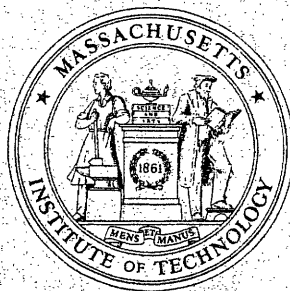


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OF TECHNOLOGY**

Tailoring Benders Decomposition
For Uncapacitated Network Design

Thomas L. Magnanti
Richard T. Wong
Paul Mireault

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ABSTRACT

Because of its imbedded network flow structure, the generic network design problem is an attractive candidate for integer programming decomposition. This paper studies the application and acceleration of Benders decomposition for uncapacitated models from this problem class and illustrates the potential flexibility of the Benders solution strategy. In particular, it (i) shows that several lower bounding inequalities from the literature can be derived as Benders cuts; and (ii) introduces new Benders cuts for the network design problem.

The paper also reports on computational experience in using Benders decomposition with a dual ascent and variable elimination preprocessing procedure to solve uncapacitated network design problem with up to 90 binary variables and 15,080 continuous variables, or 45 binary variables and 105,600 continuous variables.

KEY WORDS: Network Design, Benders Decomposition, Dual Ascent, Variable Elimination, Integer Programming.

ABBREVIATED TITLE: Benders Decomposition for Network Design

INTRODUCTION

Conceptually, integer programming models of network design are very simple. They address the following basic question: what configuration of the network minimizes the sum of the fixed costs of arcs chosen to be in the network (a set of binary decisions) and the costs of routing goods through the network defined by these arcs? And yet, when specialized, even the simplest version of this problem with linear routing costs and without arc capacities models many of the most well-known problems in combinatorial optimization. These special cases include shortest path problems, minimal spanning tree problems, optimal branching problems, and the celebrated traveling salesman problem and network Steiner problem (Magnanti and Wong [1984a]).

Moreover, as illustrated by several books and edited collections of papers (e.g., Boesch [1975], Boyce [1979], and Mandl [1981]), the generic network design problem has numerous applications. Not only does it model a variety of traditional problems in communication, transportation, and water resource planning, but it also arises in emerging problem contexts such as flexible and automated manufacturing systems (Graves and Lamar [1983] and Kimenia and Gershwin [1979]). In addition, familiar node splitting devices permit the model to represent facility location decisions that have applications in warehouse and plant location, emergency facility location, computer networking, and many other problem domains. (See Francis and White [1974], Handler and Mirchandani [1979], Larson and Odoni [1981], and Tansel, Francis and Lowe [1983] for extensive citations to this literature.) Finally, the

network design problem arises in many vehicle fleet planning applications that do not involve the construction of physical facilities, but rather model service decisions, i.e., do we dispatch a vehicle on a link or not (e.g., see Simpson [1969] or Magnanti [1981]).

Although these applications have attracted numerous algorithmic studies, even the uncapacitated network design problem remains essentially unsolved. Surprisingly, the largest problems solved to optimality (and verified as such) contain only about 10 nodes and 40-50 arcs. This experience is, in part, explainable theoretically. Not only are the simplest versions of the network design problem NP-complete (Johnson, Lenstra and Rinnooy Kan [1978], Wong [1978]), but finding even approximate solutions (to problems with budget restrictions on the fixed costs incurred) is NP-complete (Wong [1980]). Even so, there does not appear to be any compelling evidence to explain why network design remains among the most computationally elusive of all integer programs encountered in practice.

In this paper, we study Benders decomposition-based algorithms for the uncapacitated network design problem. Although Benders decomposition has proved to be a useful planning tool in several problem contexts including distribution system design (Geoffrion and Graves [1974]), aircraft routing (Richardson [1976]), and rail engine scheduling (Florian, Guerin and Bushel [1976]), its successes remain limited. We nevertheless feel that the method continues to have great potential. By studying the algorithm as applied to the uncapacitated network design problem and using it together with other integer programming solution techniques, we

- (i) report on computational results on some of the largest network design problem solved to optimality to date. The problems studied in this paper contain up to 90 binary variables and 15,080 continuous variables or 45 binary variables and 105,600 continuous variables (more compact models with as few as 480 and 6,600 continuous

variables are possible, though these alternate formulations are much less attractive computationally);

- (ii) adapt and enhance algorithmic ideas from Magnanti and Wong [1981, 1983b] that demonstrate the potential for accelerating Benders decomposition and for generating a rich variety of lower bound inequalities for the network design problem and other mixed integer programs. In particular, we show that several lower bounding procedures for the network design problem from the literature, even though originally derived by other means, can be viewed as special instances of Benders cuts; and
- (iii) illustrate the importance of using other solution procedures (bounding-based variable elimination methods, dual ascent) together with Benders decomposition as part of a comprehensive approach to solving integer programming problems.

Previously, researchers have attempted to solve uncapacitated network design problems using a variety of solution techniques. Hoang [1973], Boyce, Farhi and Weischedel [1973], Dionne and Florian [1979], Boffey and Hinxman [1979], and Los and Lardinois [1982] have all studied branch and bound algorithms for the problem. Gallo [1981] has proposed a branch and bound procedure for the related "optimal network design problem" that eliminates the fixed costs from the objective function, and instead limits fixed cost expenditures by imposing a given budget as a constraint. Although these branch and bound algorithms can successfully solve problems with a small number (in some cases up to 40-50) of arcs, their computation time grows very quickly in the problem size.

Heuristic procedures are capable of generating solutions to much larger network design problems. Billheimer and Gray [1973] describe an add-drop heuristic method with provisions for eliminating non-optimal variables. Los and Lardinois [1982] suggest improvements to this method and discuss statistical methods for analyzing the solutions that it generates. Dionne and Florian [1979] and Boffey and Hinxman [1979] propose heuristics for the optimal network design problem. Wong [1984] gives a special heuristic for optimal network problems on the Euclidean plane. He shows that with high probability,

the cost of the solution generated by this heuristic will, under certain conditions, be very close to the cost of the optimal solution.

These various heuristics can usually solve larger-sized problems (20-50 nodes) in a small amount of computation time. However, it is usually difficult to assess the quality of the heuristics' solutions since no satisfactory method is known for solving problems of this size optimally.

In one study of integer programming decomposition for the network design problem, Rardin and Choe [1979] have devised a Lagrangian relaxation algorithm. Their computational results seem to be quite promising.

Our computational results in this paper demonstrate that choosing Benders cuts judiciously can have a marked effect on the algorithm's performance. We introduce a new type of cut, which is generated by using information from solving a single shortest-path problem over all candidate arcs at the outset of the computations. This new cut and a Pareto-optimal cut generated by specializing the methodology from Magnanti and Wong [1981] are an order of magnitude more effective than cuts generated by the usual implementation of Benders method.

Our computational experience also indicates that Benders decomposition can be much more effective when used in conjunction with other integer programming techniques (dual ascent and variable elimination procedures). Equipped with these enhancements, the algorithm was able to solve to optimality 19 of 24 test problems with 45 arcs (in three cases) and 90 candidate arcs, and to find solutions to 23 of these problems that are guaranteed to be within at most 1.44% of optimality (and to within at most 5.53% for the 24th problem). The computations took from about one minute to about 1-1/2 hours on a VAX 11/780 computer, which we estimate would correspond to about 7 seconds to 10 minutes on an IBM 3033. These solution times were obtained using the branch

and bound facilities in the Land and Powell [1973] mathematical programming system. They could be reduced, and quite likely significantly, by using a commercial branch and bound code or specialized computer codes for solving the **Benders master integer program.**

This experience indicates that large-scale uncapacitated network design problems are within the reach of current integer programming capabilities; moreover, it highlights the importance of adopting a holistic view of integer programming methods, rather than treating each solution procedure in isolation.

1. MODELING AND SOLUTION APPROACH

Problem Formulation

The basic ingredients of the model are a set N of nodes and a set A of undirected arcs that are available for designing a network.

The model has multiple commodities. These might represent distinct physical goods, or the same physical good but with different points of origin and destination. We let K denote the set of commodities and for each $k \in K$, assume (by scaling, if necessary) that one unit of flow of commodity k must be shipped from its point of origin, denoted $O(k)$, to its point of destination denoted $D(k)$. This formulation can model problems in which any commodity has either multiple origins or multiple destinations by considering the flow from each origin to each destination as a separate commodity (i.e., by disaggregating flow). The formulation cannot, however, model problems with both multiple origins and multiple destinations since it then loses its imbedded shortest path structure.

The model contains two types of variables, one modeling discrete choice design decisions and the other modeling continuous flow decisions. Let y_{ij}

be a binary variable that indicates whether ($y_{ij} = 1$) or not ($y_{ij} = 0$) arc $\{i,j\}$ is chosen as part of the network's design. Let x_{ij}^k denote the flow of commodity k on arc (i,j) . Note that (i,j) and (j,i) denote directed arcs corresponding to the undirected arc $\{i,j\}$. Even though arcs in the model are undirected, we refer to these directed arcs because the flows are directed. Then, if $y = (y_{ij})$ and $x = (x_{ij}^k)$ are vectors of design and flow variables, the model becomes:

$$P_1: \quad \text{minimize} \quad \sum_{k \in K} \sum_{\{i,j\} \in A} (c_{ij}^k x_{ij}^k + c_{ji}^k x_{ji}^k) + \sum_{\{i,j\} \in A} F_{ij} y_{ij} \quad (1.1)$$

subject to:

$$\sum_{i \in N} x_{ij}^k - \sum_{\ell \in N} x_{j\ell}^k = \begin{cases} -1 & \text{if } j = O(k) \\ 1 & \text{if } j = D(k) \\ 0 & \text{otherwise} \end{cases} \quad \begin{matrix} \text{all } j \in N \\ \text{all } k \in K \end{matrix} \quad (1.2)$$

$$\left. \begin{array}{l} x_{ij}^k \leq y_{ij} \\ x_{ji}^k \leq y_{ij} \end{array} \right\} \quad \begin{matrix} \text{all } \{i,j\} \in A \\ \text{all } k \in K \end{matrix} \quad (1.3)$$

$$x_{ij}^k, x_{ji}^k \geq 0, y_{ij} = 0 \text{ or } 1 \quad \text{all } k \in K, \{i,j\} \in A \quad (1.4)$$

$$y \in Y. \quad (1.5)$$

In this formulation c_{ij}^k is the nonnegative per unit cost for routing commodity k on arc (i,j) and F_{ij} is the fixed charge design cost for arc $\{i,j\}$. In general, c_{ij}^k need not equal c_{ji}^k . Constraints (1.2) imposed upon each commodity k are the usual network flow conservation equations (multiplied through by -1 to facilitate later interpretations of our algorithmic development). The "forcing" constraints (1.3) state that if arc $\{i,j\}$ is included in the design, i.e., if $y_{ij} = 1$, then the total arc flow is unlimited (since the flow of any commodity k on arc (i,j) or (j,i) is at most one, anyway) and if arc $\{i,j\}$ is not included in the network design, i.e., if $y_{ij} = 0$, then the total arc flow must be zero. Finally, the set Y imposes any possible restrictions on the design

variables such as multiple choice constraints ($y_{ij} + y_{pq} + y_{rs} \leq 1$) or precedence constraints ($y_{ij} \leq y_{rs}$).

Like most integer programming problems, this design problem could be modeled in a variety of ways. In particular, the forcing constraints (1.3) are equivalent to the more aggregated constraints

$$\text{and } \left. \begin{array}{l} \sum_{k \in K} x_{ij}^k \leq |K| y_{ij} \\ \sum_{k \in K} x_{ji}^k \leq |K| y_{ij} \end{array} \right\} \text{ for all } \{i,j\} \in A. \quad (1.6)$$

Both versions of these constraints force each x_{ij}^k and x_{ji}^k for $k \in K$ to be zero if $y_{ij} = 0$, and become redundant if $y_{ij} = 1$.

This aggregation could substantially reduce the number of constraints in the formulation for even moderately sized problems. With 50 commodities and 100 arcs, the disaggregate formulation (1.3) contains 10,000 forcing constraints; the aggregate version contains only 200 forcing constraints.

Note that the formulation (1.1)-(1.5) includes a single commodity k for each origin-destination pair of demand. This definition of commodities gives rise to 2 constraints of the form (1.3) for each origin-destination pair. Alternatively, when the per unit routing costs are independent of the destination, we could distinguish commodities only by their point of origin (then x_{ij}^k is the total flow from origin k on arc (i,j)), and model the forcing constraints as

$$\begin{array}{l} x_{ij}^k \leq |N| y_{ij} \\ x_{ji}^k \leq |N| y_{ij} \end{array} .$$

In a problem with as few as two origins shipping goods to 25 destination nodes over 100 candidate arcs, this modified model would reduce the number of continuous variables from $2 \cdot 25 \cdot 2 \cdot 100 = 10,000$ to $2 \cdot 2 \cdot 100 = 400$.

Surprisingly, the seemingly less efficient and substantially larger disaggregate formulation with forcing constraints (1.3) instead of (1.6), and with flow variables for each origin-destination rather than for each origin, leads to more efficient algorithms. The disaggregated model is preferred computationally for two reasons. First, many techniques for solving integer programs like the network design problem first solve the linear programming relaxation of the model that results by replacing $y_{ij} = 0$ or 1 with $0 \leq y_{ij} \leq 1$. Because the linear programming version of the disaggregate formulation is much more tightly constrained than the linear programming version of the aggregate formulation, the disaggregate linear program provides a sharper lower bound on the value of the integer programming formulation; that is, it more closely approximates the integer program. A number of authors have noted the important advantages of using "tight" linear programming relaxations. (See, for example, Cornuejols, Fisher and Nemhauser [1977], Davis and Ray [1969], Beale and Tomlin [1972], Geoffrion and Graves [1974], Mairs et al. [1978], Magnanti and Wong [1981], Rardin and Choe [1979], and Williams [1974]).

In addition, solution methods such as Benders decomposition that utilize information obtained from the dual of the linear programming relaxation will also be much more effective when applied to the disaggregate formulation (1.1)-(1.5). Because the disaggregate linear programming formulation has more constraints, it has a richer collection of linear programming dual variables and, therefore, provides more flexibility in algorithmic development. Indeed, these two advantages of the disaggregate formulation are intimately related (Magnanti and Wong [1981]).

Benders Decomposition

Benders decomposition assumes a particularly simple form when applied to the uncapacitated network design problem defined by problem P_1 . The algorithm alternately solves for a tentative network configuration in the

integer variables y and a routing problem in the continuous variables x . For any particular choice of the y variables, the design model (1.1)-(1.5) reduces to a collection of independent shortest path problems, one for each commodity k . Since the mechanics of Benders decomposition is the same for one commodity as it is for many, while describing the algorithm we drop the index k and assume that there is only one commodity (and hence only one routing subproblem) for any fixed value of y . Let $S(y)$ denote this routing subproblem and let $R(y)$ denote its optimal objective value (ignoring the fixed costs). In addition, let u_i and v_{ij} (v_{ji}) denote any feasible dual variables corresponding to the constraints (1.2) and the negative of constraints (1.3), respectively, and suppose that node 1 is the origin node for this single commodity and that node n is its destination node.

By linear programming duality theory, for any fixed choice of y , the value of the dual objective function to the subproblem $S(y)$ is a lower bound on $R(y)$. That is,

$$R(y) \geq u_n - u_1 - \sum_{\{i,j\} \in A} (v_{ij} + v_{ji})y_{ij}.$$

Moreover, this inequality is satisfied as an equality if the variables u_i and v_{ij} solve the dual problem. Therefore,

$$R(y) = \text{minimum } w$$

subject to:

(1.7)

$$w \geq u_n - u_1 - \sum_{\{i,j\} \in A} (v_{ij} + v_{ji})y_{ij} \quad \text{for all } (u,v) \in D.$$

In this formulation, $u = (u_i)$ and $v = (v_{ij})$ are vectors of dual variables and D denotes the dual feasible region for the subproblem formulated with constraints (1.3) multiplied by minus one. This convention of multiplying by minus one ensures that the variables v_{ij} and v_{ji} in D are nonnegative.

Since the optimal value C^* of the design problem equals the minimum of the fixed costs $\sum_{\{i,j\} \in A} F_{ij} y_{ij}$ and routing costs $R(y)$ over all feasible network configurations,

$$C^* = \text{minimum } z$$

subject to:

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + u_n - u_1 - \sum_{\{i,j\} \in A} (v_{ij} + v_{ji}) y_{ij} \quad \text{for all } (u,v) \in D \quad (1.8)$$

$$y \in Y \text{ and } y_{ij} = 0 \text{ or } 1 \quad \text{for all } \{i,j\} \in A.$$

This reformulation of the original problem is known as the master problem.

Note that z and the y_{ij} 's are its decision variables.

Benders decomposition solves the master problem by relaxing the inequality constraints (1.8) on z for most $(u,v) \in D$ and replacing them with a finite number of constraints obtained by restricting the (u,v) 's to some small set $D' \subseteq D$. Since the constraints are relaxed, the value C' of the relaxed master problem is a lower bound on C^* . The algorithm then checks to see if the solution $C'=z'$ and $y' = (y'_{ij})$ to the relaxed master problem is feasible in the full problem by solving the routing subproblem; that is, by solving the shortest path problem defined by y' or its dual reformulation (1.7) with $y=y'$. If the optimal dual variables (u', v') to this problem satisfy (1.8), i.e., if

$$z' \geq \sum_{\{i,j\} \in A} F_{ij} y'_{ij} + u'_n - u'_1 - \sum_{\{i,j\} \in A} (v'_{ij} + v'_{ji}) y'_{ij}, \quad (1.9)$$

then (1.8) is satisfied for every other $(u,v) \in D$ and thus $C^* = C'$, and y' is an optimal configuration. Otherwise, we add the Benders cut (1.9) to the restricted master problem and repeat the procedure.

This procedure requires one further elaboration. For some values \bar{y} of y , the subproblem might be infeasible. That is, the network defined by \bar{y} might not contain any path from the commodity's origin to its destination. In this

case, there must be a cut set C separating the origin and destination with $\bar{y}_{ij} = 0$ for all arc $\{i,j\} \in C$. When this occurs, we add the feasibility cut

$$\sum_{\{i,j\} \in C} y_{ij} \geq 1$$

to the relaxed master problem and proceed as before. (The full master problem would contain all cuts of this nature.)

For multiple commodity formulations, we let $S_k(\bar{y})$ denote the routing subproblem for commodity k for a fixed value \bar{y} of y , let $R_k(\bar{y})$ denote its optimal objective value, and let $R(\bar{y}) = \sum_{k \in K} R_k(\bar{y})$. Also, let $u^k = (u_i^k)$ and $v^k = (v_{ij}^k)$ denote dual variables for the k th subproblem.

Then the method is much the same, except that the Benders cuts in (1.8) becomes

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + \sum_{k \in K} [(u_{D(k)}^k - u_{0(k)}^k) - \sum_{\{i,j\} \in A} (v_{ij}^k + v_{ji}^k) y_{ij}]. \quad (1.10)$$

This cut can be interpreted as follows. Let u_i^k denote the length of the shortest path from node $0(k)$ to node i with respect to distances c_{ij}^k and c_{ji}^k in the network defined by those arcs $\{i,j\} \in A$ with $\bar{y}_{ij} = 1$. Furthermore, let

$$v_{ij}^k = \max \{u_j^k - u_i^k - c_{ij}^k, 0\} \quad (1.11)$$

$$\text{and } v_{ji}^k = \max \{u_i^k - u_j^k - c_{ji}^k, 0\}.$$

Then $\{u_i^k\}$ and $\{v_{ij}^k\}$ solve the dual to the k th subproblem, and any positive coefficient $(v_{ij}^k + v_{ji}^k)$, and the term $(v_{ij}^k + v_{ji}^k)y_{ij}$, of (1.10) can be interpreted as a potential reduction in the shortest path distance between nodes $0(k)$ and $D(k)$ caused by setting $y_{ij} = 1$ and introducing arc $\{i,j\} \in A$. The coefficient $(\sum_k v_{ij}^k + v_{ji}^k)$ of y_{ij} represents this total potential savings (generally an overestimate) over all commodities. The reduction might not be realized because arc $\{i,j\}$ need not lie on the shortest path joining some of the origins and destinations. Moreover, the savings of introducing two arcs $\{i,j\}$ and $\{r,s\}$ might not achieve the

total potential $\sum_{k \in K} (v_{ij}^k + v_{ji}^k) + \sum_{k \in K} (v_{rs}^k + v_{sr}^k)$ because of interaction effects; that is, even if a new shortest path joining the origin and destination of some commodity k were to use both arcs $\{i,j\}$ and $\{r,s\}$, we need not realize $v_{ij}^k + v_{ji}^k + v_{rs}^k + v_{sr}^k$ in savings. For that reason, the righthand side of (1.10) is a lower bound on the total cost z , and need not necessarily be equal to this cost.

2. STRENGTHENING BENDERS CUTS

The usual (or standard) Benders cut (1.10) is but one of many possible lower bounds on the cost of the network design problem. As we have already noted, this cut is derived from dual variables to the linear programming relaxation of the problem which is a shortest path problem. Since network problems are typically degenerate, their dual problems typically have alternate optimal solutions each defining a Benders cut. Are some of these cuts to be preferred to others? How might better cuts be generated?

In this section, we discuss five other Benders cuts. The first three are known lower bounds on the network design problem's objective value derived previously in the literature from arguments unrelated to Benders decomposition. We show that each of these lower bounds is a Benders cut derived by a judicious choice of the dual variables or the tentative network configuration used to generate the cut. This demonstration illustrates the richness of Benders cuts for the network design problem as well as the fundamental role of duality (in generating the Benders cut) for developing objective function bounds.

The fourth cut, which is new, dominates (in a way to be described later) the usual cut (1.10). Like some of these other lower bounds, it sharpens the cut inequality by considering penalties for eliminating arcs from the current

configuration as well as savings attributed to adding new arcs (as in the usual cut). The fifth cut, which is of this same variety, has no explicit representation. Rather, it is generated by solving an auxiliary linear program to the subproblem which selects a "good" alternate solution to the dual.

Although this procedure is a specialization of a more general cut generation procedure (Magnanti and Wong [1981]), because of the special structure of the network design problem it can be implemented efficiently by solving one network flow problem for generating each cut.

We might note that the general conception in this section of strengthening Benders cuts applies to other problem settings as well. In particular, similar results are possible in the context of facility location problems (Magnanti and Wong [1984b]), where the Benders cuts are related to the theory of submodular inequalities (Nemhauser and Wolsey [1981]).

2.1 Interpreting Lower Bounds as Benders Cuts

Previously, Boyce et al. [1973] proposed the following lower bound on the total cost z of the network design problem:

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + R(y^a) \quad (2.1)$$

In this inequality, y^a denotes a candidate design that includes all possible arcs, i.e., $y_{ij}^a = 1$ for all $\{i,j\} \in A$. We shall refer to this special candidate network as the all-inclusive design. Recall that $R(y)$ denotes the total (minimum cost) routing cost for the given candidate network y . Therefore, since all arcs are available in the all-inclusive design, $R(y^a)$ is the minimal possible routing cost for the problem. In particular, since $R(y) \geq R(y^a)$ for any network configuration y , (2.1) is a valid lower bound on total costs. Notice that the righthand side of this lower bound is actually a lower bounding function on total cost $z(y)$ as a function of y . We have subsumed the

dependence on y to conform with standard conventions from the literature on Benders cuts.

Independently, Gallo [1983] and Magnanti and Wong [1984] proposed another lower bound:

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + \sum_{\{i,j\} \in A} y_{ij}^c \left[\sum_{k \in Q_{ij}} I_{ij}^k(y^a) \right] + R(y^a) \quad (2.2)$$

The formidable looking second term in this lower bound, which adds penalties for eliminating arcs from the all-inclusive design, requires some explanation.

The variable $y_{ij}^c \equiv 1 - y_{ij}$ equals 1 if arc $\{i,j\}$ is eliminated from the all-inclusive design and has value zero, otherwise. The term within brackets multiplying y_{ij}^c is a routing cost penalty for eliminating arc $\{i,j\}$ from the all-inclusive design. The sets Q_{ij} define any partition of the commodities and $I_{ij}^k(y^a)$ is the incremental routing cost for commodity k incurred by deleting arc $\{i,j\}$ from the all-inclusive design. Note that this incremental cost term is zero if and only if some shortest path from $O(k)$ to $D(k)$ does not contain arc $\{i,j\}$.

Notice that since each y_{ij}^c and each incremental cost $I_{ij}^k(y^a)$ is nonnegative, if any coefficient $I_{ij}^k(y^a)$ is positive the lower bound (2.2) dominates the lower bound (2.1) in the sense that the righthand side of (2.2) is as large as the righthand side of (2.1) for all values of y and is strictly larger for at least one value of y .

We next show how to interpret both lower bounds (2.1) and (2.2) as Benders cuts. For (2.1), the interpretation is easy to derive. Simply consider the Benders cut generated by the all-inclusive design. Since every arc $\{i,j\} \in A$ belongs to this network configuration, for each commodity k , the shortest path distances u_i^k from $O(k)$ to node i satisfy the optimality

conditions $c_{ij}^k + u_i^k - u_j^k \geq 0$. Consequently, the v_{ij}^k and v_{ji}^k in (1.11) are zero and the usual cut (1.10) becomes the lower bound (2.1).

Showing that (2.2) is a Benders cut is a bit more difficult. To establish this fact, consider a given arc $\{g,h\}$ and a given commodity $k \in Q_{gh}$. Suppose that we solve a shortest path problem for commodity k on the all-inclusive network with cost data specified as follows:

$$\tilde{c}_{ij}^k = c_{ij}^k \text{ and } \tilde{c}_{ji}^k = c_{ji}^k \quad \text{if } \{i,j\} \neq \{g,h\}$$

$$\tilde{c}_{gh}^k = c_{gh}^k + I_{gh}^k(y^a)$$

$$\tilde{c}_{hg}^k = c_{hg}^k + I_{gh}^k(y^a) \quad .$$

Let \tilde{u}_i^k denote the shortest path distance from node $0(k)$ to node i , with respect to these costs. For all $\{i,j\} \in A$, let \tilde{v}_{ij}^k be defined by (1.11) with $u_i^k = \tilde{u}_i^k$; i.e., $\tilde{v}_{ij}^k = \max(0, \tilde{u}_j^k - \tilde{u}_i^k - c_{ij}^k)$ and $\tilde{v}_{ji}^k = \max(0, \tilde{u}_i^k - \tilde{u}_j^k - c_{ji}^k)$.

Property 1: The \tilde{u}_i^k and \tilde{v}_{ij}^k are feasible variables in the dual of the k th subproblem $S_k(y^a)$ corresponding to the all-inclusive design.

Proof: By substitution in the dual problem. (For any values of \tilde{u}_i^k , the \tilde{v}_{ij}^k defined in this way correspond to a dual feasible solution.) \square

Property 2: The \tilde{u}_i^k and \tilde{v}_{ij}^k are optimal variables in the dual of the k th subproblem $S_k(y^a)$.

Proof: Since these variables are dual-feasible, it suffices to show that their dual objective value equals the length of the shortest path from $0(k)$ to $D(k)$ in the all-inclusive design.

First, note that by definition of the incremental cost $I_{gh}^k(y^a)$, the length $\tilde{u}_{D(k)}^k - \tilde{u}_{0(k)}^k$ of the shortest path for commodity k in the all-inclusive network with costs \tilde{c}_{ij}^k equals $I_{gh}^k(y^a)$ plus the shortest path length for costs c_{ij}^k . That is,

$$\tilde{u}_{D(k)}^k - \tilde{u}_{0(k)}^k = R_k(y^a) + I_{gh}^k(y^a). \quad (2.3)$$

Next note that since \tilde{u}_i^k are shortest path distances on the all-inclusive network, the optimality conditions $\tilde{c}_{ij}^k + \tilde{u}_i^k - \tilde{u}_j^k \geq 0$ are valid for all $\{i,j\} \in A$. But then since $c_{ij}^k = \tilde{c}_{ij}^k$ and $c_{ji}^k = \tilde{c}_{ji}^k$ for all $\{i,j\} \neq \{g,h\}$, $\tilde{v}_{ij}^k = 0$ and $\tilde{v}_{ji}^k = 0$ for all $\{i,j\} \neq \{g,h\}$.

Now consider arc $\{g,h\}$. If $I_{gh}^k(y^a) = 0$, then $\tilde{c}_{gh}^k = c_{gh}^k$, $\tilde{c}_{hg}^k = c_{hg}^k$ and, therefore, $\tilde{v}_{gh}^k = \tilde{v}_{hg}^k = 0$. If $I_{gh}^k(y^a) > 0$, then arc $\{g,h\}$ must lie in a shortest path P in the all-inclusive network with arc cost c_{ij}^k . Moreover, this path has length $R_k(y^a) + I_{gh}^k(y^a)$ in the all-inclusive network with the costs \tilde{c}_{ij}^k . Consequently, by (2.3) arc $\{g,h\}$ must also lie in a shortest path in this network and, thus, either $\tilde{c}_{gh}^k + \tilde{u}_g^k - \tilde{u}_h^k = 0$ or $\tilde{c}_{hg}^k + \tilde{u}_h^k - \tilde{u}_g^k = 0$. Assume the former (the argument is the same in either case.) Then $\tilde{v}_{gh}^k = \tilde{u}_g^k - \tilde{u}_h^k - c_{gh}^k = I_{gh}^k(y^a) > 0$. Finally, note that this conclusion implies that $\tilde{v}_{hg}^k = 0$, for otherwise summing \tilde{v}_{gh}^k and $\tilde{v}_{hg}^k = \tilde{u}_h^k - \tilde{u}_g^k - c_{hg}^k > 0$ gives $-c_{gh}^k - c_{hg}^k > 0$, which is a contradiction. Note that whether $I_{gh}^k(y^a)$ equals zero or not, we may assume that $\tilde{v}_{gh}^k = I_{gh}^k$ and $\tilde{v}_{hg}^k = 0$.

Using all of these computed values for the \tilde{v}_{ij}^k and (2.3), now evaluate the dual objective function corresponding to \tilde{u}_i^k and \tilde{v}_{ij}^k . Its value is given by

$$\begin{aligned} \tilde{u}_{D(k)}^k - \tilde{u}_{O(k)}^k - \sum_{\{i,j\} \in A} (\tilde{v}_{ij}^k + \tilde{v}_{ji}^k) y_{ij}^a &= [R_k(y^a) + I_{gh}^k(y^a)] - \tilde{v}_{gh}^k y_{gh}^a \\ &= R_k(y^a) + I_{gh}^k(y^a) - I_{gh}^k(y^a) = R_k(y^a). \end{aligned}$$

Since this value agrees with the routing cost for the k th subproblem, the \tilde{u}_i^k and \tilde{v}_{ij}^k are optimal dual variables. \square

By Property 2, we can use $\{\tilde{u}_i^k, \tilde{v}_{ij}^k\}_{k \in K}$ to produce the Benders cut

$$z \geq \sum_{k \in K} [\tilde{u}_{D(k)}^k - \tilde{u}_{O(k)}^k] - \sum_{\{i,j\} \in A} (\tilde{v}_{ij}^k + \tilde{v}_{ji}^k) y_{ij}^a + \sum_{\{i,j\} \in A} F_{ij} y_{ij}^a$$

or

$$\begin{aligned} z \geq \sum_{k \in K} R_k(y^a) + \sum_{\{i,j\} \in A} \sum_{k \in Q_{ij}^k} I_{ij}^k \\ - \sum_{\{i,j\} \in A} \sum_{k \in Q_{ij}^k} I_{ij}^k y_{ij}^a + \sum_{\{i,j\} \in A} F_{ij} y_{ij}^a \end{aligned}$$

which, upon substituting $y_{ij}^c = 1 - y_{ij}^a$, is exactly (2.2). So the lower bound inequality proposed by Gallo and Magnanti and Wong can be viewed as a Benders cut.

Hoang [1973] has proposed a cut that is an intermediary between (2.1) and (2.2); it is obtained by setting $Q_{O(k), D(k)} = \{k\}$ for all $k \in K$.

Therefore, the set Q_{ij}^k partitions the commodities into singletons and

$I_{ij}^k(y^a)$ for $i = O(k)$ and $j = D(k)$ corresponds to the incremental cost between

the origin and destination of commodity k if we eliminate the possibility of single-arc direct routing. Here, we have assumed that

$O(k_1) = D(k_2)$ and $D(k_1) = O(k_2)$ is never true; otherwise, we would set

$Q_{O(k_1), D(k_2)} = \{k_1, k_2\}$ and the interpretation would be similar.

Figure 2.1 illustrates an example of the usual and improved Benders cuts we have discussed. In this example, one unit of flow is to be sent from node 1 to node 3 and another unit from node 1 to node 4. The set A consists of all arcs drawn in the figure. The arc labels specify the arc routing costs which are the same for both commodities. Let $k=3$ and $k=4$ refer to the two destination nodes of the commodities. The optimal dual variable values for u_i^k with $k=3$ or 4 have three different values corresponding to the usual, Hoang, and Gallo-Magnanti-Wong Benders cuts.

The first set of dual variables yields Boyce et al.'s usual cut, corresponding to (2.1)

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + [(2+5) = 7].$$

The second set of dual variables indicates that deleting arc $\{1,3\}$ from the network increases the routing cost between nodes 1 and 3 from $u_1^3 + 2 = 2$ to $u_2^3 = 11$ for a net increase of 9 units (i.e., $I_{13}^3 = 9$). Thus, Hoang's cut is

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + 7 + 9(1 - y_{13}).$$

The third set of dual variables corresponds to the Gallo-Magnanti-Wong cut (2.2) where commodity 3 belongs to Q_{13} and commodity 4 belongs to Q_{34} .

This cut provides the additional information that deleting arc $\{3,4\}$ from the network increases the routing cost between nodes 1 and 4 from $u_1^4 + 5 = 5$ to $u_4^4 = 9$ for a net increase of 4 units (i.e., $I_{34}^4 = 4$ and $I_{13}^3 = 9$). The resulting cut is

$$z \geq \sum_{\{i,j\} \in A} F_{ij} y_{ij} + 7 + 9(1-y_{13}) + 4(1-y_{34}).$$

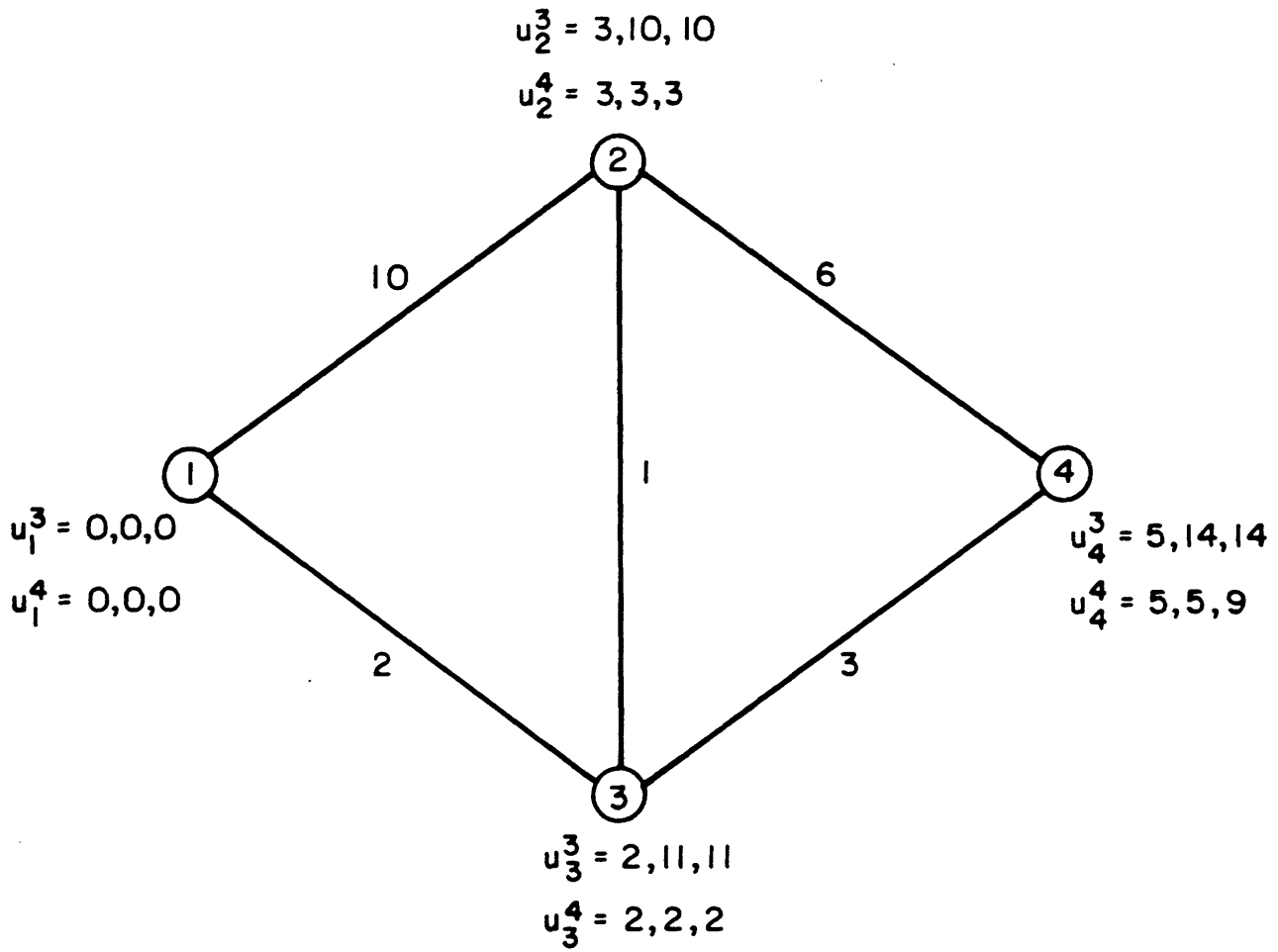


Figure 2.1 A Step of Benders Decomposition with Alternate Dual Solutions

This example illustrates the potential for strengthening the usual Benders cut by incorporating information about the penalties for deleting arcs in the current design. Although we discussed these ideas for only the specific design $y = y^a$, they can easily be adapted to any arbitrary candidate network.

2.2 Strong Benders Cuts for the Network Design Problem

In Section 1, we saw that the optimal dual variables for the usual Benders cut could be interpreted as upper bounds for estimating the decrease in network routing cost due to the addition of arcs not in the current design \bar{y} . The previous subsection demonstrated how the usual Benders cut could be improved by considering penalties caused by removing an arc already in the current design.

Next, we discuss another technique for strengthening the usual Benders cuts which utilizes more accurate estimates of the decrease in routing cost due to the addition of an arc. As before, we introduce this new Benders cut as an alternate optimal dual solution to the Benders subproblem.

Let $\{\hat{u}_i^k, \hat{v}_{ij}^k\}_{k \in K}$ be the dual variable values corresponding to the usual Benders cut for a candidate design $y = \bar{y}$. We define a "strong" Benders cut by specifying a new dual solution $\{\bar{u}_i^k, \bar{v}_{ij}^k\}_{k \in K}$ as follows:

$$\bar{u}_{D(k)}^k = \hat{u}_{D(k)}^k = \text{routing cost for commodity } k$$

$$\bar{u}_{0(k)}^k = \hat{u}_{0(k)}^k = 0$$

$$\Delta_i^k \equiv \bar{u}_{D(k)}^k - D_i^k$$

$$\bar{u}_i^k = \min(\hat{u}_i^k, \Delta_i^k) \quad i \neq 0(k) \text{ or } D(k)$$

$$\bar{v}_{ij}^k = \max\{0, \bar{u}_j^k - \bar{u}_i^k - c_{ij}^k\} \quad (2.4)$$

In the expression defining Δ_i^k , D_i^k denotes the minimum cost of routing one unit of commodity k from node i to node $D(k)$ on the all-inclusive network given by $y = y^a$.

As noted in Section 1, for the usual Benders cut with the dual variables $\{\hat{u}_i^k, \hat{v}_{ij}^k\}_{k \in K}$, we could interpret

$$\hat{v}_{ij}^k = \max(0, \hat{u}_j^k - \hat{u}_i^k - c_{ij}^k)$$

as an upper bound on the decrease in the routing cost for commodity k if arc $\{i, j\}$ is added to the current design $y = \bar{y}$.

Although the strong cut dual variables have a similar interpretation, they provide lower bound estimates that are no worse than, and usually improve upon, the ones provided by the usual cut. That is, every $\bar{v}_{ij}^k \leq \hat{v}_{ij}^k$.

Property 3: a) For every $k \in K$, $\{\bar{u}_i^k, \bar{v}_{ij}^k\}$ is an optimal solution for the dual to the k th subproblem $S_k(\bar{y})$.

b) $\bar{v}_{ij}^k \leq \hat{v}_{ij}^k$ and $\bar{v}_{ji}^k \leq \hat{v}_{ji}^k$ for all $\{i, j\} \in A$ and all $k \in K$.

Proof: We begin by proving $\bar{v}_{ij}^k \leq \hat{v}_{ij}^k$ in part b.

Case 1: Suppose $\bar{u}_i^k = \hat{u}_i^k$. Then since $\bar{u}_j^k \leq \hat{u}_j^k$, we have $\bar{u}_j^k - \bar{u}_i^k \leq \hat{u}_j^k - \hat{u}_i^k$

which by the definitions (1.11) and (2.4) of the \bar{v}_{ij}^k 's implies that

$$\bar{v}_{ij}^k \leq \hat{v}_{ij}^k.$$

Case 2: Suppose $\bar{u}_i^k = \Delta_i^k$. The triangle inequality implies that

$$D_i^k \leq D_j^k + c_{ij}^k$$

or

$$D_i^k - c_{ij}^k \leq D_j^k.$$

Consequently,

$$\Delta_j^k = \bar{u}_{D(k)}^k - D_j^k \leq \bar{u}_{D(k)}^k - (D_i^k - c_{ij}^k) = (\bar{u}_{D(k)}^k - D_i^k) + c_{ij}^k$$

and thus

$$\Delta_j^k \leq \Delta_i^k + c_{ij}^k .$$

This inequality and the fact that $\bar{u}_j^k \leq \Delta_j^k$ implies that

$$\bar{u}_j^k - \bar{u}_i^k \leq \Delta_j^k - \bar{u}_i^k = \Delta_j^k - \Delta_i^k \leq \Delta_i^k + c_{ij}^k - \Delta_i^k$$

$$\text{or } \bar{u}_j^k - \bar{u}_i^k \leq c_{ij}^k .$$

From (2.4), this last inequality implies that $\bar{v}_{ij}^k = 0$, and since $\hat{v}_{ij}^k \geq 0$,

we have $\bar{v}_{ij}^k \leq \hat{v}_{ij}^k$. A similar argument shows that $\bar{v}_{ji}^k \leq \hat{v}_{ji}^k$ and, thus,

completes the proof of part b.

In part a, substituting the values $\{\bar{u}_i^k, \bar{v}_{ij}^k\}$ into the dual of P_1 shows they are dual-feasible. By definition, $(\bar{u}_{D(k)}^k - \bar{u}_{0(k)}^k) = (\hat{u}_{D(k)}^k - 0)$ which is the optimal

objective value to the dual of $S_k(\bar{y})$. Now $\bar{y}_{ij} = 1$ implies that

$\bar{v}_{ij}^k = 0$ since $\bar{v}_{ij}^k \leq \hat{v}_{ij}^k$ by part (b) and $\hat{v}_{ij}^k = 0$ by the optimality

condition $c_{ij}^k + \hat{u}_i^k - \hat{u}_j^k \geq 0$ of the shortest path subproblem $S_k(\bar{y})$. A similar

argument shows that $\bar{y}_{ij} = 1$ also implies that $\bar{v}_{ji}^k = 0$. Therefore, the objective

function value of the dual solution $\{\bar{u}_i^k, \bar{v}_{ij}^k\}$ is $\hat{u}_{D(u)}^k$ and so the solution

is optimal as well as feasible.

Substituting $\{\bar{u}_i^k, \bar{v}_{ij}^k\}_{k \in K}$ into (1.10) yields

$$z \geq \sum_{k \in K} (\bar{u}_{D(k)}^k - \bar{u}_{0(k)}^k) - \sum_{\{i,j\} \in A} (\bar{v}_{ij}^k + \bar{v}_{ji}^k) y_{ij} + \sum_{\{i,j\} \in A} F_{ij} y_{ij}$$

or, since $\bar{u}_{D(k)}^k = \hat{u}_{D(k)}^k$ and $\bar{u}_{0(k)}^k = \hat{u}_{0(k)}^k = 0$,

$$z \geq \sum_{k \in K} (\hat{u}_{D(k)}^k - \hat{u}_{0(k)}^k) - \sum_{\{i,j\} \in A} (\bar{v}_{ij}^k + \bar{v}_{ji}^k) y_{ij} + \sum_{\{i,j\} \in A} F_{ij} y_{ij} .$$

Since $\bar{v}_{ij}^k \leq \hat{v}_{ij}^k$ and $\bar{v}_{ji}^k \leq \hat{v}_{ji}^k$ for all $\{i,j\}$, the strong cut will be no worse than the usual cut and will frequently dominate it.

Figure 2.2 depicts an example comparing the two types of cuts. The design problem is the same as the one in Figure 2.1 except that the only flow requirement is to route one unit between nodes 1 and 4. Assume that the current design consists of all arcs except for arc $\{2,3\}$.

The first set of dual variables corresponds to the usual cut

$$z \geq 5 - 7y_{23}.$$

The second set of dual variables leads to the strong cut

$$z \geq 5.$$

The usual cuts estimate that arc $\{2,3\}$ would belong to a shortest path between nodes 1 and 4 and potentially save $u_2^4 - u_3^4 - c_2^3 = 10 - 2 - 1 = 7$ units of cost. Looking ahead, though, from node 3 to 4 by computing the shortest path distance from every node to node 4, we "see" the high cost of arc $\{2,4\}$. Thus, by looking for additional costs "further down the road," we are able to produce a more accurate estimate of the value of adding arc $\{2,3\}$ to the network design.

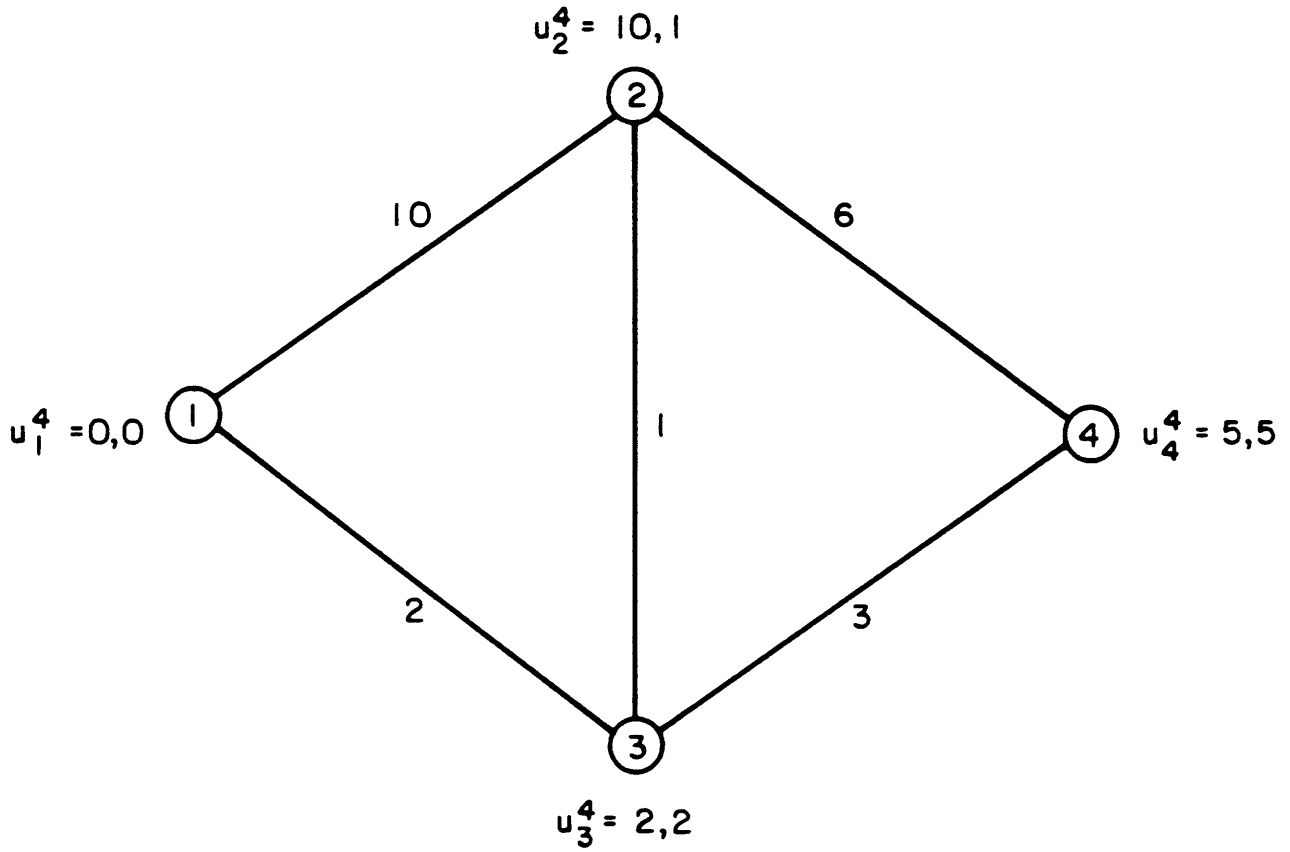


Figure 2.2 One Step of Benders Decomposition with Usual and Strong Cuts

2.3 Pareto-Optimal Cuts for Network Design Problems

To this point, we have seen several methods for generating Benders cuts for the network design problem that "improve" upon the usual ones in the sense that they are no worse than, and usually dominate, the usual cuts. In general, we would prefer to have cuts that are not dominated by any other cut. We call such undominated cuts Pareto-optimal.

Magnanti and Wong [1981] have introduced a general methodology for generating Pareto-optimal Benders cuts for any mixed integer programming problem. Their method conveniently specializes to the network design problem in the following way. Suppose that we have solved the dual to the subproblem $S_k(\bar{y})$ for the current design $y = \bar{y}$. Now consider the auxiliary linear program:

$$\begin{aligned}
 (AP_k(\bar{y})) : \quad & \text{maximize } (u_{D(k)}^k - u_{O(k)}^k) - \sum_{\{i,j\} \in A} (v_{ij}^k + v_{ji}^k) y_{ij}^0 \\
 & u_{D(k)}^k - u_{O(k)}^k - \sum_{\{i,j\} \in A} (v_{ij}^k + v_{ji}^k) \bar{y}_{ij} \geq (R_k \bar{y}) \\
 & u_j^k - u_i^k - v_{ij}^k \leq c_{ij} \quad \text{all } \{i,j\} \in A \\
 & u_i^k - u_j^k - v_{ji}^k \leq c_{ji} \quad \text{all } k \in K \\
 & v_{ij}^k, v_{ji}^k > 0
 \end{aligned}$$

where y^0 is a point in the relative interior of the convex hull of Y . Notice that $AP_k(\bar{y})$ is similar to the dual $S_k(\bar{y})$ except that it has a different objective function and an additional constraint that restricts the feasible region to the set of optimal solutions to this dual problem.

Suppose that we find an optimal solution for $AP_k(\bar{y})$. Magnanti and Wong proved that the corresponding cut is Pareto-optimal.

For the network design problem, Benders decomposition can produce Pareto-optimal cuts at the price of solving an additional $|K|$ auxiliary linear programs. In fact, it can do so by solving $|K|$ minimum cost flow problems. To demonstrate this fact, we form the linear programming dual of $AP_k(y)$.

$$\text{Min } \sum_{\{i,j\} \in A} (c_{ij}^k x_{ij} + c_{ji}^k x_{ji}) - R_k(\bar{y})x_0$$

subject to:

$$\sum_j x_{js} - \sum_j x_{sj} = -(1 + x_0) \quad \text{where } s = 0(k)$$

$$\sum_j x_{ji} - \sum_j x_{ij} = 0 \quad \text{for all } i \neq 0(k) \text{ or } D(k) \quad (2.5)$$

$$\sum_j x_{jt} - \sum_j x_{tj} = +(1 + x_0) \quad \text{where } t = D(k)$$

$$\left. \begin{aligned} x_{ij} &\leq y_{ij}^0 + x_0 \bar{y}_{ij} \\ x_{ji} &\leq y_{ij}^0 + x_0 \bar{y}_{ij} \end{aligned} \right\} \quad \text{for all } \{i,j\} \in A$$

$$x_{ij} \geq 0, \quad x_0 \geq 0.$$

In this formulation x_0 is the dual variable associated with the first constraint of problem $AP_k(\bar{y})$ and x_{ij} are dual variables associated with the other constraints. This reformulation shows that we are dealing with a parametric minimum cost flow problem in the scalar parameter x_0 which induces a variable demand of $(1 + x_0)$ units, a variable arc capacity of $y_{ij}^0 + x_0$ for the arcs in the current design (i.e., with $\bar{y}_{ij} = 1$), and a fixed arc capacity of y_{ij}^0 for the arcs not in the current design (i.e., with $\bar{y}_{ij} = 0$). We receive a rebate of $R_k(\bar{y})$ for each unit after the first one that is sent from $0(k)$ to $D(k)$ and must send a flow of at least one unit from $0(k)$ to $D(k)$. Once that unit is sent, if the marginal cost of sending

additional units is less than $R_k(\bar{y})$, there is incentive to increase the value of x_0 until the marginal cost becomes greater than or equal to $R_k(\bar{y})$. At this point, no further increase in the value of x_0 can improve the objective function of (2.5)

We claim that any value of $x_0 \geq \sum_{\{i,j\} \in A} y_{ij}^0$ is optimal in (2.5). To establish this fact, note that the total capacity $y_{ij}^0 + x_0 \bar{y}_{ij}$ for arcs not in the current design (those with $\bar{y}_{ij} = 0$) is at most $\sum_{\{i,j\} \in A} y_{ij}^0$. Also, each unit of the $(1+x_0)$ demand must flow along some path(s) from $O(k)$ to $D(k)$. At most $\sum_{\{i,j\} \in A} y_{ij}^0$ of these units can use path(s) containing an arc not in the current design defined by \bar{y} since each unit of flow along such paths must use at least one unit of capacity of arcs not in the current design. Any additional flow must use path(s) that contain arcs only in the current design and, therefore, their marginal cost will be $R_k(\bar{y})$.

So any value of $x_0 \geq \sum_{\{i,j\} \in A} y_{ij}^0$ must be optimal for (2.5).

By fixing $x_0 \geq \sum_{\{i,j\} \in A} y_{ij}^0$, we can solve (2.5) as a minimum cost flow

problem and the optimal dual variables will be an optimal solution for $AP_k(\bar{y})$.

Solving $|K|$ such minimum cost flow problems determines the coefficients $\{u_i^k, v_{ij}^k\}_{k \in K}$ for a Pareto-optimal Benders cut.

Note that like all of the procedures given in this section for generating improved cuts, the Pareto-optimal cut algorithm requires more computational effort than the usual cut algorithm. However, the improved cuts should accelerate the convergence of the master problem and thus decrease the overall computation time of Benders decomposition. Our computational results in Section 4 confirm this suspicion.

3. PROBLEM PREPROCESSING

In the past several years, researchers have made substantial progress in solving integer programs. Indeed, recent computational experience has demonstrated an important lesson--the synthesis of varied solution strategies often significantly extends solution capabilities and permits integer programming algorithms to solve large-scale applications. Moreover, the research community is witnessing a resurgence of several previously discarded methods such as cutting planes. This renaissance and the improved performance of these methods is attributable in part to the way that these methods are now being integrated with other solution approaches.

For example, Crowder and Padberg [1980], Crowder, Johnson, and Padberg [1983], Barany, Van Roy, and Wolsey [1983], and Martin and Schrage [1982] all have successfully solved large-scale integer programming problems by using cutting planes of the integer programming feasible region. These methods incorporate facet generating inequalities, logical inequalities, coefficient reduction, and/or variable elimination procedures within a branch and bound framework. Lemke and Spielberg [1967], Guignard and Spielberg [1981], and Guignard [1982], have made similar proposals, and Geoffrion and Marsten [1972] and Land and Powell [1979, 1981] have given integrating frameworks of integer programming methods and surveys of computational codes and computational experience.

For specific applications to the network design problem, Billheimer and Gray [1973] used methods for eliminating variables with a heuristic local improvement scheme.

In preliminary testing with pure Benders decomposition for the network design problem, we observed that the algorithm's performance was quite erratic

for larger-scale problems with more than 50 integer variables. This observation prompted us to consider preprocessing procedures that would

- (i) eliminate variables to reduce problem size;
- (ii) incorporate information from the linear programming relaxation for bounding purposes; and
- (iii) produce logical constraints that further restrict the integer variables.

Our preprocessing procedure works by first obtaining a feasible solution to the dual of the linear programming relaxation of the network design formulation (1.1)-(1.5). (In fact, the routine uses a slightly more complicated formulation of the network design problem adopted from Magnanti and Wong [1984c]. We will not discuss this modified integer programming formulation since it would unnecessarily encumber the presentation of our main ideas.)

Notice that any dual feasible solution (u,v) to the full linear programming relaxation is also a feasible solution to the dual for the Benders subproblem. Rewriting (1.10) using a dual feasible solution produces the Benders cut

$$z \geq a_0 + \sum_{\{i,j\} \in A} a_{ij} y_{ij} \quad (3.1)$$

where

$$a_0 = \sum_{k \in K} (u_{D(k)}^k - u_{O(k)}^k)$$

and

$$a_{ij} = F_{ij} - \sum_{k \in K} (v_{ij}^k + v_{ji}^k) \quad \text{for all } \{i,j\} \in A .$$

Inspection of (1.1) -(1.5) and the dual feasibility of (u,v) allows us to interpret a_0 as the dual objective function value of (u,v) and each a_{ij} as the nonnegative slack in the dual constraint corresponding to the variable y_{ij} .

We use a heuristic technique known as dual ascent to generate a dual feasible solution. The procedure exploits the simple form of the objective function and iteratively increases each $u_{D(k)}^k$ variable and adjusts the other variables in order to preserve feasibility. The algorithm terminates when it reaches a local optimum and no further change of a single $u_{D(k)}^k$ variable can improve the objective function. Note that by linear programming duality theory the dual objective function value a_0 is a lower bound for the value of the linear programming relaxation and consequently, for the value of the integer programming problem (1.1)-(1.5). We can also use the dual ascent solution to derive an upper bound a_1 for the optimal design cost. To do so, we form a candidate design consisting of all arcs $\{i,j\}$ whose dual slack a_{ij} is zero in the final solution. Then we improve this design by applying a simple drop-add local improvement heuristic to generate a feasible solution to the problem. The full details of this dual ascent-based algorithm for obtaining upper and lower bounds for the network design problem are rather complicated and will be given in a separate paper (Magnanti and Wong [1984c]).

A number of authors have successfully used dual ascent procedures to solve combinatorial optimization algorithms. The method has produced excellent computational results in solving large-scale problems for uncapacitated plant location (Bilde and Krarup [1977] and Erlenkotter [1978]), for plant location with side constraints (Guignard and Spielberg [1979]), for data base location (Fisher and Hochbaum [1980]), for generalized assignment (Fisher, Jaikumar and Van Wassenhove [1980]), for dynamic plant location (Van Roy and Erlenkotter [1982]), for asymmetric traveling salesman problems (Balas and Christofides [1981]), and for the Steiner tree problem on a graph (Wong [1984]). In keeping with the objectives of this study, we decided not to implement dual ascent as

a solution technique by itself, but rather to use it in a variable elimination routine for preprocessing.

The elimination routine works in the following way. We use the dual ascent-based algorithm to obtain upper and lower bounds a_1 and a_0 on the optimal objective value to the design problem as well as a Benders cut (3.1). Consider any arc $\{g,h\}$ with $a_{gh} > (a_1 - a_0)$. Then arc $\{g,h\}$ can be eliminated from the problem since no design with arc $\{g,h\}$ can be optimal. To see this, recall that the coefficients a_{ij} in the dual ascent-based Benders cut (3.1) are nonnegative. Therefore,

$$a_0 + a_{gh} > a_1$$

is a lower bound for the cost of any design containing arc $\{g,h\}$ and any design with arc $\{g,h\}$ is worse than the design corresponding to the upper bound a_1 .

The preprocessing routine also derives other information from the dual ascent solution. Let $S = \{\{i,j\} \in A : a_{ij} > (a_1 - a_0)/2\}$. Any optimal design solution must satisfy the multiple choice constraint

$$\sum_{\{i,j\} \in S} y_{ij} \leq 1 \quad . \quad (3.2)$$

Any solution containing two or more members $\{i_1, j_1\}$ and $\{i_2, j_2\}$ of S cannot be optimal since it has a cost of at least

$$a_0 + a_{i_1 j_1} + a_{i_2 j_2} > a_0 + (a_1 - a_0)/2 + (a_1 - a_0)/2 = a_1$$

Although several other inequalities like (3.2) are possible, we have limited our implementation to this multiple choice constraint.

The overall preprocessing routine works by applying the dual ascent-based algorithm to eliminate as many variables as possible. We then reapply the dual ascent routine to the reduced network design problem and try to eliminate additional variables. This process continues until no further variable elimination is possible and a minimum of 5 dual ascent iterations have been performed. The final dual solution also generates a Benders cut (3.1) and a

logical constraint (3.2). Notice how important it is for the ascent-based procedure to produce close upper and lower bounds in order for the variable elimination routine and the generated constraints (3.1) and (3.2) to be effective.

4. COMPUTATIONAL RESULTS

Our computational results tested three versions of Benders decomposition (with the usual cuts, strong cuts, and Pareto-optimal cuts) as well as the preprocessing procedures discussed in the last section. We undertook two sets of experiments:

- (1) using a pure Benders decomposition on a set of test problems with up to 45 0-1 variables; and
- (ii) using Benders decomposition with preprocessing on a set of test problems with up to 90 0-1 variables.

In each case, we implemented all algorithms in FORTRAN on a VAX 11/780 or a CYBER 76 computer. We solved master problems using a rudimentary linear programming-based branch and bound code (Land and Powell [1973]). The implementation used a state-of-the-art primal simplex code (Kennington and Helgason [1980]) to solve the minimum cost network flow problems required for generating Pareto-optimal cuts. We also used a naive implementation of Dijkstra's algorithm for finding shortest paths. Since most of the computation time is spent solving the master problem, a more efficient shortest path algorithm would not have produced any significant improvement in the overall computation time.

4.1 Benders Decomposition Without Problem Preprocessing

The first sets of experiments clearly demonstrated the superiority of both the strong and Pareto-optimal cuts over the usual cuts, though as the problem size grew, the increase in computational time led us to consider problem preprocessing.

Tables 1 and 2 specify the results for the first set of test problems. For these experiments, we used a problem generation procedure that selected nodes randomly from a 50x100 rectangle in the plane. The procedure generated the arc set A by randomly choosing arcs from the set $\{\{i,j\}: i \in N, j \in N\}$ so that each was equally likely to be chosen, using an efficient procedure discussed in Knuth [1969, section 3.4.2]. In addition, for some of the tests we discarded arcs whose distance exceeded a maximum distance problem parameter. Although this procedure does not guarantee feasibility, all of the test problems generated were feasible.

Arc Costs and Commodity Types

We selected the routing cost c_{ij} for each arc $\{i,j\}$ in several ways. In some cases, we chose the routing costs randomly using a uniform random variable $U(a,b)$ defined on the interval $[a,b]$. In other cases, we let the cost on an arc be equal to its Euclidean length, independent of commodity type. The fixed cost F_{ij} for arc $\{i,j\}$ was either a constant times c_{ij} or a constant minus c_{ij} . Thus, they were either proportional or inversely related to the routing costs. The set of commodities K consisted either of all origin-destination pairs or of all origin-destination pairs originating from two arbitrarily selected nodes.

Arcs That Are In Every Network Design

For all of the test problems, we modified the fixed costs in order to ensure that some arcs remain in all candidate designs. For these types of problems, which can be viewed as network improvement problems, Benders decomposition encounters fewer infeasible candidate solutions, and thus concentrates on cost trade-offs. The procedure specifies a set X of arcs from A by random selection, with the number #DARCS of these arcs fixed as a problem

parameter. It then redefines the fixed costs of the remaining (#ARC - #DARCS) arcs at value zero. Since all such arcs can trivially be included in an optimal design, the number of actual 0-1 variables becomes #DARCS.

Tables of Computational Results

In tables 1 and 2 the entry for each test problem is grouped together into a set of rows headed by the problem name DATAXX. The first row with the entry DATAXX describes the number of nodes, arcs and commodities and the optimal solution value for the problem. The succeeding rows summarize computational experience on the application of Benders decomposition with a particular cut type.

The column headed #ITER specifies the number of times the master problem was solved for any problem. Of these, in #ITER1 iterations, the master problem was modified by the addition of feasibility cuts. The columns headed #CUTS and #FCUTS give the total number of cuts (objective function plus feasibility) used to solve the problem. Notice that #FCUTS exceeds #ITER1 since some iterations added more than one feasibility cut to the master problem (since the network had more than one O-D pair that was disconnected). The terms CPU and MAS% describe the total CPU time and the percentage of CPU time spent on solving the master problems. For those cases when Benders algorithm did not identify an optimal solution, UB and LB give the best upper and lower bounds obtained and %DIF = $[(UB-LB)/UB] \cdot 100\%$ gives the percentage difference between the two bounds. For problems that terminated prematurely without a verified optimal solution, the letter after the CPU time indicates whether termination was due to a computer time limit (T), or due to a numerical instability and/or a choice of program parameters (e.g., maximum number of reinversions) in the Land and Powell integer programming routine that led to looping (L) in the algorithm or caused the system to terminate abnormally (A).

Looping occurs when the master problem continues to generate the same nonoptimal solution (i.e., the new cut does not eliminate the nonoptimal solution that generated it). The system terminated abnormally when it generated a lower bound on the objective function that exceeded the upper bound.

Every test problem has a COST ASSUMPTION entry that summarizes the way in which we generated the variable and fixed costs for the arcs.

Notice from Tables 1 and 2 that the problem sizes for these test problems ranged from #DARCS = 35 to #DARCS = 45 binary variables and from 2 (#ARCS) • $|K| = 810$ to 105,600 continuous variables. (The factor of 2 is included in the last count because there are two flows x_{ij}^k and x_{ji}^k for each commodity k on each undirected arc $\{i,j\}$.)

Interpreting The Computational Results

Notice from the column entitled CPU in Tables 1 and 2 that the usual cut could solve only 3 of the 16 test problems in this first group within our time limit of about 2 hours. In fact, at termination the percent difference (%DIF) between the best lower bound and best feasible solution generated by the usual cuts was as much as 20% for over a quarter of the problems. On the other hand, the strong cuts solved all these problems within the time limit (averaging 963 seconds on problems DATA01 - DATA10, and 1264 seconds on problems DATA11 - DATA13A) and the Pareto-optimal cuts solved all but one of these test problems within this time limit (averaging 128 seconds on problems DATA01 - DATA10, and 1511 seconds on problems DATA11 - DATA13A). For most of the test problems, the Pareto-cut implementation required fewer cuts (objective function cuts #CUTS plus feasibility cuts #FCUTS) than the strong cut implementation, though at the expense of larger computation time for generating each cut. In none of the cases did the usual cut outperform the other two cut strategies in solution time or number of cuts. Finally, note from the column entitled MAS% that solving the master problem consumed about 100%, 88%, and 54% of the solution time of the usual, strong, and Pareto cuts in the test problems DATA01 - DATA10.

4.2 Benders Decomposition with Problem Preprocessing

For our second set of computational experiments, we tested Benders decomposition with our problem preprocessing algorithm on the previous set of 16 problems (DATA1 to DATA13A) and on a second set of test problems that contained 90 binary variables, except for three cases that had 45 binary variables. The number $2(\#\text{ARCS}) + K$ of continuous variables was $4(\#\text{ARCS})(\#\text{NODES}-1)$ since the problems had 2 source nodes each with a commodity destined for every other node. For these test problems, the number of continuous variables ranged from 6,700 to 15,080.

For the second set of test problems, we generated the node set, arc set, and set of arcs with zero fixed costs as before. We generated arc routing costs and fixed costs in several different ways.

Arc Costs

In defining arc costs, we divided the second set of test problems into several groups. For the first group of these problems (DATA14 to DATA18), we generated the routing and fixed costs for each arc by sampling from a uniform $U(50,150)$ distribution and a uniform $U(500,1000)$ distribution, respectively. Problems 19 and 20 also used uniform distributions for both the routing and fixed costs. The third group of test problems (DATA21 to DATA26) used arc routing costs drawn from a $U(100,200)$ distribution and fixed costs equal to fixed constant times the arc routing cost. For the fourth group (DATA28 to DATA31), we let

$$\text{arc routing cost} = 10 (\text{arc length}) + b_1$$

and

$$\text{arc fixed cost} = 2(\text{routing cost}) + b_2$$

where b_1 and b_2 are values drawn from a $U(50,150)$ and a $U(-100,100)$ distribution, respectively.

The remaining problems were generated by procedures that are variants of those used for these four groups of problems.

Tables of Computational Results

Tables 3 and 4 document the computational results of Benders algorithm for our two sets of test problems. These tables have the same layout as Tables 1 and 2 except for the additional results on the dual-ascent-based preprocessing routine. Tables 3 and 4 specify the preprocessing computation time, CPU, the number #RARCS of integer variables remaining after application of the preprocessing, and #MCHU the number of variables present in the logical multiple choice constraint (3.2). The columns headed UB and LB specify the best upper and lower bounds produced by the dual ascent-based algorithm. The column %DIF indicates the percentage difference $[(UB-LB)/UB \cdot 100\%]$ between these two bounds. Notice that when $UB = LB$ for the dual ascent routine, it found and verified an optimal solution. For these cases, the use of Benders decomposition was not necessary. When $UB \neq LB$ from the dual ascent, we applied Benders decomposition using the usual strong or pareto-optimal cuts.

Interpreting the Computational Results

Table 4 shows that Benders decomposition with preprocessing was able to solve to optimality 19 out of the last group of larger 24 test problems and to find solutions to 23 of the problems that are guaranteed to be within at most 1.44% of optimality. For one particularly difficult problem, the Benders algorithm did not improve upon the bounds generated by the dual ascent procedure, which found a solution guaranteed to be within 5.53% of optimality.

For seven of the problems (15,17,18,20,28,32, and 34), the preprocessing routine itself found and verified an optimal solution. For the remaining problems, the processing routine was usually able to eliminate 75 to 95% of a problem's zero-one variables.

For the 17 of these problems that used Benders decomposition, the strong cut version clearly dominated the one with usual cuts. For the 12 problems when the strong Benders cut implementation converged within our time limit, it was as fast as the usual cut implementation in one case, at least twice as fast in every other case, and as much as 145 times as fast (even without counting the additional time that would be required for the usual cut implementation to converge.) For the five problems where Benders algorithm using strong cuts terminated prematurely due to time limitations, the strong cuts showed a moderate advantage.

The Pareto-optimal cut version was usually more effective for the more difficult problems. For the five problems that terminated prematurely, the Pareto-optimal cuts performed better than either the strong or usual cuts. For the other 13 problems, the strong cuts out-performed the Pareto-optimal cuts, but the Pareto-optimal cuts improved considerably upon the usual cuts. Recall that generating Pareto-optimal cuts requires the solution of $|K|$ minimum cost network flow problems while generating strong cuts requires the solution to only $|O|$ shortest path tree problems (where $O = \{i | i=0(k), k=1 \dots K\}$ is the set of nodes that are the origin of at least one commodity). Therefore, the Pareto-optimal cut strategy might be expected to require more solution time than the strong cut strategy. On the other hand, the Pareto cuts seem to generate somewhat better cuts and this improvement seems to be more pronounced for the more difficult problems.

We might also note that the preprocessing routine procedures are effective in reducing problem sizes considerably. Indeed, the resulting reduced problems are as small as problems that could be solved by more straightforward branch and bound procedures. However, the reduced problems appear to be much more difficult than non-preprocessed problems of comparable size (e.g., the problems in our first set of test problems).

Comparing Tables 1 and 2 with Table 4 highlights this fact. Table 3 also shows that the dual ascent procedure is very effective in solving smaller-sized problems even without using preprocessing to eliminate variables. The overall performance of the algorithm is very encouraging, especially since the test problems were generated with such a wide range of cost data relationships. These results indicate that Benders decomposition with problem preprocessing could be an effective method for solving a broad class of large-scale network design problems.

Finally, we note, as reported in Table 4, that solving the master problem consumed a large proportion of the solution time for every implementation of Benders decomposition that we tested. In particular, it consumed over 90% of the computation time for the five most difficult problems--numbers 29, 30, 35, 36, and 37 (an appendix gives the data for these problems). Consequently, solving the master integer programming problem with a commercial branch and bound code or by a specialized implicit enumeration algorithm could conceivably have dramatic effects upon the algorithm's performance, and might enable the procedures described in this paper to solve very large-scale applications.

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TABLE 1 -- USUAL, STRONG, AND PARETO CUTS *

NAME CUT TYPE	#NODES #ITER	#ARCS #ITER1	#DARCS #CUTS	#ODs #FCUTS	CPU	MAS%	OPT VAL UB LB	%DIF	COST ASSUMPTIONS
DATA01	15	60	45	28			8258		
USUAL	73	2	75	4	7135T	??.??	8258 7551	9.56	V = U(50,150)
STRONG	10	2	12	4	10	64.17			F = 10*V
PARETO	7	2	9	4	12	13.33			
DATA02	15	60	45	28			7024		
USUAL	76	3	78	5	7005T	??.??	7024 5963	15.11	V = U(50,150)
STRONG	26	3	28	5	201	95.55			F = 10*V
PARETO	19	3	21	5	125	76.17			
DATA03	15	60	45	28			8444		
USUAL	75	2	76	3	7130T	??.??			V = U(50,250)
STRONG	19	2	20	3	64	88.31			F = U(500,1000)
PARETO	10	2	11	3	23	35.88			
DATA04	15	60	45	28			10071		
USUAL	61	0	61	0	7124T	??.??	10071 7749	23.06	V = U(50,250)
STRONG	25	0	25	0	240	95.80			F = U(500,1000)
PARETO	14	0	14	0	71	65.50			
DATA05	15	60	45	28			9683		
USUAL	66	1	66	1	6792T	??.??	9925 7093	28.53	V = U(100,200)
STRONG	38	1	38	1	829	98.29			F = 1000 - 2*V
PARETO	20	1	20	1	139	76.28			
DATA06	15	60	45	28			10858		
USUAL	55	2	56	3	6860T	??.??	11165 7695	31.08	V = U(100,200)
STRONG	60	2	61	3	6645	99.66			F = 1000 - 2*V
PARETO	26	2	27	3	724	93.81			
DATA07	10	45	35	18			3988		
USUAL	58	1	58	1	2514	99.84			V = U(50,150)
STRONG	12	1	12	1	11	81.86			F = 10*V
PARETO	8	1	8	1	8	24.47			
DATA08	10	45	35	18			4304		
USUAL	33	1	33	1	571	99.59			V = U(50,150)
STRONG	7	1	7	1	3	55.24			F = 10*V
PARETO	7	1	7	1	7	20.06			
DATA09	15	60	45	28			23808		
USUAL	63	2	64	3	7086T	??.??	24046 20738	13.76	V = 10*D + U(0,50)
STRONG	25	2	26	3	170	94.35			F = 2*V + U(250,750)
PARETO	16	1	16	1	73	62.03			
DATA10	15	60	45	28			22259		
USUAL	65	1	65	1	7134	??.??			V = 10*D + U(0,50)
STRONG	44	1	44	1	1453	98.73			F = 2*V + U(250,750)
PARETO	18	1	18	1	102	69.91			

COMPUTATIONS (IN SECONDS) ON A VAX 11/780
T = TIME LIMIT

*EACH GROUP OF ROWS SPECIFIES RESULTS FOR A SINGLE PROBLEM. THE FIRST ROW GIVES THE PROBLEM DESCRIPTION IN THE FORMAT OF THE FIRST ROW IN THE HEADING. THE REMAINING ROWS WITHIN EACH GROUP GIVE THE COMPUTATIONAL RESULTS IN THE FORMAT OF THE SECOND ROW OF THE HEADING.

TABLE 2 -- VARYING NUMBER OF OD PAIRS *

NAME CUT TYPE	#NODES #11ER	#ARCS #11ERI	#DARCS #CUTS	#ODS #FCUTS	CPU	MAS%	OPT VAL		%DIF	COST ASSUMPTIONS
							UB	LB		
DATA11	10	45	35	18	7090T	?? ??	1465	1289	12.01	V = D F = 110 - V
USUAL	95	0	95	0	173	97.60	1443			
STRONG	23	0	23	0	42	76.07				
PARETO	13	0	13	0						
DATA11A	10	45	35	45	7001T	?? ??	3459	2662	23.04	V = D F = 110 - V
USUAL	77	0	77	0	1599	99.14	3152			
STRONG	34	0	34	0	1176	95.36				
PARETO	29	0	29	0						
DATA12	15	75	45	28	6517	99.70	2048			V = D F = 110 - V
USUAL	135	0	135	0	4	18.78				
STRONG	5	0	5	0	11	5.17				
PARETO	5	0	5	0						
DATA12A	15	75	45	105	6985T	?? ??	7377	5906	19.94	V = D F = 110 - V
USUAL	74	0	74	0	492	90.44	6684			
STRONG	25	0	25	0	545	67.40				
PARETO	25	0	25	0						
DATA13	33	100	45	64	7115T	?? ??	5407	5115	5.40	V = D F = 110 - V
USUAL	126	1	126	1	28	25.05	5382			
STRONG	11	1	11	1	148	14.69				
PARETO	17	1	17	1						
DATA13A	33	100	45	528	7163T	?? ??	35229	33683	6.44	V = D F = 110 - V
USUAL	111	1	111	1	5290	82.49	36001			
STRONG	55	1	55	1	7144T	?? ??				
PARETO	68	0	68	0			35238	35171	0.19	

COMPUTATIONS (IN SECONDS) ON A CYBER 76

D = DISTANCE BETWEEN POINTS

T = TIME LIMIT

*EACH GROUP OF ROWS SPECIFIES RESULTS FOR A SINGLE PROBLEM. THE FIRST ROW GIVES THE PROBLEM DESCRIPTION IN THE FORMAT OF THE FIRST ROW IN THE HEADING. THE REMAINING ROWS WITHIN EACH GROUP GIVE THE COMPUTATIONAL RESULTS IN THE FORMAT OF THE SECOND ROW OF THE HEADING.

TABLE 3 -- BENDERS DECOMPOSITION WITH PREPROCESSING*

NAME	#NODES	#ARCS	#DARCS	#ODs			OPT VAL				
ASCENT			#RARCS	#MCHV	CPU		UB	LB	%DIF	COST	
CUT TYPE	#ITER	#ITERI	#CUTS	#FCUTS	CPU	MAS%	UB	LB	%DIF	ASSUMPTIONS	
DATA01 ASCENT	15	60	45 4	28 0	14		8258	8258	0.00	V = U(50, 150) F = 10*V	
DATA02 ASCENT	15	60	45 7	28 0	19		7024	6923	1.44		
USUAL	46	3	48	5	81	96.07				V = U(50, 150)	
STRONG	14	3	16	5	10	73.20				F = 10*V	
PARETO	18	3	20	5	36	38.78					
DATA03 ASCENT	15	60	45 6	28 1	15		8444	8360	0.99		
USUAL	28	2	29	3	24	92.18				V = U(50, 250)	
STRONG	8	2	10	3	7	54.92				F = U(500, 1000)	
PARETO	12	2	13	3	19	27.60					
DATA04 ASCENT	15	60	45 4	28 0	15		10071	10071	0.00	V = U(50, 250) F = U(500, 1000)	
DATA05 ASCENT	15	60	45 3	28 0	14		9683	9683	0.00	V = U(100, 200) F = 1000 - 2*V	
DATA06 ASCENT	15	60	45 4	28 0	14		10858	10858	0.00	V = U(100, 200) F = 1000 - 2*V	
DATA07 ASCENT	10	45	35 2	18 0	5		3988	3988	0.00	V = U(50, 150) F = 10*V	
DATA08 ASCENT	10	45	35 2	18 0	5		4304	4304	0.00	V = U(50, 150) F = 10*V	
DATA09 ASCENT	15	60	45 6	28 0	15		23808	23808	0.00	V = 10*D + U(0, 50) F = 2*V + U(250, 750)	
DATA10 ASCENT	15	60	45 6	28 0	14		22259	22259	0.00	V = 10*D + U(0, 50) F = 2*V + U(250, 750)	

COMPUTATIONS (IN SECONDS) ON A VAX 11/780

*EACH GROUP OF ROWS SPECIFIES RESULTS FOR A SINGLE PROBLEM. THE FIRST ROW GIVES THE PROBLEM DESCRIPTION IN THE FORMAT OF THE FIRST ROW IN THE HEADING. THE SECOND ROW GIVES THE RESULTS OF THE DUAL ASCENT USING THE FORMAT OF THE SECOND ROW OF THE HEADING. THE REMAINING ROWS, IF NEEDED, USE THE FORMAT OF THE THIRD ROW OF THE HEADING TO GIVE THE COMPUTATIONAL STATISTICS FOR COMPLETING THE OPTIMIZATION USING BENDERS' DECOMPOSITION.

TABLE 4 -- RENDERS DECOMPOSITION WITH PREPROCESSING*

NAME CUT TYPE	#NODES #ITER	#ARCS #ITER1	#DARCS #RARCS #CUTS	#DDs #ACUV #FCUTS	CPU	MAS%	OPT VAL		%DIF %DIF	COST ASSUMPTIONS
							UB UB	LB LB		
DATA14 ASCENT	30	130	90	58			19179			
USUAL	198	3	12	3	159		19179	19042	0.71	V = U(50,150)
STRONG	17	3	200	5	5764L	99.19	19179	19172	0.04	F = U(500,1000)
PARETO	26	3	19	5	50	69.87				
			28	5	239	33.27				
DATA15 ASCENT	30	130	90	58			17096			V = U(50,150)
			7	0	148		17096	17096	0.00	F = U(500,1000)
DATA16 ASCENT	25	120	90	48			12702			V = U(50,150)
USUAL	62	0	7	1	79		12702	12530	1.35	F = U(500,1000)
STRONG	15	0	52	0	170	93.93				
PARETO	19	0	15	0	22	50.81				
			19	0	113	18.47				
DATA17 ASCENT	25	120	90	48			15257			V = U(50,150)
			6	0	87		15257	15257	0.00	F = U(500,1000)
DATA18 ASCENT	25	120	90	48			13120			V = U(50,150)
			5	0	80		13120	13120	0.00	F = U(500,1000)
DATA19 ASCENT	25	120	90	48			13449			V = U(50,150)
USUAL	54	2	9	1	89		13493	13424	0.51	F = U(650,850)
STRONG	10	2	65	3	198	94.62				
PARETO	12	2	11	3	10	41.66				
			13	3	58	12.24				
DATA20 ASCENT	25	120	90	48			17484			V = U(50,250)
			5	0	76		17484	17484	0.00	F = U(650,850)
DATA21 ASCENT	25	120	90	48			23076			V = U(100,200)
USUAL	18	2	7	0	86		23076	23049	0.12	F = 10*V
STRONG	10	2	20	4	14	79.29				
PARETO	10	2	12	4	9	29.73				
			12	4	42	9.08				
DATA22 ASCENT	25	120	90	48			31890			V = U(100,200)
USUAL	152	4	20	7	90		31890	31270	1.94	F = 20*V
STRONG	18	5	155	7	6999T	?? ??	31890	31810	0.25	
PARETO	23	5	21	8	48	74.19				
			26	8	185	45.97				
DATA23 ASCENT	25	120	90	48			28310			V = U(100,200)
USUAL	30	1	6	1	88		28310	28117	0.68	F = 20*V
STRONG	11	1	30	1	26	81.44				
PARETO	16	1	11	1	11	32.49				
			16	1	85	10.50				
DATA24 ASCENT	25	120	90	48			28522			V = U(100,200)
USUAL	58	3	15	3	93		28522	27211	4.60	F = 20*V
STRONG	12	3	60	5	585	98.30				
PARETO	13	3	14	5	17	59.63				
			15	5	68	25.39				
DATA25 ASCENT	25	120	90	48			28641			V = U(100,200)
USUAL	18	2	9	1	97		28861	28316	1.89	F = 20*V
STRONG	3	2	20	4	15	79.88				
PARETO	12	2	10	4	7	25.19				
			14	4	56	10.57				
DATA26 ASCENT	25	120	90	48			29157			V = U(100,200)
USUAL	12	2	5	0	84		28157	28077	0.28	F = 20*V
STRONG	7	2	13	3	5	65.02				
PARETO	8	2	8	3	5	21.57				
			9	3	33	4.09				
DATA27 ASCENT	25	110	90	48			67315			V = U(25,325)
USUAL	37	10	15	2	94		67315	66721	0.88	F = 30*V + U(-100,100)
STRONG	18	10	51	24	194	97.41				
PARETO	24	9	32	24	39	80.66				
			38	23	176	60.32				

(TABLE CONTINUED)

TABLE 4 (CONTINUED)*

NAME	#NODES	#ARCS	#DARCS	#ODS	CPU	MASK	OPT VAL		%DIF	COST ASSUMPTIONS
							UB LB	LB LB		
DESCENT	ITER	ITERI	#PARCS	#MCHV						
OUT TYPE			#CUTS	#FCUTS						
DATA28	25	120	90	48				44118		V = 10*D + U(50,150)
ASCENT			12	0	113			44118 44118	0.00	F = 2*V + U(-100,100)
DATA29	25	120	90	48				?		V = 10*D + U(50,150)
ASCENT			42	8	147			47166 46691	1.01	F = 2*V + U(-100,100)
USUAL	131	0	131	0	3169L	98.96		47166 46691	1.01	
STRONG	27	0	27	0	1488A	98.22		47166 46691	0.89	
PARETO	28	0	28	0	2106A	91.97		47166 46768	0.84	
DATA30	25	120	90	48				?		V = 10*D + U(50,150)
ASCENT			23	2	131			51489 51313	0.34	F = 2*V + U(-100,100)
USUAL	199	2	200	3	3495L	98.84		51489 51313	0.34	
STRONG	57	2	58	3	4178A	??.??		51489 51356	0.26	
PARETO	51	2	52	3	4484A	94.00		51489 51356	0.26	
DATA31	25	120	90	48				43024		V = 10*D + U(50,150)
ASCENT			15	3	87			43024 42897	0.30	F = 2*V + U(-100,100)
USUAL	200	1	200	1	6911L	99.48				
STRONG	41	1	41	1	395	91.93				
PARETO	57	1	57	1	1122	74.23				
DATA32	25	100	45	48				42388		V = U(25,325)
ASCENT			2	0	96			42388 42388	0.00	F = 20*V + U(-100,100)
DATA33	25	70	45	48				63407		V = U(25,325)
ASCENT			18	2	92			63407 62254	1.82	F = 20*V + U(-75,75)
USUAL	172	4	176	8	7164T	??.??		63407 62418	1.56	
STRONG	29	4	33	8	205	89.16				
PARETO	17	4	21	8	99	51.03				
DATA34	25	70	45	48				43391		V = U(25,325)
ASCENT			9	0	68			43391 43391	0.00	F = 3000 - 10*V - U(0,150)
DATA35	25	105	90	48				?		V = U(30,90)
ASCENT			90	11	368			17381 16420	5.53	F = 10*V + U(100,200)
USUAL	74	20	97	43	2747A	??.??		17381 16420	5.53	
STRONG	43	33	75	65	7071T	??.??		17381 16420	5.53	
PARETO	51	39	82	80	7087T	??.??		17381 16420	5.53	
DATA36	25	120	90	48				?		V = U(60,140)
ASCENT			21	4	127			39422 39131	0.74	F = 1000 - V + U(-50,50)
USUAL	160	2	161	3	7103T	??.??		39422 39131	0.74	
STRONG	71	2	72	3	7031T	??.??		39422 39131	0.74	
PARETO	70	2	71	3	7141T	??.??		39422 39131	0.74	
DATA37	25	120	90	48				?		V = D + U(50,150)
ASCENT			45	21	137			19115 18677	2.29	F = 5*V + 5*D
USUAL	128	3	130	5	7171T	??.??		19115 18677	2.29	
STRONG	67	3	69	5	7008T	??.??		19115 18791	1.70	
PARETO	48	2	50	4	7051T	??.??		19115 18840	1.44	

COMPUTATIONS (IN SECONDS) ON A VAX 11/780

! = SYSTEM TERMINATED ABNORMALLY

! = SYSTEM LOOPED

! = TIME LIMIT

*EACH GROUP OF ROWS SPECIFIES RESULTS FOR A SINGLE PROBLEM. THE FIRST ROW GIVES THE PROBLEM DESCRIPTION IN THE FORMAT OF THE FIRST ROW IN THE HEADING. THE SECOND ROW GIVES THE RESULTS OF THE USUAL ASCENT USING THE FORMAT OF THE SECOND ROW OF THE HEADING. THE REMAINING ROWS, IF NEEDED, USE THE FORMAT OF THE THIRD ROW OF THE HEADING TO GIVE THE COMPUTATIONAL STATISTICS FOR COMPLETING THE OPTIMIZATION USING BENDERS' DECOMPOSITION.

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APPENDIX

THIS APPENDIX CONTAINS DATA FOR THE FIVE TEST PROBLEMS (29, 30, 35, 36, AND 37) THAT THE ALGORITHMS DESCRIBED IN THIS PAPER DID NOT SOLVE TO OPTIMALITY.

TABLE A.1 DATA FOR PROBLEM DATA29

V = 10*D + U(50,150)
 F = 2*V + U(-100,100)
 D ≤ 50
 # NODES = 25
 # ARCS = 120

ARC NODES		VARIABLE COST	FIXED COST	ARC NODES		VARIABLE COST	FIXED COST
1	3	418	909	10	17	293	634
1	4	263	494	10	18	432	953
1	5	469	919	10	20	546	1119
1	6	298	584	10	21	543	0
1	8	444	0	10	23	559	0
1	10	483	1042	11	12	342	0
1	11	465	839	11	15	336	729
2	3	460	849	11	17	369	0
2	4	273	483	11	19	424	841
2	5	477	0	11	23	512	1009
2	9	385	738	11	24	587	1082
2	10	474	1013	12	13	569	0
3	4	272	616	12	14	489	995
3	5	223	346	12	15	305	606
3	10	448	0	12	17	293	532
3	12	524	0	12	18	320	614
4	5	307	0	12	19	488	0
4	6	180	0	12	25	607	1309
4	7	353	729	13	18	606	1256
4	8	256	611	13	19	363	0
4	9	228	415	13	22	467	1011
4	10	309	519	14	15	419	856
4	11	346	0	14	16	483	1006
4	12	448	840	14	18	529	1053
4	14	573	0	14	21	536	978
4	16	563	1189	14	22	359	762
5	6	321	659	14	23	539	1156
5	7	190	326	14	24	417	0
5	8	216	408	14	25	565	1226
5	11	370	750	15	16	225	522
5	12	344	632	15	20	482	965
5	15	454	891	15	22	358	0
5	17	510	936	15	23	304	593
6	7	421	804	15	24	439	0
6	9	208	358	16	17	187	434
6	10	275	590	16	18	158	341
6	12	439	882	16	19	372	824
6	14	421	885	16	22	442	0
6	15	535	1142	16	23	351	0
7	8	271	631	16	25	468	1006
7	9	362	700	17	21	301	0
7	10	318	651	17	24	394	713
7	16	491	0	18	19	374	778
7	17	501	906	18	20	550	0
7	21	609	1245	18	21	203	309
8	9	224	0	18	22	424	821
8	11	291	0	18	23	237	425
8	18	479	961	18	24	426	841
9	10	243	438	18	25	372	679
9	12	382	727	19	20	266	0
9	13	333	641	19	21	360	817
9	14	359	0	19	22	176	388
9	17	474	883	19	25	392	0
9	18	482	970	20	25	494	932
9	19	518	996	21	22	382	669
9	20	537	0	21	23	162	349
9	21	612	1254	21	24	393	783
10	11	200	334	22	25	337	664
10	13	383	770	23	24	291	483
10	15	314	0	23	25	247	544

TABLE A.2 DATA FOR PROBLEM DATA30

V = 10*D + U(50, 150)
 F = 2*V + U(-100, 100)
 D ≤ 50
 # NODES = 25
 # ARCS = 125

ARC NODES		VARIABLE COST	FIXED COST	ARC NODES		VARIABLE COST	FIXED COST
1	2	129	209	9	24	593	1126
1	5	278	573	10	11	119	306
1	12	581	0	10	12	313	680
1	13	597	1216	10	14	255	539
2	3	239	450	10	16	410	915
2	4	482	948	10	21	547	0
2	5	257	0	11	12	367	748
2	7	499	1056	11	13	419	851
2	12	585	1235	11	14	248	539
2	13	576	0	11	17	313	671
3	5	303	530	11	22	518	1054
3	7	376	841	12	15	538	1164
3	8	468	1033	12	17	387	757
3	9	573	1069	12	18	274	636
3	10	432	873	12	20	556	0
3	11	444	0	12	21	497	1086
3	12	539	990	12	22	525	0
4	7	487	946	12	23	506	1039
4	8	478	860	12	24	572	1054
4	10	419	0	13	14	568	0
4	11	520	1070	13	15	623	0
4	14	549	1187	13	16	289	663
5	6	365	0	13	17	407	0
5	7	508	1089	13	18	342	0
5	9	413	903	13	19	450	0
5	12	455	892	14	16	469	841
5	16	557	1152	14	18	365	737
6	9	377	755	14	22	489	1003
6	10	274	470	14	24	510	981
6	13	334	649	14	25	516	997
6	14	425	797	15	16	522	0
6	15	540	1146	15	17	308	672
6	17	464	892	15	20	268	531
6	18	483	867	15	21	521	999
6	19	515	998	15	22	436	802
7	10	193	327	15	23	590	0
7	12	259	474	15	24	530	1016
7	13	419	910	16	18	191	340
7	14	377	659	16	21	342	615
7	18	364	655	16	22	437	814
7	21	606	0	16	23	336	683
8	9	434	0	16	25	637	1227
8	12	309	0	17	18	155	361
8	15	342	0	17	20	311	678
8	17	402	0	17	22	368	797
8	18	352	0	17	23	340	775
8	19	449	914	17	24	352	753
8	20	540	1009	17	25	453	965
8	21	593	0	18	19	286	635
8	22	592	1201	18	20	390	0
9	10	453	962	18	22	299	559
9	11	383	689	18	23	345	748
9	12	212	0	18	25	552	1089
9	13	122	216	19	23	307	596
9	14	559	0	20	23	478	0
9	16	245	416	20	24	446	798
9	17	395	744	21	23	90	226
9	18	359	0	22	25	287	504
9	19	466	900	23	24	95	0
9	22	637	0	24	25	401	797

TABLE A.3 DATA FOR PROBLEM DATA35

V = U(30,90)
 F = 10*V + U(100,200)
 # NODES = 25
 # ARCS = 105

ARC NODES	VARIABLE COST	FIXED COST	ARC NODES	VARIABLE COST	FIXED COST
1 2	39	578	13 16	52	703
1 4	61	0	13 17	67	771
1 5	86	1023	13 18	53	684
1 6	72	900	13 20	69	864
1 7	64	803	13 21	67	770
1 9	60	707	13 22	66	839
2 4	46	0	13 24	73	843
2 5	66	806	14 15	65	821
2 6	45	629	14 18	52	707
2 7	47	0	14 19	76	0
2 8	85	1015	14 20	56	687
2 9	81	923	14 21	47	589
3 4	57	686	14 22	71	815
3 5	49	600	14 23	78	975
3 6	65	785	14 24	80	954
3 7	83	0	14 25	64	782
3 8	59	756	15 18	72	837
3 10	74	911	15 19	59	757
3 11	77	911	15 21	71	836
4 5	68	856	15 23	43	0
4 6	50	0	15 25	59	725
4 7	56	745	16 17	52	639
4 8	67	783	16 18	70	0
4 9	66	803	16 20	50	0
5 6	57	766	16 21	77	933
5 7	63	812	16 22	71	0
5 8	48	582	16 24	60	774
5 10	81	955	17 18	76	893
5 11	60	711	17 20	54	677
6 7	40	555	17 21	63	746
6 8	64	761	17 22	61	797
6 9	72	845	17 24	50	641
7 8	60	700	18 19	70	874
7 9	50	610	18 20	58	0
7 13	81	1004	18 21	50	0
8 10	59	699	18 22	47	654
8 11	61	779	18 24	66	775
8 12	74	844	18 25	58	764
9 13	60	0	19 21	70	896
9 16	79	939	19 23	34	527
9 17	77	967	19 25	70	863
10 11	52	715	20 21	65	779
10 12	53	0	20 22	38	544
10 14	80	966	20 24	41	572
10 15	69	875	21 22	48	615
11 12	56	738	21 23	65	808
11 14	68	800	21 24	65	847
11 15	70	895	21 25	63	782
12 14	73	0	22 24	49	590
12 15	72	857	22 25	85	982
12 18	80	946	23 25	56	668
12 19	67	797	24 25	75	923
13 14	84	1023			

TABLE A.4 DATA FOR PROBLEM DATA36

V = U(60,140)
 F = 1000 - V + U(100,200)
 # NODES = 25
 #ARCS = 125

ARC NODES	VARIABLE COST	FIXED COST	ARC NODES	VARIABLE COST	FIXED COST
1 3	128	0	12 19	267	787
1 6	257	810	12 20	211	847
2 3	61	955	12 22	332	0
2 4	221	0	12 23	366	717
2 5	217	875	13 14	222	859
2 6	251	845	13 16	96	0
3 4	213	877	13 17	129	955
3 6	202	0	13 20	202	847
3 7	348	676	13 21	357	651
4 6	167	846	13 22	318	715
4 7	211	799	13 23	327	0
4 8	273	813	13 24	344	0
4 9	386	634	13 25	360	715
4 10	393	619	14 15	257	0
5 8	262	809	14 16	144	913
6 7	140	876	14 17	266	760
6 8	220	855	14 18	128	962
6 9	330	715	14 22	366	693
6 10	255	771	14 24	295	0
6 11	279	792	15 16	112	902
6 13	380	652	15 18	303	739
6 14	376	0	15 19	184	0
7 8	82	979	15 20	178	914
7 12	298	763	15 21	354	738
7 13	282	0	15 22	286	0
7 14	365	718	15 25	334	0
7 15	317	708	16 17	134	920
7 16	336	0	16 18	193	817
8 9	106	0	16 19	104	897
8 11	211	839	16 20	202	0
8 13	242	0	16 21	264	804
8 15	273	790	16 22	271	751
8 16	313	0	16 23	268	802
8 17	366	0	16 24	272	794
9 10	241	855	17 19	149	876
9 11	230	776	17 20	89	966
9 12	153	866	17 21	300	0
9 13	205	842	17 22	195	825
9 16	298	778	17 23	218	867
9 17	290	710	17 24	262	758
9 19	382	675	18 19	146	895
10 14	159	854	18 21	113	971
10 16	151	881	18 22	302	743
10 17	260	773	18 23	242	0
10 18	270	0	18 24	194	0
10 19	257	772	19 21	177	832
10 20	344	722	19 23	166	0
10 21	371	692	19 24	165	847
11 12	218	0	19 25	277	749
11 15	176	861	20 21	314	742
11 16	134	915	20 22	127	934
11 19	239	781	20 23	171	870
11 20	322	690	20 24	240	0
11 21	362	655	20 25	163	0
11 22	403	0	21 22	253	801
11 24	398	628	21 23	188	888
12 13	103	996	21 24	100	952
12 15	80	958	22 23	89	0
12 17	143	939	22 24	165	911
12 18	388	691	24 25	230	855

TABLE A.5 DATA FOR PROBLEM DATA37

V = D + U(50, 150)
 F = 5*V + 5*D
 D < 50
 # NODES = 25
 # ARCS = 120

ARC NODES		VARIABLE COST	FIXED COST	ARC NODES		VARIABLE COST	FIXED COST
1	2	117	715	11	18	119	720
1	5	79	450	11	19	124	775
1	6	167	0	11	21	171	0
1	7	90	590	11	25	160	1020
1	8	91	0	12	13	164	940
1	9	128	820	12	14	89	495
2	3	114	0	12	16	88	590
2	6	107	590	12	17	129	755
2	7	154	850	12	20	168	965
2	8	98	610	12	22	126	830
2	10	143	935	12	23	150	0
2	11	150	990	13	14	182	1080
3	5	186	1125	13	15	69	405
3	6	129	685	13	16	95	530
3	7	72	440	13	18	148	0
3	8	81	510	13	19	138	910
3	11	128	870	13	20	159	955
4	5	72	370	13	22	158	950
4	6	96	0	13	23	148	950
4	8	159	0	14	15	108	755
4	10	151	950	14	17	151	895
4	11	125	860	14	18	91	610
5	6	144	895	14	19	83	0
5	7	79	535	14	22	163	1045
5	8	144	0	14	23	129	805
5	9	129	775	14	24	146	0
5	10	170	1040	14	25	144	0
5	13	134	890	15	19	181	0
6	9	97	660	15	20	108	710
6	11	140	890	15	25	115	780
6	12	99	695	16	17	86	505
7	8	60	345	16	18	109	620
7	9	169	0	16	19	113	0
7	10	145	860	16	20	173	1005
7	12	123	0	16	21	123	730
7	13	156	990	16	22	81	0
8	10	150	865	16	23	115	0
8	12	109	680	17	19	117	720
8	13	141	900	17	20	124	685
8	16	125	875	17	22	126	720
9	13	97	0	17	23	76	495
9	14	112	740	17	25	115	700
9	15	165	970	18	19	137	0
9	18	90	640	18	20	138	755
9	20	116	810	18	21	140	745
9	22	153	0	18	24	152	915
10	11	120	640	19	21	123	745
10	12	86	0	19	22	149	0
10	13	145	0	19	24	134	750
10	14	101	615	19	25	114	700
10	17	157	915	20	23	157	835
10	19	181	0	20	24	105	0
10	21	126	820	20	25	92	540
10	23	126	850	21	22	121	665
10	25	130	900	21	24	147	0
11	12	115	600	21	25	145	790
11	13	102	610	22	24	145	890
11	14	136	0	23	24	101	545
11	15	107	690	23	25	145	0
11	16	175	0	24	25	113	640