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TANGENT CONES, STARSHAPE AND CONVEXITY

J. M. BORWEIN

Department of Mathematics
Dalhousie University
Halifax, Nova Scotia
Canada, B3H 4H8

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ABSTRACT. In the last few years various infinite dimensional extensions to Krasnoselski's Theorem on starshaped sets [14] have been made. These began with a paper by Edelstein and Keener [8] and have culminated in the papers by Borwein, Edelstein and O'Brien [3] [4] by Edelstein, Keener and O'Brien [9] and finally by O'Brien [16].

Unrelatedly, Borwein and O'Brien [5] posed a question which arises in optimization [2] [11] of when a closed set is pseudoconvex at all its members.

In this paper we show that these two questions can be handled simultaneously through a slight refinement of the powerful central result in [16] with attendant strengthening of the results in [5] [16]. This in turn leads to some interesting characterizations of convexity, starshape and of various functional conditions.

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1. INTRODUCTION

We will suppose throughout that X is a (real Hausdorff) locally convex (topological vector) space with continuous linear dual X'. Most of our results will require in fact that X be a Banach space. If $A \subset X$ is an arbitrary set, $\bar{A}$, A^0, A^{ri} and co A denote respectively the closure, interior, interior relative to the closed affine span of A, and the convex hull of A. Aff(A) and span(A) will denote the affine and linear spans of A respectively. Any other usage is in accord with [7] and [12].

In accord with previous work [7] [12] we will say x sees a in A if the line segment $[x,a]$ (= co{x,a}) is contained in A. We say A has visibility of cardinality α (α-visibility) if every subset of A of cardinality α is simultaneously seen by some point $a \in A$. If $\alpha = \text{card}(A)$ we say A has starshape. If A has k-visibility for every natural number k we say A has finite visibility. We define

$$\text{star}_A(a) = \{x \in A: [x,a] \subset A\} \qquad (1.1)$$

and

$$\text{star } A = \cap\{\text{star}_A(a) : a \in A\} \qquad (1.2)$$

Then A has starshape if and only if star $A \neq \phi$ and α-visibility is the requirement that $\{\text{star}_A(a) : a \in A\}$ have nonempty intersection for subfamilies of cardinality α. We also recall that $a \in A$ is said to be a cone point [9] if there is a nonzero $x' \in X'$ such that $x'(a) = \sup\{x'(x): x \in \text{star}_A(a)\}$.

Krasnoselski [14] showed that if A is a compact subset of R^n and every n+1 cone points can be seen by some point in A, then A is starshaped. This was generalized in [4] to show that if A is a closed relatively weakly compact subset of a Fréchet space with finite visibility it has starshape. The proof method here is entirely different from Krasnoselski's but recently O'Brien [16] has shown that if A is a closed relatively weakly compact subset of a Banach space and every finite set of cone points is visible in A that A is starshaped.

Another object of our investigations will be the <u>tangent cone</u> to A at a [19] defined by

$$T(A,a) = \{h \in X: h = \lim t_n(a_n - A) \ , \ a_n \in A \ , \ a_n \to a \ , \ t_n \geqq 0\} . \qquad (1.3)$$

Here convergence is in the initial topology on X. If A is a convex set or a smooth manifold in R^n , T(A,a) is the standard tangent cone. In general, it is a closed cone but need not be convex. Following Guignard [11] we call the closed convex hull of T(A,a) the <u>pseudo-tangent cone</u> to A at a

$$P(A,a) = \overline{co}\ T(A,a) . \qquad (1.4)$$

It is always a closed convex cone and if $a \in \overline{A}$ is nonempty. In R^n $P(A,a) = \{0\}$ if and only if a is an isolated point of A. This is not true in general [2]. Also if $a \in A^0$, $P(A,a) = X$.

These cones have proven useful in optimization for the formulation of necessary and sufficient conditions for constrained optima to exist ([1],[2], [11],[19]) . When looking at sufficiency one says that a set A is <u>pseudo-convex</u> at a if

$$A - a \subset P(A,a) . \qquad (1.5)$$

Borwein and O'Brien [5] called a closed set <u>pseudoconvex</u> if it is pseudoconvex at all its members. They showed among other things that a closed bounded subset of a super reflexive Banach space [7] is pseudoconvex if and only if it is convex. If we let $P_A(a) = P(A,a) + a$ and

$$P(A) = \cap\{P_A(a): a\in A\}, \tag{1.6}$$

pseudoconvexity is the requirement that $A \subset P(A)$. Finally we will examine one other entity

$$R(A) = \cap\{P(A,a): a\in A\}, \tag{1.7}$$

which again is clearly a nonempty convex cone.

2. CONVEXITY AND STARSHAPE

We begin with some elementary relationships that will be essential to our development.

PROPOSITION 1. In any locally convex space

(i) $\operatorname{star}_A(a) \subset P_A(a)$ and (ii) $\operatorname{star}(A) \subset P(A)$.

Since (ii) follows from (i) and the relevant definitions it suffices to show (i). Let $x \in \operatorname{star}_A(a)$. Then $a_n = \frac{1}{n} x + \frac{n-1}{n} a \in A$ and since $a_n \to a$, $\lim n(a_n - a) = x-a \in T(A,a)$. Thus $x \in P_A(a)$. □

In particular, if $a \in \operatorname{star} A$, A is pseudoconvex at a (The reverse implication fails trivally as is shown by $A = \{\frac{1}{n}: n\in N\} \cup \{0\}$ and $a = 0$). It follows that a closed convex set is pseudoconvex since $\operatorname{star} A = A$ in this case.

Let $I(x) = x/||x||^2$. We will say a norm closed set A is nice if (a) $\overline{\operatorname{span}}\, A$ is a closed subspace of a weakly compactly generated Banach space X and (b) $I(A-x)$ is relatively weakly compact for all x not in A. (Recall that X is weakly compactly generated if $X = \overline{\operatorname{span}}\, K$ for some weakly compact set K [7]). The nice sets turn out to be exactly the boundedly relatively weakly compact sets as the next Lemma shows.

LEMMA 1. A is boundedly relatively weakly compact exactly when $I(A-x)$ is relatively weakly compact for each $x \notin A$.

PROOF. Since $A - x$ is boundedly relatively weakly compact exactly when A is, we may assume $x = 0$. Let $\{x_n\} = \{I(a_n)\}$ be a sequence in $I(A)$. Then $x_n = a_n/||a_n||^2$. Now either the sequence $\{a_n\}$ has an unbounded subsequence and equivalently $\{x_n\}$ has a subsequence convergent to zero, or $\{a_n\}$ being bounded has a weakly convergent subsequence (call it $\{a_n\}$ again) with limit x. Since we may assume $\{||a_n||^2\}$ converges to some positive α (because $0 \notin A$), we see that $\{x_n\}$ converges weakly to $\alpha^{-1}x$. The converse is similar. Since, by the Eberlein-Smulian Theroem [7], weak compactness and weak sequential compactness agree, we are done. □

Moreover, any boundedly relatively weakly compact set A has $\overline{\text{span}}\, A$ weakly compactly generated. To see this let $B_n = A \cap nB$ for each $n \in N$ where B is the unit ball in X. Let $K_n = \frac{1}{n^2} B_n$ and set

$$K = \cup\{K_n : n \in N\}.$$

Each B_n is relatively weakly compact and another application of the Eberlein-Smulian Theorem shows that K is also. Thus the weak closure of K generates a weakly compactly generated space and since $A \subset \text{span}\, K$, $\overline{\text{span}}\, A$ has the desired property. It follows that any closed set A in a reflexive Banach space is nice as is any weakly compact set in a normed space (we may embed it in its completion without changing its closure) and any boundedly relatively weakly compact set in a Banach space. A non-trivial example of the later class is provided by $A = \{\lambda b: b \in B, \lambda \geq 0\}$, where B is any closed relatively weakly compact set not containing the origin. An examination of the proof of the central result in [16] shows that it holds in the following form.

PROPOSITION 2. [16]. Suppose A is a nice closed set in a Banach space X and, for some $a \in A$, $[x,a] \notin A$. Then there exists a convex set U with interior, a smooth point $\bar{a} \in U$ and a functional $x' \in X'$ such that

(i) $$A \cap U = \{\bar{a}\}$$

(ii) $$x'(\bar{a}) = \sup\{x'(u): u \in U\} > x'(\bar{x})$$

Recall that u is a <u>smooth</u> point of U if there is a unique support functional f for U at u with $f(u) = 1$.

THEOREM 1. If A is a nice closed subset of a Banach space

$$\text{star } A = \cap\{P(A,a) + a: a \in A\} = P(A) \tag{2.1}$$

PROOF. By Proposition 1 (ii) it suffices to show that given any $\bar{x} \notin \text{star } A$, $\bar{x} \notin P_A(\bar{a})$ for some $\bar{a} \in A$. Since $\bar{x} \notin \text{star } A$ we can find $a \in A$ such that $[\bar{x},a] \notin A$. Let $\bar{a}$ be the smooth point guaranteed by Proposition 2. Suppose $h \in T(A,\bar{a})$. Then if h is nonzero we can find a sequence

$$a_n \to \bar{a}\ ,\ a_n \in A\ ,\ a_n \neq \bar{a}\ ,\ t_n > 0 \quad \text{with} \quad h_n = t_n(a_n - \bar{a}) \to h$$

It suffices to show that $A(h) = \{\bar{a} + \lambda h: \lambda \geqq 0\}$ and U^0 are disjoint as then they can be separated by a linear functional which supports U at $\bar{a}$. Since $\bar{a}$ is a smooth point this functional can be taken to be x' whence it follows that

$$x'(\bar{a} + h) \geqq x'(\bar{a}) > x'(\bar{x}) \qquad \forall\, h \in T(A,x)$$

As $P_A(\bar{a}) = \bar{a} + \overline{\text{co}}\, T(A,\bar{a})$ it is clear that $\bar{x} \notin P_A(\bar{a})$.

It remains to show that $A(h) \cap U^0 = \phi$. If not, $\lambda h_n + \bar{a} \in U^0$ for $n \geqq n_0$ and some $\lambda > 0$. Then assuming, as we may, that $\lambda = 1$ and $t_n > 1$ we have

$$a_n = (1 - \frac{1}{t_n})\bar{a} + \frac{1}{t_n}(t_n(a_n - \bar{a}) + a) = (1 - \frac{1}{t_n})a + \frac{1}{t_n}(h_n + \bar{a})$$
$$\subset (1 - \frac{1}{t_n})U^0 \in \frac{1}{t_n} U \subset U^0$$

since U is convex and $0 < \frac{1}{t_n} < 1$. This contradicts $A \cap U = \{\bar{a}\}$. So $A \cap U^0 = \phi$, and one is done. □

Before proceeding to derive consequences of this result let us introduce the notion of a proper point. We will say that a is a proper point of A if $P(A,a) \neq X$. An application of the Hahn-Banach theorem shows that if a is proper it is a cone point (but not conversely) since $\operatorname{star}_A(a) \subset P(A,a)$. We note that by Theorem 1 each nice closed subset $A \subsetneq X$ has a proper point. In fact, a simple adaptation of the proof of Proposition 2 in [16] will show that in this case the proper points are dense in the boundary of A.

THEOREM 2. Suppose A is a nice closed set in a Banach space and W is a weakly compact subset of X such that either

(i) every finite family $F(m) = \{P_A(a_i): 1 \leqq i \leqq m\}$ intersects in W, or

(ii) W is convex and has dimension n and every finite family with $n+1$ members intersects in W.

Then A is starshaped and $\operatorname{star} A \cap W \neq \phi$.

PROOF. We let $Q_A(a) = P_A(a) \cap W$. Then in (i) $\{Q_A(a): a \in A\}$ are weakly compact with the finite intersection property and hence

$$\phi \neq \cap\{Q_A(a): a \in A\} = P(A) \cap W = \operatorname{star} A \cap W$$

on applying Theorem 1 . Case (ii) follows similarly from Helly's Theorem [18] since the $Q_A(a)$ are now convex subsets of R^n. □

In particular if A is relatively weakly compact we may let $W = \overline{co}\ A$. So if every finite family of proper points can be seen in A (i) holds. Similar remarks apply to (ii). Thus as special cases, Theorem 2 includes Krasnoselski's theorem and the result in [16] with cone points replaced by the smaller set of proper points. In this framework the naturalness of using cone (proper) points is obvious since the improper points play no role in defining P(A) or in any intersection property. Finally, we observe that if X is separable a closed set is starshaped if every countable collection of $\{P_A(a): a \in A\}$ has intersection. This of course is guaranteed by countable visibility.

At least in incomplete normed space or Frèchet space, Theorem 1 can fail since it is then possible for there to be no proper points in which case P(A) = X It is easy to show [5] that if A is pseudoconvex at a , a is proper if and only if a is a support point for $\overline{co}\ A$. Since in the above mentioned cases closed, bounded convex sets with no support points exist [5], [13], the Theorem fails for these sets.

We turn now to psuedoconvexity of A . Since this is equivalent to $A \subset P(A)$ and since $P(A) \subset$ star A is guaranteed by Theorem 1 we have $A \subset$ star A and so:

COROLLARY 1. A nice closed set in a Banach space is convex if and only if it is pseudoconvex.

Thus to try and find a pseudoconvex non-convex subset in Banach space one must look for badly non weakly compact subsets of non reflexive spaces. In a normed space however, any non-convex set with no proper points is pseudoconvex. If we take the union of two separated copies of a convex set with no proper points the ensuing set S is disconnected and pseudoconvex and both Corollary 1 and Theorem 2 (i) fail for S .

The only other condition which one seems to be able to impose on a pseudoconvex set to make it convex is the following:

PROPOSITION 3. A pseudoconvex set A is convex if (i) $A^{ri} \neq \phi$ and (ii) its proper points are dense in the relative boundary of A .

PROOF. Suppose A is not convex and that a_1 and a_2 lie in A and $a_t = ta_1 + (1-(-t)a_2 \notin A$ for some $t \in [0,1]$. Let $a_3 \in A^{ri}$ and choose a relatively open set U in A with $a_3 + U \subset A$. Consider the set of segments $S = \{[a_t , u + a_3]: u \in U\}$. S is a relatively open set and thus some point $a_4 \in S$ is a proper point in the boundary of A . Since $A - a_4 \subset P(A,a_4) \neq X$, a_4 is a support of A with support functional x' . Thus

$$x'(a_t) = tx'(a_1) + (1-t)x'(a_2) \geqq x'(a_4) .$$

However, the line segment through a_t and a_4 extends into A^{ri} , whence it follows that $x'(a_t) < x'(a_4)$. This contradiction means that a_t fails to exist and A is convex. □

3. UNBOUNDEDNESS OF STAR A

We now turn to conditions which further specify the form that star A takes. We first examine the set R(A) .

PROPOSITION 4. In any locally convex space X

$$R(A) = R(\alpha A + x) \qquad \forall x \in X , \alpha > 0 . \tag{3.1}$$

$$P(A) + R(A) = P(A). \tag{3.2}$$

PROOF. If $x \in R(A)$, $x \in P(A,a) = P(\alpha A, \alpha a) = P(\alpha A+x, \alpha a+x)$. Thus

$x \in \cap\{P(\alpha A+x,a'): a' \in \alpha A+x\} = R(\alpha A+x)$, and (3.1) holds. Since $0 \in R(A)$, $P(A) \subset R(A) + P(A)$. Conversely if $a \in A$, $p \in P(A)$, $r \in R(A)$ then $p \in P(A,a) + a$, $r \in P(A,a)$. Thus $p + r \in P(A,a) + P(A,a) + a \subset P(A,a) + a$ since $P(A,a)$ is a convex cone. So $p + r \in P(A)$ and (3.2) holds. □

Recall that a convex set C is said to <u>recede in direction</u> h if $c + \lambda h \in C$ for $\lambda \geq 0$. The set of all directions h is called the <u>recession cone</u> of C , rec C [17].

THEOREM 3. If A is a nice closed starshaped set in a Banach space then $R(A)$ is the recession cone of a star A .

PROOF. If $s \in$ star A and $h \in R(A)$ we have for any $t > 0$

$$s + th \in \text{star } A + R(A) \subset P(A) + R(A) \subset P(A) \subset \text{star } A ,$$

where the containments follow from Proposition 1, (3.2) and (2.1) in that order. Thus star A recedes in direction h and $R(A) \subset \text{rec } C$. Conversely, if h is a recession direction, $s \in$ star A and $n \in N$

$$s + nh \in P(A,a) + a \qquad \forall\, a \in A$$

and

$$h = \lim \frac{1}{n} (s-a) \in P(A,a) \qquad \forall\, a \in A ,$$

because $P(A,a)$ is a closed cone. Thus $h \in R(A)$. □

It is well known that a finite dimensional closed convex set is bounded if and only if it has no recession directions. Theorem 3 says that in finite dimensions a closed starshaped set has an unbounded star exactly when $R(A)$ is non trivial. This can be guaranteed by requiring that for every finite set $\{a_1,\ldots,a_m\} \subset A$ one can find a point $h \neq 0$ with $th + a_i \in A$ for $1 \leqq i \leqq m$, $0 \leqq t \leqq 1$. This "finite ray" condition imposes the finite intersection property on $\{P(A,a): a \in A\} \cap S$ where $S = \{x: ||x|| = 1\}$. Since in finite dimensions this last set is compact, $R(A) \neq 0$.

Implicit in Theorem 3 is the proof that, if $P(A) \neq \phi$, $R(A)$ is the recession cone for $P(A)$. If star $A = \phi$, it is possible that $R(A)$ is nonzero as is shown by

$$A = \{(x,y): x \in R, y = 0 \text{ or } 1\} \cup \{(x,y): x \geqq o , 0 \leqq y \leqq 1\}$$

which has $R(A) = \{(x,y): x \geqq o , y = 0\}$. It seems reasonable therefore to consider $R(A)$ as a general cone of recession directions for any closed set. This is at least in part born out by the next proposition.

We recall that a functional x' strongly exposes a convex set C at $\bar{c}$ if $x'(\bar{c}) = \sup\{x'(x): c \in C\}$ and if whenever $x'(c_n) \to x'(\bar{c})$ and $c_n \in C$, $c_n \to \bar{c}$. We recall that a Banach space has the Radon-Nikodym property [7] if and only if every closed bounded convex set is the convex hull of its strongly exposed points. These spaces include dual spaces which are subspaces of weakly compactly generated spaces [7].

PROPOSITION 5. The following all imply $R(A) = 0$.

(i) A is closed relatively weakly compact in a Banach space.

(ii) A is closed and bounded in a Radon-Nikodym space.

(iii) A is closed bounded and convex in a Banach space.

(iv) A is compact in a locally convex space.

PROOF. Let $\bar{a} \in A$ and let $C = \overline{co}(A-\bar{a})$ and suppose $\bar{x} \notin C$. In case(i) C is weakly compact and thus $\bar{x}$ can be separated from C by a strongly exposing functional x' (the strongly exposing functionals are a dense subset in the dual space [6]). Let x' expose $\bar{c}$ strongly. Then $\bar{c} \in A - \bar{a}$ as A is norm closed , $\bar{c} + \bar{a} = \hat{a} \in A$ and

$$x'(\bar{x}) > x'(\bar{c}) \geq x'(a-\bar{a}) \qquad \forall\, a \in A .$$

Thus $x'(\bar{x}) > 0$ while $x'(a-\hat{a}) = x'(a-\bar{a}-\bar{c}) \leq 0 \quad \forall\, a \in A$. It follows that $\bar{x} \notin P(A,\hat{a})$. In particular $R(A) \subset C$ and because C is a bounded set and $R(A)$ is a cone $R(A) = 0$. Case (ii) follows similarily since now the strongly exposing functionals of any closed bounded convex set are dense in the dual [7]. Case (iii) follows from the Bishop-Phelps Theorem since now $C = A - \bar{a}$ and density of the support functionals suffices. In case (iv) any separating functional is a support functional and supports C at an extreme point of C by the Krein-Milman Theorem [12]. Milman's converse [12, page 132] implies that the extreme point belongs to $A - \bar{a}$ and one proceeds as above. □

It is not clear that parts of (i) and (ii) can be deduced from Proposition 2 and it remains unanswered as to whether a closed bounded set in a Banach space can have $R(A) \neq 0$.

We next collect a number of characterizations of star A under one roof:

PROPOSITION 6. If A is a nice closed set in a Banach space X

(i) $0 \in \text{star } A$ and $A \subset R(A) \iff A$ is a convex cone.

(ii) $0 \in \text{star } A$ and $\text{star } A \subset R(A) \iff \text{star } A$ is a convex cone.

(iii) $\text{span } A \subset R(A) \iff A$ is a subspace.

PROOF . (i) $\Rightarrow$: By Theorem 3 and the hypotheses,

$$A \subset R(A) \subset \text{star } A + R(A) \subset \text{star } A \subset A ,$$

Thus $A = R(A)$. $\Leftarrow$: Since A is a convex cone $0 \in \text{star } A = A$. Also $A \subset R(A)$

since star $A + A \subset$ star A . (ii) is analogous.

(iii) => : We may assume $X = \overline{\text{span}}\, A$ and then, by Theorem 1, since $R(A) = X$, $P(A) = X$ which implies star $A = X$. □

The case in which star A is nonempty but $0 \notin$ star A can be handled by translation as can the case that aff $A \subset R(A)$. In general $A \subset R(A)$ does not imply A is a cone as is shown by

$$A = \{(x',y): |y| \leqq x , x \geqq 1\} \text{ in } R^2 .$$

4. SPECIAL SETS

There are cases in which the individual structure of the pseudotangent cones can be specified more closely. Specifically one has some extra information if A is somehow connected to a Fréchet differentiable function g between Banach spaces X and Y .

PROPOSITION 6. (i) $g'(x)P(A,x) \subset P(g(A), g(x))$.

(ii) If B is a closed convex set, $A = g^{-1}(B)$, and g is continuously differentiable at x with $g'(x)$ surjective, then

$$g'(x)^{-1}P(B,g(x)) = P(A,x) \text{ and } P(g(A),g(x)) = P(B,g(x)) .$$

PROOF . (i) This is proven in [11].

(ii) Under the <u>regularity</u> conditions imposed on g it is essentially proven in [10] that

$$g(x + \phi(y)) - g(x) = g'(x)y \tag{4.1}$$

has a solution $\phi(y)$ for all sufficiently small y and that $\varepsilon^{-1}\phi(\varepsilon y)$ converges to y as $\varepsilon > 0$ goes to zero. Suppose that $g'(x)h \in P(B,g(x))$. One can show that there is a sequence $h_m \to h$ with $g'(x)h_m = \lambda_m(b_m - g(x))$, $b_m \in B$, $\lambda_m > 0$.

It suffices to show that $h_m \in P(A,x)$. Let $k = h_m/\lambda_m$. Then

$$g'(x)\frac{k}{n} + g(x) = \frac{1}{n}(g'(x)k + g(x)) + \frac{n-1}{n} g(x) \in B .$$

For n sufficiently large $\phi(\frac{k}{n})$ exists and $g(x+\phi(\frac{k}{n})) \in B$. Since $n^{-1}(x + \phi(\frac{k}{n}) - x) \to k$, $k \in P(A,x)$ and so $h_m \in P(A,x)$. The conclusions of (ii) now follows from (i). □

It follows that if B is a subspace, $P(g^{-1}(B),x)$ is a subspace whenever x is a regular point. If every $x \in g^{-1}(B)$ is regular (g is continuously differentiable at x and $g'(x)$ is surjective) it follows by using (2.1) that star A is an affine subspace (whenever A is nice) . In particular, if $g^{-1}(0)$ is a smooth surface in R^n it has such a star. Our next proposition substantiates the intuition that a nontrivial smooth surface cannot simultaneously be bounded and starshaped. Indeed much more is true.

PROPOSITION 7. Suppose either A is closed and bounded in a Radon-Nikodym space or A is relatively weakly compact in a Banach space and star A is nonempty. Either there is some point $a \in A$ with $P(A,a)$ not a subspace, or A is singleton.

PROOF. As in the proof of Proposition 5 (i), (ii) there is a strongly exposed point $a_0 \in A$ and a strongly exposing functional x' such that

$$x'(a_0) < x'(a) \qquad \forall\, a \in A \,,\ a \neq a_0 \,.$$

It follows that $x'(h) \geqq 0$ for all $h \in P(A,a_0)$. If $a_1 \neq a_0$, $a_1 \in$ star A we see that $x'(a_1-a_0) > 0$ and, since $a_1 - a_0$ lies in $P(A,a_0)$, $x'\, P(A,a_0) = [0,\infty)$ which implies $P(A,a_0)$ is not a subspace. On the other hand if, $a_0 \in$ star A then $A - a_0 \in P(A,a_0)$ and again unless $\{a_0\} \equiv A$ we can find $a_2 \neq a_0$ with $a_2 - a_0 \in P(A,a_0)$. Again since $x'(a_2 - a_0) > 0$,

$P(A,a_0)$ is not a subspace. □

If A is unbounded the proposition is certainly not true as is shown by $A = \{(x,y) \mid x = 0 \text{ or } y = 0\}$ in R^2. In any case, if all the pseudotangent cones are subspaces and star A is bounded and non empty it is a singleton.

As immediate corollaries to Theorem 1 and Proposition 6 (ii), we have in reflexive spaces:

COROLLARY 2. (i) If B is a closed convex set , $A = g^{-1}(B)$ and g is regular at every point of A

$$\text{star } A = \{x + g'(x)^{-1}P(B,g(x)) : x \in A\} \tag{4.2}$$

(ii) If in addition $g(A)$ is closed

$$\text{star } g(A) = \cap\{g(x) + g'(x)P(A,x) : x \in A\} \tag{4.3}$$

For example, if $b = [0,\infty)$ and $g(x) = x^2-1$, $A = \{x : |x| \geq 1\}$ and g is regular on A . Since star $A = \phi$ we know that the entities on the right hand side of (4.2) fail to intersect. Indeed, when $x = 1$ we get $[1,\infty)$ and when $x = -1$ we get $(-\infty,-1]$. From (4.3) we may deduce that if $g(X)$ is closed and g is always regular $g(X) = Y$, since this is the right hand side of (4.3) when $A = X$, $B = Y$.

5. FUNCTIONAL CHARACTERIZATIONS

We first recall certain classes of functions. A function $f: X \to R \cup \{\infty\}$ is convex if $f(tx+(1-t)y) \leq tf(x) + (1-t)f(y)$ for x,y in X and $0 \leq t \leq 1$. Similarily, $f: X \to Y$ is affine if $f(tx + (1-t)y) = tf(x) + (1-t)f(y)$ for x,y in X and $0 \leq t \leq 1$. If S is a convex cone in a space Y, $f: X \to Y$ is said to be S-quasiconvex at $\bar{x}$ if $f(x) - f(\bar{x}) \in -S$ implies that $f'(\bar{x}) \cdot (x-\bar{x}) \in -S$. We will also say $f: X \to R$ is star-like at $\bar{x}$ if, for every $x \in X$,

$$(x-\bar{x}, f(x)-f(\bar{x})) \in P(\text{Epi}f, (\bar{x}, f(\bar{x})) \,. \tag{5.1}$$

Here $\text{Epi}f = \{(x,y): y \geqq f(x)\}$ is the epigraph of f. If (5.1) holds for all $\bar{x}$, we say f is star-like.

THEOREM 4. If X is reflexive and f is continuous and starlike, it is convex.

PROOF. Note first that (5.1) is a condition close to pseudoconvexity of Epif. Once we show that in fact Epif is pseudoconvex it will follow from Theorem 1 that Epif is convex and hence f is a convex function. To this end we observe that since f is continuous any point on the boundary of Epif is on the form $(x,f(x))$. Thus Epif is trivially pseudoconvex at any point not of this form. We note now that if $s \geqq 0$, $(0,s) \in P(\text{Epi}f, (\bar{x}, f(\bar{x}))$ since

$$(o,s) = \lim_{n\to\infty} (n(\bar{x}-\bar{x}),\ n(f(\bar{x}) + \frac{1}{n} s - f(\bar{x})) \in T(\text{Epi}f,\ (\bar{x}, f(\bar{x})$$

and so

$$(x-\bar{x},\ f(x) + s - f(\bar{x})) \in P(\text{Epi}f,\ (\bar{x}, f(\bar{x}))) \quad \forall\ x \in X\ ,\ s \geqq 0\ ,$$

because (5.1) holds and $P(\text{Epi}f, (\bar{x}, f(\bar{x}))$ is a convex cone. This last condition says Epif is pseudoconvex at $(\bar{x}, f(\bar{x}))$ and since this holds for each $\bar{x} \in X$, Epif is convex. □

In the next corollary $\overline{d^+} f(x,h) = \limsup\limits_{t\to 0^+} \frac{f(x+th)-f(x)}{t}$ is the upper one sided derivative of f.

COROLLARY 3. Suppose f is continuous and satisfies

$$f(x+h) - f(x) \geqq \overline{d^+} f(x,h) \qquad \forall\ h \in X\ , \tag{5.2}$$

for each $x \in X$, then f is convex.

PROOF. We show (5.2) implies (5.1). Let $h \in X$ and let $s_n > 0$, $s_n \to 0$, be chosen so that

$$\overline{d^+}f(x,h) = \lim_{n\to\infty} \frac{f(x+s_n h)-f(x)}{s_n} .$$

Then, setting $x_n = x + s_n h$, $t_n = \frac{1}{s_n}$, we have

$$(t_n(x_n-x),\ t_n(f(x_n)-f(x))) \to (h,\ \overline{d^+}f(x,h))$$

and, since $f(x_n) \to f(x)$, $(h,\overline{d^+}f(x,h)) \in P(\text{Epi}f,\ (x,f(x))$. As before, $(0,f(x+h)-f(x)-\overline{d^+}f(x,h)) \in P(\text{Epi}f,\ (x,f(x))$ and we deduce that $(h,f(x+h)-f(x)) \in P(\text{Epi}f,\ (x,f(x))$ which since h is arbitrary is (5.1). □

The inequality (5.2), of course, easily implies convexity if $d^+f(x,h)$ exists and is convex in h.

COROLLARY 4. If $f: X \to Y$ has a closed pseudoconvex graph, f is affine.

PROOF. By Theorem 1, the graph of f is a convex set. This implies immediately that f is affine. □

We finish by giving a condition for the "level sets" of a function to be pseudoconvex at a point. This is again relevant for sufficiency of optimality conditions [2], [11].

THEOREM 5. Suppose f is a continuous mapping between two Banach spaces X and Y and f is S-quasiconvex at x_0. If x_0 is a regular point, $\{x: f(x)-f(x_0) \in -S\}$ is pseudoconvex at x_0.

PROOF. We must show that $A = f^{-1}(f(x_0)-S)$ is pseudoconvex at x_0.

The S-quasiconvexity says that

$$f^{-1}(f(x_0)-S) \subset x_0 + f'(x_0)^{-1}(-S)$$

Since x_0 is regular, Proposition 6 (ii) implies that

$$A \subset x_0 + P(f^{-1}(S-f(x_0)),x_0) = P_A(x_0) .$$

□

In case $y = R$ and $S = [0,\infty)$, for regularity continuous differentiability is unnecessary and it suffices that $f'(x_0) \neq 0$. Since S-quasiconvexity reduces to the usual notion of quasiconvexity at x_0 [15] we derive (assuming X is reflexive):

COROLLARY 5. Suppose $f: X \to R$ is quasiconvex at x and $f'(x)$ is nonzero when $f(x) = f(x_0)$. Then $\{x: f(x) \leq f(x_0)\}$ is a convex set.

PROOF. Since f is continuous, $\{x: f(x) = f(x_0)\}$ contains the boundary of $\{x: f(x) \leq f(x_0)\}$. By hypothesis, therefore, this later set is pseudoconvex and in turn, convex. □

In particular, a quasiconvex function with a nonzero derivative globally has all its level sets convex. One can derive this last result directly without asking for a non vanishing derivative.

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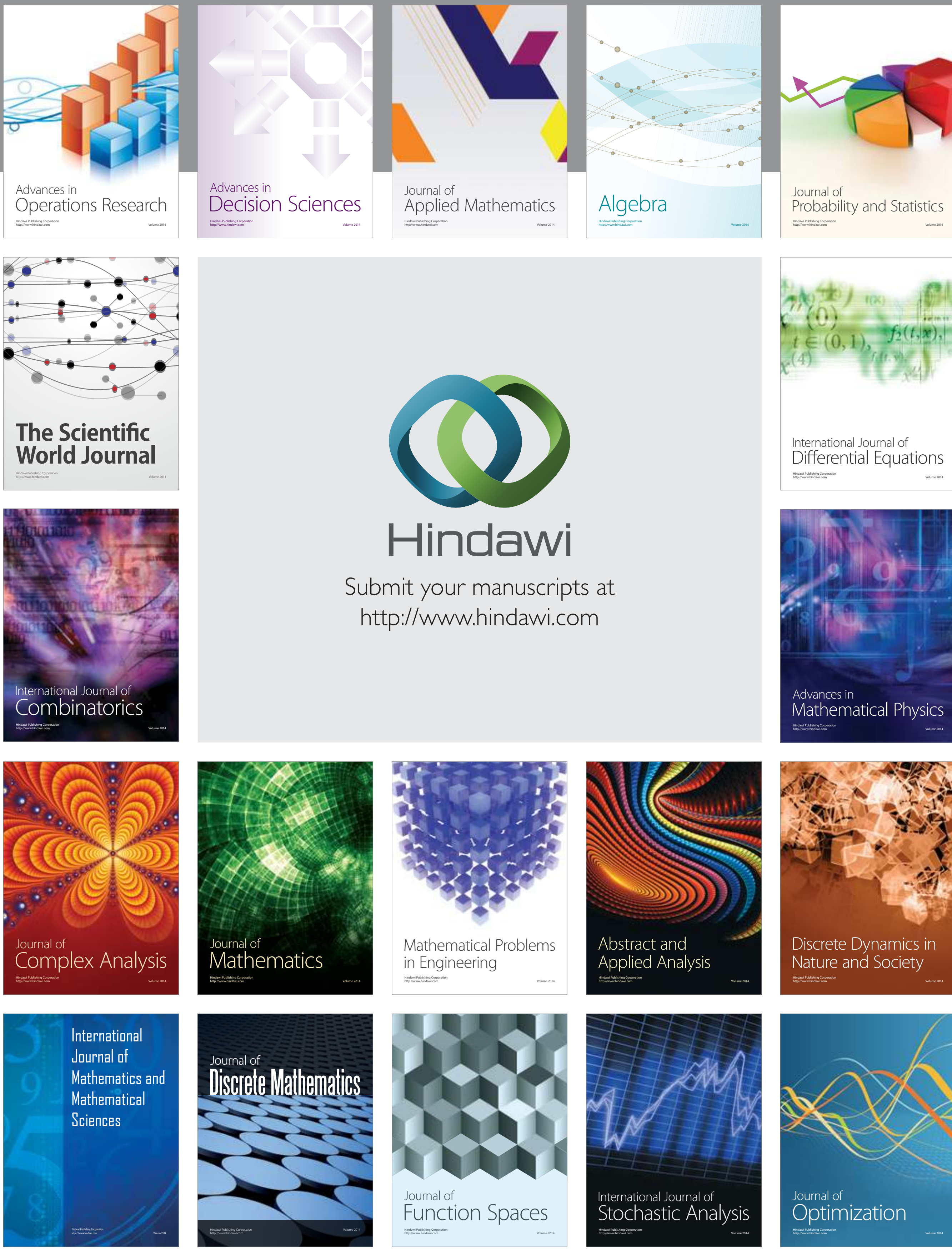
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