpand the external metric in the near zone (in fact in $D_i\cap D_e$) or, in other words, we take the post-Newtonian expansion of the (post-Minkowskian) external metric. In Sec. VI we match the near-zone-expanded external metric to an inner metric and compute the lowest-order hereditary piece in the inner field. Finally, in Sec. VII we discuss the effect of this hereditary piece on the dynamics of the source and study its "sensitivity" on the remote past history of the system. We discuss the appearance of some logarithms of c in the Appendix.

III. THE EXTERNAL METRIC

As announced in the previous section, the first step of the method consists of constructing the "external metric," i.e., the gravitational field in the weak-field domain outside the source $D_e=\{(\mathbf{x},t)\,|\,r>r_0\}$. This task has been achieved in paper I (Ref. 16), building on foundations laid by Bonnor and co-workers,[11–13] and

Thorne,[15] i.e., within what we call the multipolar-post-Minkowskian (MPM) framework. Let us summarize (and adapt to our problem) the construction of the general metric satisfying the assumptions stated in Sec. II.

It has been proven in paper I (Theorem 4.5) that the most general MPM external metric can be (functionally) parametrized by a "skeleton" of "algorithmic multipole moments," namely, by a set of time-dependent symmetric trace-free (STF) Cartesian tensors:

$$\mathcal{M}=\{M,M_i,S_i,M_{i_1i_2}(t),S_{i_1i_2}(t),\ldots,M_L(t),S_L(t),\ldots\}\ , \tag{3.1}$$

where L denotes the spatial multi-index $i_1i_2\cdots i_l$. The only constraints implied by the field equations that the M_L's and S_L's must satisfy are that M, ${}^{(1)}M_i=dM_i/dt$, and S_i be time independent. However, we make the further assumption (hopefully to be relaxed in future work) that the metric is stationary in the past ($t\le -T$). This implies that all the M_L's and S_L's are constant when $t\le -T$ and that M, S_i, and also M_i are always constant ($\forall t$). We should make it clear that the "algorithmic moments" have no direct physical meaning, and need not have one [apart from M, which is the Arnowitt-Deser-Misner (ADM) (rest) mass of the system]. They play the role of arbitrary functional parameters in the construction of the external metric. They will however acquire later an *indirect* physical meaning when we relate them to observable quantities at infinity, or to the matter distribution (in anticipation of this we choose, with Thorne,[15] to give them the physical dimensions appropriate for "multipole moments").

Given the functional skeleton $\mathcal{M}=\{M_L(t),S_L(t)\}$, the (gothic) external metric reads

$$\mathfrak{g}^{\alpha\beta}_{\rm ext}[\mathcal{M}]=f^{\alpha\beta}+Gh_1^{\alpha\beta}[\mathcal{M}]+G^2h_2^{\alpha\beta}[\mathcal{M}]+\cdots+G^nh_n^{\alpha\beta}[\mathcal{M}]+\cdots\ , \tag{3.2}$$

where the linearized external metric h_1 is given explicitly as [see Eqs. (2.32) of paper I]

$$h_1^{00}[\mathcal{M}]=-\frac{4}{c^2}\sum_{l\ge 0}\frac{(-)^l}{l!}\partial_L[r^{-1}M_L(t-r/c)]\ , \tag{3.3a}$$

$$h_1^{0i}[\mathcal{M}]=\frac{4}{c^3}\sum_{l\ge 1}\frac{(-)^l}{l!}\partial_{L-1}[r^{-1}{}^{(1)}M_{iL-1}(t-r/c)]+\frac{4}{c^3}\sum_{l\ge 1}\frac{(-)^l l}{(l+1)!}\epsilon_{iab}\partial_{aL-1}[r^{-1}S_{bL-1}(t-r/c)]\ , \tag{3.3b}$$

$$h_1^{ij}[\mathcal{M}]=-\frac{4}{c^4}\sum_{l\ge 2}\frac{(-)^l}{l!}\partial_{L-2}[r^{-1}{}^{(2)}M_{ijL-2}(t-r/c)]-\frac{8}{c^4}\sum_{l\ge 2}\frac{(-)^l l}{(l+1)!}\partial_{aL-2}[r^{-1}\epsilon_{ab(i}{}^{(1)}S_{j)bL-2}(t-r/c)]\ , \tag{3.3c}$$

where the superior prefix (n) denotes the nth derivative, and where $T_{(ij)}\equiv\frac{1}{2}(T_{ij}+T_{ji})$. The explicit powers of c^{-1} in Eqs. (3.3) come from our choice of physical dimensions for

$$[M_L]=[\text{mass}][\text{length}]^l$$

("mass" or "electric-type" moments) and

$$[S_L]=[\text{mass}][\text{velocity}][\text{length}]^l$$

("current" or "magnetic-type" moments). Then the nonlinear pieces $h_2,h_3,\ldots$ are recursively constructed, starting from h_1, as follows.

First, if one replaces the formal post-Minkowskian expansion (3.2) into the Einstein tensor density $2gE^{\alpha\beta}=2g(R^{\alpha\beta}-\frac{1}{2}Rg^{\alpha\beta})$, one obtains a formal expansion of the type

$$2gE^{\alpha\beta}=\sum_{n=1}^{\infty}G^n[\partial_{\mu\nu}H^{\alpha\mu\beta\nu}(h_n)-N_n^{\alpha\beta}(h_m;m<n)]\ , \tag{3.4a}$$

where

$$H^{\alpha\mu\beta\nu}=f^{\mu\nu}h_n^{\alpha\beta}+f^{\alpha\beta}h_n^{\mu\nu}-f^{\alpha\nu}h_n^{\beta\mu}-f^{\beta\mu}h_n^{\alpha\nu}\ . \tag{3.4b}$$

The N_n's (nonzero only if $n\ge 2$) are some nonlinear polynomials of the "previous" h_m's ($m<n$) and their first and second partial derivatives. For instance, we have

$$\begin{aligned}N_2^{\alpha\beta}(h)=&-\partial^2_{\mu\nu}(h^{\alpha\beta}h^{\mu\nu})+h^{\mu\alpha}\partial^2_{\mu\nu}h^{\beta\nu}+h^{\mu\beta}\partial^2_{\mu\nu}h^{\alpha\nu}+\partial_\mu h^{\alpha\beta}\partial_\nu h^{\mu\nu}\\&-\tfrac{1}{4}\partial^\alpha h^\mu_\mu\partial^\beta h^\nu_\nu+\tfrac{1}{2}\partial^\alpha h_{\mu\nu}\partial^\beta h^{\mu\nu}-\partial^\alpha h_{\mu\nu}\partial^\mu h^{\beta\nu}-\partial^\beta h_{\mu\nu}\partial^\mu h^{\alpha\nu}\\&+\partial_\nu h^{\alpha\mu}\partial^\nu h^\beta_\mu+\partial_\nu h^{\alpha\mu}\partial_\mu h^{\beta\nu}+f^{\alpha\beta}(\tfrac{1}{8}\partial_\mu h^\nu_\nu\partial^\mu h^\rho_\rho-\tfrac{1}{4}\partial_\mu h_{\nu\rho}\partial^\mu h^{\nu\rho}+\tfrac{1}{2}\partial_\mu h_{\nu\rho}\partial^\nu h^{\mu\rho})\ .\end{aligned} \tag{3.5}$$

Now, the definition of the recursive nonlinear h_n ($n\ge 2$) is

$$h_n^{\alpha\beta}[\mathcal{M}]\equiv p_n^{\alpha\beta}+q_n^{\alpha\beta}\ , \tag{3.6a}$$

where

$$p_n^{\alpha\beta} \equiv \text{ finite part}_{B=0} \Box_R^{-1}[(r/r_1)^B N_n^{\alpha\beta}] \; , \tag{3.6b}$$

$$q_n^{00} \equiv -cr^{-1(-1)}A(u) - c\partial_a[r^{-1(-1)}A_a(u)] + c^2\partial_a[r^{-1(-2)}C_a(u)] \; , \tag{3.6c}$$

$$q_n^{0i} \equiv -cr^{-1(-1)}C_i(u) - c\epsilon_{iab}\partial_a[r^{-1(-1)}D_b(u)] - \sum_{l\geq 2} \partial_{L-1}[r^{-1}A_{iL-1}(u)] \; , \tag{3.6d}$$

$$\begin{aligned} q_n^{ij} \equiv & -\delta_{ij}\{r^{-1}B(u) + \partial_a[r^{-1}B_a(u)]\} \\ & + \sum_{l\geq 2} \left\{ \partial_{L-2}\left[r^{-1}\left[\frac{1}{c}{}^{(1)}A_{ijL-2}(u) + \frac{3}{c^2}{}^{(2)}B_{ijL-2}(u) - C_{ijL-2}(u) \right]\right] \right. \\ & \left. + 2\delta_{ij}\partial_L[r^{-1}B_L(u)] - 6\partial_{L-1(i}[r^{-1}B_{j)L-1}(u)] - 2\partial_{aL-2}[\epsilon_{ab(i}r^{-1}D_{j)bL-2}(u)] \right\} . \end{aligned} \tag{3.6e}$$

The symbols appearing in the definitions (3.6) have the following meaning. $\Box_R^{-1}$ denotes the usual retarded integral operator

$$(\Box_R^{-1}f)(\mathbf{x}',t') \equiv -\frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{d^3\mathbf{x}}{|\mathbf{x}'-\mathbf{x}|} \times f\left[\mathbf{x}, t' - \frac{1}{c}|\mathbf{x}'-\mathbf{x}|\right] . \tag{3.7}$$

The letter B denotes a complex number and r_1 is some arbitrary (but fixed) length scale, to be chosen later. Then the right-hand side of Eq. (3.6b) is defined by analytic continuation in B in the following sense. First, choosing the real part of B small enough, one defines the function of B (considered at a fixed "field point" $\mathbf{x}',t'$),

$$F'_A(B) \equiv \Box_R^{-1}[(r/r_1)^B N_n(\mathbf{x},t) Y(|\mathbf{x}| - A)] \; , \tag{3.8a}$$

where A is some fixed radius and where Y denotes the Heaviside step function. Second, choosing now the real part of B large enough, one defines the function

$$F''_A(B) \equiv \Box_R^{-1}[(r/r_1)^B N_n(\mathbf{x},t) Y(A - |\mathbf{x}|)] \; . \tag{3.8b}$$

Then, one extends the definition of both $F'_A(B)$ and $F''_A(B)$ all over $\mathbb{C}' \equiv \mathbb{C} - \mathbb{Z}$ by analytic continuation in B. Then the function

$$F(B) = \Box_R^{-1}[(r/r_1)^B N_n]$$

appearing in Eq. (3.6b) is defined as

$$F(B) \equiv F'_A(B) + F''_A(B) \; . \tag{3.8c}$$

It is defined all over $\mathbb{C}'$ and is independent of A. Near $B=0$, $F(B)$ admits a Laurent expansion $F(B) = \sum_{-k_0}^{+\infty} C_k B^k$. Then the "finite part (FP) at $B=0$" of $F(B)$, which appears in the definition (3.6b) of $p_n^{\alpha\beta}(\mathbf{x}',t')$, is defined to be the coefficient $C_0 = C_0(\mathbf{x}',t')$ of B^0 in the Laurent expansion of $F(B)$. The fact that these constructions are mathematically well defined is nontrivial, and has been proven by induction in paper I. Actually, in paper I it was found convenient to use a somewhat different presentation of the algorithm based on decomposing N_n in a "stationary" and a "dynamic" part. But, using the results proven there, it is easy to show that the presentation given here is strictly equivalent. Finally the functions $A_L(u)$, $B_L(u)$, $C_L(u)$, and $D_L(u)$ appearing in Eqs. (3.6c), (3.6d), and (3.6e) are functions of the retarded time $u = t - r/c$, which are symmetric trace-free (STF) tensor functions. They are uniquely defined as the "multipole moments" of the following decomposition of $\partial_\beta p_n^{\alpha\beta}$:

$$\partial_\beta p_n^{0\beta} = \sum_{l\geq 0} \partial_L[r^{-1}A_L(u)] \; , \tag{3.9a}$$

$$\begin{aligned} \partial_\beta p_n^{i\beta} = & \sum_{l\geq 0} \partial_{iL}[r^{-1}B_L(u)] \\ & + \sum_{l\geq 1} \{\partial_{L-1}[r^{-1}C_{iL-1}(u)] \\ & + \epsilon_{iab}\partial_{aL-1}[r^{-1}D_{bL-1}(u)]\} \; . \end{aligned} \tag{3.9b}$$

The following notation has been used:

$$\begin{aligned} {}^{(-1)}A(u) &= \int_{-\infty}^{u} dx\, A(x) \; , \\ {}^{(-2)}A(u) &= \int_{-\infty}^{u} dx\, {}^{(-1)}A(x) \; , \end{aligned} \tag{3.10}$$

and, for $n \geq 0$, ${}^{(n)}A(u) = d^n A(u)/du^n$ [from the results of Appendix C of paper I, the functions $A_L(u), \ldots, D_L(u)$ are all zero when $u \leq -T$].

It has been shown in paper I that h_n defined by Eqs. (3.3) and (3.6) is a solution of the nth-order post-Minkowskian expanded [see Eq. (3.4)] vacuum Einstein equations in harmonic coordinates. In other words, it satisfies (in the external domain $r > r_0 \geq 0$)

$$\Box h_n^{\alpha\beta} = N_n^{\alpha\beta} \; , \tag{3.11}$$

$$\partial_\beta h_n^{\alpha\beta} = 0 \; . \tag{3.12}$$

Therefore, the definitions (3.2), (3.3), and (3.6) constitute an algorithm (called "canonical" in paper I) which generates a formal MPM vacuum metric starting from an arbitrary set of "algorithmic multipole moments" $\mathcal{M}$. Note that there is nothing unique about this algorithm, one can devise many other algorithms constructing vacuum metrics in harmonic coordinates, or in other coordinate systems. For example, one can define[53,54] a related algorithm which generates a vacuum metric expressed in "radiative coordinates" which are well suited to analyzing the asymptotic behavior of the radiative gravitational field. On the other hand, what is important for our purpose is that all these algorithms are (geometrically and physically) equivalent, and construct the *most general* metric outside a past-stationary isolated matter distribution. This has been proven, within the MPM framework, under weak technical assumptions, in paper I (Theorem 4.5) (see also Sec. IV of Ref. 53).

IV. FUNCTIONAL DEPENDENCE OF THE EXTERNAL METRIC

In the previous section we recalled (and adapted) the algorithmic construction of paper I of the external metric in terms of the set of algorithmic multipole moments $\mathcal{M}=\{M_L(t),S_L(t)\}$. The first step of this construction is the "linearized" external metric h_1 given by Eqs. (3.3).

Let us start our discussion by considering h_1. The spatial derivatives ∂_L in Eqs. (3.3) ($\partial_L=\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$ with $\partial_i=\partial/\partial x^i$) act on quantities such as $r^{-1}M_L(t-r/c)$ and thus act both on the prefactor r^{-1} and also on the retardation r/c. If we expand these derivatives [using Eq. (A35a) of paper I], we find that h_1 can be written as a sum of terms of the form

$$h_1^{\alpha\beta}(\mathbf{x},t)=\sum c^{-p}\hat{n}_L r^{-j}F_L^{\alpha\beta}(t-r/c)\ . \tag{4.1}$$

Here $F_L^{\alpha\beta}(u)$ denotes a function of u which is a contraction between some constant Cartesian tensor $K_{LL_1}^{\alpha\beta}$ (made of Kronecker or Levi-Civita symbols) and some time derivative of one algorithmic multipole moment taken at the *same* instant u:

$$F_L^{\alpha\beta}(u)=K_{LL_1}^{\alpha\beta}\,{}^{(a_1)}M_{L_1}(u)\ \text{ or }\ K_{LL_1}^{\alpha\beta}\,{}^{(a_1)}S_{L_1}(u)\ . \tag{4.2}$$

The structure of Eqs. (4.1) and (4.2) makes manifest that $h_1(\mathbf{x},t)$, considered as a functional of the algorithmic multipole moments, depends only on the values of the moments and of their time derivatives taken at *one instant:* the "retarded time" $u=t-r/c$. Our purpose in this section is to make the distinction between this type of structure and the more complicated one (which will arise at nonlinear stages) involving a dependence on the values of the algorithmic moments on the time interval $]-\infty,t-r/c]$ (actually $[-T,t-r/c]$). In order to clarify this distinction we introduce some special notation and terminology.

We shall henceforth restrict the use of the three letters F, G, and H to denoting functions of one variable, such that $F(u)$ [and $G(u)$, $H(u)$] is algebraically constructed from the values of a finite number of algorithmic moments and their derivatives taken at the *same "instant"* u. For instance this is the case of $F_L^{\alpha\beta}(u)$ given by Eq. (4.2) or, more generally, of

$$F_L^{\alpha\beta}(u)=\sum K_{LL_1\cdots L_n}^{\alpha\beta}\,{}^{(a_1)}M_{L_1}(u)\cdots{}^{(a_n)}S_{L_n}(u)\ , \tag{4.3}$$

where as above $K_{LL_1\cdots L_n}^{\alpha\beta}$ is a constant Cartesian tensor made of δ's and ϵ's. We shall say that such F's, G's, and H's are "instantaneous functionals" of the algorithmic moments. And we shall extend this denomination to fields in spacetime, say $\varphi(\mathbf{x},t)$, whose value at the field point $(\mathbf{x},t)$ depends only on the value of such F's taken at *one instant* $u(\mathbf{x},t)$ (e.g., $u=t$ or $u=t-r/c$). For example, φ given by Eq. (5.11) below is an instantaneous functional of the algorithmic multipole moments (taken at $u=t$). And, as Eq. (4.1) shows, $h_1^{\alpha\beta}(\mathbf{x},t)$ also is an instantaneous functional of the moments (taken at $u=t-r/c$). Note that the fact that the common "instant" u which rules the instantaneous dependence is $u=t$ or $u=t-r/c$ does not matter. Our terminology is only aimed at distinguishing the "snap-shot" functional dependences (whether simultaneous or simply retarded) from the dependences on the full past history of the algorithmic moments. We shall qualify the latter dependences as "hereditary functionals" of the algorithmic moments. For example, we shall say that

$$X_L^{\alpha\beta}(u)=\int_{-\infty}^{u}dv\,\ln(u-v)K_{LL_1}^{\alpha\beta}\,{}^{(a_1)}M_{L_1}(v) \tag{4.4}$$

as well as a field $k^{\alpha\beta}(\mathbf{x},t)$ with structure, say,

$$k^{\alpha\beta}(\mathbf{x},t)=\sum\hat{n}^L r^{-j}X_L^{\alpha\beta}(t-r/c) \tag{4.5}$$

are both hereditary functionals of the algorithmic multipole moments. Finally, we shall also introduce a special terminology for a particularly simple type of hereditary functional: if some derivative of an hereditary functional $T(u)$ is actually an instantaneous functional, for instance,

$$T(u)={}^{(-1)}F(u)=\int_{-\infty}^{u}dv\,F(v)\ , \tag{4.6a}$$

or

$$T(u)={}^{(-2)}F(u)=\int_{-\infty}^{u}dv\,{}^{(-1)}F(v)\ , \tag{4.6b}$$

we shall say that $T(u)$ is a "semihereditary functional" of $\mathcal{M}$.

Note that we introduced the preceding terminology to characterize some properties of a functional of the algorithmic moments. However, the terminology can be applied to any functional of any independent functions (of one variable). For instance, in Sec. VI below, we shall apply the terminology to characterize functionals of the source variables, instead of the algorithmic moments. One should carefully specify what are the independent functions because the property of "instantaneity" is clearly not invariant under a general functional change of independent functions. Only when the context is unambiguous shall we simply use the adjectives "instantaneous" or "hereditary" without further qualification.

Armed with this convenient terminology, we now study the main aspects of the functional structure of the external metric and start with the quadratic external metric, h_2. As we recalled in Sec. III, h_2 is a sum of two contributions:

$$h_2^{\alpha\beta}=p_2^{\alpha\beta}+q_2^{\alpha\beta}\ . \tag{4.7}$$

Let us first consider p_2 (we shall deal with q_2 afterwards). According to Eq. (3.6b), p_2 is the finite part of the retarded integral of the source N_2, i.e.,

$$p_2^{\alpha\beta}=\mathrm{FP}_{B=0}\Box_R^{-1}(r^BN_2^{\alpha\beta})\ . \tag{4.8}$$

(We choose $r_1=1$ in all this section for simplicity's sake.) The source N_2 is the quadratic nonlinear piece

$N_2(h_1)$ of Einstein's equations, Eq. (3.5), computed with h_1, Eqs. (3.3). The general structure of $N_2(h_1)$ reads symbolically as

$$N_2(h_1) \sim h_1 \partial^2 h_1 + \partial h_1 \partial h_1 \ . \quad (4.9)$$

We insert into N_2 the explicit form (3.3) of the multipole expansion of h_1, rearrange the terms (using the rules of manipulation of formal series), and obtain the multipole expansion of N_2. Since the multipole expansion of h_1 is an expansion in "outward spherical waves" of the type $\partial_P[r^{-1}F(t-r)]$ for some multi-index P and some "instantaneous" function $F(u)$ [skipping all indices on $F(u)$ and using $c=1$], the expansion of N_2 will be made of quadratic products of such spherical waves. We write this expansion as (with $c=1$)

$$N_2 = \sum \partial_P[r^{-1}F(t-r)]\partial_Q[r^{-1}G(t-r)] \ , \quad (4.10)$$

for some multi-indices P and Q and some instantaneous functions F and G.

Let us now perform a series of "operations by parts" on each of the terms of (4.10). We mean by this a series of operations of the type

$$\partial_i A \partial_j B = \partial_i(A \partial_j B) - A \partial_{ij} B \ , \quad (4.11)$$

by which the derivatives on the left can be shifted in front and to the right (say). This leads to the structure

$$N_2 = \sum \partial_M \{ r^{-1}F(t-r)\partial_N[r^{-1}G(t-r)] \} \ , \quad (4.12)$$

for some multi-indices $M = i_1 i_2 \cdots i_m$ and $N = i_1 i_2 \cdots i_n$. We can now use the formulas in Appendix A of paper I, first to replace the derivative operator ∂_N, which appears within the square brackets of (4.12), by its expression in terms of trace-free operators $\hat{\partial}_L$ (with $l \le n$), and then to expand the latter $\hat{\partial}_L$'s onto trace-free unit vectors $\hat{n}_L$. The resulting structure is

$$N_2 = \sum_{2 \le k \le l+2} \partial_M[\hat{n}_L r^{-k} H_k(t-r)] \ , \quad (4.13)$$

for some instantaneous functions H_k which are products of F with time derivatives of G.

The powers k of r^{-1} in the square brackets of (4.13) range from 2 to $l+2$, where l is the "multipolarity" of the angular dependence of the square brackets [see Eq. (A35a) of paper I]. Let us first consider the terms with $k=2$, for instance the term $\partial_M[\hat{n}_L r^{-2} H_2(t-r)]$. By expanding the derivative ∂_M in the latter term, we get a term proportional to r^{-2}, namely,

$$(-)^m n_M \hat{n}_L r^{-2(m)} H_2(t-r) \ ,$$

plus other terms proportional to r^{-k} with $k \ge 3$. Let us show that these other terms (with $k \ge 3$) can always be recombined to form new terms of the type of the right-hand side (RHS) of (4.13), with now $k=3$, but still $k \le l+2$. Indeed, in the simplest case $m=1$ (only one space derivative $\partial_M = \partial_i$) we get

$$\partial_i[\hat{n}_L r^{-2} H_2(t-r)] = -n_i \hat{n}_L r^{-2(1)} H_2(t-r) + H_2(t-r)\partial_i(\hat{n}_L r^{-2}) \ . \quad (4.14)$$

The second term on the RHS of (4.14), which has $k=3$, will satisfy the constraint $k \le l+2$ if and only if its "multipolarity," l, is ≥ 1, or equivalently if its monopolar part (i.e., its angular average) is zero. That this is so follows from the fact that $\partial_i(\hat{n}_L r^{-2})$ has, when $l \ge 2$, multipolarity $l-1$ and $l+1$, while in the dangerous case $l=1$ ($\hat{n}_L = n_j$), $\partial_i(n_j r^{-2}) = -3\hat{n}_{ij} r^{-3}$ has multipolarity 2 and *not* zero. [In the case $l=0$, $\partial_i(\hat{n}_L r^{-2})$ has multipolarity 1.] Hence our statement is true for $m=1$. For higher values of m we proceed by induction on m, noticing that at each step the only "dangerous" terms that arise, i.e., the terms which are not manifestly of the type of the RHS of (4.13), are in fact of the above discussed type $\partial_i(\hat{n}_{L'} r^{-2})H_2'(t-r)$. Finally, we can write N_2 as

$$N_2 = r^{-2} Q_2(t-r, \mathbf{n}) + \sum_{3 \le k \le l+2} \partial_M[\hat{n}_L r^{-k} H_k'(t-r)] \ , \quad (4.15a)$$

with some instantaneous functions H_k' (distinct from H_k), and where $Q_2(t-r,\mathbf{n})$ is the r^{-2} part of N_2, i.e., with the notation of Eq. (4.13),

$$Q_2(t-r,\mathbf{n}) = \sum (-)^m n_M \hat{n}_L{}^{(m)} H_2(t-r) \ . \quad (4.15b)$$

As we shall see below, the expression (4.15a) is extremely convenient for handling the retarded integration of N_2 and for separating the hereditary components in the retarded integral. However, before taking the retarded integral of both sides of (4.15a) we must, following the definition (3.6b), multiply both sides by the analytic-continuation factor r^B. Then, as it will be convenient to commute the retarded integral with the derivative operator ∂_M, we need to introduce r^B inside the square brackets in (4.15a). This produces many extra terms coming from the derivation of r^B, but the point is that these extra terms will all have at least a power of B as a factor (coming from $\partial_i r^B = B r^{B-1} n_i$). This leads to

$$r^B N_2 = r^{B-2} Q_2(t-r,\mathbf{n}) + \sum_{3 \le k \le l+2} \partial_M[\hat{n}_L r^{B-k} H_k'(t-r)] + \sum_{k \ge 3; p \ge 1} B^p \hat{n}_L r^{B-k} H_{kp}''(t-r) \ , \quad (4.16)$$

with some other instantaneous functions H_{kp}''. Note that in the last term the powers of r^{-1} are not *a priori* bounded by $l+2$. Next we apply the operator $\Box_R^{-1}$ to both sides of (4.16), and we commute it with ∂_M (this is allowed because of our analytic continuation procedure). This yields

$$\Box_R^{-1}(r^B N_2) = \Box_R^{-1}[r^{B-2} Q_2(t-r,\mathbf{n})] + \sum_{3 \le k \le l+2} \partial_M \{ \Box_R^{-1}[\hat{n}_L r^{B-k} H_k'(t-r)] \} + \sum_{k \ge 3; p \ge 1} \Box_R^{-1}[B^p \hat{n}_L r^{B-k} H_{kp}''(t-r)] \ . \quad (4.17)$$

From Eq. (4.17) we see that the problem of computing p_2, namely, the finite part of $\Box_R^{-1}(r^B N_2)$, is reduced to the problem of computing (1) the finite part at $B=0$ of the retarded integral $\Box_R^{-1}[\hat{n}_L r^{B-2} H(t-r)]$, i.e., when $k=2$, (2) the finite part at $B=0$ of the retarded integrals $\Box_R^{-1}[\hat{n}_L r^{B-k} H(t-r)]$ in the case $3 \leq k \leq l+2$, and (3) the *pole part* at $B=0$ (because of the factor B^p with $p \geq 1$) of the same retarded integrals in the case $k \geq 3$ (without any l-dependent upper bound for k).

We shall deal with these retarded integrals by using an explicit expression, valid for all k and l, which has been derived in paper I [Eq. (6.9a)]. This explicit expression is

$$\Box_R^{-1}[\hat{n}_L r^{B-k} H(t-r)] = \frac{1}{K(B)} \int_r^\infty dz\, H(t-z) \hat{\partial}_L \left[\frac{(z-r)^{B-k+l+2} - (z+r)^{B-k+l+2}}{r} \right], \tag{4.18a}$$

where $K(B)$ is given by

$$K(B) = 2^{B-k+3}(B-k+2)(B-k+1)\cdots(B-k-l+2) . \tag{4.18b}$$

Actually, Eq. (4.18a) has been proven in paper I for a past-zero $H(u)$, but it is easily checked to hold also for a past-stationary $H(u)$, which is the case of interest here. We consider separately the three cases of interest to us [items (1), (2), and (3) above].

When $k=2$ the latter expression becomes

$$\Box_R^{-1}[\hat{n}_L r^{B-2} H(t-r)] = \frac{1}{2^{B+1} B(B-1)\cdots(B-l)} \int_r^\infty dz\, H(t-z) \hat{\partial}_L \left[\frac{(z-r)^{B+l} - (z+r)^{B+l}}{r} \right] . \tag{4.19}$$

We know, from considering the retarded integral in its usual triple-integral form, that $\Box_R^{-1}[\hat{n}_L r^{B-2} H(t-r)]$ is convergent when $B=0$. We then would like to simply set $B=0$ on the RHS of (4.19). However, this cannot be done directly because there is an apparent pole at $B=0$ coming from the factor in front of the integral. That this pole is only apparent, in accordance with the fact that the retarded integral is convergent at $B=0$, follows from the identity

$$\hat{\partial}_L \left[\frac{(z-r)^l - (z+r)^l}{r} \right] \equiv 0 , \tag{4.20a}$$

which is a consequence of the identity

$$\hat{\partial}_L r^{2j} \equiv 0 \quad \text{if } j=0,1,\ldots,l-1 . \tag{4.20b}$$

[See Eqs. (A33) and (A36) of paper I.] The identity (4.20a) shows that the integrand on the RHS of (4.19) vanishes when $B=0$, thereby killing the pole in the coefficient. Expanding near $B=0$ the RHS of (4.19) (using $x^B = 1 + B \ln x + \cdots$) we then get

$$\Box_R^{-1}[\hat{n}_L r^{-2} H(t-r)] = \frac{(-)^l}{2(l!)} \int_r^\infty dz\, H(t-z) \hat{\partial}_L \left[\frac{(z-r)^l \ln(z-r) - (z+r)^l \ln(z+r)}{r} \right] . \tag{4.21}$$

Note, remembering the instantaneous character of H, the obvious *hereditary* character of this expression. [Note also that, thanks to the identity (4.20), the RHS of (4.21) is independent of any scale, such as r_1, that we could introduce in the logarithms.]

Let us now consider the case $3 \leq k \leq l+2$ [item (2) above]. In this case it is convenient to integrate by parts the integral on the RHS of Eq. (4.18a). Using the properties of the analytic continuation we find that the upper limit $z=\infty$ gives no contribution, while the lower limit $z=r$ gives rise to some integrated terms which can be computed by means of the formula

$$\left[\hat{\partial}_L \left[\frac{(z+r)^A}{r} \right] \right] \Bigg|_{z=r} = 2^{A-l}(A-l-1)(A-l-2)\cdots(A-2l) \hat{n}_L r^{A-l-1} \tag{4.22}$$

(where we set $z=r$ in the LHS after application of the operator $\hat{\partial}_L$). After $k-2$ integrations by parts, we get

$$\Box_R^{-1}[\hat{n}_L r^{B-k} H(t-r)] = \sum_{i=0}^{k-3} \alpha_i(B) \hat{n}_L r^{B-k+i+2} H^{(i)}(t-r) + \lambda(B) \int_r^\infty dz\, {}^{(k-2)}H(t-z) \hat{\partial}_L \left[\frac{(z-r)^{B+l} - (z+r)^{B+l}}{r} \right] , \tag{4.23a}$$

where the coefficients are given by

$$\alpha_i(B)=\frac{2^i(B-k+2+i)\cdots(B-k+3)}{(B-k+2-l+i)\cdots(B-k+2-l)(B-k+3+l+i)\cdots(B-k+3+l)}\ , \tag{4.23b}$$

$$\lambda(B)=\frac{1}{2^{B-k+3}(B-k+2)\cdots(B-k-l+2)(B+l)\cdots(B-k+l+3)}\ . \tag{4.23c}$$

(The dots indicate a product of factors decreasing by steps of one unit from left to right.) Now, when $3\leq k\leq l+2$, one sees that none of the coefficients has a pole at $B=0$. Therefore, we find that in this case the retarded integral is analytic (no pole) at $B=0$. Furthermore, using the identity (4.20a), we find that the last term in (4.23a), which includes all hereditary contributions to the retarded integral, in fact *vanishes* at $B=0$. So we arrive, rather remarkably, at the following "instantaneous" expression (for $3\leq k\leq l+2$):

$$\Box_R^{-1}(\hat{n}_L r^{B-k}H(t-r))\,|_{B=0}=-\frac{2^{k-3}(k-3)!(l+2-k)!}{(l+k-2)!}\hat{n}_L\sum_{j=0}^{k-3}\frac{(l+j)!}{2^j j!(l-j)!}\frac{{}^{(k-3-j)}H(t-r)}{r^{1+j}}\ . \tag{4.24}$$

Finally we treat the last case [item (3) above] as follows. We need only to consider $k\geq l+3$, since we have just seen that the retarded integrals are analytic when $2\leq k\leq l+2$. Now one checks that the possible poles on the RHS of (4.18a) will come only from the lower integration limit $z=r$ and from the first part of the integral, corresponding to $(z-r)^{B-k+l+2}$. We then take the derivative operator $\hat{\partial}_L$ outside this first part of the integral [we know from paper I, Eq. (6.8), that this is permissible] and operate by parts $k-l-2$ times similarly to Eqs. (4.23). By analytic continuation all integrated terms vanish at $z=r$ and we get

$$\int_r dz\,H(t-z)\hat{\partial}_L\left[\frac{(z-r)^{B-k+l+2}}{r}\right]$$
$$=\text{terms analytic at } B=0+\frac{1}{B(B-1)\cdots(B-k+l+3)}\hat{\partial}_L\left[\frac{1}{r}\int_r dz^{(k-l-2)}H(t-z)(z-r)^B\right]\ , \tag{4.25}$$

from which we deduce, taking into account the coefficient $1/K(B)$, the pole part at $B=0$ of the retarded integral (for $k\geq l+3$):

$$\text{pole part}\,|_{B=0}\Box_R^{-1}[\hat{n}_L r^{B-k}H(t-r)]=\frac{1}{B}\frac{(-)^k 2^{k-3}(k-3)!}{(k+l-2)!(k-l-3)!}\hat{\partial}_L\left[\frac{{}^{(k-l-3)}H(t-r)}{r}\right]\ . \tag{4.26}$$

(As we already knew from paper I, only simple poles arise.) The expression (4.26) is again an "instantaneous" expression.

With this knowledge of the retarded integrals, we can now come back to our problem of determining the structure of p_2, namely, the finite part at $B=0$ of $\Box_R^{-1}(r^B N_2)$ which has been put under the form (4.17). We now understand why it is so advantageous to use the form (4.17). Indeed, by Eqs. (4.24) and (4.26), we see that both the second and the third terms on the RHS of (4.17) are finite at $B=0$ and, most importantly, are (at $B=0$) instantaneous functionals of $\mathcal{M}$. On the contrary, the first term $\Box_R^{-1}(r^{B-2}Q_2)$ is still finite but yields at $B=0$ a sum of integrals such as the RHS of (4.21), which are of the hereditary type. We can thus write for p_2 an expression of the type

$$p_2=\Box_R^{-1}[r^{-2}Q_2(t-r,\mathbf{n})]+\sum_{j\geq 1}\hat{n}_L r^{-j}F_L(t-r)\ , \tag{4.27}$$

where the functions $F_L(u)$ are instantaneous functionals of the algorithmic multipole moments [of the type (4.3)]. Thereby we have completely delimited the hereditary components of p_2: they come from the retarded integration of $r^{-2}Q_2$. We recall that $r^{-2}Q_2$ is the r^{-2} piece (dominant at infinity) of the "effective source" N_2 evaluated in harmonic coordinates and at the second post-Minkowskian approximation. We shall explicitly compute Q_2 in Sec. V, but now we must turn to the second (and last) contribution to the quadratic external metric, q_2 [see Eq. (4.7)].

This q_2 is given by Eqs. (3.6c)–(3.6e) where the STF tensors A_L, B_L, C_L, and D_L are uniquely defined by the multipole expansion of the divergence $\partial_\beta p_2^{\alpha\beta}$ of $p_2^{\alpha\beta}$ [Eqs. (3.9)]. Since we have seen that the pole part of the retarded integrals is instantaneous [Eq. (4.26)] and since $\partial_\beta p_2^{\alpha\beta}$ is precisely a sum of such pole parts [see Eq. (4.10) of paper I] we deduce that the tensors $A_L, B_L, \ldots$ are in fact instantaneous functionals of $\mathcal{M}$ ($A_L=F'_L$, $B_L=F''_L, \ldots$). Thus all terms in q_2 which involve derivatives of these tensors are instantaneous, while the ones which involve antiderivatives [e.g., the term $-cr^{-1(-1)}A(u)$ in q_2^{00}] are what we have called *semihereditary*. (The latter terms have "multipolarity" $l=0$ or $l=1$.) Hence we can write

$$q_2=\sum_{l=0,1}\partial_L[r^{-1}T_L(t-r)]+\sum\partial_L[r^{-1}F_L(t-r)]\ , \tag{4.28}$$

for some semihereditary functionals $T_L(u)$ [of the type (4.6)] and some instantaneous functionals $F_L(u)$.

From Eqs. (4.27) and (4.28) we finally get the structure of h_2:

$$h_2 = \sum_{j\ge 1} \hat{n}_L r^{-j} F_L(t-r) + \sum_{l=0,1} \partial_L[r^{-1}T_L(t-r)] + \Box_R^{-1}[r^{-2}Q_2(t-r,\mathbf{n})] \,, \tag{4.29}$$

where the first term is instantaneous, the second term is semihereditary, and the third term, which is the retarded integral of the r^{-2} piece in the source N_2, is of the hereditary type

$$\Box_R^{-1}[r^{-2}Q_2(t-r,\mathbf{n})] = \sum \int_r^{+\infty} dz\, H_L(t-z)\hat{\partial}_L \left[\frac{(z-r)^l \ln(z-r) - (z+r)^l \ln(z+r)}{r} \right] . \tag{4.30}$$

Let us now give some insight on the higher nonlinear $(h_3, h_4, \dots)$ contributions to the external metric. First, let us call the noninstantaneous component of h_2, namely, the sum of the last two terms in (4.29), the *tail* t_2 of h_2, so that

$$h_2 = \sum_{j\ge 1} \hat{n}_L r^{-j} F_L(t-r) + t_2 \,, \tag{4.31a}$$

where

$$t_2 = \sum_{l=0,1} \partial_L[r^{-1}T_L(t-r)] + \Box_R^{-1}[r^{-2}Q_2(t-r,\mathbf{n})] \,. \tag{4.31b}$$

Following the external algorithm, we consider the cubic source N_3 computed with h_1 and h_2. Using the structures (4.1) and (4.31) of h_1 and h_2, and the general structure of N_3 [see Eq. (4.4) of paper I], we find that N_3 has the form

$$N_3 = \sum_{k\ge 2} \hat{n}_L r^{-k} H_L(t-r) + \sum \partial\partial\{[\hat{n}_L r^{-j} F_L(t-r)] t_2\} \,, \tag{4.32}$$

where the H_L's are instantaneous and where the second term denotes a sum of products of some $\hat{n}_L r^{-j} F_L$ with t_2, with two partial derivatives, $\partial\partial$, being distributed among $\hat{n}_L r^{-j} F_L$ and t_2. We must take the finite part of the retarded integral $\mathrm{FP}\Box_R^{-1}$ of (4.32). The retarded integration of the first, instantaneous, terms in (4.32) is treated using the formula (4.23) above. Note, however, that since $k \ge 2$ is not *a priori* bounded by $l+2$, the coefficients $\alpha_i(B)$ and $\lambda(B)$ in this formula can now have simple poles at $B=0$. The poles in the α_i's will generate logarithms in the finite part (indeed $r^B/B = 1/B + \ln r + \cdots$) and the poles in the λ's will generate some hereditary integrals of the type

$$\Box_R^{-1}[\hat{n}_L r^{-2(k-2)} H(t-r)] \,.$$

Next, the application of the operator $\mathrm{FP}\Box_R^{-1}$ to the second, noninstantaneous, term in (4.32) generates hereditary terms whose structure is more complex. We shall not try here to fully explicate this structure, although we could do it, in principle, using the explicit expression (6.4) of paper I of the retarded integral of an extended multipolar source. Finally, we must include the contribution of q_3 which gives, besides some instantaneous terms, some hereditary or semihereditary terms of the type $\partial_L[r^{-1}X_L(t-r)]$ that we also do not attempt to control explicitly. Therefore, the structure of h_3 is

$$h_3 = \sum_{\substack{j\ge 1\\ p=0,1}} \hat{n}_L r^{-j} (\ln r)^p F_L(t-r) + t_3 \,, \tag{4.33}$$

where the cubic "tail" t_3 has now the structure

$$t_3 = \Box_R^{-1}[r^{-2}Q_3(t-r,\mathbf{n})] + \sum \mathrm{FP}\Box_R^{-1}\{\partial\partial[(\hat{n}_L r^{-j} F_L) t_2]\} + \sum \partial_L[r^{-1}X_L(t-r)] \,. \tag{4.34}$$

(Beware that here Q_3 is *not* the r^{-2} piece in the instantaneous part of the source N_3.)

The same reasoning applies as well to arbitrarily high approximations [except that in order to handle the logarithms we must use not only Eq. (4.23) but the formulas obtained by repeatedly differentiating (4.23) with respect to B]. We then get for h_n an expression of the type

$$h_n = \sum_{\substack{j\ge 1\\ p\le n-2}} \hat{n}_L r^{-j} (\ln r)^p F_L(t-r) + t_n \,, \tag{4.35}$$

where the nth order post-Minkowskian tail t_n is given by an hereditary expression similar to (4.34) but more complex: the first term becomes a sum of terms of the type $\Box_R^{-1}[r^{-2}(\ln r)^p Q_n]$, while the second term now involves contributions coming from the "interaction" of p (with $p \ge 1$) tails $t_{k_1}, t_{k_2}, \dots, t_{k_p}$ (with $k_1 + k_2 + \cdots + k_p \le n-1$) between themselves, and with instantaneous terms of the type $\hat{n}_L r^{-j} (\ln r)^p F_L$. Note that, making use of Eq. (6.4) of paper I, we could, in principle, reduce t_n to a sum of integrals of the type

$$t_n = \sum \hat{n}_L \int_{-\infty}^{t-r} du_1 \cdots \int_{-\infty}^{t-r} du_n \mathcal{K}_{LL_1\cdots L_n}(t,r,u_1,\dots,u_n) \overset{(a_1)}{M}_{L_1}(u_1) \cdots \overset{(a_n)}{S}_{L_n}(u_n) \,, \tag{4.36}$$

where $\mathcal{K}_{LL_1\cdots L_n}(t,r,u_1,\dots,u_n)$ is a quite intricate kernel.

This completes our discussion of the functional dependence of the external metric.

V. FUNCTIONAL DEPENDENCE AS SEEN IN THE NEAR ZONE

We follow the next step of our method (see Sec. II) and reexpand the external metric in the exterior near zone, i.e., in the overlap region $D_i \cap D_e$ ($r_0 < r < \kappa r_0$). This means that we consider the post-Newtonian expansion, or c^{-1} expansion, of the (post-Minkowskian) external metric. This step will allow us to perform the matching to the source (Sec. VI).

Let us restore in the external metric all explicit powers of c^{-1}. At this point, it is convenient to choose the normalizing constant r_1 in (3.6b) to be $r_1 = cP$, where P is some characteristic time scale of the internal motion of the source. Then, following paper I [see also Eq. (4.36)], we notice that h_n can be decomposed in a sum of independent pieces each of which is built from an element, say, E_n, of the nth tensorial power of the set $\mathcal{M}$ of algorithmic multipole moments [Eq. (3.1)], namely, a tensorial product of n algorithmic multipole moments chosen among the M_L's and S_L's:

$$E_n = M_{L_1} M_{L_2} \cdots S_{L_n} . \tag{5.1}$$

For each moment M_L or S_L we define $\underline{l}$ to be the number of indices on M_L or $\varepsilon_{ijk} S_{kL-1}$ (endowing each S_L with its natural ε), i.e., $\underline{l} = l$ for "mass moments" M_L and $\underline{l} = l+1$ for "current moments" S_L. Then for each E_n, $\sum_{i=1}^{n} l_i$ is the total number of indices on the M_L's and S_L's composing E_n, and $\sum_{i=1}^{n} \underline{l}_i$ is $\sum_{i=1}^{n} l_i$ plus the number of current moments S_L in E_n. [In the notation of paper I, Eq. (5.2b), we have $\sum_{i=1}^{n} \underline{l}_i = b(E_n)$.] With this notation we can rewrite Eq. (4.35) (which gave h_n as a function of space and time) in a form which makes explicit the dependence of h_n on c:

$$h_n(\mathbf{x},t,c) = \sum_{E_n} \frac{1}{c^{3n+\sum_{i=1}^{n}\underline{l}_i}} \sum_{\substack{j \geq 1 \\ p \leq n-2}} \hat{n}_L \left[\frac{r}{c} \right]^{-j} [\ln(r/cP)]^p F_L(t-r/c) + t_n(\mathbf{x},t,c) . \tag{5.2}$$

Our task is now to control the dependence of the tail term t_n in Eq. (5.2) on c, when $c \to \infty$. We have seen in Sec. IV that t_n becomes very intricate when n gets large. Fortunately, the following theorem shows that t_n quickly becomes negligible (in the near zone) when n increases.

Theorem: When $c \to \infty$, the tail $t_n(c)$ ($n \geq 2$) is of order[55]

$$t_n(c) = O\left[\frac{(\ln c)^{n-1}}{c^{2n+4}} \right] . \tag{5.3}$$

The proof of this theorem is done by induction on n. Let us first consider the case $n=2$. In this case we have seen [Eq. (4.31b)] that t_2 is made of two contributions:

$$t_2 = s_2 + u_2 . \tag{5.4}$$

The first one is semihereditary,

$$s_2 = \sum_{E_2} \frac{1}{c^{6+\underline{l}_1+\underline{l}_2}} \sum_{l=0,1} c^{l+1} \partial_L [r^{-1} T_L(t-r/c)] , \tag{5.5}$$

while the other one is hereditary,

$$u_2 = \Box_R^{-1} [r^{-2} Q_2(t-r/c,\mathbf{n},c)] , \tag{5.6a}$$

where $Q_2(u,\mathbf{n},c)$, the r^{-2} piece in the source N_2, has the structure

$$Q_2(u,\mathbf{n},c) = \sum_{E_n} \frac{1}{c^{6+\underline{l}_1+\underline{l}_2}} \sum_{l} \hat{n}_L H_L(u) . \tag{5.6b}$$

We deal first with the semihereditary contribution s_2. From Eq. (5.5), taking into account the fact that $l=0$ or $l=1$, we find that s_2 is $\sum_{E_2} O(1/c^{4+\underline{l}_1+\underline{l}_2})$. Now, T_2 must, by definition, consist of (first- or second-order) antiderivatives of products of derivatives of two multipole moments (e.g., $\overset{(a_1)}{M}_{L_1} \overset{(a_2)}{M}_{L_2}$). Dimensional analysis then shows that this is possible only if both derivative orders a_1 and a_2 are strictly positive. Therefore, both moments must be nonstationary: hence $\underline{l}_1 \geq 2$ and $\underline{l}_2 \geq 2$. Thus s_2 is (in the near zone) of order

$$s_2 = O(1/c^8) . \tag{5.7}$$

Let us now consider the hereditary contribution u_2. From Eq. (5.6b) we find that Q_2 is $\sum_{E_2} O(1/c^{6+\underline{l}_1+\underline{l}_2})$. Now dimensional analysis shows that at least one of the two derivative orders must be strictly positive; therefore, at least one of the two interacting moments must be nonstationary: hence $\underline{l}_1 \geq 2$ or $\underline{l}_2 \geq 2$. Thus Q_2 is of order $O(1/c^8)$. In other words, this means that u_2 has the structure

$$u_2 = \sum_{p \geq 8} \frac{1}{c^p} \Box_R^{-1} \left[\frac{\hat{n}_L}{r^2} H(t-r/c) \right] . \tag{5.8}$$

The explicit integration formula (5.18) below [evaluated at $B=0$, and with $H(u)$ zero in the past] enables us to compute in a straightforward way the post-Newtonian expansion of each element of the RHS of Eq. (5.8). Now, when $c^{-1} \to 0$ the first term on the RHS of Eq. (5.18), which seems to be $O(c^{+(l+1)})$, is however only of order $O(c^{-l})$ because of the identity (4.20b) [see Eq. (5.22)]. As for the second term on the RHS of (5.18) [which seems to be $O(c^0)$] it is easily found to be $O(\ln c)$ [because of the pole coming from $K(B)$].

Introducing this information into Eq. (5.8), we get

$$u_2 = O\left[\frac{\ln c}{c^8} \right] . \tag{5.9}$$

Equations (5.7) and (5.9) imply that the theorem is true for $n=2$. Let us now show on the next iteration step, $n=3$, how the induction proceeds. From Eq. (4.34) we see that the cubically nonlinear tail t_3 is now made of three contributions: a hereditary contribution of the same type as u_2, say u_3 [first term on the RHS of (4.34)], some hereditary or semihereditary terms of the type $\partial_L[r^{-1}X(t-r/c)]$, say s_3 [last term of (4.34)], and a more intricate tail-generated hereditary piece, say τ_3 [second term on the RHS of (4.34)]. The contribution u_3 can be handled by the same method as used for u_2. This leads to $u_3=O(\ln c/c^{11})$. The contribution s_3 is made of two kinds of terms. Some are semihereditary and are treated by the same method as used for s_2: they are $O(c^{-11})$. The other ones are purely hereditary and come from hereditary terms in the effective source N_3: these are found to be $O(c^{-10})$ using the defining formulas (3.6c)–(3.6e) for $q^{\alpha\beta}$, and the fact that, for a term of "multipolarity" l, composed with multipoles l_i and with S free spatial indices (S being the number of spatial indices among α and β) we always have $\sum l_i - l + S = 2k$, where $k \in \mathbb{N}$. Hence we find $s_3=O(c^{-10})$. The more complex tail-generated tail τ_3 requires some tools which have been developed in paper I. Indeed, we have proven there (in Secs. V, IV, and III, respectively) the three following results.

(i) The nth-order post-Minkowskian metric h_n can be written as a sum of contributions having the form $c^{-k}f_k(\rho,t,\mathbf{n})$, where $k=3n+\sum l_i$ and ρ denotes the dimensionless ratio r/cP.

(ii) The functions $f_k(\rho)$ belong to a special class of functions (called the L^{n-1} class in paper I) which admit, when $\rho\to 0$, an asymptotic expansion along the scale functions $\rho^a(\ln\rho)^p$, where a is a (relative) integer and p a positive integer bounded upwards by $n-1$.

(iii) One can deduce the asymptotic expansion of $f_k(\rho)$ from the corresponding one of its "source" in N_n by means of Eqs. (3.23) and (3.5) of paper I.

Combining these three results with the fact that

$$\mathrm{FP}\Box_R^{-1}[(cP)^{-2}n_k(\rho,t,\mathbf{n})]=f_k(\rho,t,\mathbf{n}) ,$$

we find that if a "source" contribution, $c^{-k-2}n_k(\rho)$ (belonging to the class L^{n-2}) is $O((\ln c)^{n-2}/c^s)$ when $c\to\infty$ (and $\rho\to 0$), then its corresponding "solution," $c^{-k}f_k(\rho)$, is

$$O((\ln c)^{n-1}/c^s)+O(1/c^k)$$

(where the first term comes from the Δ^{-1} terms in Eq. (3.23) of paper I, while the second one comes from the $\Box_R^{-1}[O^N(r^N)]$ one). For instance, from Eqs. (5.7)–(5.9) one finds that the "source" of τ_3 is $O(\ln c/c^{10})$ (and belongs to L^1); therefore, τ_3 itself is $O((\ln c)^2/c^{10}) + O(1/c^{9+\sum l_i})$, which is in fact $O((\ln c)^2/c^{10})$ as $\sum l_i \geq 2$ in a tail term. This proves our theorem for $n=3$. Now, noticing that each new nonlinear order brings in a new factor G/c^2, and increases the maximal power of the logarithms of ρ, it is easy to see, reasoning by induction on n, and using the just proven general result on the propagation of post-Newtonian order from "source to field," that the theorem (5.3) holds true for any n.

This theorem will be essential for knowing how many nonlinear iterations must be considered to get the dominant (i.e., lowest order in $1/c$) hereditary piece in the near-zone expanded external metric. Let us first consider in detail the hereditary structure of the second iteration, $t_2=s_2+u_2$, as seen in the (exterior) near zone. First, it is easy to show that, although $s_2^{\alpha\beta}$ [Eq. (5.5)] is on the whole $O(c^{-8})$ according to Eq. (5.7), in fact s_2^{00} is $O(c^{-9})$, s_2^{0i} is $O(c^{-8})$ and $s_2^{ij}\equiv 0$ [as follows from Eq. (3.6e)]. Therefore, we see that $s_2^{\alpha\beta}$ contributes to the external metric $g_{\alpha\beta}$ (coming back to the usual covariant metric) semihereditary terms which are c^{-9} in g_{00}, c^{-8} in g_{0i}, and c^{-9} in g_{ij}. This means that semihereditary terms in the external metric arise (dominantly) at the $3\frac{1}{2}$ *post-Newtonian* level.[56] Let us denote by ${}_9g_{00}$, ${}_8g_{0i}$, and ${}_7g_{ij}$ the $3\frac{1}{2}$PN approximation, namely, the coefficients of c^{-9}, c^{-8}, and c^{-7} in g_{00}, g_{0i}, and g_{ij}. (For simplicity's sake, we are keeping the powers of $\ln c$ in each ${}_pg_{\alpha\beta}$; see Ref. 56 and Sec. VI below.) Then we find (adding a superscript "ext" to remember that we are considering the external metric)

$${}_9g_{00}^{\mathrm{ext}}=2r^{-1}m(t)-2(\partial_i r^{-1})m_i(t)+{}_9\varphi_{00} , \tag{5.10a}$$

$${}_8g_{0i}^{\mathrm{ext}}=-2\varepsilon_{iab}(\partial_a r^{-1})s_b(t)+{}_8\varphi_{0i} , \tag{5.10b}$$

$${}_7g_{ij}^{\mathrm{ext}}={}_7\varphi_{ij} , \tag{5.10c}$$

where the functions $m(t)$, $m_i(t)$, and $s_i(t)$ are some semihereditary functions [of the type (4.6a) or (4.6b)] and the φ's have the following "instantaneous" structure:

$$\varphi=\sum \hat{n}_L r^k \left[\ln\left[\frac{r}{cP}\right]\right]^p F_L(t) , \tag{5.11}$$

where we recall that $F_L(t)$ denotes an instantaneous functional of the algorithmic multipole moments [cf. Eq. (4.3)]. The coefficients in (5.10) have been chosen so that the functions $m(t)$, $m_i(t)$, and $s_i(t)$ appear as quadratic "corrections" to the algorithmic moments M, M_i, and S_i. For instance a straightforward application of the algorithm (3.6)–(3.9) shows that the function $m(t)$ is given by

$$m(t)=-\tfrac{1}{5}\int_{-\infty}^{t} dv\,{}^{(3)}M_{ij}(v)\,{}^{(3)}M_{ij}(v) , \tag{5.12}$$

a result first obtained by Bonnor.[11] Note that on the LHS of Eqs. (5.10) appears the $3\frac{1}{2}$PN term of the *full* external metric because, as the theorem above shows [Eq. (5.3)], the cubic and higher nonlinear tails t_n ($n\geq 3$) do not contribute to the $3\frac{1}{2}$PN approximation (they are at least $c^{-2\times 3-4}=c^{-10}$), while the hereditary tail u_2 turns out to contribute at the 4PN level only, as we now show.

To compute u_2 we first need to get Q_2, the r^{-2} part in the second-order post-Minkowskian source N_2. This Q_2 can be computed straightforwardly by inserting the r^{-1} part of our linearized metric h_1 [Eqs. (3.3)] into the expression of N_2 [given by Eq. (3.5)]. The result is

$$Q_2^{\alpha\beta}(t-r/c,\mathbf{n})=\frac{k^\alpha k^\beta}{c^2}\left(\tfrac{1}{2}{}^{(1)}z^{\mu\nu}\,{}^{(1)}z_{\mu\nu}-\tfrac{1}{4}{}^{(1)}z^{\mu}_{\mu}\,{}^{(1)}z^{\nu}_{\nu}\right) + \frac{4M}{c^4}\,{}^{(2)}z^{\alpha\beta} . \tag{5.13}$$

(See also Ref. 43.) In Eq. (5.13) we have denoted by $k^{\alpha} \equiv f^{\alpha\beta} k_{\beta} \equiv (+1, n^i)$ the radially outgoing Minkowskian null direction, while $z^{\alpha\beta}(t - r/c, \mathbf{n})$ is given explicitly by

$$z^{00}(u,\mathbf{n}) = -4 \sum_{l \geq 2} \frac{\hat{n}_L}{l! c^{l+2}} {}^{(l)}M_L(u) , \tag{5.14a}$$

$$z^{0i}(u,\mathbf{n}) = -4 \sum_{l \geq 2} \frac{\hat{n}_{L-1}}{l! c^{l+2}} {}^{(l)}M_{iL-1}(u) + 4 \sum_{l \geq 2} \frac{l}{(l+1)! c^{l+3}} \varepsilon_{iab} \hat{n}_{aL-1} {}^{(l)}S_{bL-1}(u) , \tag{5.14b}$$

$$z^{ij}(u,\mathbf{n}) = -4 \sum_{l \geq 2} \frac{\hat{n}_{L-2}}{l! c^{l+2}} {}^{(l)}M_{ijL-2}(u) + 8 \sum_{l \geq 2} \frac{l}{(l+1)! c^{l+3}} \hat{n}_{aL-2} \varepsilon_{ab(i} {}^{(l)}S_{j)bL-2}(u) . \tag{5.14c}$$

Let us now evaluate the retarded integral of $r^{-2}Q_2$ by using the formula (4.19), namely (with $r_1 = cP$),

$$\Box_R^{-1} \left[\left[\frac{r}{cP} \right]^B \frac{\hat{n}_L}{r^2} H(t - r/c) \right] = \frac{1}{K(B)} \int_r^{+\infty} dz\, H(t - z/c) \hat{\partial}_L \left[\frac{(z-r)^{B+l} - (z+r)^{B+l}}{r} \right] , \tag{5.15a}$$

where

$$K(B) = 2(2cP)^B B(B-1) \cdots (B-l) . \tag{5.15b}$$

The formula (5.15a) is not very convenient to get the asymptotic expansion of u_2 when $c \to \infty$, so let us first transform the RHS of (5.15a). We can first split the RHS of (5.15a) into two integrals from r to $+\infty$ corresponding to the two terms $(z \pm r)^{B+l}$. [This is possible because in the case considered here $H(u)$ vanishes for $u \leq -T$.] We then take the derivative operator $\hat{\partial}_L$ outside the first integral [corresponding to $(z-r)^{B+l}$] [this is allowed, as proven in paper I, Eq. (6.8)]. Then we rewrite the second integral [corresponding to $(z+r)^{B+l}$] as the sum of an integral from r to $-r$ plus an integral from $-r$ to $+\infty$. We again take the derivative $\hat{\partial}_L$ outside the latter integral. (This is allowed for the same reasons as before.) As a result we have

$$\Box_R^{-1} \left[\left[\frac{r}{cP} \right]^B \frac{\hat{n}_L}{r^2} H(t - r/c) \right] = \frac{1}{K(B)} \int_{-r}^{r} dz\, H(t - z/c) \hat{\partial}_L \left[\frac{(z+r)^{B+l}}{r} \right] + \frac{1}{K(B)} \hat{\partial}_L \left[\frac{1}{r} \int_r^{+\infty} dz\, H(t - z/c)(z-r)^{B+l} - \frac{1}{r} \int_{-r}^{+\infty} dz\, H(t - z/c)(z+r)^{B+l} \right] . \tag{5.16}$$

Next we introduce the function

$$X_B(u) = \frac{c^B}{K(B)} \int_0^{+\infty} dv\, v^{B+l} H(u - v) . \tag{5.17}$$

Our final result is then

$$\Box_R^{-1} \left[\left[\frac{r}{cP} \right]^B \frac{\hat{n}_L}{r^2} H(t - r/c) \right] = c^{l+1} \hat{\partial}_L \left[\frac{X_B(t - r/c) - X_B(t + r/c)}{r} \right] + \frac{1}{K(B)} \int_{-r}^{r} dz\, H(t - z/c) \hat{\partial}_L \left[\frac{(z+r)^{B+l}}{r} \right] . \tag{5.18}$$

This form is very convenient for our purpose because, when performing the near-zone expansion, among the two terms on the RHS of (5.18), only the first one yields a *hereditary functional* of the algorithmic multipole moments. Indeed, by expanding in the second term, $H(t - z/c)$ in powers of z/c we easily find that this second term admits, when $c \to \infty$, an instantaneous expansion of the type

$$\sum \hat{n}_L r^{B+k(i)} H(t)/c^{B+i}$$

(see Ref. 57). Note that, because of the pole at $B=0$ brought about by $1/K(B)$, the finite part of the second term admits a post-Newtonian expansion along the $(\ln c)^1/c^n$'s. Taking the finite part at $B=0$ of $X_B(u)$ [the pole parts of the first and second terms on the RHS of (5.18) cancel each other], we get

$$X(u)=\frac{(-)^l}{2(l!)}\int_0^\infty dv\, v^l\left[\ln\left[\frac{v}{2P}\right]+\sum_{i=1}^{l}\frac{1}{i}\right]H(u-v) \tag{5.19}$$

or, after l integrations by parts,

$$X(u)=\frac{(-)^l}{2}\int_0^\infty dv\left[\ln\left[\frac{v}{2P}\right]+2\sum_{i=1}^{l}\frac{1}{i}\right]{}^{(-l)}H(u-v)\,, \tag{5.20}$$

[where ${}^{(-l)}H(u-v)$ is the lth antiderivative of H which is zero when $u\le -T$]. Hence we conclude that the part of the retarded integral of $\hat{n}_L r^{-2}H(t-r/c)$, which, when viewed in the near zone, explicitly depends on all the past history of $H(u)$, is

$$\Box_R^{-1}\left[\frac{\hat{n}_L}{r^2}H(t-r/c)\right]\Bigg|_{\text{hereditary}}=\frac{(-)^l c^{l+1}}{2}\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]\hat{\partial}_L\left[\frac{{}^{(-l)}H(t-v-r/c)-{}^{(-l)}H(t-v+r/c)}{r}\right] \tag{5.21}$$

(which is a solution of the homogeneous wave equation). Expanding the RHS of (5.21) in powers of r/c [and using Eq. (A33) of paper I], we obtain the following near-zone *dominant* hereditary contribution:

$$\Box_R^{-1}\left[\frac{\hat{n}_L}{r^2}H(t-r/c)\right]\Bigg|_{\text{hereditary}}=\frac{(-)^{l+1}}{(2l+1)!!}\frac{\hat{x}_L}{c^l}\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(l+1)}H(t-v)+O(1/c^{l+2}) \tag{5.22}$$

[with $\hat{x}_L=\hat{n}_L r^l$ and $(2l+1)!!=(2l+1)(2l-1)\cdots 1$]. Note that this dominant hereditary contribution is, in order of magnitude, smaller by a factor c^{-l} than the "source" $\hat{n}_L r^{-2}H(t-r/c)$.

We can now find the lowest-order terms in the near-zone external metric, which are hereditary functionals of the algorithmic multipole moments. We have already determined, Eq. (5.10), the lowest-order semihereditary terms. However, we shall see below (Sec. VI) that, after having expressed the *algorithmic* multipole moments as functionals of some (instantaneous) *source* moments, the latter semihereditary terms will combine with other terms to form instantaneous functionals of the source. We now look for the lowest-order fully hereditary terms (they will turn out to outweigh the next-order semihereditary terms). Using Eqs. (5.3), (5.6a), (5.13), (5.14), and (5.22) we find that fully hereditary terms first arise at order c^{-10} in g_{00}, c^{-9} in g_{0i}, and c^{-8} in g_{ij} or, using the post-Newtonian language, that they arise at the 4PN *approximation* level. Furthermore, ${}_{10}g_{00}^{\text{hereditary}}$ comes both from the second and third iterations ($n=2$ and 3), while ${}_9g_{0i}^{\text{hereditary}}$ and ${}_8g_{ij}^{\text{hereditary}}$ come only from the second iteration:

$$g_{00}^{\text{ext}}\,|^{\text{fully hereditary}}=c^{-10}({}_{10}g_{2\,00}^{\text{hereditary}}+{}_{10}g_{3\,00}^{\text{hereditary}})+O\left[\frac{\ln c}{c^{11}}\right], \tag{5.23a}$$

$$g_{0i}^{\text{ext}}\,|^{\text{fully hereditary}}=c^{-9}{}_9g_{2\,0i}^{\text{hereditary}}+O\left[\frac{\ln c}{c^{10}}\right], \tag{5.23b}$$

$$g_{ij}^{\text{ext}}\,|^{\text{fully hereditary}}=c^{-8}{}_8g_{2\,ij}^{\text{hereditary}}+O\left[\frac{\ln c}{c^{9}}\right]. \tag{5.23c}$$

Now, using Eqs. (5.13), (5.14), and (5.22), we can compute the 4PN-hereditary terms in the *quadratic* external metric $g_{2\alpha\beta}^{\text{ext}}$. We can express the result as

$${}_{10}g_{2\,00}^{\text{ext}}=-\frac{8M}{5}x_{ab}\int_0^\infty dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}M_{ab}(t-v)$$

$$+2\partial_t({}_9\eta_0)+{}_{10}\varphi_{2\,00}\,, \tag{5.24a}$$

$${}_9g_{2\,0i}^{\text{ext}}=\partial_t({}_8\eta_i)+\partial_i({}_9\eta_0)+{}_9\varphi_{2\,0i}\,, \tag{5.24b}$$

$${}_8g_{2\,ij}^{\text{ext}}=\partial_i({}_8\eta_j)+\partial_j({}_8\eta_i)+{}_8\varphi_{2\,ij}\,, \tag{5.24c}$$

where $x_{ab}=x^{ab}=x_a x_b$ and where we have set

$${}_9\eta_0=\frac{2M}{3}x_{ab}\int_0^\infty dv\,\ln\left[\frac{v}{2P}\right]{}^{(6)}M_{ab}(t-v)$$

$$+\tfrac{2}{5}\int_0^\infty dv\,\ln\left[\frac{v}{2P}\right]{}^{(3)}M_{ab}(t-v){}^{(3)}M_{ab}(t-v)\,, \tag{5.24d}$$

$${}_8\eta_i=-4Mx_a\int_0^\infty dv\,\ln\left[\frac{v}{2P}\right]{}^{(5)}M_{ai}(t-v)\,, \tag{5.24e}$$

and where the φ's are some *instantaneous* expressions of the type of Eq. (5.11). It remains to include the hereditary contribution to the 4PN level coming from the *cubic* external metric, i.e., ${}_{10}g_{3\,00}^{\text{hereditary}}$ in Eq. (5.23a). It is easily checked that this contribution can only come from the coupling between the hereditary c^{-8} term in g_{2ij}^{ext} [Eq. (5.23c)] and the (instantaneous) c^{-2} term in $g_{1\,00}^{\text{ext}}$, namely, ${}_2g_{1\,00}^{\text{ext}}={}_2g_{00}^{\text{ext}}$, which is given by

$$ {}_{2}g^{\text{ext}}_{00}=2\sum_{l\geq 0}\frac{(-)^l}{l!}(\partial_L r^{-1})M_L(t) \ . \tag{5.25}$$

Using Eq. (5.24c) and the expression (3.5) of the source N_2, we find that the equation satisfied by ${}_{10}g^{\text{ext}}_{300}$ can be written as

$$\Delta({}_{10}g^{\text{ext}}_{300})=\Delta({}_{8}\eta_i\partial_i({}_{2}g^{\text{ext}}_{00}))+{}_{10}\phi_{300} \ , \tag{5.26}$$

where ${}_{10}\phi_{300}$ is of the instantaneous type (5.11). From Eq. (5.26) we deduce that ${}_{10}g^{\text{ext}}_{300}$ will be the sum of ${}_{8}\eta_i\partial_{i2}g^{\text{ext}}_{00}$, plus some instantaneous terms, plus some hereditary solution of the homogeneous Laplace equation. Using dimensional analysis, we find that this latter solution of the Laplace equation must be of the type $\partial_L r^{-1}$ (and not of the type $\hat{x}_L$) with $l\geq 1$. Hence we can write

$$\begin{aligned}{}_{10}g^{\text{ext}}_{300}&={}_{8}\eta_i\partial_i({}_{2}g^{\text{ext}}_{00})\\&\quad+2\sum_{l\geq 1}\frac{(-)^l}{l!}(\partial_L r^{-1})Y_L(t)+{}_{10}\varphi_{300} \ ,\end{aligned} \tag{5.27}$$

for some hereditary tensorial functions $Y_L(t)$ and some instantaneous ${}_{10}\varphi_{300}$ of the type of Eq. (5.11). As expressed by Eqs. (5.23) we must add up Eqs. (5.24) and (5.27) to get the complete 4PN hereditary component in the near-zone-expanded external metric. In other words, using the notation (5.11) we can write

$$\begin{aligned}{}_{10}g^{\text{ext}}_{00}&=-\frac{8M}{5}x_{ab}\int_0^\infty dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}M_{ab}(t-v)\\&\quad+\sum_{l\geq 1}\frac{2(-)^l}{l!}(\partial_L r^{-1})Y_L(t)+2\partial_t({}_{9}\eta_0)\\&\quad+{}_{8}\eta_i\partial_i({}_{2}g^{\text{ext}}_{00})+{}_{10}\varphi_{00} \ ,\end{aligned} \tag{5.28a}$$

$${}_{9}g^{\text{ext}}_{0i}=\partial_t({}_{8}\eta_i)+\partial_i({}_{9}\eta_0)+{}_{9}\varphi_{0i} \ , \tag{5.28b}$$

$${}_{8}g^{\text{ext}}_{ij}=\partial_i({}_{8}\eta_j)+\partial_j({}_{8}\eta_i)+{}_{8}\varphi_{ij} \ . \tag{5.28c}$$

Let us recall that up to now the concepts of "simultaneous" versus "hereditary" were characterizing a functional dependence on the *algorithmic* multipole moments $\mathcal{M}$, Eq. (3.1). We need now to find the functional relationship between $\mathcal{M}$ and the structure and evolution of the source. This will allow us to extract some physical significance which is, for the moment, only implicitly contained in the main results of this section Eqs. (5.10) and (5.28). We turn to this task in the next section.

VI. MATCHING TO THE SOURCE

Let us consider a material system located within the region $r<r_0$. Following Sec. II, we assume that this system is *weakly self-gravitating* and *slowly moving*. Thus there exists an overlap region between the "exterior" domain $D_e=\{(\mathbf{x},t)\mid r>r_0\}$ around the system, and an "inner" domain $D_i=\{(\mathbf{x},t)\mid r<\kappa r_0\}$ with $\kappa>1$. The overlap region $D_i\cap D_e$ will be sometimes referred to as the matching region.

Let us further assume, for the sake of definiteness, that the system is made of some *isentropic perfect fluid* described by the stress-energy tensor

$$T^{\mu\nu}=[\rho c^2+\rho\Pi(\rho)+p(\rho)]u^\mu u^\nu+p(\rho)g^{\mu\nu} \ . \tag{6.1}$$

We use the following notation. u^μ denotes the four-velocity of the fluid ($g_{\mu\nu}u^\mu u^\nu=-1$), while ρ, Π, and p denote, respectively, the proper rest-mass density, proper specific internal energy density and proper pressure [$\epsilon=\rho c^2(1+\Pi/c^2)$ would then be the proper energy density]. The previous variables, $u^\mu(x')$,$\rho(x')$, . . . , are all expressed in some "inner" coordinate system x'^μ which is *a priori* different from the exterior coordinate system x^μ. These variables are linked by the thermodynamic relation $p=\rho^2 d\Pi/d\rho$ and they must satisfy both the local energy-momentum conservation law

$$T^{\mu\nu}{}_{;\nu}=0 \tag{6.2}$$

and the continuity equation

$$(\rho u^\mu)_{;\mu}=0 \ . \tag{6.3}$$

We shall use as minimal set of physical variables describing our system ("source variables") the proper rest-mass density ρ, and the coordinate three-velocity

$$v^i=c\frac{u^i}{u^0}=\frac{dx'^i}{dt'} \ , \tag{6.4}$$

associated with the coordinate time $t'\equiv x'^0/c$. In the following we shall indifferently write the "Cartesian" three-index of v^i up or down. The "inner" metric generated by the system in D_i is a functional of the source variables ρ and v_i. It is also a function of the inner coordinates x'^μ and a function of c (and G). We assume (see Sec. II) that, by means of some post-Newtonian iteration scheme to be discussed later, the inner metric can be written as the formal expansion

$$g^{\text{in}}_{\mu\nu}(x',c)=f_{\mu\nu}+\sum_{p,q}\frac{(\ln c)^q}{c^p}\,{}_{p,q}g^{\text{in}}_{\mu\nu}(x') \tag{6.5}$$

(for $p,q\in\mathbb{N}$ with $p\geq 2$), where each ${}_{p,q}g^{\text{in}}_{\mu\nu}(x')$ is a functional of the source variables ρ, v_i, and where, in turn, ρ and v_i satisfy some corresponding post-Newtonian equations of evolution obtained by inserting (6.5) into (6.2) and (6.3). We know already, from paper I, that the external metric also admits an expansion of the type (6.5) [see, e.g., Eq. (5.2) above and the proof of Eq. (5.3)]. However, the coefficients of the latter expansion are not given as functionals of ρ and v_i but as functionals of the algorithmic multipole moments. It is then natural to assume that the algorithmic moments admit expansions of the type

$$M_L(t)=\sum_{p,q}\frac{(\ln c)^q}{c^p}\,{}_{p,q}I_L(t) \ , \tag{6.6a}$$

$$S_L(t)=\sum_{p,q}\frac{(\ln c)^q}{c^p}\,{}_{p,q}J_L(t) \ , \tag{6.6b}$$

for $p,q\in\mathbb{N}$, where the coefficients ${}_{p,q}I_L$ and ${}_{p,q}J_L$ are functionals of ρ and v^i, the functions $y^\mu\to\rho(y^\mu),v^i(y^\mu)$

being the ones corresponding to the inner coordinate system x'^{μ}. Inserting the expansions (6.6) into the expansion (5.2) of the external metric [with the previously investigated tails, see the proof of Eq. (5.3)], and performing the near-zone reexpansion, we get, in $D_i \cap D_e$, an expansion of the type

$$\hat{g}^{\text{ext}}_{\mu\nu}(x,c)=f_{\mu\nu}+\sum_{p,q}\frac{(\ln c)^q}{c^p}\,{}_{p,q}\hat{g}^{\text{ext}}_{\mu\nu}(x)\ , \tag{6.7}$$

where now each ${}_{p,q}\hat{g}^{\text{ext}}_{\mu\nu}(x)$ is some functional of ρ and v_i (to remind us of this new functional dependence we have added a caret on g^{ext}). By our assumption, both the inner metric and the external metric are valid in the overlap region $D_i \cap D_e$; therefore, these metrics must be *isometric* in this region. Hence there must exist a coordinate transformation, going from the external harmonic coordinates x^{α} which have been used so far (in Secs. III–V) to the inner coordinates (x'^{α}), such that the coordinate transform of the asymptotic reexpansion (6.7) of the external metric coincides with the asymptotic expansion (6.5) of the inner metric. (Note that the coordinates x'^{α} need not be well behaved outside D_i.) This requirement is our variant of the "matching" of the initial asymptotic expansions (2.3) and (2.5). We assume for consistency that the coordinates x'^{α} are related (in $D_i \cap D_e$) to the coordinates x^{α} by some expansions along the same $(\ln c)^q/c^p$, namely,

$$x'^{\alpha}(x,c)=x^{\alpha}+\sum_{p,q}\frac{(\ln c)^q}{c^p}\,{}_{p,q}x'^{\alpha}(x)\ , \tag{6.8}$$

where each ${}_{p,q}x'^{\alpha}(x)$ is a functional of ρ and v^i. From Eq. (6.8) we get for the external metric (6.7) in the *new coordinates* (6.8) again an expression of the type

$$\hat{g}'^{\text{ext}}_{\mu\nu}(x',c)=f_{\mu\nu}+\sum_{p,q}\frac{(\ln c)^q}{c^p}\,{}_{p,q}\hat{g}'^{\text{ext}}_{\mu\nu}(x')\ , \tag{6.9}$$

where each ${}_{p,q}\hat{g}'^{\text{ext}}_{\mu\nu}(x')$ is a functional of the source variables. Now the "matching" between the "outer" and "inner" metrics (6.9) and (6.5) gives us an infinite set of equations:

$$\forall(p,q):\ \ {}_{p,q}\hat{g}'^{\text{ext}}_{\mu\nu}(x')={}_{p,q}g^{\text{in}}_{\mu\nu}(x')\ . \tag{6.10}$$

As we shall see in the following, the "matching conditions" (6.10) are doubly useful. On the one hand, they will allow us to define and implement an "inner" post-Newtonian scheme yielding an expansion of the form (6.5) where all the coefficients are functionals of the source variables. On the other hand, they will also determine the explicit functional relationship between the algorithmic moments and the source variables. Taken together, these two results mean that we have a complete solution of our problem: we end up knowing the metric everywhere, inside and outside the source, as a functional of the source variables. However, a word of caution is necessary. We shall not prove here the consistency of our approach to all orders in $c^{-p}(\ln c)^q$, but rather assume it. Indeed, we shall use only the minimal number of "matching conditions," allowing us to answer the question addressed in this paper. We leave to future work a more complete study of the consistency of the form of matching used here.

Let us now investigate what is the lowest-order piece in the "inner" metric, which is a *hereditary* functional of the source variables. We need first to transform the external metric (as determined in Sec. V) into coordinates x'^{α} which permit a direct matching with an "inner'" metric. Let us assume beforehand that this coordinate transformation [given in general form by (6.8)] is of the following type:

$$x'^0=x^0+\frac{1}{c^3}{}_3\varphi^0+\frac{1}{c^5}{}_5\varphi^0+\frac{1}{c^6}{}_6\varphi^0+\frac{1}{c^7}{}_7\varphi^0+\frac{1}{c^8}{}_8\varphi^0+\frac{1}{c^9}({}_9\eta^0+{}_9\varphi^0)+\cdots\ , \tag{6.11a}$$

$$x'^i=x^i+\frac{1}{c^2}{}_2\varphi^i+\frac{1}{c^4}{}_4\varphi^i+\frac{1}{c^5}{}_5\varphi^i+\frac{1}{c^6}{}_6\varphi^i+\frac{1}{c^7}{}_7\varphi^i+\frac{1}{c^8}({}_8\eta^i+{}_8\varphi^i)+\cdots\ , \tag{6.11b}$$

where, following the notation of Sec. V [Eq. (5.11)], the φ's denote *instantaneous* functionals of the algorithmic moments, and where ${}_9\eta^0$ and ${}_8\eta^i$ are given by the explicit hereditary expressions (5.24d) and (5.24e). Our assumption (6.11) will be (partially) justified later when we show that it leads to a consistent matching. Furthermore, let us choose,[58] in Eqs. (6.11),

$${}_6\varphi^0=-\tfrac{1}{6}x_{ab}\,{}^{(4)}M_{ab}(t)\ , \tag{6.12a}$$

$${}_5\varphi^i=-x_a\,{}^{(3)}M_{ai}(t)\ . \tag{6.12b}$$

The latter choice is made here for convenience to transform the external (harmonic) coordinates into Burke-type[47] inner coordinates [see, e.g., Eqs. (9) of Ref. 58]. The coordinate transformation (6.11) has the effect of canceling all terms involving the η's in the 4PN-hereditary components of the external metric [Eqs. (5.28)]. Then all 4PN-hereditary terms in the transformed external metric $g'^{\text{ext}}_{\mu\nu}$ are now contained in the 00 component of the metric g'^{ext}_{00}. Bringing together Eqs. (5.10), (5.25), and (5.28) we find the following expression for g'^{ext}_{00} (which is not yet $\hat{g}'^{\text{ext}}_{00}$ because it is still expressed in terms of the algorithmic moments):

$$\begin{aligned}g'^{\text{ext}}_{00}=&-1+\frac{1}{c^2}\left[\sum_{l\geq 0}\frac{2(-)^l}{l!}(\partial_L r^{-1})M_L(t)\right]+\frac{1}{c^4}{}_4\varphi_{00}+\frac{1}{c^6}{}_6\varphi_{00}+\frac{1}{c^7}{}_7\varphi_{00}\\&+\frac{1}{c^8}{}_8\varphi_{00}+\frac{1}{c^9}[2r^{-1}m(t)-2(\partial_i r^{-1})m_i(t)+{}_9\varphi_{00}]\\&+\frac{1}{c^{10}}\left[-\frac{8M}{5}x_{ab}\int_0^{\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}M_{ab}(t-v)+\sum_{l\geq 1}\frac{2(-)^l}{l!}(\partial_L r^{-1})Y_L(t)+{}_{10}\varphi_{00}\right]+\cdots\ ,\end{aligned} \tag{6.13}$$

where the φ's have the instantaneous structure (5.11). Note that for simplicity's sake, we henceforth suppress all "primes" on the new coordinate system x'^{α}, thinking of x^i, t as dummy variables. With our choice (6.12), ${}_7\varphi_{00}$, namely, the first "odd" term in the external metric, has (as shown in Ref. 58) the expression

$$
{}_7\varphi_{00}=-\tfrac{2}{5}x_{ab}{}^{(5)}M_{ab}(t)+\sum_{l\geq 1}\frac{2(-)^l}{l!}(\partial_L r^{-1})H_L(t)\ , \tag{6.14}
$$

where the functions $H_L(t)$ are quadratically nonlinear instantaneous functionals of the algorithmic moments.

We now consider the inner metric $g^{\rm in}_{\mu\nu}$, which must satisfy the inhomogeneous Einstein equations with $T_{\mu\nu}$, Eq. (6.1), as source. Let us first see in detail how the use of the matching equations (6.10) works at the lowest post-Newtonian order (i.e., $p=2$ and $q=0$; of course the ln$c$'s will turn out to be absent at the c^{-2} level). The coefficient of $(\ln c)^0 c^{-2}$ in $g^{\rm in}_{00}$, namely, ${}_{2,0}g^{\rm in}_{00}$, satisfies, in D_i, the Poisson equation

$$
\Delta({}_{2,0}g^{\rm in}_{00})=-8\pi\rho\ . \tag{6.15}
$$

(Henceforth we shall often use units such that $G=1$.) The most general solution of Eq. (6.15) in D_i is equal to the Poisson integral of the "source" $-8\pi\rho$ plus some *regular* solution of the Laplace equation. This means that there exists a set of STF tensors ${}_{2,0}A_L(t)$ such that, in D_i,

$$
{}_{2,0}g^{\rm in}_{00}(\mathbf{x},t)=2\int\frac{d^3\mathbf{y}\,\rho(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}+\sum_{l\geq 0}\hat{x}_L[{}_{2,0}A_L(t)]\ . \tag{6.16}
$$

In particular, the latter expression is valid in the matching region $D_i\cap D_e$ outside the source where we can use the expansion ($|\mathbf{x}|>|\mathbf{y}|$)

$$
\frac{1}{|\mathbf{x}-\mathbf{y}|}=\sum_{l\geq 0}\frac{(-)^l}{l!}(\partial_L r^{-1})\hat{y}_L\ . \tag{6.17}
$$

Hence we get, in $D_i\cap D_e$,

$$
{}_{2,0}g^{\rm in}_{00}=\sum_{l\geq 0}\left[\frac{2(-)^l}{l!}(\partial_L r^{-1})I_L(t)+\hat{x}_L[{}_{2,0}A_L(t)]\right]\ , \tag{6.18}
$$

where we denote by $I_L(t)$ the usual "mass" multipole moment of the distribution of the density ρ. The "mass" and "current" multipole moments of ρ are, respectively, given by

$$
I_L(t)=\int d^3\mathbf{y}\,\hat{y}_L\rho(\mathbf{y},t)\ , \tag{6.19a}
$$

$$
J_L(t)=\int d^3\mathbf{y}\,\varepsilon_{ab\langle i_l}y_{L-1\rangle}y_a v_b\rho(\mathbf{y},t) \tag{6.19b}
$$

[where we recall that the function $(\mathbf{y},t)\to\rho(\mathbf{y},t)$ is the one corresponding to the inner coordinate system x'^{μ}]. Now the coefficient of $(\ln c)^0 c^{-2}$ in the near-zone-expanded external metric (6.13), after replacement of the algorithmic moments by their source expansions (6.6), reads

$$
{}_{2,0}\hat{g}'^{\rm ext}_{00}=\sum_{l\geq 0}\frac{2(-)^l}{l!}(\partial_L r^{-1})_{0,0}I_L(t)\ . \tag{6.20}
$$

Then the matching equation

$$
{}_{2,0}\hat{g}'^{\rm ext}_{00}={}_{2,0}g^{\rm in}_{00} \tag{6.21}
$$

gives us two pieces of information. First, that each function ${}_{2,0}A_L(t)$ must be zero, so that we now know that, all over D_i,

$$
{}_{2,0}g^{\rm in}_{00}(\mathbf{x},t)=2U(\mathbf{x},t)\ , \tag{6.22a}
$$

where

$$
U[\rho](\mathbf{x},t)=\int\frac{d^3\mathbf{y}\,\rho(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|} \tag{6.22b}
$$

denotes the Newtonian potential of ρ. Second, that the lowest-order expansion coefficient of M_L in Eq. (6.6a) must be equal to the corresponding usual Newtonian mass multipole moment of ρ, (6.19a), namely,

$$
{}_{0,0}I_L(t)=I_L(t)\ . \tag{6.23a}
$$

Similarly the matching of ${}_{3,0}g_{0i}$ shows that the leading term in the expansion of the current algorithmic moment S_L [Eq. (6.6b)] is the usual current multipole moment of the density ρ, namely,

$$
{}_{0,0}J_L(t)=J_L(t)\ . \tag{6.23b}
$$

The matching also shows that the logarithms are absent at the c^{-2} level so that ${}_{2,q}g^{\rm in}_{00}=0$ and ${}_{0,q}I_L(t)={}_{0,q}J_L(t)=0$ when $q\geq 1$. In the same way, in the case $p=3$ and q arbitrary, we get also ${}_{3,q}g^{\rm in}_{00}=0$ and ${}_{1,q}I_L(t)={}_{1,q}J_L(t)=0$.

Let us now consider the coefficient of $(\ln c)^0 c^{-4}$ in the inner metric $g^{\rm in}_{00}$, namely, ${}_{4,0}g^{\rm in}_{00}$. This ${}_{4,0}g^{\rm in}_{00}$ must satisfy in D_i (using harmonic inner coordinates) the Poisson equation

$$
\Delta({}_{4,0}g^{\rm in}_{00}+2U^2)=2\partial_t^2U[\rho]-16\pi\rho\left[\mathbf{v}^2+U[\rho]+\frac{\Pi[\rho]}{2}+\frac{3p[\rho]}{2\rho}\right] \tag{6.24}
$$

($\mathbf{v}^2$ denoting $\delta_{ij}v_iv_j$). The general solution of this equation in D_i is given by a particular one plus a general *regular* harmonic function. Hence

$$
{}_{4,0}g^{\rm in}_{00}=\partial_t^2X-2U^2+4\int\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\left[\rho\left[\mathbf{v}^2+U+\frac{\Pi}{2}+\frac{3p}{2\rho}\right]\right](\mathbf{y},t)+\sum_{l\geq 0}\hat{x}_L\,{}_{4,0}A_L(t)\ , \tag{6.25}
$$

where we have set

$$X(\mathbf{x},t)=\int d^3\mathbf{y}\,\rho(\mathbf{y},t)\,|\mathbf{x}-\mathbf{y}| \ . \tag{6.26}$$

Outside the source (in the matching region $D_i \cap D_e$), ${}_{4,0}g^{\rm in}_{00}$ can be expanded as

$$_{4,0}g^{\rm in}_{00}=\sum_{l\geq 0}\left[\sum_k \hat{n}_L r^k{}_{4,0}\hat{F}_L(t)+\hat{x}_L\,{}_{4,0}A_L(t)\right] , \tag{6.27}$$

where the functions ${}_{4,0}\hat{F}_L(t)$ are some *instantaneous* functionals of ρ and v_i [hence the caret to distinguish from Eq. (5.11) where the instantaneous functional dependence referred to the algorithmic moments]. On the other hand, the coefficient of $(\ln c)^0 c^{-4}$ in the external metric, namely, ${}_{4,0}\hat{g}'^{\rm ext}_{00}$ (expanded in the same region $D_i\cap D_e$) can be found by inserting into (6.13) the expansions (6.6) of the M_L's and S_L's (with the now known values of ${}_{0,q}I_L$, ${}_{0,q}J_L$, and ${}_{1,q}I_L$, ${}_{1,q}J_L$). The result is ${}_{4,0}\hat{\varphi}_{00}$ [by which we mean the ${}_{4,0}\varphi_{00}$ of Eq. (6.13) evaluated with I_L, J_L instead of M_L, S_L, so that ${}_{4,0}\hat{\varphi}_{00}$ is instantaneous with respect to the source variables] plus a contribution coming from the $(\ln c)^0c^{-2}$ correction ${}_{2,0}I_L$ in the algorithmic moment M_L:

$$_{4,0}\hat{g}'^{\rm ext}_{00}={}_{4,0}\hat{\varphi}_{00}+\sum_{l\geq 0}\frac{2(-)^l}{l!}(\partial_L r^{-1})_{2,0}I_L(t) \ . \tag{6.28}$$

We must then equate [by the matching conditions (6.10)] the RHS of (6.28) with the RHS of (6.27). Note that the only terms in (6.27) and (6.28) which may have, *a priori*, a hereditary character are the two homogeneous Laplace solutions (of the type $\hat{x}_L$ in the inner metric and $\partial_L r^{-1}$ in the external metric) involving the unknown quantities ${}_{4,0}A_L(t)$ and ${}_{2,0}I_L(t)$. However, the following happens: these two types of Laplace solution do not "match" together. Therefore, each of them must necessarily match some other terms, namely $\hat{x}_L\,{}_{4,0}A_L(t)$ must match some term in ${}_{4,0}\hat{\varphi}_{00}$, while $(\partial_L r^{-1})_{2,0}I_L(t)$ must match some $\hat{n}_L r^{-l-1}{}_{4,0}\hat{F}_L(t)$. This proves, therefore, that the unknown quantities ${}_{4,0}A_L(t)$ and ${}_{2,0}I_L(t)$ are in fact instantaneous functionals of ρ and v_i, and thus that the internal metric coefficient ${}_{4,0}\hat{g}'^{\rm in}_{00}$ is itself an *instantaneous* functional of ρ and v_i. A closer study shows that each ${}_{4,0}A_L$ is zero; hence we recover, in D_i, the known result

$$_{4,0}g^{\rm in}_{00}=\partial_t^2 X-2U^2+4\int\frac{d^3\mathbf{y}}{|\mathbf{x}-\mathbf{y}|}\left[\rho\left[\mathbf{v}^2+U+\frac{\Pi}{2}+\frac{3p}{2\rho}\right]\right](\mathbf{y},t) \tag{6.29}$$

(and ${}_{4,q}g^{\rm in}_{00}=0$ for $q\geq 1$). Similarly one finds that the other 1PN metric coefficients (i.e., ${}_{3,q}g^{\rm in}_{0i}$ and ${}_{2,q}g^{\rm in}_{ij}$) are instantaneous.

Now the 0PN and 1PN instantaneous inner metric coefficients generate some instantaneous source for the next 2PN approximation (${}_6g^{\rm in}_{00}$, ${}_5g^{\rm in}_{0i}$, ${}_4g^{\rm in}_{ij}$) and since the external metric (6.13) is, at the 2PN level, an instantaneous functional of the M_L's and S_L's, we find, repeating exactly the same reasoning, that the 2PN approximation is still an instantaneous functional of ρ and v_i. [Note that the reasoning assumes that one can construct some particular instantaneous solution, similar to (6.29), of the Poisson equation with an instantaneous source, see Refs. 29–38, and that we can expand this solution outside the source and consistently match it to $\hat{g}'^{\rm ext}$.] Similarly, by the same reasoning, one finds that the $2\frac{1}{2}$PN, 3PN, and $3\frac{1}{2}$PN approximations (see below for the latter approximation) are again given by some instantaneous functionals of the source variables. For instance, at the $2\frac{1}{2}$PN approximation, with the help of Eq. (6.14) we find that ${}_{7,0}g^{\rm in}_{00}$ is given by the usual Burke-Thorne expression

$$_{7,0}g^{\rm in}_{00}(\mathbf{x},t)=-\tfrac{2}{5}x_{ab}{}^{(5)}I_{ab}(t) \tag{6.30}$$

(and ${}_{7,q}g^{\rm in}_{00}=0$ if $q\geq 1$), while the quantity H_L appearing in Eq. (6.14) yields ${}_{5,0}I_L(t)=-\hat{H}_L(t)$, where the caret over H_L means that we have replaced in the expression of H_L the algorithmic moments M_L, S_L by the Newtonian source moments I_L, J_L. In fact, in the case of the $3\frac{1}{2}$PN approximation, the previous reasoning is slightly more complicated due to the presence in the external metric of the semihereditary terms involving the functions $m(t)$ and $m_i(t)$ [see Eq. (6.13)]. Indeed, we must notice that, since these terms have a form $\partial_L r^{-1}$ which cannot match terms of the type $\hat{x}_L$ in the inner metric (which are the only terms that could be semihereditary with respect to the source variables), they must be counterbalanced by contributions at the level c^{-7} in the expansions (6.6a) of the algorithmic mass M and mass dipole M_i as functionals of the source variables. Namely, we must have

$$_{7,0}I(t)=-\hat{m}(t)+{}_7\hat{F}(t) \ , \tag{6.31a}$$

$$_{7,0}I_i(t)=-\hat{m}_i(t)+{}_7\hat{F}_i(t) \ , \tag{6.31b}$$

where ${}_7\hat{F}(t)$ and ${}_7\hat{F}_i(t)$ are some source-instantaneous functionals and the caret over m and m_i means that they must be computed in terms of I_L, J_L, instead of M_L, S_L. The same happens for the hereditary functions $Y_L(t)$ appearing in the c^{-10} coefficient of the external metric [4PN level, see Eq. (6.13)]: these functions must be counterbalanced by c^{-8} contributions in the functional relationships $M_L=M_L[\text{source}]$, Eq. (6.6a), namely,

$$\sum_q(\ln c)^q{}_{8,q}I_L(t)=-\hat{Y}_L(t)+{}_8\hat{F}_L(t,\ln c) \ , \tag{6.32}$$

for some source-instantaneous functional ${}_8\hat{F}_L(t,\ln c)$. (This quantity could depend on $\ln c$, see below and the Appendix.)

However the situation changes at the 4PN level because of the algorithmic-hereditary term of the type $\hat{x}_{ab}$ in the c^{-10} coefficient of the external metric [see Eq. (6.13)]. Indeed this term cannot be absorbed in a c^{-8} correction in M_L[source], but must match some $\hat{x}_{ab\,10,0}A_{ab}(t)$ in the inner metric. Therefore, we see that, at the 4PN approximation level, the inner metric gets a *source-hereditary* contribution. Gathering all the information gained, we end up with the central result of this paper, namely, the following metric (valid all over D_i), with the dominant source-hereditary contribution ("tail") explicitly appearing in the coefficient of c^{-10}:

$$g^{\rm in}_{00}(\mathbf{x},t)=-1+\frac{1}{c^2}\left[2\int \frac{d^3y\,\rho(\mathbf{y},t)}{|\mathbf{x}-\mathbf{y}|}\right]+\frac{1}{c^4}\left[\partial_t^2X-2U^2+4\int\frac{d^3y}{|\mathbf{x}-\mathbf{y}|}\rho\left[\mathbf{v}^2+U+\frac{\Pi}{2}+\frac{3p}{2\rho}\right]\right]$$
$$+\frac{1}{c^6}{}_6\hat{\Phi}_{00}+\frac{1}{c^7}[-\tfrac{2}{5}x_{ab}{}^{(5)}I_{ab}(t)]+\frac{1}{c^8}{}_8\hat{\Phi}_{00}+\frac{1}{c^9}{}_9\hat{\Phi}_{00}$$
$$+\frac{1}{c^{10}}\left[-\tfrac{8}{5}x_{ab}I(t)\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}I_{ab}(t-v)+{}_{10}\hat{\Phi}_{00}\right]+\cdots\,. \qquad (6.33)$$

In Eq. (6.33) the ${}_n\hat{\Phi}_{00}$'s denote some instantaneous functionals of ρ and v_i similar to, albeit more complex than, the coefficients of c^{-2}, c^{-4}, or c^{-7}.

Besides the inner metric (6.33), we also obtain the following relations between the mass algorithmic moments and the source variables. For $l=0$ and $l=1$ we have [see Eqs. (6.31)]

$$M=I(t)+\frac{1}{c^2}{}_2\hat{F}(t)+\frac{1}{c^4}{}_4\hat{F}(t)+\frac{1}{c^6}{}_6\hat{F}(t,\ln c)+\frac{1}{c^7}[-\hat{m}(t)+{}_7\hat{F}(t)]+\cdots\,, \qquad (6.34a)$$

$$M_i=I_i(t)+\frac{1}{c^2}{}_2\hat{F}_i(t)+\frac{1}{c^4}{}_4\hat{F}_i(t)+\frac{1}{c^5}{}_5\hat{F}_i(t)+\frac{1}{c^6}{}_6\hat{F}_i(t,\ln c)+\frac{1}{c^7}[-\hat{m}_i(t)+{}_7\hat{F}_i(t)]+\cdots\,, \qquad (6.34b)$$

and, for $l\geq 2$ [see Eq. (6.32)],

$$M_L=I_L(t)+\frac{1}{c^2}{}_2\hat{F}_L(t)+\frac{1}{c^4}{}_4\hat{F}_L(t)+\frac{1}{c^5}{}_5\hat{F}_L(t)+\frac{1}{c^6}{}_6\hat{F}_L(t,\ln c)+\frac{1}{c^7}{}_7\hat{F}_L(t)+\frac{1}{c^8}[-\hat{Y}_L(t)+{}_8\hat{F}_L(t,\ln c)]+\cdots\,, \qquad (6.35)$$

where all $\hat{F}$'s are source instantaneous (as well as, of course, I_L). (There are similar relations for S_L whose lowest-order term is J_L.) From these equations we see that the dominant noninstantaneous contribution to the algorithmic moments (when expressed in terms of the source variables) arises at the level c^{-7} in M and M_i and at the level c^{-8} in M_L $(l\geq 2)$. Notice that we may interpret Eq. (6.34a) as the decomposition of the (constant by definition) initial mass (which is also the total mass seen at spatial infinity or ADM mass) into the sum of a time-dependent "source mass"

$$\hat{M}(t)=I(t)+\frac{1}{c^2}{}_2\hat{F}(t)+\frac{1}{c^4}{}_4\hat{F}(t)+\frac{1}{c^6}{}_6\hat{F}(t,\ln c)+\frac{1}{c^7}{}_7\hat{F}(t) \qquad (6.36)$$

(which is linked to the instantaneous state of the system) plus a (positive) "radiation mass" which has been "carried off" by the gravitational radiation field at all past epochs. Using Eq. (5.12) we find that the rate of decrease of $\hat{M}(t)$ is given by

$$\frac{d\hat{M}(t)}{dt}=-\frac{1}{5c^7}{}^{(3)}I_{ab}(t){}^{(3)}I_{ab}(t)+O(c^{-8})\,. \qquad (6.37)$$

Equation (6.37) constitutes a well-defined "quadrupole equation of the second kind" in the terminology of Ref. 17.

The appearance in the inner metric, Eq. (6.33), of the hereditary (or "tail") term

$$(\delta g^{\rm in}_{00})^{\rm hereditary}=-\frac{8}{5c^{10}}x^ax^bI(t)\times\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}I_{ab}(t-v) \qquad (6.38)$$

signals a breakdown of the post-Newtonian tenets at the 4PN level. This can also be viewed as a breakdown of the concept of near zone (see the discussion in Sec. I).

Associated with the hereditary term (6.38), there must exist an instantaneous term in ${}_{10}\hat{\Phi}_{00}$ involving the logarithm of cP (this is found necessary when considering that an instantaneous term must be constructed as some spatial integral, and requiring that the sum $\delta g^{\rm hereditary}+\hat{\Phi}$ does not depend on the arbitrary time scale P). Hence we find that the lowest-order term, in the inner metric, violating the standard post-Newtonian power-series assumption is

$$(\delta g^{\rm in}_{00})^{\rm lnc}=-\frac{8}{5}\frac{\ln c}{c^{10}}x^a x^b I(t)^{(6)}I_{ab}(t)\ . \tag{6.39}$$

This result confirms the corresponding result of Anderson *et al.*[42] [their Eq. (5.16)]. These authors also found some ln$c$, at the 3PN level, but in the external metric. We recover this result in the Appendix and show that these logarithms appear [at order $(\ln c)c^{-6}$] in the relationships (6.34)–(6.35) between the algorithmic moments and the source variables [as already indicated in Eqs. (6.34)–(6.35)].

We discuss in the next section the physical consequences of the appearance of the hereditary, or tail, term (6.38) in the inner metric.

VII. INFLUENCE OF THE HEREDITARY TERM ON THE INTERNAL DYNAMICS

Let us recall that, in a suitable coordinate system [such as the one of Eq. (6.33)] the lowest-order term in the inner metric which violates time-reversal invariance is

$$c^{-7}{}_7g^{\rm in}_{00}=-\frac{2}{c^2}V_R[\bar I_{ab}]\ , \tag{7.1}$$

where $V_R[Q_{ab}]$ denotes the usual Burke-Thorne[47,48] "radiation reaction potential,"

$$\{V_R[Q_{ab}]\}(\mathbf{x},t)=\frac{1}{5c^5}x^a x^b\frac{d^5}{dt^5}Q_{ab}(t)\ , \tag{7.2}$$

considered as a functional of some "quadrupole moment" $Q_{ab}(t)$ to be inserted in (7.2). The quadrupole moment, $\bar I_{ab}$, appearing in (7.1) is the "usual" Newtonian quadrupole moment of the "coordinate-rest-mass density," $\bar\rho$:

$$\bar\rho\equiv\rho\sqrt{g}\,u^0\ , \tag{7.3}$$

which is defined so as to have a constant zeroth-order moment,

$$\bar I=\int d^3\mathbf{x}\,\bar\rho=\text{total rest mass}=\text{const}\ . \tag{7.4}$$

Explicitly, the quadrupole moment of $\bar\rho$ reads

$$\bar I_{ab}(t)=\int d^3\mathbf{x}\,\bar\rho(\mathbf{x},t)(x^a x^b-\tfrac{1}{3}\mathbf{x}^2\delta^{ab})\ . \tag{7.5}$$

Note that, up to now, we have used as source multipole moments the moments of the proper rest-mass density ρ, see Eqs. (6.19). However, for the following discussion it will be more convenient to take as "source variables" $\bar\rho$ and v^i instead of ρ and v^i. This modifies, for instance, the explicit expression of the c^{-4} term in Eq. (6.33), but leaves invariant the functional structure of Eq. (6.33).

With this new notation the lowest-order hereditary, or tail, term in the inner metric [see Eq. (6.38)] reads

$$\delta g^{\rm in}_{00}\,|^{\rm hereditary}=-\frac{2}{c^2}V_R[\delta\bar I_{ab}]\ , \tag{7.6}$$

where

$$\delta\bar I_{ab}(t)=\frac{4}{c^3}\bar I\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(2)}\bar I_{ab}(t-v)\ . \tag{7.7}$$

Note that we have obtained (7.7) by integrating five times the last factor on the RHS of Eq. (6.38). Actually the quantity that directly appears in the RHS of Eq. (7.6) is

$$\frac{d^5}{dt^5}\delta\bar I_{ab}(t)=\frac{4}{c^3}\bar I\int_0^{+\infty}dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}\bar I_{ab}(t-v)\ . \tag{7.8}$$

Let us remark that while the term (7.1), involving the fifth time derivative of $\bar I_{ab}(t)$, was "time-odd" (i.e., it changed sign under time reversal), the hereditary ("tail") term (7.6) cannot be considered as having a well-defined parity under time reversal (because the range of integration in Eq. (7.8) changes during the latter operation). However, the latter term is certainly not time even, i.e., it is "time asymmetric."

The equations ruling the local evolution of our isentropic perfect-fluid source, as obtained from Eqs. (6.2), (6.3), and (6.33), can be written as

$$\frac{\partial}{\partial t}\bar\rho+(\bar\rho v^i)_{,i}=0\ , \tag{7.9}$$

$$\bar\rho\left[\frac{\partial v^i}{\partial t}+v^j v^i{}_{,j}\right]=-\bar p_{,i}+\bar\rho\bar U_{,i}+\bar{\mathcal{F}}^{\rm even}_i+\mathcal{R}_i\ , \tag{7.10}$$

where we have distinguished a time-even (or time-symmetric) "force density,"

$$\bar{\mathcal{F}}^{\rm even}_i=\frac{1}{c^2}{}_2\bar{\mathcal{F}}_i+\frac{1}{c^4}{}_4\bar{\mathcal{F}}_i+\frac{1}{c^6}{}_6\bar{\mathcal{F}}_i+\frac{1}{c^8}{}_8\bar{\mathcal{F}}_i\ , \tag{7.11}$$

and a non-time-even (or time-asymmetric) one,

$$\mathcal{R}_i=\mathcal{R}'_i+\mathcal{R}''_i\ , \tag{7.12a}$$

$$\mathcal{R}'_i=-\frac{2}{5c^5}\bar\rho x^j\frac{d^5}{dt^5}(\bar I_{ij}+\delta\bar I_{ij})\ , \tag{7.12b}$$

$$\mathcal{R}''_i=\frac{1}{c^7}{}_7\bar{\mathcal{F}}_i+O(1/c^9)\ . \tag{7.12c}$$

In Eqs. (7.11) and (7.12) the quantities ${}_n\bar{\mathcal{F}}_i$ denote some instantaneous functionals of our new source variables, $\bar\rho$ and v^i. In the terminology of Ref. 17 (Secs. 4.2 and 4.15) Eq. (7.12b) constitutes a well-defined "quadrupole equation of the third kind," or "radiation reaction quadrupole equation" in which the "radiation reaction" quadrupole moment is given by

$$\bar I'_{ab}(t)=\bar I_{ab}(t)+\delta\bar I_{ab}(t)\ , \tag{7.13}$$

where $\delta\bar I_{ab}(t)$ is the hereditary term (7.7). Let us now examine the effects that the hereditary-modified "force" (7.12b) is likely to cause in the actual evolution of the source.

First of all, we shall *admit* that the truncated equations of motion

$$\frac{\partial}{\partial t}\bar\rho+(\bar\rho v^i)_{,i}=0\ , \tag{7.14a}$$

$$\bar\rho\left[\frac{\partial v_i}{\partial t}+v_j v_{i,j}\right]=-\bar p_{,i}+\bar\rho\bar U_{,i}+\bar{\mathcal{F}}^{\rm even}_i\ , \tag{7.14b}$$

which are, at once, source instantaneous and time symmetric, form a fully *conservative* system of equations of motion (we are here following the discussion of "radiation damping" of Refs. 17 and 38). This assumption implies, in particular, that there exists a source-instantaneous first integral of (7.14) of the type

$$\bar{E}[\bar{\rho}(t),v_i(t)]=\int d^3\mathbf{x}\,\bar{\rho}\left[\frac{1}{2}\mathbf{v}^2+\Pi-\frac{\bar{U}}{2}\right]+\frac{1}{c^2}{}_2\bar{E}+\frac{1}{c^4}{}_4\bar{E}+\frac{1}{c^6}{}_6\bar{E}+\frac{1}{c^8}{}_8\bar{E}\ . \tag{7.15}$$

Now, in considering $\mathcal{R}_i$ as a "perturbation" superimposed on the conservative system (7.14), we can neglect its square, which is $O(c^{-10})$, while $\mathcal{R}_i$ is defined only up to $O(c^{-9})$. Therefore, we can add up linearly the effects caused by the various terms making up $\mathcal{R}_i$. We leave a study of the effects associated with $\mathcal{R}_i''$, Eq. (7.12c), to future work. Hence we shall consider only the heredity-modified "quadrupole radiation reaction force" $\mathcal{R}_i'$, Eq. (7.12b).

The method of variation of constants shows that the effect of adding up $\mathcal{R}_i'$ to the time-symmetric evolution (7.14) is to cause a slow change of $\bar{E}(t)$ given, to first order in $\mathcal{R}_i'$, by

$$\frac{d\bar{E}}{dt}=\int d^3\mathbf{x}\,v^i\mathcal{R}_i'\ . \tag{7.16}$$

Hence

$$\frac{d\bar{E}}{dt}=-\frac{2}{5c^5}\int d^3\mathbf{x}\,\bar{\rho}v^ix^j\frac{d^5}{dt^5}(\bar{I}_{ij}+\delta\bar{I}_{ij})$$
$$=-\frac{1}{5c^5}\frac{d}{dt}(\bar{I}_{ij})\frac{d^5}{dt^5}(\bar{I}_{ij}+\delta\bar{I}_{ij})\ . \tag{7.17}$$

Operating twice by parts on the time derivatives we can rewrite (7.17) as

$$\frac{d\bar{E}}{dt}=-\frac{1}{5c^5}\frac{d}{dt}\left[\frac{d}{dt}\bar{I}_{ab}\frac{d^4}{dt^4}\bar{I}_{ab}-\frac{d^2}{dt^2}\bar{I}_{ab}\frac{d^3}{dt^3}\bar{I}_{ab}\right]-\frac{1}{5c^5}\frac{d}{dt}\left[\frac{d}{dt}\bar{I}_{ab}\frac{d^4}{dt^4}\delta\bar{I}_{ab}-\frac{d^2}{dt^2}\bar{I}_{ab}\frac{d^3}{dt^3}\delta\bar{I}_{ab}\right]-\frac{1}{5c^5}\left[\frac{d^3}{dt^3}(\bar{I}_{ab}+\tfrac{1}{2}\delta\bar{I}_{ab})\right]^2\ . \tag{7.18}$$

Let us now set

$$\tilde{E}=\bar{E}+\frac{1}{5c^5}\left[\frac{d}{dt}\bar{I}_{ab}\frac{d^4}{dt^4}\bar{I}_{ab}-\frac{d^2}{dt^2}\bar{I}_{ab}\frac{d^3}{dt^3}\bar{I}_{ab}\right]+\frac{1}{5c^5}\left[\frac{d}{dt}\bar{I}_{ab}\frac{d^4}{dt^4}\delta\bar{I}_{ab}-\frac{d^2}{dt^2}\bar{I}_{ab}\frac{d^3}{dt^3}\delta\bar{I}_{ab}\right]\ . \tag{7.19}$$

Note that $\tilde{E}$ is not an instantaneous functional of the source variables because the last set of parentheses in (7.19) depends, via $\delta\bar{I}$, on all the past evolution of the source. However, we shall see below that, with an accuracy much better than the formal c^{-8} order of magnitude of this term, it depends only on the *recent* past evolution of the source. Furthermore, we can insert in this last term, with the required accuracy, the recent-past evolution of the time-symmetric system (7.14). We shall further admit that we can reexpress, with the required accuracy, the quantity $\tilde{E}$ as a *time-symmetric* functional, $\tilde{E}^{\text{even}}$, of the recent evolution of the source, i.e., as a functional both of the recent-past evolution and recent-future evolution of the source [the time symmetry of the evolution of the system (7.14) makes this probable; without, however, proving it as time-symmetric equations can admit time-asymmetric solutions, see also the discussion in Sec. 4.15 of Ref. 17]. From (7.18), we see that $\tilde{E}^{\text{even}}$ slowly but *monotonically* decreases at the rate

$$\frac{d\tilde{E}^{\text{even}}}{dt}=-\frac{1}{5c^5}\left[\frac{d^3}{dt^3}(\bar{I}_{ab}+\tfrac{1}{2}\delta\bar{I}_{ab})\right]^2\ , \tag{7.20}$$

thereby clearly displaying the *irreversible* nature of the effects caused by $\mathcal{R}_i'$. In the terminology of Ref. 17, Eq. (7.20) constitutes a well-defined "quadrupole equation of the second kind" or "energy-loss quadrupole equation." We can therefore interpret $\mathcal{R}_i'$ [Eq. (7.12b)] as a tail-modified quadrupole *damping* force. Finally, note that the "effective" quadrupole moment appearing in (7.20) is

$$\bar{I}''_{ab}(t)=\bar{I}_{ab}(t)+\tfrac{1}{2}\delta\bar{I}_{ab}(t)\ , \tag{7.21}$$

which differs through the factor $\frac{1}{2}$ in its second term from the quadrupole moment (7.13) appearing in the damping force (7.12b).

Intuitively we may regard the past-dependent (tail) contribution to the radiation damping force as follows. The outgoing waves emitted by the system at all times in the past scatter off the curvature of spacetime generated by the system itself. This produces incoming secondary waves which converge back on the system, and which act on its present dynamics.

Let us now estimate, in a quantitative way, how sensitive is the hereditary term (7.6) to the *remote* past behavior of the system. To do this, let us first split the integral (7.8) into an integral between 0 and P plus an integral between P and $+\infty$, and let us integrate the latter integral five times by parts. The result is

$$\frac{d^5}{dt^5}\delta\bar{I}_{ab}(t)=\frac{4\bar{I}}{c^3}\left[-\ln 2\,{}^{(6)}\bar{I}_{ab}(t-P)+\frac{1}{P}{}^{(5)}\bar{I}_{ab}(t-P)-\frac{1}{P^2}{}^{(4)}\bar{I}_{ab}(t-P)+\frac{2}{P^3}{}^{(3)}\bar{I}_{ab}(t-P)-\frac{6}{P^4}{}^{(2)}\bar{I}_{ab}(t-P)+\int_0^P dv\,\ln\left[\frac{v}{2P}\right]{}^{(7)}\bar{I}_{ab}(t-v)+24\int_P^\infty\frac{dv}{v^5}{}^{(2)}\bar{I}_{ab}(t-v)\right]\ . \tag{7.22a}$$

Note that a simpler, though less explicit, form of Eq. (7.22a) is

$$\frac{d^5}{dt^5}\delta\bar{I}_{ab}(t)=\frac{4\bar{I}}{c^3}\frac{1}{(2P)^5}\left[24\mathrm{FP}_{B=0}\int_0^\infty dv\left[\frac{v}{2P}\right]^{B-5}{}^{(2)}\bar{I}_{ab}(t-v)-\tfrac{25}{12}{}^{(6)}\bar{I}_{ab}(t)\right] . \tag{7.22b}$$

The fact that the very early times enter the formulas (7.22) only through the integral

$$\int_{-\infty} d\tau \frac{{}^{(2)}\bar{I}_{ab}(\tau)}{(t-\tau)^5} , \tag{7.23}$$

indicates that $(d/dt)^5\delta\bar{I}_{ab}$ is probably only weakly sensitive to the remote-past history of the system [a fact which was not apparent in Eq. (7.8) where one had a logarithmically increasing kernel $\ln(t-\tau)$]. To be more precise we must now make some assumption about the behavior of the system in the past. Up to now we assumed that the system had been stationary when $t\leq -T$. This is not a very realistic assumption, so let us first admit that our results still hold in the limit $-T\rightarrow -\infty$, i.e., for a system that has always been nonstationary. We shall now impose some conditions of "moderation" of the gravitational wave emission during the entire past. This is in order to preclude, for instance, the emission of an extremely strong burst of gravitational radiation around some very remote epoch which could make an appreciable contribution to the integral (7.23). As it is the second time derivative of $\bar{I}_{ab}$ that gives the order of magnitude of the transverse-traceless metric form of the gravitational wave emission, we shall assume that there exists a numerical constant, k, of order unity, such that

$$\forall v\geq 0,\quad |{}^{(2)}\bar{I}_{ab}(t-v)|\leq k\,|{}^{(2)}\bar{I}_{ab}(t)| . \tag{7.24}$$

Note that we impose no condition of boundedness (in the mathematical sense) for ${}^{(1)}\bar{I}_{ab}$ or $\bar{I}_{ab}$. For instance, our system could be gravitationally bound now, and still have been formed by the capture of free incoming objects. Then in the very remote past $\bar{I}_{ab}(t)$ would blow up like t^2, while ${}^{(2)}\bar{I}_{ab}(t)$ will still satisfy (7.24) (if the incoming velocities in the past were not much larger than the present orbital ones).

From Eq. (7.22a) and our assumption (7.24) we find that the quantitative influence on ${}^{(5)}(\delta\bar{I}_{ab})$ of the history of the system before some time $t'<t$ ("remote past") is smaller, in absolute magnitude, than

$$|[{}^{(5)}\delta\bar{I}_{ab}(t)]^{\text{remote past}}|\leq\frac{24k}{(t-t')^4}\frac{\bar{I}}{c^3}|{}^{(2)}\bar{I}_{ab}(t)| . \tag{7.25a}$$

Relatively to the lowest-order damping, the remote past history brings a modification

$$\frac{|[{}^{(5)}\delta\bar{I}_{ab}(t)]^{\text{remote past}}|}{|{}^{(5)}\bar{I}_{ab}(t)|}\leq 24k\left[\frac{P}{t-t'}\right]^4\frac{Gm}{c^3P} , \tag{7.25b}$$

where we used $m=\bar{I}$ (characteristic mass), $|d^n\bar{I}/dt^n|\sim|\bar{I}|/P^n$ (P referring to the characteristic time scale or principal period divided by 2π) and where we have put back G. The RHS of Eq. (7.25b) clearly shows the fact that, if the system has never been much more "active" than now, the remote past contributes only very little to the present damping. For instance, let us consider the binary pulsar system PSR 1913 + 16 (assuming that our results are applicable to this system, although it contains regions with strong gravitational fields). Then the characteristic time scale P is (orbital period)$/2\pi=74$ minutes, while we can take as remote past the span of time extending before the discovery of the binary pulsar,[59] say, before September 1974, i.e., $(t-t')/P\simeq 9.2\times 10^4$. We then find (using $m=2.8$ solar masses) that the RHS of (7.25b) is $3.7k\,10^{-28}$, so that we do not expect the remote past to affect the present secular acceleration of the mean orbital motion of the binary pulsar before the 27th digit.

Therefore, the main hereditary influence will come from the "recent past" contribution, $0\leq v\leq t-t'$, or $0\leq v\leq NP$, where N is such that we can confidently neglect $N^{-4}(Gm/c^3P)$. Further assuming that the system has been, during the recent past, more or less in the same dynamic state as now, we can make a crude quantitative estimate of this recent-past contribution. Using Eq. (7.22a) we find

$$\frac{|[{}^{(5)}\delta\bar{I}_{ab}(t)]^{\text{recent past}}|}{|{}^{(5)}\bar{I}_{ab}(t)|}\sim\frac{Gm}{c^3P}=\beta\gamma , \tag{7.26}$$

where β and γ are defined in Sec. II. [The RHS of Eq. (7.26) is $\approx 3\times 10^{-9}$ in the binary pulsar case.]

In conclusion, the effect of heredity, although very small in absolute magnitude [$O(c^{-8})$ in the equations of motion, i.e., 4PN level], is a rather large relative modification of the quadrupole radiation damping [$O(c^{-3})$, i.e., $1\frac{1}{2}$PN relative level]. As damping forces can often be accurately measured through their secular effects even when their absolute magnitude is very small, the effect of heredity might be important for some gravitationally interacting systems. This might be the case during the late stage of inspiralling neutron star binaries. However, the treatment of such cases will necessitate a generalization of our method to systems containing some strong field regions. We however expect that our final result will remain unchanged (via some "effacement"[38] of the strong field effects).

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APPENDIX: LOGARITHMS OF c AT 3PN IN THE EXTERNAL METRIC

Anderson *et al.*[42] have pointed out the appearance of a logarithm of c at the 3PN approximation in the external metric. We wish here to confirm and generalize their result. After matching, these logarithms enter the relations (6.34) and (6.35) between the algorithmic moments and the source variables.

We can easily see these logarithms by inspecting Eq. (5.2). Indeed from this equation we see that at the cubic iteration ($n=3$) logarithms can appear in the (post-Minkowskian) "instantaneous" terms of $h_3^{\alpha\beta}$. Let us denote by $A^{\alpha\beta}$ the coefficient of $\ln(r/cP)$ in these (algorithmic-)instantaneous terms. Then we easily find that $A^{\alpha\beta}$ [if larger than the tail $\sim O(\ln c/c^{10})$] must be a homogeneous solution of the d'Alembert equation. Then we can always write

$$A^{\alpha\beta}=\sum_{E_3}\frac{1}{c^{9+\underline{l}_1+\underline{l}_2+\underline{l}_3}}\sum_{l\geq 0}c^{l+1}\partial_L[r^{-1}G_L^{\alpha\beta}(t-r/c)]\ , \tag{A1}$$

from which we deduce (using $\underline{l}_1+\underline{l}_2+\underline{l}_3-l\geq -S$, where S is the number of spatial indices among $\alpha\beta$) that $A^{\alpha\beta}=O(1/c^{8-S})$ and thus that logarithms can indeed arise at the 3PN approximation in the near-zone re-expanded external metric. A closer study of $A^{\alpha\beta}$ (using the "logarithm-free" stationary metric of the Appendix C of paper I) shows that, in fact, A^{00} is of the order $O(c^{-8})$, A^{0i} is $O(c^{-7})$, and A^{ij} is $O(c^{-8})$. Let us denote by ${}_8A^{00}$, ${}_7A^{0i}$, and ${}_8A^{ij}$ the coefficients of c^{-8}, c^{-7}, and c^{-8} in the $A^{\alpha\beta}$'s. These coefficients are some harmonic functions of the type $\partial_L r^{-1}$. It is now straightforward to show that there must exist a vector σ^α (itself a harmonic function) and some (algorithmic-instantaneous) tensors $H_L(t)$ such that

$$ {}_8A^{00}=-\partial_t\sigma^0+\partial_i\sigma^i -4\sum_{l\geq 0}\frac{(-)^l}{l!}(\partial_L r^{-1})H_L(t)\ , \tag{A2a}$$

$$ {}_7A^{0i}=\partial_i\sigma^0\ , \tag{A2b}$$

$$ {}_8A^{ij}=\partial_i\sigma^j+\partial_j\sigma^i-\delta_{ij}(\partial_t\sigma^0+\partial_k\sigma^k)\ . \tag{A2c}$$

Then if we perform on the external metric the coordinate transformation

$$x'^0=x^0-\frac{1}{c^7}\sigma^0\ln(r/cP)\ , \tag{A3a}$$

$$x'^i=x^i-\frac{1}{c^8}\sigma^i\ln(r/cP) \tag{A3b}$$

[see Eqs. (6.11)] the only remaining logarithms will appear in the 00 component of the transformed c^{-8}-external metric, which will be given [after having inserted the source expansions (6.6) of the algorithmic moments] by

$$ {}_8\hat{\mathscr{G}}'^{\text{ext}}_{00}=\sum_{l\geq 0}\frac{2(-)^l}{l!}(\partial_L r^{-1}) \times[\ln(r/cP)\hat{H}_L(t)+(\ln c)_{6,1}I_L(t)]+{}_8\hat{\varphi}'_{00}\ , \tag{A4}$$

where ${}_8\hat{\varphi}'_{00}$ is some "logarithm-free" source-instantaneous quantity. Now, through matching, we shall have

$$ {}_{6,1}I_L(t)=\hat{H}_L(t)\ . \tag{A5}$$

Therefore, we find that these logarithms of c, which were originally present in the c^{-8} coefficient of the PN-re-expanded MPM external metric, do not explicitly appear in the inner metric, but arise, at the c^{-6} level, in Eqs. (6.34)–(6.35) relating the algorithmic moments to the source variables.

The functions $H_L(t)$ are some complicated sums of cubic products of the derivatives of the M_L's and S_L's. In the case of the interaction (mass monopole M)$\times$(mass monopole M)$\times$[mass multipole $M_L(t)$] we have found that $H_L(t)$ is given by

$$H_L(t)=-2\frac{15l^4+30l^3+28l^2+13l+24}{l(l+1)(2l+3)(2l+1)(2l-1)} \times M^{2}\,{}^{(2)}M_L(t)\ . \tag{A6}$$

We then recover, in the quadrupole case ($l=2$), the value $H_{ij}=-\frac{214}{105}M^2\,{}^{(2)}M_{ij}(t)$ obtained by Anderson *et al.*[42] [see their Eq. (A11)].

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[46]Our conventions and notation are the following: signature $-+++$; greek indices$=0,1,2,3$; latin indices$=1,2,3$; $g=-\det(g_{\mu\nu})$; $f_{\alpha\beta}=f^{\alpha\beta}=$flat metric$=\mathrm{diag}(-1,+1,+1,+1)$; $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, are the usual sets of non-negative integers, integers, real numbers, and complex numbers; $r=(x_1^2+x_2^2+x_3^2)^{1/2}$; $n^i=n_i=x_i/r$; $\partial_i=\partial/\partial x^i$; $n^L=n_L=n_{i_1}n_{i_2}\cdots n_{i_l}$ and $\partial_L=\partial_{i_1}\partial_{i_2}\cdots\partial_{i_l}$, where $L=i_1i_2\cdots i_l$ is a multi-index with l indices ($L-1$ denoting $i_1i_2\cdots i_{l-1}$, etc); $\hat{n}_L$ and $\hat{\partial}_L$ are the (symmetric) trace-free parts of n_L and ∂_L (we freely raise or lower spatial indices by means of $f^{ij}=f_{ij}=\delta_i^j$).

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[55]By $f(\epsilon)=O(g(\epsilon))$ we mean that when $\epsilon\to 0$ there exists a constant A such that $|f(\epsilon)|<A\,|g(\epsilon)|$.

[56]Let us recall that the "n post-Newtonian level" traditionally refers to the terms in the metric which contribute $1/c^{2n}$ corrections to the Newtonian equations of motion; this means the orders $1/c^{2n+2}$ in g_{00}, $1/c^{2n+1}$ in g_{0i}, and $1/c^{2n}$ in g_{ij}. We shall here extend this terminology in the presence of powers of $\ln c$ by still factorizing the powers of $1/c$ while keeping the powers of $\ln c$ in the coefficients: ${}_p g_{\mu\nu}=\sum_q(\ln c)^q{}_{p,q}g_{\mu\nu}$ using the notation of Sec. VI.

[57]One can show that the post-Newtonian expansion of this second term is equal to the formal "time-symmetric" solution of the wave equation with source

$$(r/cP)^B\hat{n}_L r^{-2}\left[H(t)-\frac{r}{c}{}^{(1)}H(t)+\frac{r^2}{2c^2}{}^{(2)}H(t)+\cdots\right]$$

obtained by means of the Green's function

$$\Box^{-1}=\Delta^{-1}(1-c^{-2}\Delta^{-1}\partial_t^2)^{-1}=\Delta^{-1}+c^{-2}\Delta^{-2}\partial_t^2+\cdots ,$$

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