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TANGENT BUNDLE CONNECTIONS AND THE GEODESIC FLOW

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1. **Introduction.** If M is a Riemannian manifold there are several ways to induce a pseudo-Riemannian metric on the total space TM of the tangent bundle of M ([**10**], [**12**]). The present paper is concerned with the use of these natural structures to study the geodesic flow on the unit sphere bundle of M.

Our point of view is to study the dynamical properties of the geodesic flow in terms of certain spectral properties of the operator, "Lie differentiation in the direction of the geodesic vector field," defined in an appropriate space of sections. This operator decomposes into a sum of operators, one of which is the covariant derivative associated with the tangent bundle connection defined by an induced pseudo-Riemannian metric on TM, and the spectral properties follow from this decomposition.

In §2 and §3 we collect notation and previous results. §4 contains the decomposition of the Lie differentiation operator. The final sections, 5 and 6, contain the applications. In particular, we prove that the geodesic flow in the unit tangent bundle of a compact manifold of constant negative curvature is infinitesimally ergodic.

2. **The geodesic flow, spaces of sections and the adjoint representation.** Let M denote a smooth compact connected Riemannian manifold with metric tensor g. The geodesic flow G_t on the unit tangent bundle T_1M generates the geodesic vector field X which has local form

$$X(x, v) = (x, v, v, -\Gamma(v, v))$$

where $(x, v) \in T_1M$ and Γ is the vector valued bilinear form defined by the Levi-Civita connection.

If $T\pi\colon T^2M \to TM$ dedotes the derivative of $\pi\colon TM \to M$ we have the familiar commutative diagram of bundle maps:

$$\begin{array}{ccc} T^2M & \xrightarrow{T\pi} & TM \\ \Big\downarrow{\scriptstyle \pi_{TM}} & & \Big\downarrow{\scriptstyle \pi} \\ TM & \xrightarrow{\pi} & M \end{array}$$

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In natural coordinates, if $(x, v, u, w) \in T^2M$, then $T\pi(x, v, u, w) = (x, u)$. The kernel of $T\pi$ on each fiber defines the vertical subbundle of T^2M. Using the Levi-Civita connection define also the connection map K: $T^2M \to TM$ given by $K(x, v, u, w) = (x, w + \Gamma(u, v))$. The kernel of K on each fiber defines the horizontal subbundle of T^2M. Moreover, the bundle map $F: T^2M \to TM \oplus TM$ given by $F(A) = (T\pi(A), K(A))$ defines an isomorphism of vector bundles:

$$\begin{tikzcd} T^2M \arrow[rr, "F"] \arrow[dr, "\pi_{TM}"'] & & TM \oplus TM \arrow[dl] \\ & TM & \end{tikzcd}$$

Identifying the image of $T\pi$ with the horizontal subbundle and the image of K with the vertical subbundle decomposes T^2M into horizontal and vertical components and provides TM with the Sasaki metric (**[10]**) given by

$$S(A, B) = g(T\pi A, T\pi B) + g(KA, KB).$$

The geodesic vector field X generates a line bundle $[X]$ over T_1M. If E denotes the orthogonal complement of $[X]$ in $T(T_1M)$ with respect to the Sasaki metric, then $T(T_1M)$ is isomorphic to $E \oplus [X]$.

PROPOSITION 2.1. *If A is a vector field on TM given by $A(v) = (v, a(v), b(v))$ in horizontal and vertical components, then A is a section of $E \to T_1M$ if and only if $g(a, v) = 0$ and $g(b, v) = 0$.*

PROOF. See **[8]**.

Since T_1M is compact, the space $C^0(E)$ of all continuous sections of E has the structure of a Banach space with norm

$$\|A\| = \sup\{S_v(A, A)^{1/2} \mid v \in T_1M\}.$$

Also, G_t preserves a volume which defines a Borel measure μ on T_1M and hence an inner product on $C^0(E)$ given by

$$\langle A, B\rangle = \int_{T_1M} S(A, B)d\mu.$$

This inner product extends naturally to all complex sections of E and defines a pre-Hilbert space structure whose completion is the Hilbert space $H^0(E)$ of all square integrable sections. In the standard manner (see **[3]** for details), one defines, for $r > 0$, $H^r(E)$, the Sobolev space of all sections of E with r square integrable derivatives in $H^0(E)$. Finally, define $H^{-r}(E)$ to be the dual space of $H^r(E)$.

3. **The adjoint representation of G_t.** For a vector field A on T_1M define Λ_t, the adjoint representation of G_t, by $\Lambda_t A = TG_{-t} \circ A \circ G_t$ where TG_t is the derivative of G_t. We view Λ_t as a group of transformations in the various spaces of sections. For the definitions and standard theorems of semi-group theory see [2] and [5]. In particular, we state the following theorem.

THEOREM 3.1. *If $T(t)$ is a strongly continuous semi-group of bounded linear operators in a Banach space B (strongly continuous means* $\lim_{t\to 0} \|T(t)x - x\| = 0$ *for all $x \in B$), then the infinitesimal generator A defined by*

$$Ax = \lim_{h\to 0} \frac{1}{h}(T(h) - I)x$$

is a closed linear operator in B with dense domain.

For the 1-parameter group Λ_t we have the following proposition.

PROPOSITION 3.2. *In $C^0(E)$ or $H^r(E)$*

(a) *Λ_t is a strongly continuous group of bounded operators, and*

(b) *The infinitesimal generator L_X of Λ_t is the closed extension of the operator given by Lie differentiation in the direction of the geodesic vector field.*

PROOF: (a) is proved for $C^0(E)$ in Ôtsuki [8]. The proof of part (a) for $H^r(E)$ is standard and left to the reader.

Part (b) follows from theorem 3.1 and the definition of Lie differentiation:

$$L_X A = \lim_{t\to 0} \frac{1}{t}(TG_{-t} \circ A \circ G_t - A).$$

The spectral properties of the adjoint representation reflect important dynamical properties of the flow G_t. In particular, recall the definition of an Anosov flow.

DEFINITION 3.3. A flow G_t on a manifold M is called an *Anosov flow* if there is a continuous splitting of the tangent bundle $TM = E^s \oplus E^u \oplus [X]$ such that

(a) $[X]$ is the line bundle generated by the tangent field to the flow G_t,

(b) E^s and E^u are TG_t invariant subbundles, and

(c) for some Riemannian metric $\| \;\|$ on M there exist positive constants c and w so that for $t \geqq 0$

$$\|TG_t A\| \leqq C\, e^{-wt}\, \|A\| \text{ for all } A \in E^s, \text{ and}$$

$$\|TG_{-t} A\| \leqq C\, e^{-wt}\, \|A\| \text{ for all } A \in E^u.$$

Also, if T is an operator in a Banach space B, the spectrum $\sigma(T)$ of T is the set of complex numbers λ for which the resolvent $R(\lambda, T) = (\lambda I - T)^{-1}$ does not exist as a bounded operator. With these definitions we state the first theorem of the subject.

THEOREM 3.4. (Mather [7], Ôtsuki [8]). *The geodesic flow G_t on T_1M is Anosov if and only if $\sigma(\Lambda_1)$ lies off the unit circle for Λ_t considered as an operator in $C^0(E)$.*

4. **Decomposition of L_X.** Since formally $\Lambda_t = \exp(tL_X)$, the spectral analysis of Λ_t is closely related to that of L_X. While this fact is the motivation for this section, our purpose is to provide a general setting for studying operator theoretic questions related to both of the operators Λ_t and L_X. Specific application of the results of this section will be made in §5 and §6.

In order to study the operator L_X we take advantage of the differential geometric structure associated with a Riemannian metric. Whenever ∇ is a covariant derivative $L_X - \nabla_X$ is a vector valued tensor. We exploit this fact by making a judicious choice for the covariant derivative on TM.

DEFINITION 4.1. If (M, g) is a Riemannian manifold with the tangent bundle $\pi: TM \to M$ and connection map $K: T^2M \to TM$, then the *Vilms metric* ([12]) V on TM is the pseudo-Riemannian metric given by

$$V(X, Y) = g(T\pi X, KY) + g(KX, T\pi Y).$$

THEOREM 4.2. *If $\tilde{\nabla}$ is the Levi-Civita connection for the Vilms metric V, $X(x, v) = (x, v, u, w)$ and $Y(x, v) = (x, v, \alpha, \beta)$ are vector fields on TM represented in natural coordinates and $\tilde{\nabla}_X Y = XY + \tilde{\Gamma}(X, Y)$, then $\tilde{\Gamma}$ is the symmetric vector valued tensor defined by*

$$\tilde{\Gamma}_{(x,v)}((u, w), (\alpha, \beta)) = (\Gamma_x(\alpha, u), D_1\Gamma_x(\alpha, u)v + \Gamma_x(\alpha, w) + \Gamma_x(\beta, u))$$

where $\nabla_u v = \Gamma(u, v)$ is the Levi-Civita connection associated with the metric g on M, XY and uv denote the vector directional derivatives and D_1 denotes the derivative with respect to the manifold variable "x".

PROOF. See Vilms [12].

THEOREM 4.3. *If X is the geodesic vector field and $\Omega = L_X - \tilde{\nabla}_X$ then Ω is the vector valued tensor represented in horizontal and vertical components by*

$$\Omega_{x,v} = \begin{pmatrix} 0 & -I \\ R_x(\cdot, v)v & 0 \end{pmatrix}$$

where I is the $n \times n$ identity matrix and R is the Riemann curvature tensor associated with g.

PROOF: By the definition of Ω

$$\begin{aligned} \Omega(Y) &= L_X Y - \tilde{\nabla}_X Y \\ &= XY - YX - XY - \tilde{\Gamma}(X, Y) \\ &= -YX - \tilde{\Gamma}(X, Y). \end{aligned}$$

To express Ω in horizontal and vertical components apply the change of coordinates $F\colon T^2M \to TM \oplus TM$. For this let Y be expressed in horizontal and vertical components as $Y(x, v) = (x, v, a, b)$ and compute $F\Omega F^{-1}(Y)$.

$$\begin{aligned} F\Omega F^{-1}(Y) &= F\Omega F^{-1}(x, v, a, b) \\ &= F\Omega_{x,v}(a, b - \Gamma_x(z, v)). \end{aligned}$$

Using Theorem 4.2 the definition of directional derivative compute

$$\begin{aligned} -\tilde{\Gamma}(X, Y) &= (-\Gamma_x(a, v), -\Gamma_x(b - \Gamma_x(a, v), v) \\ &\qquad - D_1\Gamma_x(a, v)v + \Gamma_x(a, \Gamma_x(v, v))) \\ &= (-\Gamma_x(a, v), -\Gamma_x(b, v) + \Gamma_x(\Gamma_x(a, v), v) \\ &\qquad - D_1\Gamma_x(a, v)v + \Gamma_x(a, \Gamma_x(v, v))) \end{aligned}$$

and

$$\begin{aligned} -YX &= (-b + \Gamma_x(a, v), D_1\Gamma_x(v, v)a + 2\Gamma_x(b - \Gamma_x(a, v), v)) \\ &= (-b + \Gamma_x(a, v), D_1\Gamma_x(v, v)a + 2\Gamma_x(b, v) - 2\Gamma_x(\Gamma_x(a, v), v)). \end{aligned}$$

Then,

$$\begin{aligned} \Omega(Y) &= -YX - \tilde{\Gamma}(X, Y) \\ &= (-b, D_1\Gamma_x(v, v)a - D_1\Gamma_x(a, v)v + \Gamma_x(b, v) + \Gamma_x(a, \Gamma_x(v, v)) \\ &\qquad - \Gamma_x(\Gamma_x(a, v), v)). \end{aligned}$$

Hence,

$$\begin{aligned} F\Omega(Y) &= (-b, D_1\Gamma_x(v, v)a - D_1\Gamma_x(a, v)v + \Gamma_x(a, \Gamma_x(v, v)) - \Gamma_x(\Gamma_x(a, v)v)) \\ &= (-b, R_x(a, v)v). \end{aligned}$$

Let $\mathfrak{X}$ denote either $C^0(E)$ or $H^0(E)$ and define an indefinite innerproduct on $\mathfrak{X}$ by $(Y, Z) = \int_{T_1M} V(Y, Z)d\mu$.

THEOREM 4.4. *Ω extends uniquely to a bounded operator on $\mathfrak{X}$ such that $V(\Omega Y, Z) = V(Y, \Omega Z)$ for all Y and Z in $\mathfrak{X}$. Also, $(\Omega Y, Z) = (Y, \Omega Z)$.*

PROOF. If $A \in C^0_R(E)$ is given by $A(x, v) = (x, v, a, b)$ in horizontal and vertical components, then by proposition 3.1 $g(a, v)$ and $g(b, v)$ vanish. As $\Omega(A) = (-b, R_x(a, v)v)$, $\Omega(A) \in C^0_R(E)$ if and only if $g(R_x(a, v)v, v) = 0$, but this follows immediately from the classical symmetries of the Riemann-Christoffel curvature tensor. This proves that $\Omega(C^0_R(E)) \subset C^0_R(E)$.

For $Y \in C^0(E)$ let $Y = A + iB$ and define $\Omega Y = \Omega A + i\Omega B$. Clearly $\Omega C^0(E) \subset C^0(E)$. Since Ω is tensorial and T_1M is compact, Ω is continuous on $C^0(E)$. Moreover, the estimate

$$\begin{aligned}\langle \Omega Y, Y\rangle &= \int_{T_1M} S(\Omega Y, \Omega Y)d\mu \\ &\leqq \sup_{T_1M} S(\Omega Y, \Omega Y) \cdot \mu(T_1M)\end{aligned}$$

implies that Ω is bounded on a dense subset of $H^0(E)$. Hence, Ω extends uniquely to a bounded operator on $H^0(E)$.

For the second assertion, let $A(x, v) = (x, v, a, b)$ and $B(x, v) = (x, v, c, d)$ be elements of $C^0_R(E)$ expressed in horizontal and vertical components. Again, from the classical symmetries of the Riemann-Christoffel curvature tensor we have $g(R(a, v)v, c) = g(a, R(c, v)v)$. Now compute

$$\begin{aligned}V(\Omega A, B) &= g(-b, d) + g(R(a, v)v, c) \\ &= g(b, -d) + g(a, R(c, v)v) \\ &= V(A, \Omega B).\end{aligned}$$

Integrating the equality over T_1M yields $(\Omega A, B) = (A, \Omega B)$ and the same equalities hold for $A, B \in \mathfrak{X}$ by linearity.

LEMMA 4.5. *If X is the geodesic vector field and f is a function on T_1M, then $\int_{T_1M} Xf\, d\mu = 0$.*

PROOF. By definition $Xf(p) = \lim_{h\to 0} 1/h(f(G_h(p)) - f(p))$. In view of the compactness of T_1M the Lebesgue dominated convergence theorem gives

$$\begin{aligned}&\int_{T_1M} \lim_{h\to 0} \frac{1}{h}(f(G_h(p) - f(p))d\mu \\ &\quad = \lim_{h\to 0} \frac{1}{h}\left(\int_{T_1M} f(G_h(p))d\mu - \int_{T_1M} f(p)d\mu\right).\end{aligned}$$

Since μ is G_t invariant, $\int_{T_1M} f(G_h(p))d\mu = \int_{T_1M} f\, d\mu$ and therefore, $\int_{T_1M} Xf\, d\mu = 0$.

Although $\tilde{\nabla}$ is the Levi-Civita connection for the Vilms metric, the next theorem shows that the operator $\tilde{\nabla}_X$ behaves well with respect to the Sasaki metric.

THEOREM 4.6. *$\tilde{\nabla}_X$ extends to a densely defined closed operator on $\mathfrak{X}$ such that $\langle \tilde{\nabla}_X Y, Z\rangle = -\langle Y, \tilde{\nabla}_X Z\rangle$.*

PROOF. By proposition 3.2 and theorem 4.4, $L_X - \Omega$ extends to a densely defined closed operator on $\mathfrak{X}$, hence $\tilde{\nabla}_X$ has the same property.

To obtain the second assertion we first show $XS(Y, Z) = S(\tilde{\nabla}_X Y, Z)$

$+ S(Y, \tilde{\nabla}_X Z)$ when X is the geodesic vector field. If $\bar{\nabla}$ denotes the Levi-Civita connection on T_1M for S, Vilms **[12]** proves

$$\begin{aligned} B_{(x,v)}(Y, Z) &= \bar{\nabla}_Y Z - \tilde{\nabla}_Y Z \\ &= \frac{1}{2}(R(KY, v)T\pi Z + R(KZ, v)T\pi Y)^H \\ &\quad + \frac{1}{2}(R(v, T\pi Y)T\pi Z + R(v, T\pi Z)T\pi Y)^V \end{aligned}$$

where H and V denote the horizontal and vertical lifts. When $X(x, v) = (x, v, v, -\Gamma(v, v))$ and $Y(x, v) = (x, v, u, w)$ compute

$$B(X, Y) = \frac{1}{2}(R(w, v)v + R(\Gamma(u, v), v)v, R(v, u)v)$$

expressed in horizontal and vertical components. Since $\bar{\nabla}$ is the Levi-Civita connection for S, we have $XS(Y, Z) = S(\bar{\nabla}_X, Z) + S(Y, \bar{\nabla}_X Z)$. Hence, $XS(Y, Z) = S(\tilde{\nabla}_X Y, Z) + S(Y, \tilde{\nabla}_X Z)$ will follow at once provided $E = S(B(X, Y), Z) + S(Y, B(X, Z)) = 0$. Using the definition of S we obtain for $Z(x, v) = (x, v, \alpha, \beta)$

$$\begin{aligned} E = {} & (g(R(w, v)v, \alpha) + g(w, R(v, \alpha)v)) \\ & + (g(R(\Gamma(u, v), v)v, \alpha) + g(\Gamma(u, v), R(v, \alpha)v)) \\ & + (g(R(v, u)v, \beta) + g(u, R(\beta, v)v)) \\ & + (g(R(v, u)v, \Gamma(\alpha, v)) + g(u, R(\Gamma(\alpha, v), v)v)). \end{aligned}$$

Using the symmetries of the Riemann-Christoffel curvature tensor, the terms cancel in pairs as indicated by the parentheses.

By the lemma $\int_{T_1M} XS(Y, Z)d\mu = 0$ and therefore $\langle \tilde{\nabla}_X Y, Z\rangle + \langle Y, \tilde{\nabla}_X Z\rangle = 0$.

5. **Curvature and the H^0 spectrum of L_X.** In this section we make a preliminary application of the decomposition of L_X. As we have seen, the Anosov hyperbolicity condition is equivalent to the C^0 spectrum of Λ_t being disjoint from the unit circle. It follows easily that the C^0 spectrum of L_X is disjoint from the imaginary axis. We will show that when the sectional curvature is negative, the H^0 spectrum of L_X is disjoint from the imaginary axis.

In view of theorem 4.3, $\tilde{\nabla}_X$ is skew symmetric in $H^0(E)$ and, in fact, generates a one parameter group of unitary transformations given by parallel transport along geodesics. By Stone's theorem $\sigma(\tilde{\nabla}_X)$ is pure imaginary. Hence, we have the following theorem.

THEOREM 5.1. *$\sigma(\tilde{\nabla}_X)$ for $\tilde{\nabla}_X$ considered as an unbounded operator in $H^0(E)$ is pure imaginary.*

To compute the spectrum of Ω we first prove the following lemma.

LEMMA 5.2. *If M is a manifold of negative curvature, then $V(\Omega Y, Y) \leqq 0$ for all $Y \in C^0(E)$ with equality if and only if $Y = 0$.*

PROOF: As usual let $A \in C^0_R(E)$ with $A(x, v) = (c, v, a, b)$ in horizontal and vertical components. We have

$$V(\Omega A, A) = g(-b, b) + g(R(a, v)v, a) \leqq 0$$

with equality if and only if $A = 0$. For $Y = A + iB$ compute

$$\begin{aligned} V(\Omega Y, Y) &= V(\Omega A, A) + V(\Omega B, B) + i(V(\Omega B, A) - V(B, \Omega A)) \\ &= V(\Omega A, A) + V(\Omega B, B). \end{aligned}$$

Hence, the result follows for $Y \in C^0(E)$.

THEOREM 5.3. *If M is a manifold of negative curvature, then $\sigma(\Omega)$ consists of nonzero real numbers.*

PROOF. The theorem follows immediately from Theorem 4.4 and Lemma 5.2.

With the structure of $\tilde{\nabla}_X$ and Ω provided by the theorems of this section consider an analogous situation given by the following example.

EXAMPLE 5.4. Let $C(S^1, \mathbf{C})$ be the collection of continuous functions on the unit circle and let a denote an element of $C(S^1, \mathbf{C})$ with real range. Define for $f \in C(S^1, \mathbf{C})$ an operator L given by $Lf = \nabla f + \Omega f$ where $\nabla f = f'$ and $\Omega f = af$. If $Lf = \lambda f$ for $f \neq 0$, then

$$\lambda = \frac{1}{2\pi}\int_0^{2\pi} f'/f d\theta + \frac{1}{2\pi}\int_0^{2\pi} a d\theta = iN + \frac{1}{2\pi}\int_0^{2\pi} a d\theta$$

for some integer N. Hence, $\sigma(L) = \sigma(\nabla) + \text{Average } \sigma(\Omega)$.

This example motivates the next definition.

DEFINITION 5.5. If M is a Riemannian manifold, the H^0 *curvature* of M, $K_0(M)$ is defined by $K_0(M) = \sup_{Y \in B}(\Omega Y, Y)$ where B is the unit ball in $H^0(E)$.

PROPOSITION 5.6. *If M is a Riemannian manifold of negative curvature, then $K_0(M) < 0$.*

PROOF. The sectional curvature ρ of each tangent two plane is bounded above by zero. Since M is compact, there is a real number δ such that $\rho < \delta < 0$ for each tangent two plane. For $A \in C^0_R(E)$ we have

$$\begin{aligned} V(\Omega A, A) &= -g(b, b) + g(R(a, v)v, a) \\ &= -g(b, b) + \rho(a, v)(g(a, a)g(v, v) - g(a, v)^2) \\ &= -g(b, b) + \rho(a, v)g(a, a) \\ &< -g(b, b) + \delta g(a, a) \\ &\leqq -\min(1, |\delta|)S(A, A). \end{aligned}$$

Hence, $(\Omega A, A) < -\min(1, |\delta|) \cdot \langle A, A\rangle$.

Now, if $Y \in C^0(E)$ and $Y = A + iB$, then as before $V(\Omega Y, Y) = V(\Omega A, A) + V(\Omega B, B)$ and therefore

$$\begin{aligned} (\Omega Y, Y) &< -c^2(\langle A, A\rangle + \langle B, B\rangle) \\ &= -c^2\langle Y, Y\rangle. \end{aligned}$$

Since $H^0(E)$ is the completion of $C^0(E)$ in the metric given by $\langle\ \rangle$, the same inequality holds for $Y \in H^0(E)$ and then $K_0(M) < -c^2 < 0$ as required.

We now present the main theorem in abstract form. Our theorem shows that the decomposition $L_X = \tilde{\nabla}_X + \Omega$ is analogous to the decomposition of a complex number into its real and imaginary parts.

THEOREM 5.7. *Let $(H, \langle\ \ \rangle)$ be a Hilbert space, A a bounded self-adjoint operator H with a bounded inverse and define $(x, y) = \langle x, Ay\rangle$ for $x, y, \in H$. If $L = iD + B$ where*

(a) D is a closed densely defined operator on H such that $(Dx, y) = (x, Dy)$ for all x, y in the domain of D,

(b) B is a bounded operator on H such that $(Bx, y) = (x, By)$ for all x, $y \in H$, and

(c) $\sup_{\|x\|=1}(Bx, x) < 0$,

then $\sigma(L) \cap \{i\beta | \beta \in \mathbf{R}\} = \emptyset$.

PROOF. The identity $(x, y) = \langle x, Ay\rangle = \langle Ax, y\rangle = \overline{\langle y, Ax\rangle} = \overline{(y, x)}$ proves (x, x) is real and $(Bx, x) = (x, Bx) = \overline{(Bx, x)}$. In particular, the supremum in (c) is taken over a set of real numbers. Set $T = i\beta I - L$ and assume there is a sequence x_n in H such that $\|x_n\| = 1$ and $\lim_{n\to\infty} Tx_n = 0$. The Schwarz inequality yields

$$|(Tx_n, x_n)| = |\langle Tx_n, Ax_n\rangle| \leqq \|Tx_n\| \cdot \|A\|$$

and therefore $\lim_{n\to\infty}(Tx_n, x_n) = 0$. Since (Tx_n, x_n) is a sequence of complex numbers, the sequence of real parts must also converge to zero. As

$$(Tx_n, x_n) = i\beta(x_n, x_n) - i(Dx_n, x_n) - (Bx_n, x_n)$$

and (x_n, x_n), (Dx_n, x_n) and (Bx_n, x_n) are real, the real part of (Tx_n, x_n) is $-(Bx_n, x_n)$. Since $\|x_n\| = 1$

$$\lim_{n\to\infty} (Bx_n, x_n) \leqq \sup_{\|x\|=1} (Bx, x) < 0$$

and this contradicts $\lim_{n\to\infty}(Bx_n, x_n) = 0$. Hence $i\beta$ does not belong to the continuous spectrum of L. By choosing the sequence $x_n = x$ the same argument implies $Tx = 0$ for $\|x\| = 1$ is impossible, and therefore $i\beta$ does not belong to the point spectrum of L.

For the residual spectrum notice that the pairing (,) identifies H and H^* with a conjugate linear isomorphism which represents the adjoint of L as $-iD + B = L^*$. The standard fact that a point in the residual spectrum of an operator is a point in the point spectrum of its adjoint translates into λ in the residual spectrum of L implies $\bar{\lambda}$ in the point spectrum of L^*. But, if $i\beta$ belongs to the point spectrum of L^* there is an x in H such that $\|x\| = 1$ and

$$i\beta(x, x) + i(Dx, x) - (Bx, x) = 0.$$

Since $(Bx, x) < 0$, we again have a contradiction.

In view of the theorem we have the following interesting results.

THEOREM 5.8. *If $K_0(M) < 0$, then $\sigma(L_X)$ for L_X considered as an operator in $H^0(E)$ lies off the imaginary axis.*

COROLLARY 5.9. *If $K_0(M) < 0$, then the geodesic flow is Anosov.*

PROOF: See [1].

6. **Infinitesimal ergodicity.** In this section we study the operator Λ_t on the Sobolev space $H^1(E)$.

DEFINITION 6.1. The flow G_t is *infinitesimally ergodic* if the operator $\Lambda_t - I$, for fixed $t > 0$, has dense range as an operator on $H^1(E)$.

DEFINITION 6.2. The flow G_t is *ergodic* if the only L^2-functions f for which $f \circ G_t = f$ are constants.

The concept of infinitesimal ergodicity was introduced by J. Robbin [9] for discrete dynamical systems. We next prove the analogue of his theorem in the flow case.

THEOREM 6.3. *If G_t is infinitesimally ergodic, then G_t is ergodic.*

PROOF. Assume $f \in L^2(T_1M)$ and $G_t^*f = f$. $H^{-1}(E)$, the dual space of $H^1(E)$, may be represented as the space of H^{-1} 1-forms which are complementary to the flow field X, i.e., 1-forms α such that $\alpha(X) = 0$. With this representation the adjoint of Λ_t is G_t^* the pull back operator on forms. As $G_t^* f = f$, we have $df = G_t^* df$ with $df \in H^{-1}(E)$. Since $\Lambda_t - I$ has dense range, $G_t^* - I$ has no kernel and $df = 0$. Hence, f is constant almost everywhere.

For the remainder of this section assume that M is an n-dimensional Riemannian manifold with constant negative curvature k. In this case, the curvature tensor R has the form

$$R(a, v)v = k(g(v, v)a - g(v, a)v).$$

Hence, the operator Ω expressed in horizontal and vertical components is given by

$$\Omega = \begin{pmatrix} 0 & -I \\ kI & 0 \end{pmatrix}.$$

The operator $\tilde{\nabla}_X$ generates a 1-parameter group P_t which operates on vector fields by parallel transport. In particular, the value of P_t on a vector field A at the point $(x, v) \in T_1 M$ is the parallel transport of the vector A at $G_t(x, v)$ along the curve $G_s(x, v)$ to the point (x, v).

The key observation for the analysis to follow is the obvious fact that the operators $\tilde{\nabla}_X$ and Ω commute when the curvature is constant. Exponentiation of the generator L_X yields

$$\Lambda_t = P_t \exp(t\Omega) = \exp(t\Omega) P_t.$$

Each fiber of E splits as the direct sum of the two eigenspaces of Ω corresponding to the eigenvalues $\pm\alpha$ where $\alpha = (-k)^{1/2}$. In fact, this is the Anosov splitting $E = E^+ \oplus E^-$. Clearly, each summand is preserved by the operators L_X, $\tilde{\nabla}_X$ and Ω. The space $H^1(E)$ splits into a direct sum of subspaces $H^1(E^+) \oplus H^1(E^-)$ and each operator is represented on the splitting as a direct sum of operators. We have the equalities $\Omega A = -\alpha A$ for $A \in E^+$ and $\Omega A = \alpha A$ for $A \in E^-$.

THEOREM 6.4. *The geodesic flow G_t is infinitesimally ergodic.*

PROOF. The H^1 norm for $A \in H^1(E)$ is given by

$$\|A\|_1^2 = \|A\|_0^2 + \|DA\|_0^2$$

where D denotes the derivative and the norms on the right are taken, respectively, in $H^0(E)$ and $H^0(TE)$. Assume for the moment the estimates

$$e^{-2\alpha t} \|DA\|_0^2 \leqq \|DP_t A\|_0^2 \leqq e^{2\alpha t} \|DA\|_0^2. \tag{6.5}$$

Now, choose $A \in H^1(E^+)$ and compute

$$\|\Lambda_t A\|_1^2 = \|P_t \exp(t\Omega) A\|_0^2 + \|\exp(t\Omega) DP_t A\|_0^2.$$

By theorem 4.6, P_t is unitary on $H^0(E)$ and therefore

$$\|P_t \exp(t\Omega)A\|_0^2 \leqq e^{-2\alpha t}\|A\|_0^2.$$

For the second term we have

$$\|\exp(t\Omega)DP_tA\|_0^2 \leqq e^{-2\alpha t}\,\|DP_tA\|_0^2 \leqq \|DA\|_0^2.$$

Combining these estimates we obtain the strict inequality $\|\Lambda_t A\|_1^2 < \|A\|_1^2$ for $A \in H^1(E^+)$. In a similar manner one shows $\|\Lambda_t A\|_1^2 > \|A\|_1^2$ for $A \in H^1(E^-)$.

If $\Lambda_t - I$ fails to have dense range on $H^1(E^+)$, there is an element $A \in H^1(E^+)$ orthogonal to the range. In particular, $\langle(\Lambda_t - I)A, A\rangle_1 = 0$. This leads to a contradiction

$$\|A\|_1^2 = \langle \Lambda_t A, A\rangle_1 \leqq \|\Lambda_t A\|_1\|A\|_1 < \|A\|_1^2.$$

A similar argument shows $\Lambda_t - I$ has dense range on $H^1(E^-)$. Hence, the proof will be complete when 6.5 is established.

Each point $p \in T_1M$ is contained in a chart U which admits $2n - 1$ vector fields H_i which are linearly independent at each point of U and parallel along the G_t orbits in U. By the compactness of T_1M there is a $t > 0$ such that each point p admits a chart with the additional property that $G_s(p) \in U$ for $0 \leqq s \leqq t$.

Let A be a vector field and observe that on U $A = \sum f_iH_i$ for some choice of functions $F = (f_1, \ldots, f_{2n-1})$. Clearly, the parallel transport is represented by

$$P_tA = \sum(f_i \circ G_t)H_i = F \circ G_t$$

for points sufficiently close to p. At these points the derivative DP_tA is computed as $DP_tA = DF \circ TG_t$. Also,

$$TG_{-t} \circ A \circ G_t = \exp(t\Omega)F \circ G_t$$

and it follows that near p, TG_{-t} is represented as $\exp(t\Omega)$. Hence, we obtain the estimates

$$e^{-\alpha t}|DF| \leqq |DF \circ TG_t| \leqq e^{\alpha t}|DF|$$

at p. Integration gives (6.5).

Does 6.4 hold when the curvature is negative but not constant? This is probably false without some regularity assumption on the curvature. The strongest negative evidence is provided in **[11]**.

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