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TANGENT BUNDLES OF HOMOGENEOUS SPACES ARE HOMOGENEOUS SPACES

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ABSTRACT. In this paper we describe how the tangent bundle of a homogeneous space can be viewed as a homogeneous space.

The purpose of this note is to establish a simple result on the structure of the tangent bundle of a homogeneous space. Even though it is both natural and elementary it does not appear to be in the literature.

We shall associate with every Lie group G another Lie group G^*, constructed as a semidirect product of G with the Lie algebra of G (the precise definition is given below).

Our result is:

THEOREM. *If a Lie group G acts transitively and with maximal rank on a differentiable manifold X, then G^* acts transitively and with maximal rank on the tangent bundle of X.*

Clearly, our result implies that the tangent bundle of a coset space G/H is again a coset space and moreover, is of the form G^*/K for some closed subgroup K of G^*. We will compute K below.

We now define G^* and prove the theorem. Let L be the Lie algebra of G, thought of as the tangent space of G at the identity. For each $g \in G$, we let $ad(g)$ denote the differential at the identity of the inner automorphism $x \to gxg^{-1}$ of G. Thus ad is a (not necessarily one-to-one) homomorphism of G into the group of linear automorphisms of L. We define G^* as the product manifold $L \times G$, with the group operation given by

$$(a, g) \cdot (a', g') = (a + ad(g)(a'), gg'). \tag{1}$$

The verification that G^* is a group is trivial and will be omitted. Also, it is clear that the operation defined by (1) is differentiable, so that G^* is a Lie group.

Now, let G act differentiably on a manifold X. For each $x \in X$, let $\theta_x : G \to X$ be defined by $\theta_x(g) = gx$.

If $x \in X$, then the differential of θ_x at the identity maps L into X_x (the tangent space of X at x). If $a \in L$, we let $\bar{a}(x) = d\theta_x(a)$. It is easy to see that $\bar{a}$ is a smooth vector field on X. Use $\tau(X)$ to denote the tangent bundle of X.

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We define a left action of G^* on $\tau(X)$ by

$$(a, g) \cdot v = d\sigma_g(v) + \bar{a}(g \cdot \pi(v)) \quad \text{for } v \in \tau(X). \tag{2}$$

Here π denotes the natural projection from $\tau(X)$ onto X (i.e. $\pi(v)=x$ if and only if $v \in X_x$) and $\sigma_g: X \to X$ is the map $x \to gx$. Clearly, both $d\sigma_g(v)$ and $\bar{a}(g \cdot \pi(v))$ belong to $X_{g \cdot \pi(v)}$, so that the sum is defined. We omit the trivial verification that (2) satisfies

$$(0, e) \cdot v = v \quad \text{for all } v \in \tau(X)$$

and

$$((a, g) \cdot (a', g')) \cdot v = (a, g) \cdot ((a', g') \cdot v).$$

Also, it is clear that the action of G^* on $\tau(X)$ defined by (2) is differentiable.

Now assume that G acts transitively and with maximal rank on X. If v and v' belong to $\tau(X)$, then there exists $g \in G$ such that $g \cdot \pi(v) = \pi(v')$ (by the transitivity). Since G acts with maximal rank, there is an $a \in L$ such that $d\theta_{\pi(v')}(a) = v' - d\sigma_g(v)$.

Therefore $(a, g) \cdot v = v'$. This shows that G^* acts transitively on $\tau(X)$. We now show that G^* acts with maximal rank. Let G_0 be the connected component of the identity element of G. Then G_0 acts with maximal rank on X. Therefore the G_0-orbits are open submanifolds of X. If Y is a G_0-orbit, then G_0 acts transitively and with maximal rank on Y. We have already shown that this implies that G_0^* acts transitively on $\tau(Y)$. Since G_0^* is obviously connected, it follows that G_0^* acts with maximal rank on $\tau(Y)$. Now $\tau(X)$ is obviously the union of the sets $\tau(Y)$, where Y is a G_0-orbit in X. These sets are open submanifolds of $\tau(X)$. It follows that G_0^* acts with maximal rank on $\tau(X)$. Then, necessarily, G^* also acts with maximal rank on $\tau(X)$.

REMARKS. (A) If G acts transitively on X it does not follow that G^* acts transitively on $\tau(X)$ (let X = the real line with its usual one-dimensional differentiable structure and G—the real line considered as a discrete group).

(B) If H is a closed subgroup of G, then H^* can be identified, in an obvious way, with a closed subgroup of G^*. One verifies easily that the isotropy group of $0(\varepsilon X_x)$ corresponding to the action of G^* on $\tau(X)$ is precisely H_x^*, where H_x is the isotropy group of x corresponding to the action of G on X. In particular, we have the diffeomorphism $\tau(G/H) \simeq G^*/H^*$.

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