

TAME ARCS ON WILD CELLS

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ABSTRACT. We prove here that, for $n \geq 5$, every cell in E^n contains a tame arc and that, for product cells $B^{n-k} \times I^k \subset E^{n-k} \times E^k = E^n$, every k -dimensional polyhedron $P \subset B^{n-k} \times I^k$ is tame in E^n .

1. Introduction. Bing showed in [1] that every 2-cell in 3-dimensional Euclidean space contains a tame arc and in [2] that there is a 2-sphere that is wild but for which all subarcs are tame. We obtain here analogous results in higher dimensions (≥ 5). First we show that for $n \geq 5$, any subarc of any k -cell in E^n can be approximated by subarcs tame in E^n . Then we show that if C is any $(m-k)$ -cell in E^{n-k} , $I^k \subset E^k$ is the k -fold product of the unit interval I , $m \leq n-2$, and $n \geq 5$, then every sub k -cell of $C \times I^k \subset E^{n-k} \times E^k$ is tame in E^n . Since there are cells in this class of factored cells that are wild at every point [10] we have a generalization of Bing's example [2] to higher dimensions.

2. The approximation theorems. First we give a few definitions. Let $X \subset M$ be closed subsets of E^n . Let d denote the usual metric on E^n . A homeomorphism h of M is an ϵ -push of (M, X) if there is an isotopy h_t of M such that $h_0 = \text{Identity}$, $h_1 = h$, $d(h_t(x), x) < \epsilon$ for each $t \in I$ and each $x \in M$, and h_t is the identity outside the ϵ -neighborhood of X in M for each t . If P is a polyhedron and $h: P \rightarrow E^n$ is an embedding we say that h is tame if there is a homeomorphism H of E^n such that $H \cdot h$ is piecewise linear (PL).

LEMMA 1. *Suppose X is a compact subset of E^n , $\text{Int } X = \emptyset$, X does not locally separate E^n , G is a compact 1-dimensional subpolyhedron of E^n , $n \geq 4$, and $\epsilon > 0$. Then there is an ϵ -push h of $(E^n, G \cap X)$ such that $h(G) \cap X = \emptyset$.*

PROOF. The proof is an immediate consequence of general position and Corollary 5.6 of [3].

LEMMA 2. *Suppose X is a compact subset of E^n , $\text{Int } X = \emptyset$, X does not locally separate E^n , P is a 2-dimensional subpolyhedron of E^n , $n \geq 4$,*

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and $\epsilon > 0$. Then there is an ϵ -push h of $(E^n, P \cap X)$ such that $h(P) \cap X$ is totally disconnected.

PROOF. Let K be a triangulation of P and $\{K_i | i = 1, 2, \dots\}$ the sequence of i th derived barycentric subdivisions of K . We shall use Lemma 1 to construct an ϵ -push h of $(E^n, P \cap X)$ such that $h(\bigcup |K_i^1|) \cap X = \emptyset$. Clearly h then satisfies the conclusion of Lemma 2.

Let $\epsilon_1 = \epsilon/2$ and apply Lemma 1 with (X, G, ϵ) replaced by $(X, |K_1^1|, \epsilon_1)$, obtaining an ϵ_1 -push h_1 of $(E^n, |K_1^1| \cap X)$ such that $h_1(|K_1^1|) \cap X = \emptyset$. Let $\delta_1 = d(h_1(|K_1^1|), X)$ and η_1 be some positive number chosen depending on h_1 . (See [8] or Theorem 3.4 of [7].) Set $\epsilon_2 = \min\{\epsilon_1/2, \delta_1/2, \eta_1\}$. As before we obtain an ϵ_2 -push h_2' of $(E^n, h_1|K_2^1| \cap X)$ such that $h_2' \cdot h_1(|K_2^1|) \cap X = \emptyset$. Set $h_2 = h_2' \cdot h_1$. Continuing in this way we obtain a sequence $\{h_i\}$ of homeomorphisms of E^n . Since $\epsilon_{i+1} < \epsilon_i/2$, $\lim_{i \rightarrow \infty} h_i = h$ is an ϵ -map of E^n supported on a compact set. Because $\epsilon_{i+j} \leq \delta_i/2^j$, $h(\bigcup |K_i^1|) \cap X = \emptyset$ and because the η_i are chosen sufficiently small depending on the h_i , h is a homeomorphism. Thus h is the required ϵ -push of (E^n, X) .

THEOREM 3. Let M be a PL m -manifold topologically embedded in E^n , G a 1-dimensional subpolyhedron of M , G' a subpolyhedron of G that is tame in E^n , $n \geq 5$, and $m \geq 2$. Then for each $\epsilon > 0$ there is an ϵ -embedding $\alpha: G \rightarrow M$ such that α is tame in E^n and $\alpha|_{G'} = \text{inclusion}: G' \rightarrow E^n$.

PROOF. It is sufficient to consider the case that M is a 2-cell, G is an arc, and $G' = \{\text{end points of } G\}$. Using Lemma 2 we shall construct an embedding $\alpha: G \rightarrow M$ fixed on G' such that $E^n - \alpha(G)$ is uniformly locally 1-connected (1-ULC). By Theorem 4.2 of [4], α is thus tame.

Let K be a triangulation of E^n and K_1, K_2, \dots the sequence of i th derived barycentric subdivisions of K . It follows from general position and Lemma 2 that there is an $\epsilon/2$ -push h_1 of $(E^n, G \cap |K_1^2|)$ such that $h_1(|K_1^2|) \cap M$ is totally disconnected and $h_1(|K_1^2|) \cap G' = \emptyset$. Thus there is an $\epsilon/2$ -embedding $\alpha_1: G \rightarrow M$ fixed on G' such that $\alpha_1(G) \cap h_1|K_1^2| = \emptyset$. Now as in the proof of Lemma 2 we set $\epsilon_1 = \epsilon/2$, $\delta_1 = d(\alpha_1(G), h_1(|K_1^2|))$, $\eta_1 > 0$ chosen depending on α_1 , $\eta_1' > 0$ chosen depending on h_1 , $\epsilon_2 = \min\{\epsilon_1/2, \delta_1/4, \eta_1\}$, and $\epsilon_2' = \min\{\epsilon_1/2, \delta_1/4, \eta_1'\}$. Then using Lemma 2 we find an ϵ_2' -push h_2' of $(E^n, h_1(|K_2^2|) \cap M)$ such that $h_2' \cdot h_1(|K_2^2|) \cap M$ is totally disconnected. Let $h_2 = h_2' \cdot h_1$. We can again find an embedding $\alpha_2: G \rightarrow M$ such that $d(\alpha_1, \alpha_2) < \epsilon_2$, $\alpha_2(G) \cap h_2(|K_2^2|) = \emptyset$, and α_2 is fixed on G' . Continuing in this way we construct a sequence $\alpha_i: G \rightarrow M$ of ϵ_i -embeddings and a sequence $\{h_i\}$ of homeomorphisms of E^n . Because $\epsilon_{i+1} \leq \epsilon_i/2$, the α_i converge to an ϵ -map $\alpha: G \rightarrow M$. The η_i can be picked so as to guarantee that α is an

ϵ -embedding. Similarly the h_i converge to an ϵ -push of (E^n, M) . Because $\max\{\epsilon_{i+j}, \epsilon'_{i+j}\} \leq \delta_i/2^{j+1}$, $\alpha(G) \cap h(\bigcup_{i=1}^{\infty} |K_i^2|) = \emptyset$. Thus $h^{-1} \cdot \alpha(G) \cap \bigcup_{i=1}^{\infty} |K_i^2| = \emptyset$, and so $E^n - h^{-1} \cdot \alpha(G)$ is 1-ULC. Therefore $h^{-1} \cdot \alpha$ and hence α is tame.

COROLLARY 3.1. *Suppose N is a PL n -manifold, M is a PL m -manifold topologically embedded in N , G is a 1-dimensional polyhedron, $G' \subset G$ is a subpolyhedron, $\beta: G \rightarrow M$ is an embedding such that $\beta|_{G'}: G' \rightarrow N$ is tame, $n \geq 5$, and $m \geq 2$. Then for each $\epsilon > 0$ there is an embedding $\alpha: G \rightarrow M$ such that $d(\alpha, \beta) < \epsilon$, $\alpha|_{G'} = \beta|_{G'}$, and $\alpha: G \rightarrow N$ is tame.*

PROOF. First take an infinite triangulation of $G - G'$ and approximate β by an embedding $\beta': G \rightarrow M$ such that $\beta'|_{G'} = \beta|_{G'}$ and β' is locally PL on $G - G'$. Then apply Theorem 3 to a sequence of compact subpolyhedra of $\beta'(G - G')$. Thus we obtain an embedding $\alpha: G \rightarrow M$ such that $\alpha|_{G'} = \beta|_{G'}$ and $\alpha|_{G - G'}$ is locally tame in E^n . Thus $\alpha: G \rightarrow N$ is tame (Theorem 4.2 of [4]).

THEOREM 4. *Suppose N is a PL n -manifold, M is a PL m -manifold topologically embedded in N , every 2-complex of M can be approximated by a 2-complex in M that is tame in N , and $5 \leq m \leq n - 2$. Then each k -dimensional polyhedron P topologically embedded in M , $k < m$, can be approximated in M by embeddings that are tame in N .*

PROOF. It follows from [5] and either [6] or [9] that an approximation of P is tame if its complement is 1-ULC. Such an approximation is found by modifying the proof of Theorem 3. Let L be a triangulation of M and L_1, L_2, \dots the sequence of barycentric subdivisions. Similarly let K_1, K_2, \dots be the sequence of barycentric subdivisions of a triangulation of N . Using techniques similar to those above it is possible to construct a homeomorphism h of N such that $h(\bigcup |K_i^1|) \cap M = \emptyset$ and $h(\bigcup |K_i^2|) \cap (UQ_i) = \emptyset$ where Q_i is a close approximation of $|L_i^2|$ for each i that is tame in N . Now for each i we can find an arc $A_i \subset M$ such that $C_i = h(|K_i^2|) \cap M \subset A_i$ and $A_i - C_i$ is locally tame in M . Since $M - C_i$ is 1-ULC, $M - A_i$ is 1-ULC and hence A_i is tame. Using A_i we can construct, for each i , a homeomorphism f_i of M moving points a distance depending on f_{i-1} so that $f_i(P) \cap C_i = \emptyset$. Thus as in the proof of Lemma 2 and Theorem 3 we can construct an ϵ -push f of (M, P) such that $f(P) \cap (UC_i) = \emptyset$. Thus $N - f(P)$ is 1-ULC and the required approximation has been found.

It is evident that we have actually proved the following.

ADDENDUM TO THEOREM 4. *Under the hypotheses of Theorem 4 it is possible to find for each $\epsilon > 0$ an ϵ -push f of (M, P) such that $f|_{P}: P \rightarrow N$ is tame.*

3. Subpolyhedra of factored cells. We say that an m -cell $C \subset E^n$ factors k -times if for some homeomorphism $h: E^n \rightarrow E^n$ and some $(m-k)$ -cell $B \subset E^{n-k}$, $h(C) = B \times I^k \subset E^{n-k} \times E^k$ where I^k is the k -fold product of the interval I naturally embedded in E^k and $B \times I^k \subset E^{n-k} \times E^k$ is the product embedding.

THEOREM 5. *Suppose C is an m -cell topologically embedded in E^n , C factors k -times, $n \geq 5$, and $m \leq n-2$. Then every embedding of any compact k -dimensional polyhedron into C is tame in E^n .*

PROOF. Let B be an $(m-k)$ -cell in E^{n-k} , P a finite k -dimensional polyhedron topologically embedded in $B \times I^k \subset E^{n-k} \times E^k$, $n \geq 5$, and $1 \leq k < m \leq n-2$. It follows from [5] and either [6] or [9] that P is tame in E^n if $E^n - P$ is 1-ULC. However, $E^n - P$ is 1-ULC if each 2-complex in E^n can be homotoped off P by arbitrarily small homotopies. Let K be a finite 2-complex. First find a very small homotopy of $|K|$ such that for some subdivision K' each 2-cell of K' either projects onto a 0- or 1-simplex of E^{n-k} or else lies in $E^{n-k} \times t$ for some $t \in E^k$. Since $n-k \geq (m-k) + 2$, $E^{n-k} - B$ is locally 0-connected. Thus it follows that any 0- or 1-simplex in E^{n-k} can be homotoped off B by a small homotopy. Thus any 2-cell of K' that projects onto a 0- or 1-cell of E^{n-k} can be homotoped off $B \times I^k$. Let σ be a 2-cell of K' , $t \in E^k$, and $\sigma \subset E^{n-k} \times t$. For $n-k \geq 4$ it follows from Lemma 2 that there is an ϵ -push h of $(E^{n-k} \times t, \sigma)$ such that $h(\sigma) \cap (B \times t)$ is 0-dimensional. For $n-k=3$ we can use the techniques of the proof of Lemma 2 to find an embedding $h: \sigma \rightarrow E^{n-k} \times t$ such that $h(\sigma) \cap (B \times t)$ is 0-dimensional and h is close to the inclusion of σ into $E^{n-k} \times t$. Let $A = h(\sigma) \cap (B \times I^k)$. A is a 0-dimensional subset of $B \times I^k$. Let P be a k -dimensional polyhedron topologically embedded into $B \times I^k$. Let $T \subset P$ be defined as follows: $x \in T$ if there is a neighborhood U of x in P and a point $y \in E^{n-k}$ such that $U \subset y \times I^k$. Then T is open in P and P is locally tame at each point of U [5]. We shall construct a map $f: B \times E^k \rightarrow B \times E^k$ such that $p_1 \cdot f = p_1$ where $p_1 = \text{projection: } B \times E^k \rightarrow B$, $f(A) \cap P \subset T$, and $d(f, \text{Id}|_{B \times E^k})$ is small. For each $(x, t) \in A \cap (P - T)$, let $\epsilon_x > 0$ be chosen so that for some $t_x \in E^k$ with $d(t_x, t) < \epsilon_x$ and for all $x' \in B$ with $d(x', x) < \epsilon_x$, $(x', t_x) \in (B \times E^k) - P$. Now for some finite number of $x \in B$, the ϵ_x -neighborhoods of the x 's cover $p_1(A \cap (P - T))$. Since A is totally disconnected it is possible to cover $p_1(A \cap (P - T))$ by closed sets B_1, \dots, B_k that are pairwise disjoint and, for each $i = 1, \dots, k$, there is an x_i such that B_i lies in the ϵ_{x_i} -neighborhood of x_i . Define $f(x, y) = (x, y + t_{x_i} - t)$ for $x \in B_i$. Then extend $p_2 \cdot f: \bigcup B_i \times E^k \rightarrow E^k$ to a map $f_2: B \times E^k \rightarrow E^k$ such that $d(f_2, p_2) < \epsilon$. Then extend f to $B \times E^k$ by setting $f = \text{Id} \times f_2: B \times E^k \rightarrow B \times E^k$. Then $f(A) \cap P \subset T$. Now f can be extended to an ϵ -map of E^n such that

$p_1 \cdot f = p_1: E^n \rightarrow E^{n-k}$. Thus $f \cdot h(\sigma) \cap P \subset T$. Since P is locally tame at each point of T there is an approximation g of $f \cdot h$ such that $g(\sigma) \cap P = \emptyset$. Thus $E^n - P$ is 1-ULC and P is tame in E^n .

COROLLARY 5.1. *Let $C \subset E^n$ be an m -cell that factors 1-time. Let P be a k -dimensional polyhedron topologically embedded in C , $k < m \leq n-2$, and $n \geq 5$. Then for each $\epsilon > 0$ there is an ϵ -push H of (C, P) such that $H(P)$ is tame in E^n .*

PROOF. This is actually a corollary to the proofs of Theorem 3 and Theorem 5. Let K be a triangulation of E^n and suppose $C = B \times I \subset E^{n-1} \times E^1$. Then there is an approximation j of the inclusion map $i: |K^2| \rightarrow E^n$ such that $j(|K^2|) \cap C$ is a 0-dimensional subset of $B \times \{t_1, \dots, t_p\}$ for some numbers $t_1, \dots, t_p \in I$. Thus for any k -dimensional polyhedron $P \subset C$, there is a small homeomorphism h of C such that $h(P) \cap j|K^2| = \emptyset$. Thus we can obtain by a sequence of such steps a small homeomorphism H of C such that $E^n - H(P)$ is 1-ULC. Thus $H(P)$ is tame.

REMARKS. Do Theorem 3 and Theorem 5 remain true if the hypothesis $n \geq 5$ is replaced by $n = 4$? Does Theorem 5 remain true if the hypothesis $m \leq n-2$ is replaced by $m = n-1$? More specifically take Bing's 2-sphere $S \subset E^3$ [2]. Are all subarcs of $S \times I \subset E^4$ tame?

Theorem 5 is sharp in the sense that there are examples of cells that factor k -times and for which some $(k+1)$ -dimensional subcell is wild.

Daverman has independently proved Theorem 3 for the case $m = 2$.

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