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KODAI MATH. J.
31 (2008), 272–306

TANGENT BUNDLE AND INDICATRIX BUNDLE OF A FINSLER MANIFOLD

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Abstract

Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold and G be the Sasaki-Finsler metric on TM°. We show that the curvature tensor field of the Levi-Civita connection on (TM°, G) is completely determined by the curvature tensor field of Vrănceanu connection and some adapted tensor fields on TM°. Then we prove that (TM°, G) is locally symmetric if and only if $\mathbf{F}^m$ is locally Euclidean. Also, we show that the flag curvature of the Finsler manifold $\mathbf{F}^m$ is determined by some sectional curvatures of the Riemannian manifold (TM°, G). Finally, for any $c \neq 0$ we introduce the c-indicatrix bundle $IM(c)$ and obtain new and simple characterizations of $\mathbf{F}^m$ of constant flag curvature c by means of geometric objects on both $IM(c)$ and (TM°, G).

Introduction

The geometry of the tangent bundle TM of a Riemannian manifold (M, g) goes back to Sasaki [20], who constructed a natural Riemannian metric G on TM. Then G was called the Sasaki metric on TM and it was the main tool in studying interrelations between the geometries of (M, g) and (TM, G). For results and references on this matter we refer to the excellent survey of Gudmundsson and Kappos [12]. Many of the research papers on this topic pointed out deep interrelations between the tangent sphere bundles and the base manifold (cf. Kowalski-Sekizawa [16], Yampolski [25]).

The purpose of the present paper is to initiate a study of interrelations between the geometries of both the tangent bundle and the indicatrix bundle of a Finsler manifold on one side, and the geometry of the manifold itself, on the other side. As we shall see later in the paper, the extension of the study from Riemannian manifolds to Finsler manifolds is not an easy task. This is because a Finsler manifold $\mathbf{F}^m = (M, F)$ does not admit a canonical linear connection on M, as it is the Levi-Civita connection on a Riemannian manifold. Thus the study of the geometry of $\mathbf{F}^m$ was done by means of some linear connections on

2000 *Mathematics Subject Classification*. 53C60, 53B21.

Key words and phrases. Finsler manifold, tangent bundle, indicatrix bundle, locally symmetric spaces, constant flag curvature.

Received July 24, 2007; revised February 13, 2008.

vector bundles over the slit tangent bundle TM° of M. Here we refer to Berwald connection, Cartan connection, Chern-Rund connection and Hashiguchi connection. To develop our study we consider the Sasaki-Finsler metric G on TM° and instead of the above Finsler connections we take the Vrănceanu connection on TM° induced by the Levi-Civita connection on (TM°, G). It is noteworthy that the local coefficients of the Vrănceanu connection give the local coefficients of all the above classical Finsler connections and viceversa. In our study we combine both the classical coordinate approach and the modern coordinate-free approach.

Now, we outline the content of the paper. In the first section we arrange some basic formulae from Finsler geometry and define the adapted tensor fields $\mathbf{R}$, B and C. Then in Section 2 we show that the curvature tensor field $\tilde{R}$ of the Levi-Civita connection $\tilde{\nabla}$ on (TM°, G) is completely determined by the curvature tensor field R of Vrănceanu connection ∇ on TM° and the above adapted tensor fields (cf. Theorem 2.2). In particular, when $\mathbf{F}^m$ is a Riemannian manifold, we obtain some well known results of Kowalski [15]. Section 3 deals with interrelations between the flag curvature of $\mathbf{F}^m$ and the curvature of (TM°, G). First, we find an interesting characterization of Riemannian manifolds among Finsler manifolds by means of the covariant derivative of $\tilde{R}$ with respect to $\tilde{\nabla}$ (cf. Theorem 3.1). Then we prove that (TM°, G) is locally symmetric if and only if $\mathbf{F}^m$ is locally Euclidean (cf. Theorem 3.3). This is an extension to Finsler manifolds of a result of Kowalski [15] for Riemannian manifolds. New characterizations of Landsberg manifolds are given in Theorem 3.4. Also, we obtain an important formula which relates the flag curvature of $\mathbf{F}^m$ with some sectional curvatures of (TM°, G) (cf. (3.32)). Then we introduce the L-horizontal and L-vertical sectional curvatures of (TM°, G) and show that the flag curvature of $\mathbf{F}^m$ is completely determined by them (cf. Corollary 3.2). Finally, we obtain new characterizations of Finsler manifolds of constant flag curvature via the L-horizontal and L-vertical curvatures (cf. Theorems 3.8 and 3.9). In the last section we define for any $c \neq 0$ the c-indicatrix bundle $IM(c)$ and construct a contact metric structure (φ, ξ, η, g) on it. Then we obtain simple characterizations of $\mathbf{F}^m$ of constant curvature c by means of the structure tensor field φ, the Levi-Civita connection $\bar{\nabla}$ on $(IM(c), g)$ and the ξ-horizontal and ξ-tangential curvatures of $(IM(c), g)$ (cf. Theorems 4.2, 4.3, 4.4).

1. Preliminaries

Let $\mathbf{F}^m = (M, F)$ be an m-dimensional Finsler manifold, where F is the fundamental function of $\mathbf{F}^m$ that is supposed to be of class C^∞ on the slit tangent bundle $TM^\circ = TM\backslash\{0\}$ (see Bao-Chern-Shen [4], p. 2). Denote by (x^i, y^i), $i = \{1, \ldots, m\}$, the local coordinates on TM, where (x^i) are the local coordinates of a point $x \in M$ and (y^i) are the coordinates of a vector $y \in T_xM$. Then the functions

$$g_{ij} = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j},$$

define a Finsler tensor field of type $(0,2)$ on TM°. The $m \times m$ matrix $[g_{ij}]$ is supposed to be positive definite and its inverse is denoted by $[g^{ij}]$. The following functions

$$G_i^j = \frac{\partial G^j}{\partial y^i}; \quad G^j = \frac{1}{4} g^{jk} \left\{ \frac{\partial^2 F^2}{\partial y^k \partial x^i} y^i - \frac{\partial F^2}{\partial x^k} \right\},$$

have an important role in Finsler geometry.

Next, we consider the kernel VTM° of the differential of the projection map $\pi : TM^\circ \to M$, which is known as *vertical bundle* on TM°. Denote by $\Gamma(VTM^\circ)$ the $\mathscr{F}(TM^\circ)$-module of sections of VTM°, where $\mathscr{F}(TM^\circ)$ is the algebra of smooth functions on TM°. The same notation will be used for any other vector bundle. Locally, $\Gamma(VTM^\circ)$ is spanned by the natural vector fields $\{\partial/\partial y^1, \dots, \partial/\partial y^m\}$. Then by using the functions G_i^j we define the nonholonomic vector fields

$$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - G_i^j \frac{\partial}{\partial y^j}, \quad i \in \{1, \dots, m\}, \tag{1.1}$$

which enable us to construct a complementary vector subbundle HTM° to VTM° in TTM° that is locally spanned by $\{\delta/\delta x^1, \dots, \delta/\delta x^m\}$. We call HTM° the *horizontal distribution* on TM°. Thus the tangent bundle of TM° admits the decomposition

$$TTM^\circ = HTM^\circ \oplus VTM^\circ. \tag{1.2}$$

Then we can define the *Sasaki-Finsler metric* G on TM° as follows (cf. Bao-Chern-Shen [4], p. 48, Bejancu-Farran [6], p. 35, Matsumoto [18], p. 136):

$$G\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = G\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = g_{ij}(x, y), \quad G\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = 0. \tag{1.3}$$

Now, we define some geometric objects of Finsler type on TM°. First, the Lie brackets of the above vector fields are expressed as follows:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = \mathbf{R}_{ij}^k \frac{\partial}{\partial y^k}; \quad \mathbf{R}_{ij}^k = \frac{\delta G_i^k}{\delta x^j} - \frac{\delta G_j^k}{\delta x^i}, \tag{1.4}$$

and

$$\left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G_{ij}^k \frac{\partial}{\partial y^k}; \quad G_{ij}^k = G_{ji}^k = \frac{\partial G_i^k}{\partial y^j}. \tag{1.5}$$

We note that $\mathbf{R}_{ij}^k$ define a skew-symmetric Finsler tensor field of type $(1,2)$ while G_{ij}^k are the local coefficients of the Berwald connection. Some other Finsler tensor fields defined by $\mathbf{R}_{ij}^k$ will be useful in a study of Finsler manifolds of constant flag curvature (see Sections 3 and 4):

$$\text{(a) } \mathbf{R}_{hij} = g_{hk} \mathbf{R}_{ij}^k, \quad \text{(b) } \mathbf{R}_{hj} = \mathbf{R}_{hij} y^i, \quad \text{(c) } \mathbf{R}_j^k = g^{kh} \mathbf{R}_{hj}. \tag{1.6}$$

From their properties we mention the following:

$$\text{(a)}\ y^h\mathbf{R}_{hij} = 0, \quad \text{(b)}\ y^h\mathbf{R}_{hj} = 0, \quad \text{(c)}\ \mathbf{R}_{ij} = \mathbf{R}_{ji},$$

$$\text{(d)}\ \mathbf{R}^k_{ij} = \frac{1}{3}\left\{\frac{\partial \mathbf{R}^k_j}{\partial y^i} - \frac{\partial \mathbf{R}^k_i}{\partial y^j}\right\}. \tag{1.7}$$

On the other hand, since F is positively homogeneous of degree one with respect to y, several geometric objects on TM° will be positively homogeneous of certain degrees. In particular, G^k_i are positively homogeneous of degree one and by (1.5) and Euler theorem we obtain

$$G^k_{ij}y^j = G^k_i. \tag{1.8}$$

Apart from G^k_{ij}, the functions F^k_{ij} given by

$$F^k_{ij} = \frac{1}{2}g^{kh}\left\{\frac{\delta g_{hi}}{\delta x^j} + \frac{\delta g_{hj}}{\delta x^i} - \frac{\delta g_{ij}}{\delta x^h}\right\}, \tag{1.9}$$

are involved in both the Chern-Rund connection and the Cartan connection.

Moreover, we have (cf. Bejancu-Farran [6], p. 46)

$$F^k_{ij}y^j = G^k_i. \tag{1.10}$$

By means of G^k_{ij} and F^k_{ij}, we define a symmetric Finsler tensor field of type $(1,2)$ whose local components are given by

$$B^k_{ij} = F^k_{ij} - G^k_{ij}. \tag{1.11}$$

As a consequence of (1.8), (1.10) and (1.11) we have

$$B^k_{ij}y^j = 0. \tag{1.12}$$

Also, the Cartan tensor field is given by its local components:

$$\text{(a)}\ C^k_{ij} = \frac{1}{2}g^{kh}\frac{\partial g_{ij}}{\partial y^h} \quad \text{or} \quad \text{(b)}\ C_{ijk} = \frac{1}{2}\frac{\partial g_{ik}}{\partial y^j}. \tag{1.13}$$

By the homogeneity condition for F we obtain

$$\text{(a)}\ C^k_{ij}y^j = 0 \quad \text{and} \quad \text{(b)}\ C_{ijk}y^j = 0. \tag{1.14}$$

Next, we denote by h and v the projection morphisms of TTM° on HTM° and VTM° respectively, with respect to the decomposition (1.2). Then by using the above Finsler tensor fields $\mathbf{R}^k_{ij}$, C^k_{ij} and B^k_{ij} we define the following adapted tensor fields:

$$\mathbf{R}\colon \Gamma(HTM^\circ) \times \Gamma(HTM^\circ) \to \Gamma(VTM^\circ), \tag{1.15}$$

$$\mathbf{R}(hX, hY) = \mathbf{R}^k_{ij}Y^iX^j\frac{\partial}{\partial y^k},$$

$$C\colon \Gamma(HTM^\circ) \times \Gamma(HTM^\circ) \to \Gamma(VTM^\circ), \tag{1.16}$$

$$C(hX, hY) = C^k_{ij}Y^iX^j\frac{\partial}{\partial y^k},$$

$$B: \Gamma(VTM^\circ) \times \Gamma(VTM^\circ) \to \Gamma(HTM^\circ), \tag{1.17}$$
$$B(vU, vW) = B_{ij}^k W^i U^j \frac{\delta}{\delta x^k},$$

where we set

$$hX = X^j \frac{\delta}{\delta x^j}, \quad hY = Y^i \frac{\delta}{\delta x^i}, \quad vU = U^j \frac{\partial}{\partial y^j} \quad \text{and} \quad vW = W^i \frac{\partial}{\partial y^i}.$$

By using (1.4) and (1.15) it is easy to check that

$$\text{(a)}\ \mathbf{R}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \mathbf{R}_{ij}^k \frac{\partial}{\partial y^k} \quad \text{and} \quad \text{(b)}\ \mathbf{R}(hX, hY) = -v[hX, hY]. \tag{1.18}$$

Thus HTM° is an integrable distribution if and only if $\mathbf{R} = 0$. For this reason $\mathbf{R}$ is known as the integrability tensor field of HTM°. On the other hand, the adapted tensor fields B and C represent the obstructions for $\mathbf{F}^m$ to be a Landsberg manifold and a Riemannian manifold, respectively. Indeed, $\mathbf{F}^m$ becomes a Landsberg (resp. Riemannian) manifold if and only if $B = 0$ (resp. $C = 0$).

Finally, we should stress that the adapted tensor field $\mathbf{R}$, B and C together with the Vrănceanu connection (which is constructed in the next section) will play an important role in the study of interrelations between the geometries of the tangent bundle and indicatrix bundle of $\mathbf{F}^m$ on one side, and the geometry of $\mathbf{F}^m$ on the other side.

2. The Levi-Civita connection and the Vrănceanu connection on (TM°, G)

Let (TM°, G) be the Riemannian manifold, where G is the Sasaki-Finsler metric on TM° given by (1.3). Then denote by $\tilde{\nabla}$ the Levi-Civita connection on (TM°, G), that is, $\tilde{\nabla}$ is given by (cf. Yano-Kon [26], p. 29)

$$\begin{aligned} 2G(\tilde{\nabla}_X Y, Z) = {} & X(G(Y, Z)) + Y(G(Z, X)) - Z(G(X, Y)) \\ & + G([X, Y], Z) - G([Y, Z], X) + G([Z, X], Y), \end{aligned} \tag{2.1}$$

for any $X, Y, Z \in \Gamma(TM^\circ)$. The *Vrănceanu connection* ∇ on TM° that is induced by $\tilde{\nabla}$ with respect to the decomposition (1.2) is defined by

$$\nabla_X Y = v\tilde{\nabla}_{vX} vY + h\tilde{\nabla}_{hX} hY + v[hX, vY] + h[vX, hY], \tag{2.2}$$

for any $X, Y \in \Gamma(TM^\circ)$. Vrănceanu [23] introduced ∇ by its local coefficients for a study of both nonholonomic manifolds and nonholonomic mechanical systems. The invariant formula (2.2) was given by Ianuş [14] in the general context of almost product manifolds. It is noteworthy that the Vrănceanu connection is one of the main tools in a study of the geometry of foliations (see Bejancu-Farran [7]). In our case, the local coefficients of ∇ with respect to the adapted frame field $\left\{\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^i}\right\}$, $i \in \{1, \dots, m\}$, on TM° are given by the local coefficients of all the classical Finsler connections: Berwald connection, Cartan connection, Chern-Rund connection and Hashiguchi connection. Indeed, by using (2.2), (2.1), (1.5), (1.9) and (1.13) we obtain

$$
\text{(2.3)} \qquad \begin{aligned} &\text{(a)}\ \nabla_{\partial/\partial y^j}\frac{\partial}{\partial y^i} = C_{ij}^k\frac{\partial}{\partial y^k}, \quad &&\text{(b)}\ \nabla_{\delta/\delta x^j}\frac{\delta}{\delta x^i} = F_{ij}^k\frac{\delta}{\delta x^k},\\ &\text{(c)}\ \nabla_{\partial/\partial y^j}\frac{\delta}{\delta x^i} = 0, &&\text{(d)}\ \nabla_{\delta/\delta x^j}\frac{\partial}{\partial y^i} = G_{ij}^k\frac{\partial}{\partial y^k}. \end{aligned}
$$

Also, we should note that ∇ is an adapted linear connection on TM° with respect to the decomposition (1.2), that is, both distributions HTM° and VTM° are parallel with respect to ∇. The main purpose of this section is to relate the curvature tensor fields of $\tilde{\nabla}$ and ∇ by means of $\mathbf{R}$, B and C. First, we prove the following propositions.

Proposition 2.1. *The Lie brackets on TM° are expressed in terms of Vrănceanu connection as follows*:

$$
\text{(2.4)} \qquad \begin{aligned} &\text{(a)}\ [hX, hY] = \nabla_{hX}hY - \nabla_{hY}hX - \mathbf{R}(hX, hY),\\ &\text{(b)}\ [hX, vY] = \nabla_{hX}vY - \nabla_{vY}hX,\\ &\text{(c)}\ [vX, vY] = \nabla_{vX}vY - \nabla_{vY}vX, \end{aligned}
$$

for any $X, Y \in \Gamma(TTM^\circ)$.

Proof. By direct calculations using (1.2), (1.18b), (2.2), and taking into account that $\tilde{\nabla}$ is torsion-free we obtain

$$
\begin{aligned} [hX, hY] &= h[hX, hY] + v[hX, hY] = h\{\tilde{\nabla}_{hX}hY - \tilde{\nabla}_{hY}hX\} - \mathbf{R}(hX, hY)\\ &= \nabla_{hX}hY - \nabla_{hY}hX - \mathbf{R}(hX, hY), \end{aligned}
$$

which proves (2.4a). Similar reason applies for proofs of (2.4b) and (2.4c). ∎

Proposition 2.2. *The adapted tensor fields B and C can be expressed in terms of Vrănceanu connection as follows*:

$$
G(B(vX, vY), hZ) = \frac{1}{2}(\nabla_{hZ}G)(vX, vY), \tag{2.5}
$$

$$
G(C(hX, hY), vZ) = \frac{1}{2}(\nabla_{vZ}G)(hX, hY), \tag{2.6}
$$

for any $X, Y, Z \in \Gamma(TTM^\circ)$.

Proof. By using (1.3) and (2.3d) we obtain

$$
(\nabla_{\delta/\delta x^k}G)\left(\frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = \frac{\delta g_{ij}}{\delta x^k} - g_{hj}G_{ik}^h - g_{ih}G_{jk}^h. \tag{2.7}
$$

Now, we denote by $g_{ij;k}$ the right part in (2.7) and observe that it is just the h-covariant derivative of g_{ij} with respect to the Berwald connection. Then by using (18.14) and (18.24) from Matsumoto [18] and (1.11) we deduce that

$$g_{ij;k} = -2C_{ijk|t}y^t = 2g_{ih}(F^h_{jk} - G^h_{jk}) = 2g_{ih}B^h_{jk} = 2B_{jik}, \tag{2.8}$$

where "|" represents the h-covariant derivative with respect to Chern-Rund connection. Thus from (2.7) and (2.8) we obtain (2.5). Next, by using (1.3) and (2.3c) we infer that

$$(\nabla_{\partial/\partial y^k} G)\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = \frac{\partial g_{ij}}{\partial y^k},$$

which implies (2.6) via (1.13). ■

Next, in order to express the Levi-Civita connection in terms of Vrănceanu connection, we define for each of the adapted tensor fields **R**, C and B a twin (denoted by the same symbol) as follows:

$$\begin{aligned} &\mathbf{R}\colon \Gamma(HTM^\circ) \times \Gamma(VTM^\circ) \to \Gamma(HTM^\circ), \\ &G(\mathbf{R}(hX, vY), hZ) = G(\mathbf{R}(hX, hZ), vY), \end{aligned} \tag{2.9}$$

$$\begin{aligned} &C\colon \Gamma(HTM^\circ) \times \Gamma(VTM^\circ) \to \Gamma(HTM^\circ), \\ &G(C(hX, vY), hZ) = G(C(hX, hZ), vY), \end{aligned} \tag{2.10}$$

$$\begin{aligned} &B\colon \Gamma(HTM^\circ) \times \Gamma(VTM^\circ) \to \Gamma(VTM^\circ), \\ &G(B(hX, vY), vZ) = G(B(vY, vZ), hX), \end{aligned} \tag{2.11}$$

for any $X, Y, Z \in \Gamma(TTM^\circ)$. Now, we can prove the following theorem.

THEOREM 2.1. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold. Then the Levi-Civita connection $\tilde{\nabla}$ and the Vrănceanu connection ∇ on (TM°, G) are related as follows:*

$$\tilde{\nabla}_{hX} hY = \nabla_{hX} hY - C(hX, hY) - \frac{1}{2}\mathbf{R}(hX, hY), \tag{2.12}$$

$$\tilde{\nabla}_{hX} vY = \nabla_{hX} vY + B(hX, vY) + C(hX, vY) + \frac{1}{2}\mathbf{R}(hX, vY), \tag{2.13}$$

$$\tilde{\nabla}_{vX} hY = \nabla_{vX} hY + C(hY, vX) + \frac{1}{2}\mathbf{R}(hY, vX) + B(hY, vX), \tag{2.14}$$

$$\tilde{\nabla}_{vX} vY = \nabla_{vX} vY - B(vX, vY), \tag{2.15}$$

for any $X, Y \in \Gamma(TTM^\circ)$.

Proof. First, by using (1.2) and (2.2) we obtain

$$G(\tilde{\nabla}_{hX} hY, hZ) = G(h\tilde{\nabla}_{hX} hY, hZ) = G(\nabla_{hX} hY, hZ). \tag{2.16}$$

Then, we use (2.1), (1.3), (2.2), (1.18b) and (2.6), and deduce that

$$
\begin{aligned}
(2.17)\quad G(\tilde{\nabla}_{hX}hY, vZ) &= \frac{1}{2}\{-vZ(G(hX,hY)) + G(h[vZ,hX],hY) \\
&\quad + G(hX, h[vZ,hY]) + G(v[hX,hY],vZ)\} \\
&= \frac{1}{2}\{-vZ(G(hX,hY)) + G(\nabla_{vZ}hX,hY) + G(hX,\nabla_{vZ}hY) \\
&\quad - G(\mathbf{R}(hX,hY),vZ)\} \\
&= -G\left(C(hX,hY) + \frac{1}{2}\mathbf{R}(hX,hY),vZ\right).
\end{aligned}
$$

Thus from (2.16) and (2.17) we obtain (2.12). Next, by using (2.1), (1.3), (2.4b), (2.6), (1.18b), (2.9) and (2.10) we infer that

$$
\begin{aligned}
(2.18)\quad G(\tilde{\nabla}_{hX}vY, hZ) &= \frac{1}{2}\{vY(G(hX,hZ)) + G([hX,vY],hZ) \\
&\quad + G(hX,[hZ,vY]) - G(v[hX,hZ],vY)\} \\
&= \frac{1}{2}(\nabla_{vY}G)(hX,hZ) + \frac{1}{2}G(\mathbf{R}(hX,hZ),vY) \\
&= G\left(C(hX,vY) + \frac{1}{2}\mathbf{R}(hX,vY),hZ\right).
\end{aligned}
$$

Also, by using (2.1), (1.3), (2.4b), (2.5), (2.11), and taking into account that VTM° is integrable, we obtain

$$
\begin{aligned}
(2.19)\quad G(\tilde{\nabla}_{hX}vY, vZ) &= \frac{1}{2}\{hX(G(vY,vZ)) + G(\nabla_{hX}vY,vZ) - G(vY,\nabla_{hX}vZ)\} \\
&= \frac{1}{2}(\nabla_{hX}G)(vY,vZ) + G(\nabla_{hX}vY,vZ) \\
&= G(\nabla_{hX}vY + B(hX,vY),vZ).
\end{aligned}
$$

Then (2.13) is a consequence of (2.18) and (2.19). Next, (2.14) is obtained from (2.13) by using (2.4b) and taking into account that $\tilde{\nabla}$ is torsion-free. Finally, we use (2.1), (1.3), (2.4b) and (2.5), and deduce that

$$
G(\tilde{\nabla}_{vX}vY, vZ) = G(v\tilde{\nabla}_{vX}vY, vZ) = G(\nabla_{vX}vY, vZ),
$$

and

$$
\begin{aligned}
G(\tilde{\nabla}_{vX}vY, hZ) &= \frac{1}{2}\{-hZ(G(vX,vY)) - G([vX,hZ],vY) \\
&\quad - G(vX,[vY,hZ])\} = -G(B(vX,vY),hZ),
\end{aligned}
$$

which prove (2.15). We should remark that in all these calculations we use the fact that ∇ is an adapted linear connection on TM°. ■

Now, we want to find interrelations between the curvature tensor fields of $\tilde{\nabla}$ and ∇. First, we denote by $\tilde{R}$ the curvature tensor field of $\tilde{\nabla}$, that is, we have

$$\tilde{R}(X, Y, Z) = \tilde{R}(X, Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X, Y]} Z, \tag{2.20}$$

for any $X, Y, Z \in \Gamma(TTM^\circ)$. The curvature tensor field of ∇ is denoted by R and it is given by a similar formula to (2.20).

In order to simplify the presentation of some long formulae we use the symbol $\mathscr{A}_{(hX, hY)}$, which means that in the expression that follows this symbol we interchange hX and hY, and then subtract, as in the following formula

$$\mathscr{A}_{(hX, hY)}\{f(hX, hY)\} = f(hX, hY) - f(hY, hX).$$

In a similar way we use the symbol $\mathscr{A}_{(vX, vY)}$. Now we can state the theorem on interrelations between $\tilde{R}$ and R.

THEOREM 2.2. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold. Then the curvature tensor field $\tilde{R}$ of the Levi-Civita connection on (TM°, G) is completely determined by the curvature tensor field R of Vrănceanu connection on (TM°, G) and the adapted tensor fields $\mathbf{R}$, C and B as follows*:

$$\begin{aligned} \tilde{R}(hX, hY, hZ) &= R(hX, hY, hZ) + B(hZ, \mathbf{R}(hX, hY)) \\ &\quad + C(hZ, \mathbf{R}(hX, hY)) + \frac{1}{2}\mathbf{R}(hZ, \mathbf{R}(hX, hY)) \\ &\quad - \mathscr{A}_{(hX, hY)}\bigg\{(\nabla_{hX} C)(hY, hZ) + \frac{1}{2}(\nabla_{hX}\mathbf{R})(hY, hZ) \\ &\quad + B(hX, C(hY, hZ)) + \frac{1}{2}B(hX, \mathbf{R}(hY, hZ)) \\ &\quad + C(hX, C(hY, hZ)) + \frac{1}{2}C(hX, \mathbf{R}(hY, hZ)) \\ &\quad + \frac{1}{2}\mathbf{R}(hX, C(hY, hZ)) + \frac{1}{4}\mathbf{R}(hX, \mathbf{R}(hY, hZ))\bigg\}, \end{aligned} \tag{2.21}$$

$$\begin{aligned} \tilde{R}(hX, hY, vZ) &= R(hX, hY, vZ) - B(\mathbf{R}(hX, hY), vZ) \\ &\quad + \mathscr{A}_{(hX, hY)}\bigg\{(\nabla_{hX} B)(hY, vZ) + (\nabla_{hX} C)(hY, vZ) \\ &\quad + \frac{1}{2}(\nabla_{hX}\mathbf{R})(hY, vZ) + B(hX, B(hY, vZ)) \\ &\quad + C(hX, B(hY, vZ)) + \frac{1}{2}\mathbf{R}(hX, B(hY, vZ)) \\ &\quad - C(hX, C(hY, vZ)) - \frac{1}{2}C(hX, \mathbf{R}(hY, vZ)) \\ &\quad - \frac{1}{2}\mathbf{R}(hX, C(hY, vZ)) - \frac{1}{4}\mathbf{R}(hX, \mathbf{R}(hY, vZ))\bigg\}, \end{aligned} \tag{2.22}$$

$$\begin{aligned}(2.23)\quad \tilde{R}(vX, vY, hZ) &= R(vX, vY, hZ)\\ &+ \mathscr{A}_{(vX, vY)}\Big\{(\nabla_{vX}C)(hZ, vY) + (\nabla_{vX}B)(hZ, vY)\\ &+ \frac{1}{2}(\nabla_{vX}\mathbf{R})(hZ, vY) + C(C(hZ, vY), vX)\\ &+ \frac{1}{2}C(\mathbf{R}(hZ, vY), vX) + \frac{1}{2}\mathbf{R}(C(hZ, vY), vX)\\ &+ \frac{1}{4}\mathbf{R}(\mathbf{R}(hZ, vY), vX) + B(C(hZ, vY), vX)\\ &+ \frac{1}{2}B(\mathbf{R}(hZ, vY), vX) - B(vX, B(hZ, vY))\Big\},\end{aligned}$$

$$\begin{aligned}(2.24)\quad \tilde{R}(vX, vY, vZ) &= R(vX, vY, vZ)\\ &- \mathscr{A}_{(vX, vY)}\Big\{(\nabla_{vX}B)(vY, vZ) + C(B(vY, vZ), vX)\\ &+ \frac{1}{2}\mathbf{R}(B(vY, vZ), vX) + B(B(vY, vZ), vX)\Big\},\end{aligned}$$

$$\begin{aligned}(2.25)\quad \tilde{R}(hX, vY, hZ) &= R(hX, vY, hZ) + (\nabla_{hX}C)(hZ, vY)\\ &+ (\nabla_{vY}C)(hX, hZ) + \frac{1}{2}(\nabla_{hX}\mathbf{R})(hZ, vY)\\ &+ \frac{1}{2}(\nabla_{vY}\mathbf{R})(hX, hZ) + (\nabla_{hX}B)(hZ, vY)\\ &- C(hX, C(hZ, vY)) - \frac{1}{2}C(hX, \mathbf{R}(hZ, vY))\\ &- \frac{1}{2}\mathbf{R}(hX, C(hZ, vY)) - \frac{1}{4}\mathbf{R}(hX, \mathbf{R}(hZ, vY))\\ &+ B(hX, B(hZ, vY)) + C(hX, B(hZ, vY))\\ &+ \frac{1}{2}\mathbf{R}(hX, B(hZ, vY))\\ &- B(vY, C(hX, hZ)) - \frac{1}{2}B(vY, \mathbf{R}(hX, hZ)),\end{aligned}$$

$$\begin{aligned}(2.26)\quad \tilde{R}(hX, vY, vZ) &= R(hX, vY, vZ) - (\nabla_{hX}B)(vY, vZ)\\ &- (\nabla_{vY}B)(hX, vZ) - (\nabla_{vY}C)(hX, vZ)\\ &- \frac{1}{2}(\nabla_{vY}\mathbf{R})(hX, vZ) + C(hX, B(vY, vZ))\end{aligned}$$

$$+\frac{1}{2}\mathbf{R}(hX, B(vY, vZ)) + B(vY, B(hX, vZ))$$
$$- C(C(hX, vZ), vY) - \frac{1}{2}C(\mathbf{R}(hX, vZ), vY)$$
$$-\frac{1}{2}\mathbf{R}(C(hX, vZ), vY) - \frac{1}{4}\mathbf{R}(\mathbf{R}(hX, vZ), vY)$$
$$- B(C(hX, vZ), vY) - \frac{1}{2}B(\mathbf{R}(hX, vZ), vY),$$

for any $X, Y, Z \in \Gamma(TTM^{\circ})$.

Proof. First, by using (2.12) and (2.13) we obtain

$$\begin{aligned}(2.27)\qquad \tilde{\nabla}_{hX}\tilde{\nabla}_{hY}hZ &= \tilde{\nabla}_{hX}(\nabla_{hY}hZ) - \tilde{\nabla}_{hX}\left\{C(hY, hZ) + \frac{1}{2}\mathbf{R}(hY, hZ)\right\} \\ &= \nabla_{hX}\nabla_{hY}hZ - C(hX, \nabla_{hY}hZ) - \frac{1}{2}\mathbf{R}(hX, \nabla_{hY}hZ) \\ &\quad - \nabla_{hX}(C(hY, hZ)) - \frac{1}{2}\nabla_{hX}(\mathbf{R}(hY, hZ)) \\ &\quad - B(hX, C(hY, hZ)) - \frac{1}{2}B(hX, \mathbf{R}(hY, hZ)) \\ &\quad - C(hX, C(hY, hZ)) - \frac{1}{2}C(hX, \mathbf{R}(hY, hZ)) \\ &\quad - \frac{1}{2}\mathbf{R}(hX, C(hY, hZ)) - \frac{1}{4}\mathbf{R}(hX, \mathbf{R}(hY, hZ)).\end{aligned}$$

Then, taking into account the decomposition (1.2) and by using (2.12), (2.14), (2.4a) and (1.18b) we deduce that

$$\begin{aligned}(2.28)\qquad \tilde{\nabla}_{[hX, hY]}hZ &= \tilde{\nabla}_{h[hX, hY]}hZ + \tilde{\nabla}_{v[hX, hY]}hZ \\ &= \nabla_{h[hX, hY]}hZ - C(h[hX, hY], hZ) - \frac{1}{2}\mathbf{R}(h[hX, hY], hZ) \\ &\quad + \nabla_{v[hX, hY]}hZ + C(hZ, v[hX, hY]) + B(hZ, v[hX, hY]) \\ &\quad + \frac{1}{2}\mathbf{R}(hZ, v[hX, hY]) = \nabla_{[hX, hY]}hZ - C(\nabla_{hX}hY, hZ) \\ &\quad + C(\nabla_{hY}hX, hZ) - \frac{1}{2}\mathbf{R}(\nabla_{hX}hY, hZ) + \frac{1}{2}\mathbf{R}(\nabla_{hY}hX, hZ) \\ &\quad - C(hZ, \mathbf{R}(hX, hY)) - B(hZ, \mathbf{R}(hX, hY)) \\ &\quad - \frac{1}{2}\mathbf{R}(hZ, \mathbf{R}(hX, hY)).\end{aligned}$$

Next, by direct calculations using (2.20) for both $\tilde{R}$ and R, and taking into account (2.27) and (2.28) we obtain (2.21). All the other equalities from the theorem are deduced by similar calculations as above. ■

In order to relate the integrability tensor field $\mathbf{R}$ of HTM° to the curvature tensor field R of the Vrănceanu connection we consider the *horizontal Liouville vector field*

$$L = y^i \frac{\delta}{\delta x^i}, \tag{2.29}$$

and prove the following.

PROPOSITION 2.3. *The horizontal Liouville vector field is parallel with respect to Vrănceanu connection along the horizontal distribution HTM°, that is, we have*

$$\nabla_{hX} L = 0, \quad \forall X \in \Gamma(TTM^\circ). \tag{2.30}$$

Proof. First, by direct calculations using (1.1) we obtain

$$\frac{\delta y^i}{\delta x^j} = -G^i_j. \tag{2.31}$$

Then by using (2.3b), (2.31) and (1.10) we deduce that

$$\nabla_{\delta/\delta x^j} L = \nabla_{\delta/\delta x^j} y^i \frac{\delta}{\delta x^i} = \frac{\delta y^i}{\delta x^j}\frac{\delta}{\delta x^i} + y^i F^k_{ij}\frac{\delta}{\delta x^k} = (-G^k_j + y^i F^k_{ij})\frac{\delta}{\delta x^k} = 0,$$

which proves (2.30). ■

We also need the almost complex structure J on TM° given by

$$\text{(a)}\ J\left(\frac{\delta}{\delta x^i}\right) = \frac{\partial}{\partial y^i}; \quad \text{(b)}\ J\left(\frac{\partial}{\partial y^i}\right) = -\frac{\delta}{\delta x^i}. \tag{2.32}$$

The main properties of J are presented in the next section. Here we prove the following.

PROPOSITION 2.4. *The integrability tensor field $\mathbf{R}$ of HTM° is related to both the curvature tensor field R and the torsion tensor field T of Vrănceanu connection as follows*:

$$\mathbf{R}(hX, hY) = JR(hX, hY, L) = T(X, Y), \quad \forall X, Y \in \Gamma(TTM^\circ). \tag{2.33}$$

Proof. By using (2.20) for R, (2.30), (1.4), (2.32a), (2.3c) and (1.18a) we obtain

$$\begin{aligned} JR\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}, L\right) &= -J\nabla_{[\delta/\delta x^j, \delta/\delta x^i]} L = -J\nabla_{\mathbf{R}^k_{ji}(\partial/\partial y^k)}\left(y^h \frac{\delta}{\delta x^h}\right) \\ &= \mathbf{R}^h_{ij}\frac{\partial}{\partial y^h} = \mathbf{R}\left(\frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right), \end{aligned}$$

which proves the first equality in (2.33). Taking into account that

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

and by using (2.4) we deduce that

$$T(X, Y) = \mathbf{R}(hX, hY),$$

which completes the proof of the proposition. ∎

Also, we need to express the local components of the curvature tensor field R of ∇ in terms of the local coefficients of ∇. First we set

$$R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = K_{ijk}^{h} \frac{\delta}{\delta x^h}, \quad R\left(\frac{\delta}{\delta x^k}, \frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = R_{ijk}^{h} \frac{\partial}{\partial y^h},$$
$$R\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}, \frac{\delta}{\delta x^i}\right) = F_{ijk}^{h} \frac{\delta}{\delta x^h}, \quad R\left(\frac{\partial}{\partial y^k}, \frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = P_{ijk}^{h} \frac{\partial}{\partial y^h},$$
$$R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}, \frac{\delta}{\delta x^i}\right) = D_{ijk}^{h} \frac{\delta}{\delta x^h}, \quad R\left(\frac{\partial}{\partial y^k}, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = S_{ijk}^{h} \frac{\partial}{\partial y^h}.$$

Then, by direct calculations using (2.20) for R, (2.3), (1.4) and (1.5) we obtain

$$\text{(2.34)} \quad \begin{aligned} &\text{(a)}\ K_{ijk}^{h} = \frac{\delta F_{ij}^{h}}{\delta x^k} - \frac{\delta F_{ik}^{h}}{\delta x^j} + F_{ij}^{t} F_{tk}^{h} - F_{ik}^{t} F_{tj}^{h}, \\ &\text{(b)}\ F_{ijk}^{h} = \frac{\partial F_{ij}^{h}}{\partial y^k}, \\ &\text{(c)}\ D_{ijk}^{h} = 0, \\ &\text{(d)}\ R_{ijk}^{h} = \frac{\delta G_{ij}^{h}}{\delta x^k} - \frac{\delta G_{ik}^{h}}{\delta x^j} + G_{ij}^{t} G_{tk}^{h} - G_{ik}^{t} G_{tj}^{h} + C_{it}^{h} \mathbf{R}_{jk}^{t}, \\ &\text{(e)}\ P_{ijk}^{h} = \frac{\partial G_{ij}^{h}}{\partial y^k} - \frac{\delta C_{ik}^{h}}{\delta x^j} + G_{ij}^{t} C_{tk}^{h} - C_{ik}^{t} G_{tj}^{h} + C_{it}^{h} G_{jk}^{t}, \\ &\text{(f)}\ S_{ijk}^{h} = \frac{\partial C_{ij}^{h}}{\partial y^k} - \frac{\partial C_{ik}^{h}}{\partial y^j} + C_{ij}^{t} C_{tk}^{h} - C_{ik}^{t} C_{tj}^{h}. \end{aligned}$$

It is easy to see that K_{ijk}^{h} and F_{ijk}^{h} are the h-curvature and hv-curvature tensors of Chern-Rund connection, while S_{ijk}^{h} is the v-curvature tensor of Cartan connection. Moreover, (2.34d) and (2.34e) can be written as follows:

$$R_{ijk}^{h} = H_{ijk}^{h} + C_{it}^{h} \mathbf{R}_{jk}^{t},$$
$$P_{ijk}^{h} = G_{ijk}^{h} - C_{ik;j}^{h},$$

where H_{ijk}^{h} and G_{ijk}^{h} are the h-curvature and hv-curvature tensors of Berwald connection, and the h-covariant derivative of C_{ik}^{h} is taken with respect to Berwald connection. Finally, by using (1.10), (1.11), (1.5) and (2.34b) we deduce that

$$B_{jk}^h = -y^i \frac{\partial F_{ij}^h}{\partial y^k} = -y^i F_{ijk}^h. \tag{2.35}$$

In the remaining part of this section we want to show that in particular, when $\mathbf{F}^m$ is a Riemannian manifold, the Theorems 2.1 and 2.2 give some well known results of Kowalski [15]. Thus, let $\mathbf{F}^m = (M, F)$ be a Riemannian manifold, that is, we have

$$F^2(x, y) = g_{ij}(x) y^i y^j,$$

where $g_{ij}(x)$ are the local components of a Riemannian metric g on M. In this case, the functions G_i^j are given by

$$G_i^j(x, y) = y^k \Gamma_{ki}^j(x), \tag{2.36}$$

where $\Gamma_{ki}^j(x)$ are the Christoffel symbols of the Levi-Civita connection D on (M, g). Denote by $\mathscr{R}$ the curvature tensor field of D and set

$$\mathscr{R}\left(\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^i}\right) = \mathscr{R}_{ijk}^h \frac{\partial}{\partial x^h}. \tag{2.37}$$

Then by using (2.36) in (1.4), and taking into account (1.1) we obtain

$$\mathbf{R}_{ij}^k(x, y) = y^r \mathscr{R}_{rij}^k(x). \tag{2.38}$$

Next, we take a vector field $X = X^i(\partial/\partial x^i)$ on M and consider its *vertical* and *horizontal lifts* $X^v = X^i(\partial/\partial y^i)$ and $X^h = X^i(\delta/\delta x^i)$, respectively. Then by using (1.15), (2.38) and (2.37) we deduce that

$$\mathbf{R}(X^h, Y^h)_{(x,y)} = \{\mathscr{R}_x(X_x, Y_x, y)\}_{(x,y)}^v, \tag{2.39}$$

where $(x, y) \in TM^\circ$, $y = y^i(\partial/\partial x^i)_x$ and $X_x, Y_x \in T_xM$. Also, by using basic properties of $\mathscr{R}$, (2.9) and (2.38) we infer that

$$\mathbf{R}(X^h, Y^v)_{(x,y)} = \{\mathscr{R}_x(y, Y_x, X_x)\}_{(x,y)}^h. \tag{2.40}$$

Finally, by using (1.5), (2.36), (1.9), (1.1) and (1.13) we obtain

$$\begin{aligned} &\text{(a)}\ \ G_{ij}^k(x, y) = F_{ij}^k(x, y) = \Gamma_{ij}^k(x), \\ &\text{(b)}\ \ C_{ij}^k(x, y) = 0, \quad \forall i, j, k \in \{1, \dots, m\}, \end{aligned} \tag{2.41}$$

which together with (2.3) imply

$$\begin{aligned} &\text{(a)}\ \ \nabla_{X^v} Y^v = 0, \quad \text{(b)}\ \ \nabla_{X^h} Y^h = \{D_X Y\}^h, \\ &\text{(c)}\ \ \nabla_{X^v} Y^h = 0, \quad \text{(d)}\ \ \nabla_{X^h} Y^v = \{D_X Y\}^v, \quad \forall X, Y \in \Gamma(TM). \end{aligned} \tag{2.42}$$

Thus, by using (2.39), (2.40) and (2.42), and taking into account that on the Riemannian manifold (M, g) we have $B = 0$ and $C = 0$, from Theorem 2.1 we deduce the following corollary.

Corollary 2.1 (Kowalski [15]). *Let G be the Sasaki metric on the tangent bundle TM of a Riemannian manifold (M, g). Then the Levi-Civita connections $\tilde{\nabla}$ and D on (TM, G) and (M, g) respectively, are related as follows*:

$$(\tilde{\nabla}_{X^h} Y^h)_{(x,y)} = \{D_{X_x} Y_x\}^h_{(x,y)} - \frac{1}{2}\{\mathscr{R}_x(X_x, Y_x, y)\}^v_{(x,y)},$$

$$(\tilde{\nabla}_{X^h} Y^v)_{(x,y)} = \{D_{X_x} Y_x\}^v_{(x,y)} + \frac{1}{2}\{\mathscr{R}_x(y, Y_x, X_x)\}^h_{(x,y)},$$

$$(\tilde{\nabla}_{X^v} Y^h)_{(x,y)} = \frac{1}{2}\{\mathscr{R}_x(y, X_x, Y_x)\}^h_{(x,y)},$$

$$(\tilde{\nabla}_{X^v} Y^v)_{(x,y)} = 0.$$

Now, we want to find interrelations between the curvature tensor fields $\tilde{R}$ and $\mathscr{R}$ of $\tilde{\nabla}$ and D, respectively. First, by using (2.40) and (1.1) in (2.34) we obtain the following proposition.

Proposition 2.5. *Let (M, g) be a Riemannian manifold. Then the curvature tensor field R of Vrănceanu connection on (TM, G) is completely determined by the following formulae*:

$$\text{(2.43)} \quad \begin{aligned} &\text{(a)}\ R(X^h, Y^h, Z^h) = \{\mathscr{R}(X, Y, Z)\}^h, \\ &\text{(b)}\ R(X^h, Y^h, Z^v) = \{\mathscr{R}(X, Y, Z)\}^v, \\ &\text{(c)}\ R(X^v, Y^h, Z^h) = R(X^v, Y^v, Z^h) = R(X^v, Y^h, Z^v) \\ &\qquad = R(X^v, Y^v, Z^v) = 0, \end{aligned}$$

for any $X, Y, Z \in \Gamma(TM)$.

Also, we need the following proposition.

Proposition 2.6. *Let D be the Levi-Civita connection on a Riemannian manifold (M, g) and ∇ the Vrănceanu connection on (TM, G) where G is the Sasaki metric on TM. Then the integrability tensor $\mathbf{R}$ of the horizontal distribution and the curvature tensor field $\mathscr{R}$ of D satisfy the equalities*:

$$\text{(2.44)} \quad \begin{aligned} &\text{(a)}\ (\nabla_{X^h}\mathbf{R})(Y^h, Z^h)_{(x,y)} = \{(D_X\mathscr{R})(Y, Z, y)\}^v_{(x,y)}, \\ &\text{(b)}\ (\nabla_{X^h}\mathbf{R})(Y^h, Z^v)_{(x,y)} = \{(D_X\mathscr{R})(y, Z, Y)\}^h_{(x,y)}, \\ &\text{(c)}\ (\nabla_{X^v}\mathbf{R})(Y^h, Z^h) = \{\mathscr{R}(Y, Z, X)\}^v, \\ &\text{(d)}\ (\nabla_{X^v}\mathbf{R})(Y^h, Z^v) = \{\mathscr{R}(X, Z, Y)\}^h, \\ &\text{(e)}\ (\nabla_{Z^h}\mathbf{R})(X^h, Y^h) = (\nabla_{Y^h}\mathbf{R})(X^h, Z^h) - (\nabla_{X^h}\mathbf{R})(Y^h, Z^h), \\ &\text{(f)}\ \mathbf{R}(X^h, \mathbf{R}(Y^h, Z^h))_{(x,y)} = \{\mathscr{R}(y, \mathscr{R}(Y, Z, y), X)\}^h_{(x,y)}, \\ &\text{(g)}\ \mathbf{R}(X^h, \mathbf{R}(Y^h, Z^v))_{(x,y)} = \{\mathscr{R}(X, \mathscr{R}(y, Z, Y), y)\}^v_{(x,y)}, \\ &\text{(h)}\ \mathbf{R}(\mathbf{R}(Z^h, Y^v), X^v)_{(x,y)} = \{\mathscr{R}(y, X, \mathscr{R}(y, Y, Z)\}^h_{(x,y)}. \end{aligned}$$

Proof. First, by using (1.15), (2.3b), (2.38), (2.41a) and (2.36) we deduce that

$$\left\{(\nabla_{\delta/\delta x^k}\mathbf{R})\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right)\right\}_{(x,y)}$$
$$=\left\{\left(\frac{\delta \mathbf{R}^h_{ij}}{\delta x^k}+\mathbf{R}^t_{ij}G^h_{tk}-\mathbf{R}^h_{tj}F^t_{ik}-\mathbf{R}^h_{it}F^t_{jk}\right)\frac{\partial}{\partial y^h}\right\}_{(x,y)}$$
$$=\left\{\left(\frac{\delta}{\delta x^k}(y^r\mathscr{R}^h_{rij})+y^r(\mathscr{R}^t_{rij}\Gamma^h_{tk}-\mathscr{R}^h_{rtj}\Gamma^t_{ik}-\mathscr{R}^h_{rit}\Gamma^t_{jk})\right)\frac{\partial}{\partial y^h}\right\}_{(x,y)}$$
$$=\left\{y^r\left(\frac{\partial \mathscr{R}^h_{rij}}{\partial x^k}+\mathscr{R}^t_{rij}\Gamma^h_{tk}-\mathscr{R}^h_{tij}\Gamma^t_{rk}-\mathscr{R}^h_{rtj}\Gamma^t_{ik}-\mathscr{R}^h_{rit}\Gamma^t_{jk}\right)\frac{\partial}{\partial y^h}\right\}_{(x,y)}$$
$$=\left\{(D_{\partial/\partial x^k}\mathscr{R})\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i},y\right)\right\}^v_{(x,y)},$$

which proves (2.44a). By similar calculations we obtain (2.44b). Next, at the first sight, it is surprising that in the right parts of both (2.44c) and (2.44d) we do not have covariant derivatives of $\mathscr{R}$. This is due to (2.42a) and (2.42c), as we see now. By using (2.42c), (2.38), (2.42a) and (2.37) we infer that

$$(\nabla_{\partial/\partial y^k}\mathbf{R})\left(\frac{\delta}{\delta x^j},\frac{\delta}{\delta x^i}\right)=\nabla_{\partial/\partial y^k}\left(\mathbf{R}^h_{ij}\frac{\partial}{\partial y^h}\right)=\nabla_{\partial/\partial y^k}\left(y^r\mathscr{R}^h_{rij}(x)\frac{\partial}{\partial y^h}\right)$$
$$=\mathscr{R}^h_{kij}(x)\frac{\partial}{\partial y^h}=\left\{\mathscr{R}\left(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^k}\right)\right\}^v,$$

which proves (2.44c). Similar reason applies to (2.44d). Next, we write the second Bianchi identity for $\mathscr{R}$:

$$(D_X\mathscr{R})(Y,Z,y)+(D_Y\mathscr{R})(Z,X,y)+(D_Z\mathscr{R})(X,Y,y)=0.$$

Then by using (2.44a) we deduce that

$$(\nabla_{X^h}\mathbf{R})(Y^h,Z^h)+(\nabla_{Y^h}\mathbf{R})(Z^h,X^h)+(\nabla_{Z^h}\mathbf{R})(X^h,Y^h)=0,$$

which implies (2.44e). Finally, by using (2.39) and (2.40) we obtain

$$\mathbf{R}(X^h,\mathbf{R}(Y^h,Z^h))_{(x,y)}=\mathbf{R}(X^h,\{\mathscr{R}(Y,Z,y)\}^v)_{(x,y)}=\{\mathscr{R}(y,\mathscr{R}(Y,Z,y),X)\}^h_{(x,y)},$$

which proves (2.44f). By similar calculations we deduce (2.44g) and (2.44h). ■

Now, we can give a new proof of the following theorem.

THEOREM 2.3 (Kowalski [15]). *Let (M,g) be a Riemannian manifold and (TM,G) be its tangent bundle endowed with the Sasaki metric G. Then the curvature tensor field $\tilde{R}$ of the Levi-Civita connection $\tilde{\nabla}$ on (TM,G) is completely determined by the curvature tensor field $\mathscr{R}$ of the Levi-Civita connection D on (M,g) as follows*:

(2.45)

$$
\begin{aligned}
&\text{(a)}\ \{\tilde{R}(X^h, Y^h, Z^h)\}_{(x,y)} = \frac{1}{2}\{(D_Z\mathcal{R})(X, Y, y)\}^v_{(x,y)} \\
&\quad + \left\{\mathcal{R}(X, Y, Z) + \frac{1}{2}\mathcal{R}(y, \mathcal{R}(X, Y, y), Z)\right. \\
&\quad \left. + \frac{1}{4}\mathcal{R}(y, \mathcal{R}(Z, Y, y), X) + \frac{1}{4}\mathcal{R}(y, \mathcal{R}(X, Z, y), Y)\right\}^h_{(x,y)}, \\
&\text{(b)}\ \{\tilde{R}(X^h, Y^h, Z^v)\}_{(x,y)} = \left\{\mathcal{R}(X, Y, Z) + \frac{1}{4}\mathcal{R}(\mathcal{R}(y, Z, Y), X, y)\right. \\
&\quad \left. - \frac{1}{4}\mathcal{R}(\mathcal{R}(y, Z, X), Y, y)\right\}^v_{(x,y)} \\
&\quad + \frac{1}{2}\{(D_X\mathcal{R})(y, Z, Y) - (D_Y\mathcal{R})(y, Z, X)\}^h_{(x,y)}, \\
&\text{(c)}\ \{\tilde{R}(X^v, Y^v, Z^h)\}_{(x,y)} = \left\{\mathcal{R}(X, Y, Z) + \frac{1}{4}\mathcal{R}(y, X, \mathcal{R}(y, Y, Z))\right. \\
&\quad \left. - \frac{1}{4}\mathcal{R}(y, Y, \mathcal{R}(y, X, Z))\right\}^h_{(x,y)}, \\
&\text{(d)}\ \tilde{R}(X^v, Y^v, Z^v) = 0, \\
&\text{(e)}\ \{\tilde{R}(X^h, Y^v, Z^h\}_{(x,y)} = \left\{\frac{1}{4}\mathcal{R}(\mathcal{R}(y, Y, Z), X, y) + \frac{1}{2}\mathcal{R}(X, Z, Y)\right\}^v_{(x,y)} \\
&\quad + \frac{1}{2}\{(D_X\mathcal{R})(y, Y, Z)\}^h_{(x,y)}, \\
&\text{(f)}\ \{\tilde{R}(X^h, Y^v, Z^v)\}_{(x,y)} \\
&\quad = -\left\{\frac{1}{2}\mathcal{R}(Y, Z, X) + \frac{1}{4}\mathcal{R}(y, Y, \mathcal{R}(y, Z, X))\right\}^h_{(x,y)},
\end{aligned}
$$

for any $X, Y, Z \in \Gamma(TM)$.

Proof. Taking into account that $B = C = 0$ on TM, we obtain (2.45) from (2.21)–(2.26) as follows. First, by using (2.44e), (2.44a), (2.43a) and (2.44f) in (2.21) we obtain

$$
\begin{aligned}
&\{\tilde{R}(X^h, Y^h, Z^h)\}_{(x,y)} \\
&\quad = \left\{R(X^h, Y^h, Z^h) + \frac{1}{2}\mathbf{R}(Z^h, \mathbf{R}(X^h, Y^h)) + \frac{1}{2}(\nabla_{Y^h}\mathbf{R})(X^h, Z^h)\right. \\
&\quad \left. - \frac{1}{2}(\nabla_{X^h}\mathbf{R})(Y^h, Z^h) - \frac{1}{4}\mathbf{R}(X^h, \mathbf{R}(Y^h, Z^h)) + \frac{1}{4}\mathbf{R}(Y^h, \mathbf{R}(X^h, Z^h))\right\}_{(x,y)}
\end{aligned}
$$

$$= \frac{1}{2}\{(D_Z\mathscr{R})(X, Y, y)\}^v_{(x,y)} + \Big\{\mathscr{R}(X, Y, Z) + \mathscr{R}(y, \mathscr{R}(X, Y, y), Z)$$

$$+ \frac{1}{4}\mathscr{R}(y, \mathscr{R}(Z, Y, y), X) + \frac{1}{4}\mathscr{R}(y, \mathscr{R}(X, Z, y), Y)\Big\}^h_{(x,y)},$$

which is just (2.45a). In a similar way, by using (2.43b), (2.44b) and (2.44g) into (2.22) we deduce (2.45b). Next, (2.45c) follows from (2.23) by using (2.43c) for $R(X^v, Y^v, Z^h)$, (2.44d) and (2.44h). As a consequence of (2.24) we obtain (2.45d) via (2.43c) for $R(X^v, Y^v, Z^v)$. Finally, by similar calculations in (2.25) and (2.26) we obtain (2.45e) and (2.45f) respectively. ■

3. Flag curvature of $\mathbf{F}^m$ and curvatures of (TM°, G)

Let $\mathbf{F}^m = (M, F)$ be an m-dimensional Finsler manifold and (TM°, G) be its slit tangent bundle endowed with the Sasaki-Finsler metric G. Then it is easy to check that J given by (2.32) is an almost Hermitian structure on (TM°, G), that is, we have

$$G(JX, JY) = G(X, Y), \quad \forall X, Y \in \Gamma(TTM^\circ). \tag{3.1}$$

The fundamental 2-form of (TM°, G, J) is denoted by Ω and it is given by

$$\Omega(X, Y) = G(X, JY). \tag{3.2}$$

Next, we consider the horizontal Liouville vector field L (see (2.29)) and define the 1-form θ by

$$\theta(X) = G(X, L), \quad \forall X \in \Gamma(TTM^\circ). \tag{3.3}$$

Also, we denote by N the vertical Liouville vector field on TM°, that is, we have

$$N = y^i \frac{\partial}{\partial y^i}. \tag{3.4}$$

Now, we express locally the twins of $\mathbf{R}$, C and B defined by (2.9), (2.10) and (2.11), respectively. Thus we put

$$\text{(a)}\ \mathbf{R}\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = \bar{R}^k_{ij}\frac{\delta}{\delta x^k}, \quad \text{(b)}\ C\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = \bar{C}^k_{ij}\frac{\delta}{\delta x^k},$$
$$\text{(c)}\ B\left(\frac{\delta}{\delta x^j}, \frac{\partial}{\partial y^i}\right) = \bar{B}^k_{ij}\frac{\partial}{\partial y^k}. \tag{3.5}$$

Taking into account that C_{ijk} and B_{ijk} from (1.13) and (2.8) respectively, are symmetric with respect to all indices, by (2.10) and (2.11) we deduce that

$$\text{(a)}\ \bar{C}^k_{ij} = C^k_{ij} \quad \text{and} \quad \text{(b)}\ \bar{B}^k_{ij} = B^k_{ij}. \tag{3.6}$$

However, this is not the case for $\bar{R}^k_{ij}$ and R^k_{ij}. Indeed, by using (3.5a), (1.18a), (2.9) and (1.6a) we obtain

$$\text{(3.7)} \qquad \text{(a)}\ \bar{\mathbf{R}}^{k}_{ij} = g_{ih}\mathbf{R}^{h}_{tj}g^{tk}, \quad \text{(b)}\ \bar{\mathbf{R}}_{kij} = \mathbf{R}_{ikj},$$

where we set

$$\bar{\mathbf{R}}_{kij} = g_{kh}\bar{\mathbf{R}}^{h}_{ij}.$$

Finally, since we have (cf. Bejancu-Farran [7], p. 234)

$$\text{(3.8)} \qquad y^{i}\mathbf{R}_{ikj} = 0,$$

by (3.7b) we see that

$$\text{(3.9)} \qquad y^{i}\bar{\mathbf{R}}_{kij} = 0.$$

Now, we prove some lemmas which refer to basic properties of the geometric objects that are investigated in the paper.

Lemma 3.1. *The adapted tensor fields* $\mathbf{R}$, C *and* B *satisfy the equalities*:

$$\text{(3.10)} \qquad \begin{aligned} &\text{(a)}\ \mathbf{R}(hX, N) = 0, \quad \text{(b)}\ \|\mathbf{R}(L, hX)\| = \|\mathbf{R}(L, JhX)\|, \\ &\text{(c)}\ C(hX, L) = C(L, hX) = C(L, vX) = C(hX, N) = 0, \\ &\text{(d)}\ B(vX, N) = B(N, vX) = B(L, vX) = B(hX, N) = 0, \end{aligned}$$

for any $X \in \Gamma(TTM^{\circ})$, *where* $\|\cdot\|$ *is taken with respect to* G.

Proof. First, by (3.4), (3.5a) and (3.9) we obtain

$$\mathbf{R}\left(\frac{\delta}{\delta x^{j}}, y^{i}\frac{\partial}{\partial y^{i}}\right) = y^{i}\bar{\mathbf{R}}^{k}_{ij}\frac{\delta}{\delta x^{k}} = g^{kh}y^{i}\bar{\mathbf{R}}_{hij}\frac{\delta}{\delta x^{k}} = 0,$$

which proves (3.10a). Then, by direct calculations using (1.18a), (1.6), (2.32), (3.5a) and (3.7) we deduce that

$$\left\|\mathbf{R}\left(L, \frac{\delta}{\delta x^{i}}\right)\right\|^{2} = \mathbf{R}_{ij}g^{jk}\mathbf{R}_{ik} = \left\|\mathbf{R}\left(L, \frac{\partial}{\partial y^{i}}\right)\right\|^{2},$$

and thus we proved (3.10b). Finally, (3.10c) and (3.10d) are direct consequences of (1.14) and (1.12) respectively, via (3.6). ∎

Lemma 3.2. *Let* ∇ *and* $\tilde{\nabla}$ *be the Vrănceanu and Levi-Civita connections on* (TM°, G). *Then we have the following equalities*:

$$\text{(3.11)} \qquad \begin{aligned} &\text{(a)}\ \nabla_{vX}L = -JvX, \quad \text{(b)}\ \nabla_{vX}N = vX, \quad \text{(c)}\ \nabla_{hX}N = 0, \\ &\text{(d)}\ \tilde{\nabla}_{vX}L = -JvX + \frac{1}{2}\mathbf{R}(L, vX), \quad \text{(e)}\ \tilde{\nabla}_{hX}L = -\frac{1}{2}\mathbf{R}(hX, L), \\ &\text{(f)}\ \tilde{\nabla}_{vX}N = vX, \quad \text{(g)}\ \tilde{\nabla}_{hX}N = 0, \quad \text{(h)}\ \tilde{\nabla}_{L}vX = \nabla_{L}vX + \frac{1}{2}\mathbf{R}(L, vX), \\ &\text{(i)}\ \nabla_{N}Y^{h} = \tilde{\nabla}_{N}Y^{h} = 0, \quad \text{(j)}\ \nabla_{N}Y^{v} = \tilde{\nabla}_{N}Y^{v} = 0, \end{aligned}$$

for any $X \in \Gamma(TTM^{\circ})$, $Y \in \Gamma(TM)$.

Proof. By using (2.3c) and (2.32b) we obtain

$$\nabla_{\partial/\partial y^j}\left(y^i\frac{\delta}{\delta x^i}\right)=\frac{\delta}{\delta x^j}=-J\left(\frac{\partial}{\partial y^i}\right),$$

which proves (3.11a). A similar reason applies for the proofs of (3.11b) and (3.11c). Now, by using (3.10c), (3.10d) and (3.11a) into (2.14) we deduce (3.11d). Analogously, by using (2.12), (2.13) and (2.15) we obtain the remaining formulae in (3.11). ■

LEMMA 3.3. *The covariant derivatives of* **R**, C *and* B *with respect to Vrănceanu connection on* (TM°, G) *satisfy the equalities*:

$$\begin{aligned}
&\text{(a)}\ (\nabla_X\mathbf{R})(L,L)=0,\quad \text{(b)}\ (\nabla_N\mathbf{R})(hX,vY)=\mathbf{R}(hX,vY),\\
&\text{(c)}\ (\nabla_X C)(L,L)=(\nabla_L C)(hX,L)=(\nabla_L C)(L,vX)=0,\\
&\text{(d)}\ (\nabla_N C)(hX,vY)=-C(hX,vY),\\
&\text{(e)}\ (\nabla_{vX}B)(N,N)=(\nabla_L B)(L,vX)=0,\\
&\text{(f)}\ (\nabla_{hX}B)(N,vY)=(\nabla_N B)(hX,vY)=(\nabla_N B)(vX,vY)=0,
\end{aligned}\tag{3.12}$$

for any $X, Y\in\Gamma(TTM^\circ)$.

Proof. Clearly, (3.12a) is a consequence of the skew-symmetry of **R**. Then, taking into account that $\bar{\mathbf{R}}_{ij}^k$ from (3.7a) are positively homogeneous of degree 1 with respect to (y^h) and by using (3.5a), (3.11i) and (3.11j) we obtain

$$(\nabla_N\mathbf{R})\left(\frac{\delta}{\delta x^j},\frac{\partial}{\partial y^i}\right)=\nabla_N\left(\bar{\mathbf{R}}_{ij}^k\frac{\delta}{\delta x^k}\right)=\bar{\mathbf{R}}_{ij}^k\frac{\delta}{\delta x^k},$$

which proves (3.12b). The formulae (3.12c) and (3.12e) are deduced by direct calculations using (3.10c), (3.10d) and taking into account that

$$\text{(a)}\ \nabla_L L=0\quad\text{and}\quad\text{(b)}\ \nabla_N N=N.\tag{3.13}$$

By similar calculations, taking into account that C_{ij}^k and B_{ij}^k are homogeneous of degrees -1 and 0 respectively, we obtain (3.12d) and (3.12f). ■

Next, for the sake of completeness, we examine the almost Hermitian structure (G,J) on TM°. First, by using (3.2) and (2.32) we deduce that

$$\Omega(hX,hY)=\Omega(vX,vY)=0\quad\text{and}\quad\Omega(hX,vY)=G(hX,JvY).\tag{3.14}$$

Then, taking into account that G is parallel with respect to $\tilde{\nabla}$ we obtain

$$d\theta(X,Y)=\frac{1}{2}\{G(Y,\tilde{\nabla}_X L)-G(X,\tilde{\nabla}_Y L)\}.\tag{3.15}$$

Now, from (3.11d) and (3.11e) we see that

$$\tilde{\nabla}_{vX}L\in\Gamma(HTM^\circ)\quad\text{and}\quad\tilde{\nabla}_{hX}L\in\Gamma(VTM^\circ).$$

Hence (3.15) implies

$$d\theta(hX, hY) = d\theta(vX, vY) = 0. \tag{3.16}$$

Finally, by using (3.11d), (3.11e) and (2.9) in (3.15) we obtain

$$\begin{aligned} d\theta(hX, vY) &= \frac{1}{4}\{G(\mathbf{R}(L, hX), vY) - G(\mathbf{R}(L, vY), hX)\} \\ &\quad + \frac{1}{2}G(hX, JvY) = \frac{1}{2}G(hX, JvY). \end{aligned} \tag{3.17}$$

Thus as a consequence of (3.14), (3.16) and (3.17) we have

$$\Omega(X, Y) = 2d\theta(X, Y), \quad \forall X, Y \in \Gamma(TTM^{\circ}), \tag{3.18}$$

which enables us to state the following proposition.

Proposition 3.1. *Let G be the Sasaki-Finsler metric on TM° given by* (1.3) *and J be the natural almost complex structure on TM° given by* (2.32). *Then (G, J) is an almost Kählerian structure on TM°.*

Remark 3.1. The above proposition represents a generalization to Finsler manifolds of a result of Tondeur [22] on the tangent bundle of a Riemannian manifold. A different proof of this result was given by Hasegawa, Yamauchi and Shimada [13]. Moreover, they proved that (G, J) becomes Kählerian if and only if $\mathbf{F}^m$ is of zero flag curvature. ■

In the remaining part of this section we present new characterizations of special Finsler manifolds: Riemannian manifolds, Landsberg manifolds, Finsler manifolds of constant flag curvature, by means of the geometry of (TM°, G). First, we prove the following surprising theorem.

Theorem 3.1. *$\mathbf{F}^m$ is a Riemannian manifold if and only if*

$$(\tilde{\nabla}_N \tilde{R})(hX, N, vY) = 0, \quad \forall X, Y \in \Gamma(TTM^{\circ}). \tag{3.19}$$

Proof. First, by using (3.12f), (3.12d), (3.12b), (3.10a), (3.10c) and (3.10d) in (2.26) we obtain

$$\tilde{R}(hX, N, vY) = R(hX, N, vY) + C(hX, vY) - \frac{1}{2}\mathbf{R}(hX, vY). \tag{3.20}$$

Next, by (1.5), (1.8) and (2.31) we deduce that

$$\left[\frac{\delta}{\delta x^i}, N\right] = 0.$$

Also, the homogeneity of G_{ij}^k and (3.11i) imply that

$$\nabla_N \nabla_{\delta/\delta x^j} \frac{\partial}{\partial y^i} = 0.$$

Then, by using again (3.11i) we infer that

$$R(hX, N, vY) = 0. \tag{3.21}$$

Thus (3.20) and (3.21) imply

$$\tilde{R}(hX, N, vY) = C(hX, vY) - \frac{1}{2}\mathbf{R}(hX, vY). \tag{3.22}$$

Now, by using (3.11i), (3.11g), (3.11j), (3.22) and the homogeneity of both C_{ij}^k and $\bar{\mathbf{R}}_{ij}^k$ we deduce that

$$\begin{aligned}(\tilde{\nabla}_N \tilde{R})\left(\frac{\delta}{\delta x^j}, N, \frac{\partial}{\partial y^i}\right) &= \tilde{\nabla}_N\left(\tilde{R}\left(\frac{\delta}{\delta x^j}, N, \frac{\partial}{\partial y^i}\right)\right) - \tilde{R}\left(\frac{\delta}{\delta x^j}, N, \frac{\partial}{\partial y^i}\right) \\ &= \tilde{\nabla}_N\left\{C_{ij}^k - \frac{1}{2}\bar{\mathbf{R}}_{ij}^k\right\}\frac{\delta}{\delta x^k} - \left\{C_{ij}^k - \frac{1}{2}\bar{\mathbf{R}}_{ij}^k\right\}\frac{\delta}{\delta x^k} = -2C_{ij}^k\frac{\delta}{\delta x^k},\end{aligned}$$

that is,

$$(\tilde{\nabla}_N \tilde{R})(hX, N, vY) = -2C(hX, vY). \tag{3.23}$$

As $\mathbf{F}^m$ is a Riemannian manifold if and only if $C = 0$, from (3.23) we obtain the assertion of the theorem. ■

Next, we recall the following well known result on the geometry of the tangent bundle of a Riemannian manifold.

THEOREM 3.2 (Kowalski [15]). *Let (M, g) be a Riemannian manifold and G be the Sasakian metric on TM. Then (TM, G) is locally symmetric if and only if (M, g) is locally Euclidean.*

Thus by combining Theorems 3.1 and 3.2 we obtain the following generalization of Theorem 3.2 to Finsler manifolds.

THEOREM 3.3. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold and G be the Sasaki-Finsler metric on TM°. Then (TM°, G) is locally symmetric if and only if $\mathbf{F}^m$ is locally Euclidean.*

Proof. Suppose (TM°, G) is locally symmetric. Then by Theorem 3.1 we deduce that $\mathbf{F}^m$ is a Riemannian manifold. Thus the remaining part of the proof is consequence of Theorem 3.2. ■

Remark 3.2. Theorem 3.3 has been stated also by Wu [24]. Unfortunately, the proof given by Wu has some mistakes. First, formula (4.3) of that paper is missing the v-curvature tensor field of Cartan connection whose local components are given by (2.34f) of the present paper. Then, as a consequence of (4.3), it is stated in Wu [24] that

$$\tilde{R}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\right)=0,$$

which must be replaced by

$$\tilde{R}\left(\frac{\partial}{\partial y^i},\frac{\partial}{\partial y^j},\frac{\partial}{\partial y^k}\right)=S^h_{kji}\frac{\partial}{\partial y^h}. \qquad \blacksquare$$

Aikou [2] has proved that $\mathbf{F}^m$ is a Landsberg manifold if and only if the leaves of vertical distribution are totally geodesic immersed in (TM°, G).

Now, we obtain new characterizations of Landsberg manifolds by means of the Levi-Civita connection $\tilde{\nabla}$ on (TM°, G).

THEOREM 3.4. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold. Then the following assertions are equivalent*:

(i) *$\mathbf{F}^m$ is a Landsberg manifold.*

(ii) $\tilde{R}(N, vY, vZ) = 0$, *for all* $Y, Z \in \Gamma(TTM^\circ)$.

(iii) $(\tilde{\nabla}_N \tilde{R})(N, vY, vZ) = 0$, *for all* $Y, Z \in \Gamma(TTM^\circ)$.

Proof. First, by using (3.10a), (3.10c), (3.10d) and (3.12f) in (2.24) we obtain

$$\tilde{R}(N, vY, vZ) = R(N, vY, vZ) + (\nabla_{vY} B)(N, vZ).$$

Then by (3.10d) and (3.11b) we deduce that

$$(\nabla_{vY} B)(N, vZ) = -B(vY, vZ).$$

Also, we have

$$R\left(N, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = y^k S^h_{ijk}\frac{\partial}{\partial y^h} = 0.$$

Hence we infer that

$$\tilde{R}(N, vY, vZ) = -B(vY, vZ). \tag{3.24}$$

Next, by using (3.24), (3.11f) and (3.11j) we obtain

$$(\tilde{\nabla}_N \tilde{R})\left(N, \frac{\partial}{\partial y^j}, \frac{\partial}{\partial y^i}\right) = -\tilde{\nabla}_N\left(B^k_{ij}\frac{\delta}{\delta x^k}\right) + B^k_{ij}\frac{\delta}{\delta x^k}.$$

Taking into account the homogeneity of B^k_{ij} and by using (3.11i) we deduce that

$$\tilde{\nabla}_N\left(B^k_{ij}\frac{\delta}{\delta x^k}\right) = 0.$$

Hence we proved that

$$(\tilde{\nabla}_B \tilde{R})(N, vY, vZ) = B(vY, vZ). \tag{3.25}$$

As $\mathbf{F}^m$ is a Landsberg manifold if and only if $B = 0$, the equivalence of the assertions follows from (3.24) and (3.25). $\blacksquare$

Next, we want to relate the flag curvature of $\mathbf{F}^m$ and the sectional curvature of (TM°, G). Let (x, y) be a point of TM°, where $x = (x^i)$ is a point of M and $y = (y^i)$ is a non-zero tangent vector to M at x. Suppose that $X = (X^i)$ is another tangent vector to M at x such that y and X are linearly independent in T_xM. Then, according to Bao-Chern-Shen [4], p. 68, we call the plane $\Pi(X) = \mathrm{span}\{y, X\}$ the flag at x with flagpole y and transverse edge X. Consider the horizontal lifts $X^h = X^i(\delta/\delta x^i)$ and L of X and y respectively, and give the following definition. The *flag curvature* of $\mathbf{F}^m$ at the point x with respect to the flag $\Pi(X)$ is the number

$$K(X) = \frac{G(R(X^h, L, L), X^h)}{\Delta(X^h, L)}, \tag{3.26}$$

where R is the curvature tensor of Vrănceanu connection on TM° and

$$\Delta(X^h, L) = G(X^h, X^h)G(L, L) - G(X^h, L)^2.$$

We may choose X such that X^h and L are orthogonal with respect to G (see a discussion in Bao-Chern-Shen [4], p. 69). For other types of curvatures of Finsler manifolds see Shen [21].

We also recall that the sectional curvature of (TM°, G) at the point (x, y) with respect to the plane $\mathrm{span}\{U, V\}$ is given by

$$\tilde{K}(U, V) = \frac{G(\tilde{R}(U, V, V), U)}{\Delta(U, V)}, \tag{3.27}$$

where

$$\Delta(U, V) = G(U, U)G(V, V) - G(U, V)^2.$$

In order to relate K and $\tilde{K}$ we prove the following theorem.

THEOREM 3.5. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold and (TM°, G) the slit tangent bundle of M endowed with the Sasaki-Finsler metric G. Then we have the following equalities*:

$$\tilde{R}(hX, L, L) = R(hX, L, L) + \frac{3}{4}\mathbf{R}(L, \mathbf{R}(hX, L)) + \frac{1}{2}(\nabla_L \mathbf{R})(hX, L), \tag{3.28}$$

$$\tilde{R}(vX, L, L) = R(vX, L, L) - \frac{1}{2}(\nabla_L \mathbf{R})(L, vX) + \frac{1}{4}\mathbf{R}(L, \mathbf{R}(L, vX)), \tag{3.29}$$

$$\tilde{R}(hX, N, N) = 0, \tag{3.30}$$

$$\tilde{R}(vX, N, N) = 0, \tag{3.31}$$

for any $X \in \Gamma(TTM^\circ)$.

Proof. First, by direct calculations using (3.10c), (3.10d), (3.12c) and (3.12a) in (2.21) we obtain (3.28). Then, by using (3.10c), (3.10d), (3.12a), (3.12c) and (3.12e) in (2.25) we deduce (3.29). Next, we use (3.10a), (3.10c), (3.10d), (3.12b), (3.12d) and (3.12f) in (2.26) and obtain

$$\tilde{R}(hX, N, N) = R(hX, N, N).$$

Then, by using (3.11c), (3.11b) and (2.4b), we obtain

$$R(hX, N, N) = 0,$$

which proves (3.30). Finally, by using (3.10c), (3.10d), (3.12e) and (3.12f) in (2.24) we deduce that

$$\tilde{R}(vX, N, N) = R(vX, N, N).$$

On the other hand, we have

$$R\left(\frac{\partial}{\partial y^k}, N, N\right) = y^i y^j S^h_{ijk} \frac{\partial}{\partial y^h} = 0,$$

and thus we obtain (3.31). ■

Now, let (x, y) be a point of TM° and X be a tangent vector to M at x such that $\text{span}\{y, X\}$ is a flag at x. Then we state the following theorem.

THEOREM 3.6. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold and (TM°, G) the slit tangent bundle of M endowed with the Sasaki-Finsler metric G. Then we have the following equalities*:

$$\tilde{K}(X^h, L) = K(X) - \frac{3}{4}\frac{\|\mathbf{R}(X^h, L)\|^2}{\Delta(X^h, L)}, \tag{3.32}$$

$$\tilde{K}(X^v, L) = \frac{1}{4}\frac{\|\mathbf{R}(X^h, L)\|^2}{\Delta(X^h, L)}, \tag{3.33}$$

$$\tilde{K}(X^h, N) = \tilde{K}(X^v, N) = 0, \tag{3.34}$$

where X^h and X^v are the horizontal and vertical lifts of X respectively, and $\|\cdot\|$ is taken with respect to G.

Proof. Taking into account that the last term in (3.28) is lying in $\Gamma(VTM^\circ)$ we obtain

$$G(\tilde{R}(X^h, L, L), X^h) = G(R(X^h, L, L), X^h) + \frac{3}{4}G(\mathbf{R}(L, \mathbf{R}(X^h, L)), X^h).$$

Then, by using (2.9), we deduce that

$$G(\tilde{R}(X^h, L, L), X^h) = G(R(X^h, L, L), X^h) - \frac{3}{4}G(\mathbf{R}(X^h, L), \mathbf{R}(X^h, L)),$$

which proves (3.32) via (3.26) and (3.27). Next, by using again (2.9) and taking into account that the first terms in (3.29) lie in $\Gamma(HTM^\circ)$ we infer that

$$G(\tilde{R}(X^v, L, L), X^v) = \frac{1}{4}G(\mathbf{R}(L, X^v), \mathbf{R}(L, X^v)).$$

Thus, by (3.10b) we obtain (3.33). Finally, (3.30) and (3.31) imply (3.34). ■

By (3.34) we can state the following corollary.

Corollary 3.1. *The tangent bundle of a Finsler manifold cannot be of positive or negative sectional curvature with respect to G.*

We call $\tilde{K}(X^h, L)$ (resp. $\tilde{K}(X^v, L)$) the *L-horizontal sectional curvature* (resp. *L-vertical sectional curvature*) of (TM°, G) at (x, y) with respect to $X \in T_xM$. Then, by using (3.32) and (3.33), we obtain the following interesting corollary.

Corollary 3.2. *The flag curvature of the Finsler manifold* $\mathbf{F}^m = (M, F)$ *is completely determined by the L-horizontal and L-vertical sectional curvatures of* (TM°, G) *as follows*:

$$K(X) = \tilde{K}(X^h, L) + 3\tilde{K}(X^v, L). \tag{3.35}$$

Remark 3.3. It is easy to see that (3.32) represents a generalization to Finsler geometry of a well known formula that relates curvatures of (TM, G) and (M, g), where (M, g) is a Riemannian manifold (see Aso [3], Gudmundsson-Kappos [12]). The latter was obtained by using a curvature formula for Riemannian submersions due to O'Neill [19], which of course does not apply to the Finsler case. ■

Next, we recall that F^m is a *Finsler manifold of scalar curvature K* if the flag curvature $K(X)$ is independent of X, that is, it is a scalar function on TM°. If, moreover, $K(X)$ is independent of both the vector X and the point (x, y), then $\mathbf{F}^m$ is called a *Finsler manifold of constant flag curvature*.

Several interesting characterizations of both classes of Finsler manifolds are presented in the book of Bao-Chern-Shen [4], pp. 76, 313. To state one characterization for each class we need to recall that h_{ij}, given by

$$h_{ij} = g_{ij} - \ell_i \ell_j, \quad \text{where } \ell_i = \frac{1}{F} g_{ik} y^k, \tag{3.36}$$

is called the *angular metric* of $\mathbf{F}^m$. In our study we need the characterizations from the next theorem (see Matsumoto [18], p. 168).

Theorem 3.7. *Let* $\mathbf{F}^m = (M, F)$ *be a Finsler manifold. Then we have the following assertions*:

(i) $\mathbf{F}^m$ *is of scalar flag curvature K if and only if*

$$\mathbf{R}_{ij} = KF^2 h_{ij}. \tag{3.37}$$

(ii) $\mathbf{F}^m$ *is of constant flag curvature c if and only if*

$$\mathbf{R}_{ij} = cF^2 h_{ij}. \tag{3.38}$$

Remark 3.4. We should mention that our $\mathbf{R}_{ij}$ is the same as R_{i0j} of Matsumoto [18], and it is F^2 times the R_{ij} of Bao-Chern-Shen [4]. ■

Now, we state new characterizations of the above Finsler manifolds by using geometry of (TM°, G).

THEOREM 3.8. *Let* $\mathbf{F}^m = (M, F)$ *be a Finsler manifold. Then we have the following assertions*:

(i) $\mathbf{F}^m$ *is of scalar flag curvature if and only if both the sectional curvature* $\tilde{K}(X^h, L)$ *and* $\tilde{K}(X^v, L)$ *of* (TM°, G) *are independent of* X.

(ii) $\mathbf{F}^m$ *is of constant flag curvature* c *if and only if the L-horizontal and L-vertical sectional curvatures of* (TM°, G) *are given by*

$$\tilde{K}(X^h, L) = c - \frac{3c^2}{4}F^2, \tag{3.39}$$

and

$$\tilde{K}(X^v, L) = \frac{c^2}{4}F^2, \tag{3.40}$$

respectively.

Proof. Suppose that $\mathbf{F}^m$ is of scalar curvature K. Then, by direct calculations using (1.15), (1.6), (3.37) and (3.36), we obtain

$$\|\mathbf{R}(X^h, L)\|^2 = K^2F^2\Delta(X^h, L).$$

Thus, from (3.32) and (3.33), we deduce that

$$\text{(a)}\ \tilde{K}(X^h, L) = K - \frac{3}{4}K^2F^2, \quad \text{(b)}\ \tilde{K}(X^v, L) = \frac{1}{4}K^2F^2, \tag{3.41}$$

that is, both $\tilde{K}(X^h, L)$ and $\tilde{K}(X^v, L)$ depend only on the point (x, y). Conversely, suppose

$$\tilde{K}(X^h, L) = f_1(x, y) \quad \text{and} \quad \tilde{K}(X^v, L) = f_2(x, y).$$

Then, by (3.35), we obtain

$$K(X) = f_1(x, y) + 3f_2(x, y),$$

which completes the proof of the assertion (i). Next, we suppose that $\mathbf{F}^m$ is of constant flag curvature c. Then, by using (3.41), we obtain (3.39) and (3.40). Conversely, suppose that $\tilde{K}(X^h, L)$ and $\tilde{K}(X^v, L)$ are given by (3.39) and (3.40), respectively. Then, by using (3.35), we deduce that $K(X) = c$. ■

In particular, we state the following theorem.

THEOREM 3.9. *Let* $\mathbf{F}^m = (M, F)$ *be a Finsler manifold. Then the following assertions are equivalent*:

(i) $\mathbf{F}^m$ *is of zero flag curvature.*

(ii) *Both the L-horizontal and L-vertical sectional curvatures of* (TM°, G) *vanish on* TM°.

(iii) *The curvature tensor field* $\tilde{R}$ *of* (TM°, G) *satisfies*

$$\tilde{R}(hX, hY, N) = 0, \quad \forall X, Y \in \Gamma(TTM^{\circ}).$$

Proof. The equivalence of (i) and (ii) is a direct consequence of the assertion (ii) of Theorem 3.8. Next, by using (3.11g) and (1.18b) we obtain

$$\tilde{R}(hX, hY, N) = \mathbf{R}(hX, hY), \quad \forall X, Y \in \Gamma(TTM^{\circ}).$$

Thus, by using (3.38), (1.6c) and (1.7d), we obtain the equivalence of (i) and (iii). ■

Remark 3.5. From the above theorem we see that $\tilde{R} = 0$ implies $K = 0$. It is interesting to note that the converse is not true. Indeed, let $\mathbf{F}^m$ be a Finsler manifold of zero flag curvature and non-Riemannian. Then, by (3.22), we conclude that there exist hX and vY such that $\tilde{R}(hX, N, vY) \neq 0$. ■

4. Indicatrix bundle and curvature of $\mathbf{F}^m$

Let $\mathbf{F}^m = (M, F)$ be an m-dimensional Finsler manifold and (TM°, G) be the slit tangent bundle of M endowed with the Sasaki-Finsler metric G. For any $c \neq 0$ we denote by $IM(c)$ the hypersurface of TM° given by the equation

$$F^2(x, y) = \frac{\varepsilon}{c}, \tag{4.1}$$

where $\varepsilon = +1$ or $\varepsilon = -1$, according to $c > 0$ or $c < 0$, respectively. We call $IM(c)$ the *c-indicatrix bundle* of $\mathbf{F}^m$, and note that $IM(c) = IM(-c)$. It is easy to check that L given by (2.29) is tangent to $IM(c)$, while $N = JL$ is a normal vector field to $IM(c)$. Then we set:

$$\text{(a) } \xi = 2L, \quad \text{(b) } \mathcal{N} = 2N, \quad \text{and} \quad \text{(c) } \eta = \frac{\varepsilon c}{2}\theta, \tag{4.2}$$

where θ is the 1-form given by (3.3). Next, we consider on TM° the Riemannian metric g given by

$$g = \frac{\varepsilon c}{4} G, \tag{4.3}$$

and denote by the same symbol g the Riemannian metric induced on $IM(c)$. Then both ξ and $\mathcal{N}$ are unit vector fields with respect to g and we have

$$\eta(X) = g(X, \xi), \quad \forall X \in \Gamma(TIM(c)). \tag{4.4}$$

Finally, for any vector field X on $IM(c)$ we put

$$JX = \varphi X + \eta(X)\mathcal{N}, \tag{4.5}$$

where J is the almost complex structure on TM° given by (2.32) and φX is a vector field that is tangent to $IM(c)$. Then we prove the following proposition.

Proposition 4.1. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold. Then (φ, ξ, η, g) is a contact metric structure on $IM(c)$.*

Proof. Since ξ is a unit vector field with respect to g, from (4.4) we obtain

$$\eta(\xi) = 1.$$

Also, by using (4.4) and (4.5) we deduce that

$$\eta \circ \varphi = 0.$$

Then we apply J to (4.5) and obtain

$$\varphi^2 = -I + \eta \otimes \xi. \tag{4.6}$$

Next, taking into account that J is an isometry with respect to G (cf. (3.1)), and using (4.5) we infer that

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \forall X, Y \in \Gamma(TIM(c)). \tag{4.7}$$

Then, according to Blair [9], p. 33, (φ, ξ, η, g) is an almost contact metric structure on $IM(c)$. Finally, we consider the 2-form

$$\Phi(X, Y) = g(X, \varphi Y),$$

and, by using (4.5), (4.3), (3.2), (3.18) and (4.2c), we deduce that $\Phi = d\eta$. Thus (φ, ξ, η, g) is a contact metric structure on $IM(c)$. ■

Remark 4.1. By using the Cartan connection on $\mathbf{F}^m$, Hasegawa, Yamauchi and Shimada [13], and Anastasiei [1] proved the existence of a contact metric structure on $IM(1)$. ■

Next, we take a vector field X on M and consider its horizontal and vertical lifts X^h and X^v on TM°, respectively. Then X^h is tangent to $IM(c)$, while X^v is expressed at the points of $IM(c)$ as follows

$$X^v = X^t + \eta(X^h)\mathcal{N}, \tag{4.8}$$

where X^t represents the part of X^v that is tangent to $IM(c)$ and it is called the *tangential lift* of X on $IM(c)$ (see Boecks-Vanhecke [10] for the sphere bundle). Taking into account (4.2b), (3.11f) and (3.11g) we deduce that

$$\text{(a)}\ \tilde{\nabla}_{X^t}\mathcal{N} = 2X^t \quad \text{and} \quad \text{(b)}\ \tilde{\nabla}_{X^h}\mathcal{N} = 0. \tag{4.9}$$

On the other hand, we have the Weingarten formula

$$\tilde{\nabla}_U \mathcal{N} = -AU, \quad \forall U \in \Gamma(TIM(c)), \tag{4.10}$$

where A is the shape operator of the immersion of $IM(c)$ in (TM°, g). Thus comparing (4.9) with (4.10) we obtain

$$\text{(a)}\ AX^t = -2X^t \quad \text{and} \quad \text{(b)}\ AX^h = 0. \tag{4.11}$$

This entitles us to state the following proposition.

PROPOSITION 4.2. *The c-indicatrix bundle has m principal curvatures equal to 0 and $m-1$ principal curvatures equal to -2. Thus $IM(c)$ is a hypersurface of (TM°, g) of constant mean curvature*

$$H = -2\frac{m-1}{2m-1}.$$

Remark 4.2. Matsumoto [17] has considered the metric G on the indicatrix bundle corresponding to $c = 1$ and obtain m principal curvatures 0 and $m-1$ principal curvatures equal to -1. ■

In the remaining part of this section we want to show that the geometry of the c-indicatrix bundle is deeply related to the study of Finsler manifolds of constant flag curvature c. First we prove the following theorem.

THEOREM 4.1. *Let $\mathbf{F}^m = (M, F)$ be a Finsler manifold and c be a non zero real number. Then $\mathbf{F}^m$ is of constant flag curvature c if and only if we have*

$$\mathbf{R}_{ij}(x,y) = \varepsilon h_{ij}(x,y), \quad \forall (x,y) \in IM(c), \tag{4.12}$$

where $\mathbf{R}_{ij}$ and h_{ij} are given by (1.6b) *and* (3.36) *respectively, and $\varepsilon = +1$ or $\varepsilon = -1$, according to $c > 0$ or $c < 0$.*

Proof. Suppose $\mathbf{F}^m$ is a Finsler manifold of constant flag curvature $c \neq 0$. Then, by (3.38) and (4.1), we have (4.12). Conversely, suppose (4.12) is satisfied. Thus, for any point of $IM(c)$, we have (3.38). It remains to show that (3.38) is still true at any other point of TM°. Let (x, y) be a point of $TM^\circ \backslash IM(c)$. Then there exists a positive number k such that $F^2(x, y) = k$. As F^2 is positively homogeneous of degree 2 with respect to y, we have

$$F^2\left(x, \sqrt{\frac{\varepsilon}{kc}}y\right) = \frac{\varepsilon}{c}.$$

Hence the point $\left(x, \sqrt{\frac{\varepsilon}{kc}}y\right)$ lies on $IM(c)$ and by (4.12) we deduce that

$$\mathbf{R}_{ij}\left(x, \sqrt{\frac{\varepsilon}{kc}}y\right) = \varepsilon h_{ij}\left(x, \sqrt{\frac{\varepsilon}{kc}}y\right).$$

Then, taking into account that $\mathbf{R}_{ij}$ and h_{ij} are homogeneous of degrees 2 and 0 respectively, we obtain

$$R_{ij}(x,y) = ckh_{ij}(x,y) = cF^2(x,y)h_{ij}(x,y).$$

Thus (3.38) is satisfied at (x, y), and therefore $\mathbf{F}^m$ is of constant flag curvature c. ■

COROLLARY 4.1. $\mathbf{F}^m$ *is of constant flag curvature $c \neq 0$ if and only if on $IM(c)$ we have*

(4.13) $$R^i_k = \varepsilon(\delta^i_k - 2y^i\eta_k),$$

where we put $\eta_k = \eta(\delta/\delta x^k)$.

Proof. By direct calculations using (1.6c), (3.36), (4.2) and (3.3) we obtain the equivalence of (4.12) and (4.13). ■

Next, we make the notations:

$$HIM(c) = HTM^\circ_{|IM(c)} \quad \text{and} \quad VIM(c) = VTM^\circ_{|IM(c)}.$$

Also, we denote by $IM(c)^t$ the complementary orthogonal vector bundle to span$\{\mathcal{N}\}$ into $VIM(c)$. Then the tangent bundle of $IM(c)$ admits the orthogonal decomposition

$$TIM(c) = HIM(c) \oplus IM(c)^t.$$

Now, apply J to (4.8) and by using (4.5), (4.4) and taking into account that $J\mathcal{N} = -\xi$ and $JX^v = -X^h$ we deduce that

(4.14) $$X^h = -\varphi X^t + \eta(X^h)\xi.$$

Thus the above decomposition becomes

(4.15) $$TIM(c) = \text{span}\{\xi\} \oplus \varphi(IM(c)^t) \oplus IM(c)^t.$$

By applying φ to (4.14) and using (4.6) we obtain

(4.16) $$\varphi X^h = X^t.$$

Finally, the integrability tensor field $\mathbf{R}$ of HTM° defines a vector bundle morhpism denoted by the same symbol $\mathbf{R}$ and given by

(4.17) $$\mathbf{R} : HIM(c) \to HIM(c) : \mathbf{R}(X^h) = J\mathbf{R}(L, X^h).$$

By (1.6) and (4.17) we infer that

(4.18) $$\mathbf{R}\left(\frac{\delta}{\delta x^j}\right) = \mathbf{R}^i_j \frac{\delta}{\delta x^i}.$$

Now, we can state the following theorem.

Theorem 4.2. *$\mathbf{F}^m = (M, F)$ is a Finsler manifold of constant flag curvature $c \neq 0$ if and only if at any point $(x, y) \in IM(c)$ and for any $X \in T_xM$, one of the following equalities is satisfied*:

(4.19) $$\varphi X^h = \varepsilon\mathbf{R}(X^h, L),$$

(4.20) $$\varphi X^t = \varepsilon\mathbf{R}(L, X^t),$$

(4.21) $$\varphi X^t = \varepsilon\mathbf{R}(\varphi X^t).$$

Proof. Take $X = (\partial/\partial x^k)_x$ in (4.19) and, by using (1.15), (1.6), (4.5) and (4.2b), we obtain the equivalence of (4.13) and (4.19). Next, we suppose that (4.19) is satisfied, and by using (2.9) and (4.7) we deduce that

$$g(\mathbf{R}(L, X^t), Y^h) = g(\mathbf{R}(L, Y^h), X^t) = -\varepsilon g(\varphi Y^h, X^t)$$
$$= g(\varepsilon\varphi X^t, Y^h), \quad \forall Y \in T_xM,$$

which proves (4.20). Conversely, suppose (4.20) is true, and by using (2.9), (4.8), (3.10a) and (4.7) we infer that

$$g(\mathbf{R}(X^h, L), Y^v) = -g(\mathbf{R}(L, X^h), Y^v) = -g(\mathbf{R}(L, Y^t + \eta(Y^h)\mathcal{N}), X^h)$$
$$= -g(\mathbf{R}(L, Y^t), X^h) = -\varepsilon g(\varphi Y^t, X^h) = g(Y^t, \varepsilon\varphi X^h)$$
$$= g(\varepsilon\varphi X^h, Y^v), \quad \forall Y \in T_xM.$$

Thus (4.19) and (4.20) are equivalent. Finally, by using (4.17) and (4.14) we deduce that (4.21) is equivalent to

$$\varepsilon\mathbf{R}(X^h, L) = -J\varphi X^t,$$

which in turn, is equivalent to (4.19) via (4.5), (4.6) and (4.16). This completes the proof of the theorem. ■

Now, let I be the identity morhpism on $\varphi(IM(c)^t)$. Then, by using (4.21), we obtain the following simple characterization of Finsler manifolds of non zero constant flag curvature.

COROLLARY 4.2. $\mathbf{F}^m$ *is of constant flag curvature* $c \neq 0$ *if and only if the restriction of* $\mathbf{R}$ *given by* (4.17) *to* $\varphi(IM(c)^t)$ *is either* I *or* $-I$, *according to* $c > 0$ *or* $c < 0$, *respectively.*

Next, let $\bar{\nabla}$ be the Levi-Civita connection on $(IM(c), g)$. Then, by using Gauss formula for the immersion of $IM(c)$ in (TM°, g), we obtain

$$\bar{\nabla}_U\xi = \tilde{\nabla}_U\xi, \quad \forall U \in \Gamma(TIM(c)), \tag{4.22}$$

since by (4.11b) we have

$$A\xi = 0. \tag{4.23}$$

THEOREM 4.3. $\mathbf{F}^m = (M, F)$ *is of constant flag curvature* $c \neq 0$ *if and only if at any point* $(x, y) \in IM(c)$ *and for any* $X \in T_xM$, *one of the following equalities is satisfied*:

$$\bar{\nabla}_{X^h}\xi = -\varepsilon\varphi X^h, \tag{4.24}$$

$$\bar{\nabla}_{X^t}\xi = (\varepsilon - 2)\varphi X^t. \tag{4.25}$$

Proof. By using (4.22), (3.11e) and (4.2a), we obtain

$$\bar{\nabla}_{X^h}\xi = -\mathbf{R}(X^h, L),$$

which proves the equivalence of (4.24) and (4.19). In a similar way, by using (4.22), (3.11d), (4.5) and (4.4), we deduce that

$$\bar{\nabla}_{X^t}\xi = -2\varphi X^t + \mathbf{R}(L, X^t).$$

Thus (4.25) is equivalent to (4.20). ■

Corollary 4.3. *$\mathbf{F}^m$ is of positive constant flag curvature c if and only if the contact metric structure (φ, ξ, η, g) on $IM(c)$ is a K-contact structure.*

Proof. By Theorem 4.3 we deduce that $\mathbf{F}^m$ is of positive constant flag curvature c if and only if

$$\bar{\nabla}_U\xi = -\varphi U, \quad \forall U \in \Gamma(TIM(c)). \tag{4.26}$$

Then, by a result of Blair [8], p. 64, we see that (4.26) is just the condition for (φ, ξ, η, g) to be a K-contact structure. ■

Remark 4.2. By Corollary 4.3 we conclude that $\mathbf{F}^m$ is of positive constant flag curvature c if and only if ξ is a Killing vector field on $IM(c)$ (cf. Bejancu and Farran [5]). In particular, for $c = 1$ it was proved by Hasegawa, Yamauchi and Shimada [13], and Anastasiei [1] that $IM(1)$ admits a Sasakian structure. ■

Now, let $\bar{R}$ and $\tilde{R}$ be the curvature tensor field of $\bar{\nabla}$ and $\tilde{\nabla}$, respectively. Then, by using (4.23) and (4.3) into the Gauss equation of the immersion of $IM(c)$ in (TM°, g) (cf. Chen [11], p. 45), we obtain

$$g(\bar{R}(U, V, \xi), W) = \frac{\varepsilon c}{4} G(\tilde{R}(U, V, \xi), W), \tag{4.27}$$

for any $U, V, W \in \Gamma(TIM(c))$. The *$\xi$-horizontal sectional curvature* (resp. *ξ-tangential sectional curvature*) of $(IM(c), g)$ at the point (x, y) with respect to $X \in T_xM$ is the sectional curvature $\bar{K}(X^h, \xi)$ (resp. $\bar{K}(X^t, \xi)$) given by a similar formula to (3.27), but with respect to g and $\bar{R}$.

Theorem 4.4. *$\mathbf{F}^m = (M, F)$ is a Finsler manifold of constant flag curvature $c \neq 0$ if and only if at any point $(x, y) \in IM(c)$ and for any $X \in T_xM$, the ξ-horizontal and ξ-tangential sectional curvatures of $(IM(c), g)$ are given by*

$$\bar{K}(X^h, \xi) = 4\varepsilon - 3, \tag{4.28}$$

and

$$\bar{K}(X^t, \xi) = 1, \tag{4.29}$$

respectively.

Proof. By using (3.27) for both $\bar{K}$ and $\tilde{K}$ and (4.27) we obtain

$$\bar{K}(X^h, \xi) = \frac{4\varepsilon}{c}\tilde{K}(X^h, \xi). \tag{4.30}$$

Next, by using (3.11i) and (3.11e) and taking into account that $[\mathcal{N}, \xi] = 0$, we deduce that

$$\tilde{\mathscr{R}}(\mathscr{N}, \xi, \xi) = 0. \tag{4.31}$$

Thus, by direct calculations using (4.27), (4.8), (4.31) and (3.27) we infer that

$$\bar{K}(X^t, \xi) = \frac{4\varepsilon}{c} \tilde{K}(X^v, \xi). \tag{4.32}$$

Now, suppose that $\mathbf{F}^m$ is of constant flag curvature $c \neq 0$. Then, by assertion (ii) of Theorem 3.8 and (4.1), we have

$$\tilde{K}(X^h, \xi) = \frac{c}{4}(4 - 3\varepsilon) \quad \text{and} \quad \tilde{K}(X^v, \xi) = \frac{\varepsilon c}{4} \quad \text{on } IM(c). \tag{4.33}$$

Thus, by using (4.30), (4.32) and (4.33) we obtain (4.28) and (4.29). Conversely, suppose (4.28) and (4.29) be satisfied on $IM(c)$. Then, by (4.30) and (4.32) , we deduce (4.33). Thus the conditions in (3.39) and (3.40) are satisfied on $IM(c)$. Hence $K(X) = c$ at any point $(x, y) \in IM(c)$ and for any $X \in T_xM$. This means that (4.12) is true, and therefore by Theorem 4.1 we conclude that $\mathbf{F}^m$ is of constant flag curvature c. ■

In particular, we obtain the following corollary.

COROLLARY 4.4. $\mathbf{F}^m$ *is of positive constant curvature if and only if we have*

$$\bar{K}(X^h, \xi) = \bar{K}(X^t, \xi) = 1, \quad \textit{on } IM(c).$$

Remark 4.3. The necessity of conditions on $\bar{K}$ in this corollary can be also deduced by using Corollary 4.3 and a general result on the curvature tensor field of a K-contact manifold (cf. Blair [8], p. 65). ■

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