Tangential Cauchy-Riemann Complexes on Distributions (*).

MAURO NACINOVICH - GIORGIO VALLI

Summary. – *Dopo aver definito il complesso di Cauchy-Riemann tangenziale su sottovarietà reali generiche, di codimensione qualsiasi, di una varietà complessa, per mezzo di una opportuna successione di Mayer-Vietoris se ne studiano i gruppi di coomologia locali. Si ottengono inoltre risultati relativi al problema dell'estensione di « distribuzioni di Cauchy-Riemann » e alla coomologia globale di sottovarietà compatte e di domini compatti con frontiera regolare a tratti.*

Introduction.

This paper is a natural continuation of [14], where the same problems are considered in the C^∞ category. Given a complex manifold X of dimension n and a real submanifold S of codimension k in X, under a genericity assumption, the Dolbeault complex on X induces a differential complex on S, called the tangential Cauchy-Riemann complex. In the case of functions, the first operator of this complex expresses the necessary conditions for a given function on S to be the trace of a holomorphic function defined on a neighborhood of S. In terms of distributions, we can no longer think in terms of traces, but it turns out that any holomorphic function that is defined in a wedge with edge S and has at S a singularity of distribution type defines in a natural way an element of the k-th Dolbeault cohomology group with supports in S. These last groups are related to those of the tangential Cauchy-Riemann complexes on distributions, by means of a statement that is the dual of a formal Cauchy-Kowalewski theorem. We exploit sistematically these correspondences, reducing the cohomology of the C.R. complex on S to the Dolbeault cohomology of some wedge-domains.

The arguments are ordered in the following way: the first paragraph (in a general framework) and the third (for complex analysis) investigate the afore mentioned relations between the different cohomology groups, relating them by long exact Mayer-Vietoris sequences. In section 2 we rehearse some results of [15] relevant for the study of cohomology groups « up to the boundary » on piece-wise smooth domains. In § 4 these results are applied to prove the vanishing of some local Dolbeault cohomology groups of wedges and in § 5 we obtain the corresponding results

(*) Entrata in Redazione il 20 ottobre 1984.

Indirizzo degli AA.: Università di Pisa, Dipartimento di Matematica, Via Buonarroti 2, 56100 Pisa, Italia.

for the tangential Cauchy-Riemann complex. In § 6 we take up the problem of extending Cauchy-Riemann distributions defined on S, proving that in general they are restrictions of holomorphic functions defined on suitable wedges. In § 7 we prove finiteness for the dimension of some cohomology groups of the tangential Cauchy-Riemann complex on compact submanifolds and of the Dolbeault complex on relatively compact piece-wise smooth domains.

We believe that in § 1 we outlined a generalization of the methods of [4] and [5] that could be fertile of further applications, also beyond the scope of complex analysis.

We also want to indicate that results similar to ours, but in different categories of functions or of generalized functions where obtained in [7], [8], [11], [12], [17], [14].

Table of contents.

1. – Mayer-Vietoris exact sequences for distributions.

A) *Regular Families of Closed Sets.*

Let X be a C^∞ differentiable manifold and let E be a smooth vector bundle over X. For any open subset U of X we denote by $\Gamma(U, E)$ the space of smooth sections of E over U and by $\mathfrak{D}(U, E)$ the subspace of $\Gamma(X, E)$ of sections of E having compact support contained in U, all endowed with the usual Schwartz topologies.

Let $\mathfrak{D}'(U, E)$ be the space of distribution sections of E over U: it is the strong dual of the space $\mathfrak{D}(U, E^*)$, where E^* is the dual bundle of E.

For a closed subset F of X we denote by $\mathfrak{D}'_F(U, E)$ the closed subspace of distribution sections in $\mathfrak{D}'(U, E)$ having support contained in $F \cap U$.

The maps $U \to \Gamma(U, E)$, $U \to \mathfrak{D}'(U, E)$ and $U \to \mathfrak{D}'_F(U, E)$ from the open sets of X to the category of vector spaces define soft sheaves over X, that we will denote by $\Gamma(E)$, $\mathfrak{D}'(E)$ and $\mathfrak{D}'_F(E)$ respectively. We will use the notation $\Gamma(E)_x$, $\mathfrak{D}'(E)_x$ and $\mathfrak{D}'_F(E)_x$ for their stalks at the point $x \in X$.

Let $\Phi = \{F_\alpha\}$ be a locally finite family of closed subsets of X. We consider the simplicial complex

$$\mathcal{N}(\Phi) = \{(\alpha_0, \ldots, \alpha_q) \colon F_{\alpha_0} \cap \ldots \cap F_{\alpha_q} \neq \emptyset;\ q = 0, 1, 2, \ldots\}$$

and we denote by $C_q(\mathfrak{D}'_\Phi(E))$ the sheaf that associates to any open U in X the space $C_q(\mathfrak{D}'_\Phi(U, E))$ of alternated q-chains $f = (f_{\alpha_0 \ldots \alpha_q})$ with

$$f_{\alpha_0 \ldots \alpha_q} \in \mathfrak{D}'_{F_{\alpha_0} \cap \ldots \cap F_{\alpha_q}}(U, E)\,.$$

We define a boundary operator

$$\partial_q = \partial \colon C_q(\mathfrak{D}'_\Phi(E)) \to C_{q-1}(\mathfrak{D}'_\Phi(E))$$

for $q \geqslant 1$ by setting

$$(\partial_q f)_{\alpha_0 \ldots \alpha_{q-1}} = \sum_\alpha f_{\alpha_0 \ldots \alpha_{q-1} \alpha}\,.$$

We extend these definitions by setting

$$C_{-1}(\mathfrak{D}'_\Phi(E)) = \mathfrak{D}'_{\cup \Phi}(E)$$

and

$$\partial_0 f = \sum_\alpha f_\alpha \qquad \text{for } f = (f_\alpha) \in C_0(\mathfrak{D}'_\Phi(E))\,.$$

As we have

$$\partial_{q+1} \circ \partial_q = 0 \qquad \text{for } q \geqslant 0$$

we obtain a complex of soft sheaves:

$$\text{(1)} \qquad \ldots \to C_q(\mathfrak{D}'_\Phi(E)) \xrightarrow{\partial} C_{q-1}(\mathfrak{D}'_\Phi(E)) \xrightarrow{\partial} C_{q-2}(\mathfrak{D}'_\Phi(E)) \to \ldots \to$$
$$\to C_0(\mathfrak{D}'_\Phi(E)) \to C_{-1}(\mathfrak{D}'_\Phi(E)) \to 0\,.$$

We recall the notion of regular closed set (cf. [18], p. 92): « a closed subset F of X is *regular* if for any fixed Riemannian distance d on X, for every point $x_0 \in F$, we

can find an open neighborhood U of x_0 in X and positive constants C, $\alpha > 0$ such that every couple of points $x, y \in U \cap F$ can be joined by a continuous rectifiable path γ in F with lenght of $\gamma \leqslant c \cdot d(x, y)^{\alpha}$. »
This definition is independent of the choice of the Riemannian distance on X, because all Riemannian metrics on X define equivalent distances.

In [18] is proved the following statement:

If Φ is a locally finite family of closed sets in X and $\cup \Phi$ is regular, then for every open subset U of X the sequence

$$C_0(\mathfrak{D}'_{\Phi}(E))(U) \xrightarrow{\partial} C_{-1}(\mathfrak{D}'_{\Phi}(E))(U) \to 0$$

is exact.

We say that a locally finite family Φ of closed sets is a *regular family* if every union of intersections of elements of Φ is regular.

From the above statements, one obtains the following

PROPOSITION 1. – *If Φ is a regular family of closed subsets of X, then* (1) *is an exact sequence of $\Gamma(\mathbb{R})$-modules* (*where* $\mathbb{R}$ *denotes the trivial real line bundle over* X).

B) Extensible Distributions.

Let X be a C^{∞} differentiable manifold and let E be a smooth vector bundle over X. Let us fix an open set $\Omega \subseteq X$ and set $F = X - \Omega$. Then the sheaf $\check{\mathfrak{D}}'_{\Omega}(E)$ of extensible distributions in Ω is defined by the exact sequence

$$0 \to \mathfrak{D}'_F(E) \to \mathfrak{D}'(E) \to \check{\mathfrak{D}}'_{\Omega}(E) \to 0 . \tag{2}$$

Then also $\check{\mathfrak{D}}'_{\Omega}(E)$ is a soft sheaf. We recall (cf. MARTINEAU [13]) that also the sequence

$$0 \to \mathfrak{D}'_{\partial\Omega}(E) \to \mathfrak{D}'_{\bar{\Omega}}(E) \to \check{\mathfrak{D}}'_{\Omega}(E) \to 0 \tag{3}$$

is exact. For any open set $U \subset X$, the space $\check{\mathfrak{D}}'_{\Omega}(E)(U) \underset{\text{def}}{=} \check{\mathfrak{D}}'_{\Omega}(U, E)$ can be identified to the space of distributions on $\Omega \cap U$ that are restrictions of distributions on U.

In [13] the following statement is proved:

if interior of $\bar{\Omega} = \Omega$, *then for every open set U in X the space $\check{\mathfrak{D}}'_{\Omega}(U, E)$ is the strong dual of the closed subspace of $\mathfrak{D}(U, E^*)$ of smooth sections of E^* over U having compact support contained in $\bar{\Omega}$.*

Let $\mathfrak{U} = \{\Omega_i\}$ be a family of open sets in X. Because the map $U \to \mathfrak{D}'_U(E)$ is a presheaf of $\Gamma(\mathbb{R})$-modules, we can consider the space $C^h(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))$ of chains of the covering $\mathfrak{U}$ with coefficients in this presheaf. Let

$$\delta\colon C^h(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \to C^{h+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))$$

denote the coboundary map.

We set for $h=1$:

$$C^{-1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) = \check{\mathfrak{D}}'_{\cup\mathfrak{U}}(E)$$

and, for $f \in \check{\mathfrak{D}}'_{\cup\mathfrak{U}}(E)$:

$$\delta f = (f|_{\Omega_i}) \in C^0(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) .$$

Then we obtain a complex

$$0 \to C^{-1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \xrightarrow{\delta} C^0(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \xrightarrow{\delta} C^1(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \to \dots . \tag{4}$$

DEFINITION. – *A family $\mathfrak{U} = \{\Omega_i\}$ of open sets is said to be co-regular if it is locally finite and for every $x \in X$ one can find a neighborhood U of x in X such that $\Phi_U = \{U - \Omega_i\}$ is a locally finite regular family of closed subsets of U.*

Then we have the following:

PROPOSITION 2. – *If $\mathfrak{U} = \{\Omega_i\}$ is a co-regular family of open sets in X, then* (4) *is an exact sequence of $\Gamma(\mathbb{R})$-modules.*

PROOF. – Exactness at $C^{-1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))$ follows from the localization principle for distributions. Also the exactness at the places $C^j(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))$ for $j > 0$ is an easy consequence of the fact that $\check{\mathfrak{D}}'(E)$ is a presheaf of $\Gamma(\mathbb{R})$-modules.

To prove exactness at $C^0(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))$, due to the local nature of the statement, we can as well assume that $\mathfrak{U}$ is finite, say $\mathfrak{U} = \{\Omega_1, \dots, \Omega_k\}$.

Let U be an open subset of X and $f \in C^0(\mathfrak{U}, \check{\mathfrak{D}}'(E))(U)$ satisfy $\delta f = 0$ on U. This means that $f = (f_i)$ with

$$f_i \in \hat{\mathfrak{D}}'_{\Omega_i}(U, E) \quad \text{and} \quad f_i = f_j \quad \text{on} \quad \Omega_i \cap \Omega_j \cap U \quad \text{for } 1 \leqslant i, j \leqslant k .$$

Let $\tilde{f}_i \in \mathfrak{D}'(U, E)$ be such that

$$\tilde{f}_i | \Omega_i \cap U = f_i \quad \text{for } 1 \leqslant i \leqslant k .$$

By hypothesis we can assume that $\{U - \Omega_1, \dots, U - \Omega_k\}$ is a regular family of closed sets. We prove by recurrence that we can find $\psi_1, \dots, \psi_k \in \mathfrak{D}'(U, E)$ such that

$$\psi_h | \Omega_i \cap U = f_i \quad \text{for } i \leqslant h .$$

Indeed, we can choose $\psi_1 = \tilde{f}_1$: If ψ_h has been found, with $1 \leqslant h < k$, we observe that

$$\psi_h = \tilde{f}_{h+1} \quad \text{on } (\Omega_1 \cup \ldots \cup \Omega_h) \cap \Omega_{h+1} \cap U .$$

Therefore $\psi_h - \tilde{f}_{h+1} \in \mathcal{D}'(U, E)$ has support in

$$[U - (\Omega_1 \cup \ldots \cup \Omega_h)] \cup (U - \Omega_{h+1}) = [(U - \Omega_1) \cap \ldots \cap (U - \Omega_h)] \cup (U - \Omega_{h+1})$$

and we can find

$$g \in \mathcal{D}'_{U-(\Omega_1 \cup \ldots \cup \Omega_h)}(U, E)$$
$$w \in \mathcal{D}'_{U-\Omega_{h+1}}(U, E)$$

such that

$$\psi_h - \tilde{f}_{h+1} = g - w .$$

Then we set

$$\psi_{h+1} = \psi_h - g = \tilde{f}_{h+1} - w .$$

Clearly

$$\psi_{h+1} | \, \Omega_i \cap U = f_i \quad \text{for } i \leqslant h + 1$$

and then the inductive statement is proved. With

$$u = \psi_k | \, (\Omega_1 \cup \ldots \cup \Omega_k) \cap U \in C^{-1}(\mathfrak{U}, \check{\mathcal{D}}'_*(E))(U)$$

we obtain $\delta u = f$.

C) Complexes of Linear Partial Differential Operators.

We introduce first some standard notations. If

$$(\mathcal{S}^*, A_*) = \{0 \to \mathcal{S}^0 \xrightarrow{A_0} \mathcal{S}^1 \xrightarrow{A_1} \mathcal{S}^2 \xrightarrow{A_2} \ldots\}$$

is a complex of sheaves of abelian groups over X, then for every open set U in X we denote by

$$(U, \mathcal{S}^*, A_*) = \{0 \to \mathcal{S}^0(U) \xrightarrow{A_0} \mathcal{S}^1(U) \xrightarrow{A_1} \mathcal{S}^2(U) \to \ldots\}$$

the corresponding complex of sections and by $H^j(U, \mathcal{S}^*, A_*)$ its j-th cohomology group ($j \geqslant 0$). For any fixed point $x \in X$ we denote by

$$(x, \mathcal{S}^*, A_*) = \{0 \to \mathcal{S}_x^0 \xrightarrow{A_0} \mathcal{S}_x^1 \xrightarrow{A_1} \mathcal{S}_x^2 \xrightarrow{A_2} \ldots\}$$

the corresponding complex of germs at x and by $H^j(x, \mathcal{S}^*, A_*)$ its j-th cohomology group ($j \geqslant 0$). We note that

$$H^j(x, \mathcal{S}^*, A_*) = \varinjlim_{U \text{ open} \ni x} H^j(U, \mathcal{S}^*, A_*) .$$

Then we consider a sequence $\{E^j: j = 0, 1, 2, \ldots\}$ of $\mathcal{C}^\infty$ smooth complex vector bundles over X and a sequence $\{A_j: \Gamma(X, E^j) \to \Gamma(X, E^{j+1}): j = 0, 1, 2, \ldots\}$ of linear partial differential operators with complex valued C^∞ coefficients on X. We assume that

$$A_{j+1} \circ A_j = 0 \quad \text{for } j \geqslant 0 .$$

Then

(5) $$\big(\Gamma(E^*), A_*\big) = \{0 \to \Gamma(E^0) \xrightarrow{A_0} \Gamma(E^1) \xrightarrow{A_1} \Gamma(E^2) \to \ldots\}$$

is a complex of sheaves on X. We will be also interested, for $F, \Omega \subset X$, F closed and Ω open, to the complexes of sheaves:

(6) $$\big(\mathfrak{D}'_F(E^*), A_*\big) = \{0 \to \mathfrak{D}'_F(E^0) \xrightarrow{A_0} \mathfrak{D}'_F(E^1) \xrightarrow{A_1} \mathfrak{D}'_F(E^2) \to \ldots\}$$

and

(7) $$\big(\check{\mathfrak{D}}'_\Omega(E^*, A_*)\big) = \{0 \to \check{\mathfrak{D}}'_\Omega(E^0) \xrightarrow{A_0} \check{\mathfrak{D}}'_\Omega(E^1) \xrightarrow{A_1} \check{\mathfrak{D}}'_\Omega(E^2) \to \ldots\} .$$

From the exact sequence (2) we obtain:

PROPOSITION 3. – *Let Ω be an open subset of X and let $F = X - \Omega$. Then for every U open in X the following long cohomology sequence is exact*:

(8) $$\begin{aligned} &0 \to H^0\big(U, \mathfrak{D}'_F(E^*), A_*\big) \to H^0\big(U, \mathfrak{D}'(E^*), A_*\big) \to H^0\big(U, \check{\mathfrak{D}}'_\Omega(E^*), A_*\big) \to \\ &\to H^1\big(U, \mathfrak{D}'_F(E^*), A_*\big) \to H^1\big(U, \mathfrak{D}'(E^*), A_*\big) \to H^1\big(U, \check{\mathfrak{D}}'_\Omega(E^*), A_*\big) \to \\ &\to H^2\big(U, \mathfrak{D}'_F(E^*), A_*\big) \to \ldots . \end{aligned}$$

D) Long Exact Sequences of cohomology associated to regular families of closed sets.

We use the notations of subsections *A*) and *C*). We set

$$Z_{-1}\big(\mathfrak{D}'_\Phi(E)\big) = C_{-1}\big(\mathfrak{D}'_\Phi(E)\big)$$

and

$$Z_q(\mathfrak{D}'_\Phi(E)) = \operatorname{Ker}\left(\partial\colon C_q(\mathfrak{D}'_\Phi(E)) \to C_{q-1}(\mathfrak{D}'_\Phi(E))\right) \quad \text{for } q \geqslant 0\,.$$

For $q \geqslant 0$ and $j \geqslant -1$ we define

$$A_j\colon C_q(\mathfrak{D}'_\Phi(E^j)) \to C_q(\mathfrak{D}'_\Phi(E^{j+1}))$$

in the obvious way if $j = -1$ and by

$$A_j f = (A_j f_{i_0 \dots i_q}) \quad \text{if} \quad f = (f_{i_0 \dots i_q}) \in C_q(\mathfrak{D}'_\Phi(E^j))(U) \quad (U \text{ open} \subset X) \text{ if } j \geqslant 0\,.$$

Because the diagram $(j \geqslant 0,\ q \geqslant 0)$

$$\begin{array}{ccc} C_q(\mathfrak{D}'_\Phi(E^j)) & \xrightarrow{\partial} & C_{q-1}(\mathfrak{D}'_\Phi(E^j)) \\ {\scriptstyle A_j}\downarrow & & \downarrow{\scriptstyle A_j} \\ C_q(\mathfrak{D}'_\Phi(E^{j+1})) & \xrightarrow{\partial} & C_{q-1}(\mathfrak{D}'_\Phi(E^{j+1})) \end{array}$$

commutes, we have

$$A_j\left(Z_q(\mathfrak{D}'_\Phi(E^j))\right) \subset Z_q(\mathfrak{D}'_\Phi(E^{j+1}))$$

and we obtain for every $q \geqslant -1$ complexes of sheaves:

$$\text{(9)} \qquad \left(C_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) = \{0 \to C_q(\mathfrak{D}'_\Phi(E^0)) \xrightarrow{A_0} C_q(\mathfrak{D}'_\Phi(E^1)) \xrightarrow{A_1} C_q(\mathfrak{D}'_\Phi(E^2)) \to \dots\}$$

and for $q \geqslant 0$:

$$\text{(10)} \qquad \left(Z_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) = \{0 \to Z_q(\mathfrak{D}'_\Phi(E^0)) \xrightarrow{A_0} Z_q(\mathfrak{D}'_\Phi(E^1)) \xrightarrow{A_1} Z_q(\mathfrak{D}'_\Phi(E^2)) \to \dots\}\,.$$

By Proposition 1, if Φ is a regular family we have for every $q \geqslant 0$ and $j \geqslant 0$ an exact sequence

$$\text{(11)} \qquad 0 \to Z_q(\mathfrak{D}'_\Phi(E^j)) \to C_q(\mathfrak{D}'_\Phi(E^j)) \to Z_{q-1}(\mathfrak{D}'_\Phi(E^j)) \to 0\,.$$

Therefore it follows:

PROPOSITION 4. – *If Φ is a locally finite regular family of closed sets in X, then for every U open in X we have the long exact cohomology sequence (for $q \geqslant 0$)*

$$\begin{aligned} \text{(12)} \qquad 0 &\to H^0\left(U, Z_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to H^0\left(U, C_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to \\ &\to H^0\left(U, Z_{q-1}(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to H^1\left(U, Z_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to \dots \to \\ &\to H^j\left(U, Z_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to H^j\left(U, C_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to \\ &\to H^j\left(U, Z_{q-1}(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to H^{j+1}\left(U, Z_q(\mathfrak{D}'_\Phi(E^*)), A_*\right) \to \dots\,. \end{aligned}$$

E) Restrictions.

Let Φ be a locally finite regular family of closed sets in X and let us set

$$F = \cup \Phi \quad \text{and} \quad \Omega = X - F\,.$$

For every open set U in X and integers $j, q \geqslant 0$ we have maps

$$H^q\big(U, Z_{j-1}(\mathcal{D}'_\Phi(E^*)), A_*\big) \to H^{q+1}\big(U, Z_j(\mathcal{D}'_\Phi(E^*)), A_*\big)\,.$$

By composition we obtain maps

$$H^q\big(U, \mathcal{D}'_F(E^*), A_*\big) = H^q\big(U, Z_{-1}(\mathcal{D}'_\Phi(E^*)), A_*\big) \to H^{q+k}\big(U, Z_{k-1}(\mathcal{D}'_\Phi(E^*)), A_*\big)$$

for $q, k \geqslant 0$.

We also have a natural map

$$H^{q-1}\big(U, \check{\mathcal{D}}'_\Omega(E^*), A_*\big) \to H^q\big(U, \mathcal{D}'_F(E^*), A_*\big)$$

and thus by composition a map

$$(*)\qquad H^{q-1}\big(U, \check{\mathcal{D}}'_\Omega(E^*), A_*\big) \to H^{q+k}\big(U, Z_{k-1}(\mathcal{D}'_\Phi(E^*)), A_*\big)\,.$$

Let us consider now the following particular case of the above construction: for real valued smooth functions

$$\varrho_1, \ldots, \varrho_k \colon X \to \mathbb{R}$$

with

$$d\varrho_1 \wedge \ldots \wedge d\varrho_k \neq 0 \quad \text{on } X$$

we set

$$F_j = \{\varrho_j(x) = 0\} \subset X \quad \text{for } j = 1, \ldots, k$$
$$\Phi = \{F_1, \ldots, F_k\}\,.$$

Then Φ is a regular family. The set

$$S = F_1 \cap \ldots \cap F_k$$

is a smooth submanifold of X of real codimension k. Then

$$Z_{k-1}\big(\mathcal{D}'_\Phi(E)\big) = C_k\big(\mathcal{D}'_\Phi(E)\big) \cong \mathcal{D}'_S(E)$$

for every smooth vector bundle E over X. Thus $(*)$ yields a map

$$(**)\qquad H^{q-1}(U, \check{\mathfrak{D}}'_{\Omega}(E^*), A_*) \overset{\text{Res}}{\rightarrow} H^{q+k}(U, \mathfrak{D}'_S(E^*), A_*)\,.$$

Given $u \in H^{q-1}(U, \check{\mathfrak{D}}'_{\Omega}(E^*), A_*)$ we call the corresponding element $\text{Res}\,(u) \in H^{q+k}(U, \mathfrak{D}'_S(E^*), A_*)$ *restriction* of u on S.

Let $G = X - S$ and $W = \{\varrho_1 > 0, \dots, \varrho_k > 0\}$. To $f \in \mathfrak{D}'_G(U, E^{q-1})$ we associate the element $f' \in \mathfrak{D}'_{\Omega}(U, E^{q-1})$ such that $f'|W = f|W$ and $f' = 0$ on $\Omega - W$. This gives a map

$$H^q(U, \check{\mathfrak{D}}'_G(E^*), A_*) \to H^q(U, \check{\mathfrak{D}}'_{\Omega}(E^*), A_*)$$

and thus by composition a map:

$$(***)\qquad H^{q-1}(U, \check{\mathfrak{D}}'_G(E^*), A_*) \xrightarrow{\pm\text{res}} H^{q+k}\big(U, (\check{\mathfrak{D}}'_S(E^*)), A_*\big)\,.$$

REMARK. – Modulo the sign, the map $(***)$ depends only on S and not on the choice of the defining functions $\varrho_1, \dots, \varrho_k$: Indeed, it is sufficient to observe that a permutation of $\varrho_1, \dots, \varrho_k$ only changes the sign of the image and that if $\sigma: X \to \mathbb{R}$ is a smooth function vanishing on S with $d\varrho_1 \wedge \dots \wedge d\varrho_{k-1} \wedge d\sigma \neq 0$ on X, then the map obtained from $\varrho_1, \dots, \varrho_{k-1}, \sigma$ is ($+$ or $-$) the same as that obtained from $\varrho_1, \dots, \varrho_{k-1}, \varrho_k$. (The choice of $\varrho_1, \dots, \varrho_k$ corresponds to choose an orientation of S.)

F) Long Exact Sequences corresponding to co-regular famili's of open sets.

We use the notations introduced in subsections B) and C). Let $\mathfrak{U} = \{\Omega_i\}$ be a locally finite co-regular family of open sets in X. If E is a smooth vector bundle over X we set

$$Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) = \text{Ker}\,\big(\delta\colon C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))\big) \to C^{q+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E))\,, \quad \text{for } q \geq -1\,.$$

We define $(q \geq -1, j \geq 0)$

$$A_j\colon C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^j)) \to C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^{j+1}))$$

by associating to $f \in C^q(\mathfrak{U}, \check{\mathfrak{D}}(E^j))(U)$, $f = (f_{i_0 \dots i_q})$, U open in X, the co-chain

$$A_j f = (A_j f_{i_0 \dots i_q}) \in C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^{j+1}))(U)\,.$$

Because the diagram

$$\begin{array}{ccc} C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^j)) & \xrightarrow{A_j} & C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^{j+1})) \\ \delta\big\downarrow & & \big\downarrow\delta \\ C^{q+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^j)) & \xrightarrow{A_j} & C^{q+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^{j+1})) \end{array}$$

commutes, we obtain complexes:

$$\left(C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) = \{0 \to C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^0)) \xrightarrow{A_0} C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^1)) \xrightarrow{A_1} C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^2)) \to \dots\}$$

and

$$\left(Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) = \{0 \to Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^0)) \xrightarrow{A_0} Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^1)) \xrightarrow{A_1} Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^2)) \to \dots\}.$$

Then, from the exact sequence of sheaves $(q \geqslant -1)$; $(Z^{-1} = 0)$:

$$(13) \qquad 0 \to Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \to C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \to Z^{q+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E)) \to 0$$

we obtain the following

PROPOSITION 5. – *If $\mathfrak{U}$ is a locally finite co-regular family of open sets in X, then for every open set U in X and every $q \geqslant 0$ we have a long exact sequence:*

$$\begin{aligned}(14) \qquad 0 &\to H^0\left(U, Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to H^0\left(U, C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to \\ &\to H^0\left(U, Z^{q+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to H^1\left(U, Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to \dots \\ &\dots \to H^j\left(U, Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to H^j\left(U, C^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to \\ &\to H^j\left(U, Z^{q+1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to H^{j+1}\left(U, Z^q(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) \to \dots .\end{aligned}$$

From this proposition we deduce:

COROLLARY 1. – *If $q \geqslant k$, U is open in X (resp. $x_0 \in X$),*

$$H^{q-k}\left(U, Z^k(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*), A_*)\right) = 0 \qquad \left(\text{resp. } H^{q-k}\left(x_0, Z^k(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) = 0\right)$$

and

$$H^{q-j}\left(U, C^j(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*), A_*)\right) = 0 \qquad \left(\text{resp. } H^{q-j}\left(x_0, C^j(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\right) = 0\right)$$

for $j = 0, 1, \dots, k$

then, setting $\Omega = \cup \mathfrak{U}$, we have:

$$H^q(U, \check{\mathfrak{D}}'_\Omega(E^*), A_*) = 0 \qquad \left(\text{resp. } H^q(x_0, \check{\mathfrak{D}}'_\Omega(E^*), A_*) = 0\right).$$

We shall also need the following

COROLLARY 2. – *Let $q \geqslant 1$ and let us assume that for $0 \leqslant r \leqslant k$ and any choice of the indices $i_0, \dots, i_r$ we have for $0 \leqslant j \leqslant k$*

$$H^{q+j}(U, \check{\mathfrak{D}}'_{\Omega_{i_0} \cup \dots \cup \Omega_{i_r}}(E^*), A_*) = 0 \qquad \left(\text{resp. } H^{q+j}(x_0, \check{\mathfrak{D}}'_{\Omega_{i_0} \cup \dots \cup \Omega_{i_r}}(E^*), A_*) = 0\right)$$

Then we also have:

$$H^q\big(U, C^k(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) = 0 \qquad \big(resp.\ H^q\big(x_0, C^k(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) = 0\big) .$$

PROOF. – We prove the statement by induction on the number k. We note that for $k = 0$ there is nothing to prove. Let us fix then $m > 0$ and assume that the statement is true for every $0 \leqslant k < m$ and every $q \geqslant 1$. We give the argument for an open set U, as the proof is analogous in the case of germs. Let us fix indices $i_0, \ldots, i_m$. Because $\mathfrak{V} = \{\Omega_{i_0}, \ldots, \Omega_{i_m}\}$ is co-regular, then we have exact sequences:

$$H^q\big(U, C^j(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \to H^q\big(U, Z^{j+1}(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \to$$
$$\to H^{q+1}\big(U, Z^j(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \to H^{q+1}\big(U, C^j(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) .$$

By the inductive assumption the first and last terms are zero, for $j < m$, so that

$$H^q\big(U, Z^{j+1}(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \cong H^{q+1}\big(U, Z^j(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) .$$

It follows that

$$H^q(U, \hat{\mathfrak{D}}'_{\Omega_{i_0} \cap \ldots \cap \Omega_{i_m}}(E^*), A_*) \cong H^q\big(U, Z^m(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \cong$$
$$\cong H^{q+m}\big(U, Z^0(\mathfrak{V}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \cong H^{q+m}(U, \check{\mathfrak{D}}'_{\Omega_{i_0} \cap \ldots \cup \Omega_{i_m}}(E^*), A_*)$$

and the last group is zero by assumption. Because

$$H^q\big(U, C^m(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*), A_*)\big) \cong \oplus \{H^q(U, \check{\mathfrak{D}}'_{\Omega_{i_0} \cap \ldots \cap \Omega_{i_m}}(E^*), A_*) : (i_0, \ldots, i_m) \in \mathcal{N}(\mathfrak{U})\}$$

the statement is proved.

COROLLARY 3. – *Let* $q \geqslant 1$ *and let us assume that*

$$H^{q+1}\big(U, C^{k-j-1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) = 0 \qquad \big(resp.\ H^{q+j}\big(x_0, C^{k-j-1}(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) = 0\big)$$
$$for\ j = 0, 1, \ldots, k .$$

Then we have also

$$H^q\big(U, Z^k(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A^*\big) = 0 \qquad \big(resp.\ H^q\big(x_0, Z^k(\mathfrak{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) = 0\big) .$$

PROOF. – Because of the long exact sequence (14) the hypothesis imply that we have for $j = 0, 1, \ldots, k$ injective maps:

$$H^{q+j}\big(U, Z^{k-j}(\mathfrak{U}, \check{\mathfrak{D}}'(E^*)), A_*\big) \to H^{q+j+1}\big(U, Z^{k-j-1}(\mathfrak{U}, \check{\mathfrak{D}}'(E^*)), A_*\big)$$

and then the statement follows because

$$H^{q+k}\big(U, Z^0(\mathcal{U}, \check{\mathfrak{D}}'(E^*)), A_*\big) \cong H^{q+k}\big(U, C^{-1}(\mathcal{U}, \check{\mathfrak{D}}'(E^*)), A_*\big) = 0 .$$

The proof for the case of germs is the same.

We also have:

COROLLARY 4. – *Let* $q \geqslant k$ *and*

$$H^{q-k+j}\big(U, C^{k-j}(\mathcal{U}, \check{\mathfrak{D}}'(E^*)), A_*\big) = 0 \quad \big(\textit{resp.}\ H^{q-k+j}\big(x_0, C^{k-1}(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) = 0\big) \quad \textit{for } j = 1, \dots, k .$$

Then the following maps are all surjective:

$$H^{q-k}\big(U, Z^k(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \to H^{q-k+1}\big(U, Z^{k-1}(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \to \dots \to$$
$$\to H^q\big(U, Z^0(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \xrightarrow{\sim} H^q(U, \check{\mathfrak{D}}'_{\cup\mathcal{U}}(E^*), A_*)$$
$$\big(\textit{resp.}\ H^{q-k}\big(x_0, Z^k(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \to H^{q-k+1}\big(x_0, Z^{k+1}(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \dots \to$$
$$\to H^q\big(x_0, Z^0(\mathcal{U}, \check{\mathfrak{D}}'_*(E^*)), A_*\big) \xrightarrow{\sim} H^q(x_0, \check{\mathfrak{D}}'_{\cup\mathcal{U}}(E^*), A_*)\big) .$$

The proof is again an easy application of Proposition 5.

2. – System of linear partial differential operators with constant coefficients.

Let $\mathcal{T} = \mathbb{C}[\xi_1, \dots, \xi_n]$ denote the ring of polynomials with complex coefficients in n indeterminates. If

(15) $$0 \to \mathcal{T}^{p_d} \xrightarrow{{}^tA_{d-1}(\xi)} \mathcal{T}^{p_{d-1}} \to \dots \xrightarrow{{}^tA_0(\xi)} \mathcal{T}^{p_0} \to M \to 0$$

is a resolution of the $\mathcal{T}$-module of finite type $M = \operatorname{coKer}\,({}^tA_0(\xi)\colon \mathcal{T}^{p_1} \to \mathcal{T}^{p_0})$, then we consider for open sets Ω, U of $\mathbb{R}^n$ the complex of differential operators

(16) $$0 \to \check{\mathfrak{D}}'_\Omega(U, \mathbb{C}^{p_0}) \xrightarrow{A_0(D)} \check{\mathfrak{D}}'_\Omega(U, \mathbb{C}^{p_1}) \to \dots \to \check{\mathfrak{D}}'_\Omega(U, \mathbb{C}^{p_d}) \to 0 ,$$

where $A_j(D) = A_j\big((1/i)(\partial/\partial x)\big)$ is obtained from $A_j(\xi)$ by the formal substitution $\xi_h \to (1/i)(\partial/\partial x_h)$.

Let Ω be an open set in $\mathbb{R}^n$ and let $x_0 \in \partial\Omega$.

We say that Ω is convex at x_0 if we can find an open neighborhood U of x_0 such that $\Omega \cap U$ is convex.

We say that Ω is strictly concave at x_0 if for an open neighborhood U of x_0 in $\mathbb{R}^n$ the set $(\mathbb{R}^n - \Omega) \cap \overline{U}$ is strictly convex.

Because convex sets satisfy the condition of Martineau given in § 1, *B*), we obtain the following (cf. [15]).

PROPOSITION 6. – *With the notations introduced above*

(*a*) *If* Ω *is convex at* x_0, *then*

$$H^j(x_0, \check{\mathfrak{D}}'_\Omega(\mathbf{C}^{p_\bullet}), A_*) = 0 \qquad \text{for } j > 0 .$$

(*b*) *If* Ω *is strictly concave at* x_0 *and* $\operatorname{Ext}^{j+1}_{\mathcal{T}}(M, \mathcal{T}) = 0$, *then*

$$H^j(x_0, \check{\mathfrak{D}}'_\Omega(\mathbf{C}^{p_\bullet}), A_*) = 0 .$$

We will need in the applications a corollary of this proposition in the case in which the differential operators $A_j(D)$ only depend on derivatives in the first m coordinates in $\mathbb{R}^n$.

So we assume that, for $j = 0, 1, \ldots, d-1$,

(*) $A_j(D)$ depends only upon the derivatives $\partial/\partial x_1, \ldots, \partial/\partial x_m \quad (m < n)$.

PROPOSITION 7. – *With the notations introduced above and under the assumption* (*): *Let* $\varrho_1, \ldots, \varrho_k$ *be real valued* C^∞ *function defined on an open neighborhood* U *of* 0 *in* $\mathbb{R}^n$ *such that*

(i) *we can find* $N \in \mathbb{R}^n$ *such that*

$$< \operatorname{grad} \varrho_j(0) , N > 0 \qquad \text{for } j = 1, \ldots, k ;$$

(ii) *there is a constant* $\nu > 0$ *such that*

$$\sum_{\alpha,\beta=1}^{m} \frac{\partial^2 \varrho_j(0)}{\partial x_\alpha \partial x_\beta} u^\alpha u^\beta \geq \nu \sum_{\alpha=1}^{m} u^{\alpha^2} \qquad \text{for every } j = 1, \ldots, k \text{ and } u^1, \ldots, u^m \in \mathbb{R} .$$

Let

$$\Omega = \{x \in U : \varrho_1(x) < 0, \ldots, \varrho_k(x) < 0\} , \qquad G = \bigcup_{j=1}^{k} \{x \in U : \varrho_j(x) > 0\} .$$

Then

(*a*) $H^j(0, \mathfrak{D}'_\Omega(\mathbf{C}^{p_\bullet}), A_*) = 0 \quad$ *for* $j > 0$.

(*b*) *If* $\operatorname{Ext}^{j+1}_{\mathcal{T}}(M, \mathcal{T}) = 0$, *then*

$$H^j(0, \check{\mathfrak{D}}'_G(\mathbf{C}^{p_\bullet}), A_*) = 0 .$$

PROOF. – Let us consider the C^∞ change of coordinates about 0 given by

$$x = y + \lambda\Big(\sum_{h=m+1}^{n} y_h^2\Big)N .$$

This change of coordinates transforms the complex (16) into a complex of linear partial differential operators still having constant coefficients, while, if $\lambda > 0$ and large, Ω and G are transformed into sets respectively convex and strictly concave at 0. The statement follows then from proposition 6.

REMARK. – In [15] the following stronger version of Proposition 6, (*a*) is proved:

« *For every convex set* U, *we have* $H^j(\mathbb{R}^n, \check{\mathfrak{D}}'_U(\mathbf{C}^{p_\bullet}), A_*) = 0$ *for* $j > 0$ ».

From this one deduces the following improvement of Proposition 7 (*a*):

« *There exists* $\varepsilon_0 > 0$, *only depending on* $\langle \operatorname{grad} \varrho_j(0), N\rangle$, *on* ν *and on the complex* (16), *such that*

$$H^j(\{|x| < \varepsilon\}, \check{\mathfrak{D}}'_\Omega(\mathbf{C}^{p_\bullet}), A_*) = 0 \qquad \textit{for } j > 0 \textit{ if } 0 < \varepsilon \leqslant \varepsilon_0 \text{ ».}$$

3. – Tangential Cauchy-Riemann complexes on distributions.

A) *Preliminaries.*

Let X be a complex manifold of dimension n and let S be a closed connected real submanifold of X of real dimension $2n - k$.

If $J: TX \to TX$ is the complex structure of X, then for every point $x \in S$ we denote by

$$H_x S = T_x S \cap J T_x S$$

the complex analytic tangent space to S at x.

We say that S is *generic* at x if

$$\dim_{\mathbf{C}} H_x S = n - k$$

and we say simply that S is generic if this property holds at every $x \in S$. In this case

$$HS = \bigcup_{x\in S} H_x S$$

is a subbundle of TS.

We denote by A the vector bundle of complex valued exterior differential forms on X and we set $\mathcal{A} = \Gamma(A)$. Then we define the sheaf of ideals $\mathfrak{J}_S$ in $\mathcal{A}$ generated by the germs of functions vanishing on S and by their antiholomorphic differentials.

The quotient sheaf

$$\mathfrak{Q}_S = \mathcal{A}/\mathfrak{J}_S$$

is concentrated on S. We have the exact sequence of sheaves:

$$0 \to \mathfrak{J}_S \to \mathcal{A} \to \mathfrak{Q}_S \to 0 . \tag{17}$$

Let $A^{p,q}$ be the bundle of exterior forms of type p, q (i.e. homogeneous of degree p in the holomorphic and of degree q in the anti-holomorphic differentials) and let $\mathcal{A}^{p,q} = \Gamma(A^{p,q})$. We set

$$\mathfrak{J}_S^{p,q} = \mathfrak{J}_S \cap \mathcal{A}^{p,q}$$

$$\mathfrak{Q}_S^{p,q} = \mathfrak{J}^{p,q}/\mathcal{A}_S^{p,q} .$$

From the Dolbeault complex

$$(\mathcal{A}^{p,*}, \bar{\partial}_*) = \{0 \to \mathcal{A}^{p,0} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,2} \xrightarrow{\bar{\partial}} \dots \to \mathcal{A}^{p,n} \to 0\}$$

and the sub-complex

$$(\mathfrak{J}_S^{p,*}, \bar{\partial}_*) = \{0 \to \mathfrak{J}_S^{p,0} \xrightarrow{\bar{\partial}} \mathfrak{J}_S^{p,1} \xrightarrow{\bar{\partial}} \mathfrak{J}_S^{p,2} \xrightarrow{\bar{\partial}} \dots \to \mathfrak{J}_S^{p,n} \to 0\}$$

passing to the quotient we obtain the complex

$$(\mathfrak{Q}_S^{p,*}, \bar{\partial}_{S*}) = \{0 \to \mathfrak{Q}_S^{p,0} \xrightarrow{\bar{\partial}_S} \mathfrak{Q}_S^{p,1} \xrightarrow{\bar{\partial}_S} \mathfrak{Q}_S^{p,2} \to \dots \to \mathfrak{Q}_S^{p,n-k} \to 0\} . \tag{18}$$

(We note that $\mathfrak{Q}_S^{p,q} = \Gamma(Q_S^{p,q})$ for a smooth vector bundle $Q_S^{p,q}$ over S.)

This is a complex of differential operators of first order on S, that is called tangential Cauchy-Riemann complex on S (cf. [2]).

Let us denote by $\mathfrak{C}$ the sheaf of currents on X and by $\mathfrak{C}^{p,q}$ the subsheaf of those currents that are of type (p, q). Taking the natural orientation in X, choosing a smooth density on X, we can define a duality pairing between $A^{p,q}$ and $A^{n-p,n-q}$. Then

$$\mathfrak{C}^{p,q} = \mathfrak{D}'(A^{p,q}) .$$

Given $T \in \mathfrak{C}^{p,q}(U)$ and $u \in \mathfrak{D}(U, A^{n-p,n-q})$, with U open in X, we denote by

$$\langle T, u\rangle = T(u)$$

the action of T on u. We note that

$$\langle\bar{\partial}T, u\rangle = (-1)^{p+q}\langle T, \bar{\partial}u\rangle \quad \text{for } T \in \mathfrak{C}^{p,q}(U),\ u \in \mathfrak{D}(U, A^{n-p,n-q}),\ q < n,$$

$$\bar{\partial}T = 0 \quad \text{if } q = n .$$

We have a complex of sheaves (Dolbeault complex on currents)

$$(\mathcal{C}^{p,*}, \bar\partial_*) = \{0 \to \mathcal{C}^{p,0} \xrightarrow{\bar\partial} \mathcal{C}^{p,1} \xrightarrow{\bar\partial} \mathcal{C}^{p,2} \to \dots \to \mathcal{C}^{p,n} \to 0\}.$$

For $0 \leqslant p \leqslant n$ and $0 \leqslant q \leqslant n$ we define ${}^0\mathfrak{J}_S^{p,q}$ as the subsheaf of $\mathcal{C}^{p,q}$ such that, for U open in X, $T \in {}^0\mathfrak{J}_S^{p,q}(U)$ if and only if $\langle T, \varphi\rangle = 0$ for every $\varphi \in \mathfrak{J}_S^{n-p,n-q}(U) \cap \mathfrak{D}(U, A^{n-p,n-q})$. Because $\mathfrak{J}_S^{n-p,n-q} = \mathcal{A}^{n-p,n-q}$ when $q < k$, then ${}^0\mathfrak{J}_S^{p,q} = 0$ if $q < k$. Moreover we notice that ${}^0\mathfrak{J}_S^{p,q} \subset \mathfrak{D}'_S(A^{p,q})$.

LEMMA 1. - *We have*

$$\bar\partial\, {}^0\mathfrak{J}_S^{p,q} \subset {}^0\mathfrak{J}_S^{p,q+1}.$$

PROOF. - Let U be open in X and let $T \in {}^0\mathfrak{J}_S^{p,q}(U)$. Let $\varphi \in \mathfrak{J}_S^{n-p,n-q-1}(U) \cap \mathfrak{D}(U, A^{n-p,n-q-1})$. Then

$$\langle \bar\partial T, \varphi\rangle = (-1)^{p+q}\langle T, \bar\partial\varphi\rangle = 0$$

because $\bar\partial\varphi \in \mathfrak{J}_S^{n-p,n-q}(U) \cap \mathfrak{D}(U, A^{n-p,n-q})$.

By the above lemma, we have a complex of sheaves:

$$({}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) = \{0 \to {}^0\mathfrak{J}_S^{p,k} \xrightarrow{\bar\partial} {}^0\mathfrak{J}_S^{p,k+1} \xrightarrow{\bar\partial} {}^0\mathfrak{J}_S^{p,k+2} \to \dots \to {}^0\mathfrak{J}_S^{p,n} \to 0\} \tag{19}$$

that we call the tangential Cauchy-Riemann complex on distributions.

By the exact sequence (17), for every open set U in X the space ${}^0\mathfrak{J}_S^{p,q}(U)$ is the dual of $\mathfrak{D}(U, Q_S^{n-p,n-q})$ and hence ${}^0\mathfrak{J}_S^{p,q} = \mathfrak{D}'\big((Q_S^{n-p,n-q})^*\big)$ where $(Q_S^{n-p,n-q})^*$ is the dual bundle of $Q_S^{n-p,n-q}$. If S is orientable, we can associate to any smooth density on S a natural isomorphism $(Q_S^{n-p,n-q-k})^* \cong Q_S^{p,q}$ and (19) is isomorphic to the complex:

$$(\mathfrak{D}'(Q_S^{p,*}), \bar\partial_{S*}) = \{0 \to \mathfrak{D}'(Q_S^{p,0}) \xrightarrow{\bar\partial_S} \mathfrak{D}'(Q_S^{p,1}) \xrightarrow{\bar\partial_S} \mathfrak{D}'(Q_S^{p,2}) \to \dots \to \mathfrak{D}'(Q_S^{p,n-k}) \to 0. \tag{19'}$$

B) A Mayer-Vietoris Exact Sequence for Distributions.

We have the following exact sequences of sheaves on X:

$$0 \to \mathfrak{D}'_S(A) \to \mathfrak{D}'(A) \to \check{\mathfrak{D}}'_{X-S}(A) \to 0 \tag{20}$$

$$0 \to {}^0\mathfrak{J}_S \to \mathfrak{D}'(A) \to \mathfrak{D}'(A)/{}^0\mathfrak{J}_S \to 0 \tag{21}$$

$$0 \to \mathfrak{D}'_S(A)/{}^0\mathfrak{J}_S \to \mathfrak{D}'(A)/{}^0\mathfrak{J}_S \to \check{\mathfrak{D}}'_{X-S}(A) \to 0. \tag{22}$$

Denoting by $\mathcal{F}_S^{p,q}$ the subsheaf of $\mathcal{A}^{p,q}$ of germs of smooth exterior forms vanishing of infinite order on S, for every open set U in X the quotient space $\mathfrak{D}'_S(U, A^{p,q})/{}^0\mathfrak{J}_S^{p,q}(U)$

can be identified to the strong dual of the quotient space

$$(\mathfrak{I}_S^{n-p,n-q}(U) \cap \mathfrak{D}(U, A^{n-p,n-q}))/(\mathcal{F}_S^{n-p,n-q}(U) \cap \mathfrak{D}(U, A_S^{n-p,n-q}) \overset{\text{def}}{=} \mathcal{K}_S^{n-p,n-q}(U) .$$

If S is generic, it was proven in [2] that (Formal Cauchy-Kowalewski Theorem)

$$0 \to \mathcal{K}_S^{p,0}(U) \xrightarrow{\bar\partial|} \mathcal{K}_S^{p,1}(U) \xrightarrow{\bar\partial|} \mathcal{K}_S^{p,2}(U) \to \dots \to \mathcal{K}_S^{p,n}(U) \to 0$$

(where the vertical bar means restriction to the quotient) is exact for every open U in X and $p = 0, \dots, n$. Then by duality we obtain (cf. also [13]):

PROPOSITION 8. – (*Dual Formal Cauchy-Kowalewski Theorem.*) *If S is generic, then*

$$H^q(U, \mathfrak{D}'_S(A^{p,*})/{}^0\mathfrak{I}_S^{p,*}, \bar\partial_*) = 0$$

for every U open in X and every $0 \leqslant p, q \leqslant n$.

From this result we deduce the following

COROLLARY. – *If S is generic, then for every open $U \subset X$ and $0 \leqslant p, q \leqslant n$ we have an isomorphism*

$$H^q(U, \mathfrak{D}'(A^{p,*})/{}^0\mathfrak{I}_S^{p,*}, \bar\partial_*) \cong H^q(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) .$$

PROOF. – The statement follows from the previous proposition and the long exact cohomology sequences deduced from the short exact sequence (22).

Then we have:

PROPOSITION 9. – (*Mayer-Vietoris Exact Sequence for Distributions.*) *If S is generic, then we have for every open set U in X and every $0 \leqslant p \leqslant n$ the long exact sequences:*

$$(23) \quad 0 \to H^{k-1}(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \to H^0(U, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) \to H^k(U, \mathfrak{D}'(A^{p,*}), \bar\partial_*) \to$$
$$\to H^k(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \to H^1(U, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) \to \dots \to H^{q-1}(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \to$$
$$\to H^{q-k}(U, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) \to H^q(U, \mathfrak{D}'(A^{p,*}), \bar\partial_*) \to H^q(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \to$$
$$\to H^{n-k}(U, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) \to H^n(U, \mathfrak{D}'(A^{p,*}), \bar\partial_*) \to H^n(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \to 0 .$$

PROOF. – Indeed it is sufficient to apply the above corollary to the long exact sequence deduced from the short exact sequence (21).

PROPOSITION 10. – *If S is generic, then we have for every open set U in X and every $0 \leqslant p, q \leqslant n$, $q \geqslant k$, isomorphisms:*

$$H^{q-k}(U, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) \cong H^q(U, \mathfrak{D}'_S(A^{p,*}), \bar\partial_*) .$$

PROOF. – The statement follows from Proposition 8 and the long exact cohomology sequence deduced from the short exact sequence

$$0 \to {}^0\mathfrak{J}_S \to \mathfrak{D}'_S(A) \to \mathfrak{D}'_S(A)/{}^0\mathfrak{J}_S \to 0 \,. \tag{24}$$

COROLLARY. – *With the assumptions of the previous proposition, if U is a Stein open set, then*

$$H^q(U, \mathfrak{D}'_S(A^{p,*}), \bar\partial_*) \cong H^{q-k}(U, {}^0\mathfrak{J}_S^{p,k+*}), \bar\partial_*) \cong H^{q-1}(U, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*)$$

for $p \geq 0$ *and* $q > k$.

4. – Vanishing theorems for the local Dolbeault cohomology at edge points.

A) Preliminary Lemmas.

Let X be a complex manifold of dimensions n and let $\varrho_1, \ldots, \varrho_k$ be real valued smooth functions defined on X. Let

$$\Omega = \{x \in X \colon \varrho_1(x) < 0, \ldots, \varrho_k(x) < 0\}$$

and let $x_0 \in \partial\Omega$ be a point where

$$\varrho_1(x_0) = \ldots = \varrho_k(x_0) = 0$$

and $d\varrho_1(x_0), \ldots, d\varrho_k(x_0)$ belong to a same open half space of $T_{x_0}X$: this means that we can find $N \in T_{x_0}X$ such that

$$\langle d\varrho_j(x_0), N\rangle > 0 \qquad \text{for } j = 1, \ldots, k \,.$$

We assume moreover that the Taylor series of the ϱ_j's be linearly independent in the space of formal power series. Then we have:

LEMMA 2. – *Assume that we can find a complex linear subspace V of complex dimension s of $T_{x_0}X$ such that, for every $j = 1, \ldots, k$, the Levi form of ϱ_j at x_0 is positive definite on the intersection of V with the tangent analytic space to the hypersurface $\{\varrho_j = 0\}$:*

$$i\langle \partial\bar\partial\varrho_j(x_0), v \wedge Jv\rangle > 0 \quad \text{if } v \neq 0,\ v \in V \text{ and } \langle d\varrho_j(z_0), v\rangle = \langle d\varrho_j(z_0), Jv\rangle = 0 \,. \tag{25}$$

Then we have:

$$H^q(x_0, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar\partial_*) = 0 \qquad \text{for } q > n - s \,.$$

PROOF. - Substituting $\exp[c\varrho_j]-1$, with $c>0$ sufficiently large, for ϱ_j, we can assume that the Levi forms (25) are positive definite on V.

By a convenient choice of holomorphic coordinates $z^1, \ldots, z^n$ on a neighborhood of x_0 we can achieve the following:

(i) x_0 corresponds to the origin of the coordinates.

(ii) In the corresponding tangential coordinates $u^1, \ldots, u^n$ the subspace V has equations:

$$u^{s+1} = \ldots = u^n = 0 .$$

Moreover we make the additional assumption that the coordinates can be chosen to satisfy:

(iii) There is a convex open neighborhood ω_1 of 0 in $\mathbf{C}^s$ and an open neighborhood ω_2 of 0 in $\mathbf{C}^{n-s}$ such that for every $(z^{s+1}, \ldots, z^n) \in \omega_2$, the map

$$\omega_1 \in (z^1, \ldots, z^s) \to \varrho_j(z^1, \ldots, z^i, z^{i+1}, \ldots, z^n)$$

has a positive hessian on ω_1.

(We note that condit on (iii) can always be fulfilled if the edge is « generic », i.e. if

$$\partial\varrho_1(x_0) \wedge \ldots \wedge \partial\varrho_k(x_0) \neq 0) .$$

Let us set

$$\bar\partial = \bar\partial' + \bar\partial''$$

where $\bar\partial'$ and $\bar\partial''$ denote anti-holomorphic differentiation with respect to $z^1, \ldots, z^s$ and with respect $z^{s+1}, \ldots, z^n$ respectively.

Let $A_r^{p,q}$ denote the subbundle of $A^{p,q}$ of forms homogeneous of degree r in $d\bar z^{s+1}, \ldots, d\bar z^n$. We have $A_r^{p,q} = 0$ if $r > q$.

Because $\bar\partial'$ defines in the given coordinates differential operators with constant coefficients without derivatives in the coordinates corresponding to $z^{s+1}, \ldots, z^n$, then by proposition 7 (a) the complex

$$(0, \check{\mathfrak{D}}'_\Omega(A_r^{p,r+*}), \bar\partial'_*) = \{0 \to \check{\mathfrak{D}}'_\Omega(A_r^{p,r})_0 \xrightarrow{\bar\partial'} \check{\mathfrak{D}}'_\Omega(A_r^{p,r+1})_0 \to \ldots \to \check{\mathfrak{D}}'_\Omega(A_r^{p,n})_0 \to 0\}$$

is acyclic in positive dimensions:

$$H^j(0, \check{\mathfrak{D}}'_\Omega(A_r^{p,r+*}), \bar\partial'_*) = 0 \quad \text{for } j > 0 .$$

Let us set

$$A_{(r)}^{p,q} \cong \bigoplus_{j \geq r} A_j^{p,q} .$$

Then we have the exact sequence of (trivial) vector bundles on $\omega_1 \times \omega_2$:

$$0 \to A^{p,q}_{(r+1)} \to A^{p,q}_{(r)} \to A^{p,q}_{r} \to 0$$

where the first map is the inclusion and the last in the natural projection. We obtain therefore a commutative diagram of sheaf complexes with exact columns:

$$\begin{array}{ccccccccc}
 & 0 & & 0 & & 0 & & 0 & \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & 0 & \to & \check{\mathfrak{D}}'_\Omega(A^{p,r+1}_{(r+1)}) & \xrightarrow{\bar\partial} & \check{\mathfrak{D}}'_\Omega(A^{p,r+2}_{(r+1)}) & \xrightarrow{\bar\partial} & \check{\mathfrak{D}}'_\Omega(A^{p,r+3}_{(r+1)}) & \to \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \check{\mathfrak{D}}'_\Omega(A^{p,r}_{(r)}) & \xrightarrow{\bar\partial} & \check{\mathfrak{D}}'_\Omega(A^{p,r+1}_{(r)}) & \xrightarrow{\bar\partial} & \check{\mathfrak{D}}'_\Omega(A^{p,r+2}_{(r)}) & \xrightarrow{\bar\partial} & \check{\mathfrak{D}}'_\Omega(A^{p,r+3}_{(r)}) & \to \dots \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 \to & \check{\mathfrak{D}}'_\Omega(A^{p,r}_{r}) & \xrightarrow{\bar\partial'} & \check{\mathfrak{D}}'_\Omega(A^{p,r+1}_{r}) & \xrightarrow{\bar\partial'} & \check{\mathfrak{D}}'_\Omega(A^{p,r+2}_{r}) & \xrightarrow{\bar\partial'} & \check{\mathfrak{D}}'_\Omega(A^{p,r+3}_{r}) & \to \dots . \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
 & 0 & & 0 & & 0 & & 0 &
\end{array}$$

This yields an exact sequence in cohomology:

(28) $$\dots \to H^q(0, \check{\mathfrak{D}}'_\Omega(A^{p,*}_{(r+1)}), \bar\partial_*) \to H^q(0, \check{\mathfrak{D}}'_\Omega(A^{p,*}_{(r)}), \bar\partial_*) \to \\ \to H^{q-r}(0, \check{\mathfrak{D}}'_\Omega(A^{p,r+*}_{r}), \bar\partial'_*) \to \dots .$$

Because the last group is zero for $q > r$, we deduce that we have a surjective map

$$H^q(0, \check{\mathfrak{D}}'_\Omega(A^{p,*}_{(q)}), \bar\partial_*) \to H^q(0, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar\partial_*) .$$

But, if $q > n - s$, $A^{p,j}_{(q)} = 0$ for every j and thus necessarily

$$H^q(0, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar\partial_*) = 0 .$$

To complete the proof, we need now to get rid of the additional assumption (iii). To this aim, let us introduce the $k-1$ dimensional simplex:

(26) $$\Lambda = \{\lambda = (\lambda^1, \dots, \lambda^k) \in \mathbb{R}^k : \lambda^i \geq 0, \ \forall i \text{ and } \lambda^1 + \dots + \lambda^k = 1\} .$$

Then we set

(27) $$\varrho_{(\lambda)} = \lambda^1 \varrho_1 + \dots + \lambda^k \varrho_k$$

and we notice that for every $\lambda \in \Lambda$

$$i < \partial\bar{\partial}\varrho_{(\lambda)}(x_0)\,,\ v \wedge Jv > > 0 \quad \text{for } v \in V - \{0\} \quad \text{and} \quad \langle d\varrho_j(x_0), v\rangle = \\ = \langle d\varrho_j(x_0), Jv\rangle = 0\,.$$

Then we conclude by repeating the argument of the proof of Proposition 8 in [14], because (iii) can be easily fulfilled when $k = 1$ (this starts the induction on k) and when $\varrho_1, \ldots, \varrho_k$ are substituted by $\varrho_{\lambda_1}, \ldots, \varrho_{\lambda_k}$ with $\sum_{i,j=1}^{k} |\lambda_i - \lambda_j|^2$ sufficiently small.

Let us set $\Omega_i = \{x \in X\colon \varrho_i(x) < 0\}$ for $i = 1, \ldots, k$.

LEMMA 3. – *Assume that we can find a complex linear subspace V of $T_{x_0}X$ such that*

$$(29) \qquad i\langle \partial\bar{\partial}\varrho_j, v \wedge Jv\rangle < 0 \quad \textit{for } v \in V - \{0\},\ \langle d\varrho_j(x_0), v\rangle = \langle d\varrho_j(x_0), Jv) = 0\,.$$

Then we have

$$H^q(x_0, \check{\mathfrak{D}}'_{\Omega_1 \cup \ldots \cup \Omega_k}(A^{p,*}), \bar{\partial}_*) = 0 \quad \textit{for } p = 0, \ldots, n \textit{ and } 1 \leq q < \dim_{\mathbf{C}} V - 1\,.$$

PROOF. – Substituting $1 - \exp[-c\varrho_j]$, with $c > 0$ and large, we can assume that the Levi forms of the ϱ_j's are negative definite on V. Then we choose holomorphic coordinates to satisfy (i) and (ii) in the proof of the previous lemma and we make the additional assumption that:

(iii)′ There is a convex open neighborhood ω_1 of 0 in $\mathbf{C}^s$ ($s = \dim_{\mathbf{C}} V$) and an open neighborhood ω_2 of 0 in $\mathbf{C}^{n-s}$ such that, for every $(z^{s+1}, \ldots, z^n) \in \omega_2$, the maps

$$\omega_1 \in (z^1, \ldots, z^s) \to \varrho_1(z^1, \ldots, z^s, z^{s+1}, \ldots, z^n) \qquad (j = 1, \ldots, k)$$

have a strictly negative defined hessian on ω_1.

We give now explicitly the argument, analogous to that at the end of the proof of the previous lemma, by which one can get rid of this additional assumption.

Let Λ be as in (26). First we notice that for every choice of a positive integer h and of $\lambda_1, \ldots, \lambda_h \in \Lambda$, the functions $\varrho_{(\lambda_1)}, \ldots, \varrho_{(\lambda_h)}$ satisfy the same assumptions as $\varrho_1, \ldots, \varrho_k$ in the statement of the lemma.

For $(\lambda_1, \ldots, \lambda_h) = \boldsymbol{\lambda} \in \Lambda^h$ we set

$$\varepsilon(\boldsymbol{\lambda}) = \sup_{1 \leq i,j \leq h} |\lambda_i - \lambda_j|\,.$$

Given $\boldsymbol{\lambda} \in \Lambda^h$ let (i_0, j_0) be the first pair of indices (for the lexicographic order) such that

$$\varepsilon(\boldsymbol{\lambda}) = |\lambda_{i_0} - \lambda_{j_0}| .$$

Then we define

$$\begin{aligned}
&\sigma_0(\boldsymbol{\lambda}) \in \Lambda^{h-1} \quad \text{by} \quad \sigma_0(\boldsymbol{\lambda})_i = \lambda_i \quad \text{for } i \neq i_0,\ i < j_0;\ \sigma_0(\boldsymbol{\lambda})_{i_0} = \tfrac{1}{2}(\lambda_{i_0} + \lambda_{j_0});\\
&\qquad\qquad\qquad\qquad \sigma_0(\boldsymbol{\lambda})_i = \lambda_{i+1} \quad \text{if } i \geqslant j_0\\
&\sigma_1(\boldsymbol{\lambda}) \in \Lambda^h \quad \text{by} \quad \sigma_1(\boldsymbol{\lambda})_i = \lambda_i \quad \text{for } i \neq i_0,\ \sigma_1(\boldsymbol{\lambda})_{i_0} = \tfrac{1}{2}(\lambda_{i_0} + \lambda_{j_0})\\
&\sigma_2(\boldsymbol{\lambda}) \in \Lambda^h \quad \text{by} \quad \sigma_2(\boldsymbol{\lambda})_i = \lambda_i \quad \text{for } i \neq j_0,\ \sigma_2(\boldsymbol{\lambda})_{j_0} = \tfrac{1}{2}(\lambda_{i_0} + \lambda_{j_0}) .
\end{aligned}$$

Moreover we set

$$\Omega(\boldsymbol{\lambda}) = \bigcup_{i=1}^{h} \{x \in X \colon \varrho_{(\lambda_i)} < 0\} \quad \text{for } \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_h) \in \Lambda^h .$$

Then

$$\begin{aligned}
\Omega(\sigma_1(\boldsymbol{\lambda})) \cup \Omega(\sigma_2(\boldsymbol{\lambda})) &= \Omega(\boldsymbol{\lambda})\\
\Omega(\sigma_1(\boldsymbol{\lambda})) \cap \Omega(\sigma_2(\boldsymbol{\lambda})) &= \Omega(\sigma_0(\boldsymbol{\lambda})) .
\end{aligned}$$

By the assumptions we made on the ϱ_j's, the family $\{\Omega(\sigma_1(\boldsymbol{\lambda})), \Omega(\sigma_2(\boldsymbol{\lambda}))\}$ is co-regular near x_0 for every $\lambda \in \Lambda^h$, so that the sequence

$$0 \to \check{\mathfrak{D}}'_{\Omega(\boldsymbol{\lambda})}(A^{p,j})_{x_0} \to \check{\mathfrak{D}}'_{\Omega(\sigma_1(\boldsymbol{\lambda}))}(A^{p,j})_{x_0} \oplus \check{\mathfrak{D}}'_{\Omega(\sigma_2(\boldsymbol{\lambda}))}(A^{p,j})_{x_0} \to \check{\mathfrak{D}}'_{\Omega(\sigma_0(\boldsymbol{\lambda}))}(A^{p,j})_{x_0} \to 0$$

is exact for all $0 \leqslant p, j \leqslant n$ and therefore we have a long exact sequence in cohomology:

$$\begin{aligned}
0 \to H^0(x_0, \check{\mathfrak{D}}'_{\Omega(\boldsymbol{\lambda})}(A^{p,*}), \bar{\partial}_*) &\to \bigoplus_{i=1}^{2} H^0(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_i(\boldsymbol{\lambda}))}(A^{p,*}), \bar{\partial}_*) \to\\
&\xrightarrow{\alpha} H^0(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_0(\boldsymbol{\lambda}))}(A^{p,*}), \bar{\partial}_*) \to H^1(x_0, \check{\mathfrak{D}}'_{\Omega(\boldsymbol{\lambda})}(A^{p,*}), \bar{\partial}_*) \to \ldots \to\\
&\to H^{q-1}(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_0(\boldsymbol{\lambda}))}(A^{p,*}), \bar{\partial}_*) \to H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\boldsymbol{\lambda})}(A^{p,*}), \bar{\partial}_*) \to\\
&\to \bigoplus_{i=1}^{2} H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_i(\boldsymbol{\lambda}))}(A^{p,*}), \bar{\partial}_*) \to H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_0(\boldsymbol{\lambda}))}(A^{p,*}), \bar{\partial}_*) \to \ldots .
\end{aligned}$$

Because for $k = 1$ we can always choose coordinates to satisfy (iii)$'$, by induction we can assume that, given some $h > 1$, the statement holds for $1 \leqslant k \leqslant h$, so that

$$H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_0(\boldsymbol{\lambda}))}(A^{p,*}), \bar{\partial}_*) = 0 \quad \text{for } 1 \leqslant q < s .$$

Since $s \geqslant 2$ (otherwise there is nothing to prove), by Hartogs' theorem the map α is onto. Therefore we obtain

$$H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\lambda)}(A^{p,*}), \bar{\partial}_*) \simeq \bigoplus_{i=1}^{2} H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\sigma_i(\lambda))}(A^{p,*}), \bar{\partial}_*) \quad \text{for } 1 \leqslant q < s .$$

An iterated application of this argument shows that the statement reduces to show that the vanishing of the cohomology is true for groups

$$H^q(x_0, \check{\mathfrak{D}}'_{\Omega(\lambda)}(A^{p,*}), \bar{\partial}_*)$$

when $\varepsilon(\boldsymbol{\lambda})$ is arbitrarily small, and in this case we can certainly fulfill (iii)$'$.

Assume then that we have chosen coordinates $z^1, \ldots, z^n$ at x_0 to satisfy (i), (ii) and (iii)$'$ and let $\bar{\partial} = \bar{\partial}' + \bar{\partial}''$ as in the previous lemma. With $G = \Omega_1 \cup \ldots \cup \Omega_k$ we have, with the notations of the previous lemma, a complex

$$\check{\mathfrak{D}}'_G(A_r^{p,r})_0 \xrightarrow{\bar{\partial}'} \check{\mathfrak{D}}'_G(A_r^{p,r+1})_0 \xrightarrow{\bar{\partial}'} \check{\mathfrak{D}}'_G(A_r^{p,r+2})_0 \to \ldots \to \check{\mathfrak{D}}'_G(A_r^{p,r+s})_0 \to 0$$

which is acyclic at the places $r + j$ for $1 \leqslant j \leqslant s - 2$ and $j = s$. From the exact sequence (28) (with G at the place of Ω) we deduce that we have surjective maps

$$H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}_{(r+1)}), \bar{\partial}_*) \to H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}_{(r)}), \bar{\partial}_*)$$

for $1 \leqslant q - r \leqslant s - 2$. Therefore

$$H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}), \bar{\partial}_*) = H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}_{(0)}), \bar{\partial}_*) = 0$$

if $H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}_{(q)}), \bar{\partial}_*) = 0$ and $q \leqslant s - 2$.

But we have an injection

$$0 \to H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}_{(q)}), \bar{\partial}_*) \to H^0(0, \check{\mathfrak{D}}'_G(A_q^{p,q+*}), \bar{\partial}_*) \simeq H^0(0, \check{\mathfrak{D}}'(A_q^{p,q+*}), \bar{\partial}'_*)$$

(where the last isomorphism is due to Hartogs' extension theorem), that shows that $H^q(0, \check{\mathfrak{D}}'_G(A^{p,*}_{(q)}), \bar{\partial}_*)$ injects into $H^q(0, \mathfrak{D}'(A^{p,*}), \bar{\partial}_*)$ and thus is zero since this last group is zero.

By Corollary 2 to Proposition 5 and the previous lemma we obtain:

LEMMA 4. – *Under the same assumptions of lemma 3 we have*

$$H^q(x_0, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar{\partial}_*) = 0 \quad \textit{for } 0 \leqslant p \leqslant n \quad \textit{and} \quad 1 \leqslant q < \dim_{\mathbf{C}} V - k .$$

B) The notion of p-pseudoconvexity and q-pseudoconcavity at the edge points.

We use the notations of point (*A*) and in particular Λ and $\varrho_{(\lambda)}$ are defined by (26) and (27).

We set

$$H_{\lambda,x_0} = \{u \in T_{x_0}X\colon \langle \partial\varrho_{(\lambda)}(x_0), u\rangle = 0\} = \{u \in T_{x_0}X\colon \langle d\varrho_{(\lambda)}(x_0), u\rangle = \langle d\varrho_{(\lambda)}(x_0), Ju\rangle = 0\}$$

$s_\lambda =$ minimal subsimplex of Λ containing $\lambda = \{\mu \in \Lambda\colon \mu_i = 0 \text{ if } \lambda_i = 0\}$

$d_\lambda =$ dimension of $s_\lambda =$ (number of i with $\lambda_i \neq 0) - 1$

$V_{\lambda,x_0} = \cap\, \{H_{\mu,x_0}\colon \mu \in s_\lambda\}$

$\delta_\lambda = \dim_{\mathbb{C}} V_{\lambda,x_0} - n + d_\lambda + 1\,.$

Then we consider the Hermitean form on V_{λ,x_0} corresponding to the quadratic form:

$$i\langle \partial\bar\partial\varrho_{(\lambda)}(x_0), v \wedge Jv\rangle \quad \text{for } v \in V_{\lambda,x_0} \tag{30}$$

(restriction to V_{λ,x_0} of the Levi form of $\varrho_{(\lambda)}$). We define

$$e(x_0, \lambda) = (p, q)$$

if the quadratic hermitean form (30) has $p - d_\lambda + \delta_\lambda$ positive eigenvalues and $q - d_\lambda + \delta_\lambda$ negative eigenvalues.

Let

$$E(x_0) = \{e(x_0, \lambda)\colon \lambda \in \Lambda\}$$

and

$$p_0(x_0) = \inf\{p\colon (p, q) \in E(x_0)\}$$

$$q_0(x_0) = \inf\{q\colon (p, q) \in E(x_0)\}\,.$$

We say that Ω is $p_0(x_0)$-pseudoconvex and $q_0(x_0)$-pseudo-concave at x_0.

Let $\chi\colon \mathbb{R} \to \mathbb{R}$ be an increasing smooth real function with $\chi(0) = 0$ and $\chi'(0) > 0$. If we set

$$\tilde\varrho_j = \chi(\varrho_j) \qquad (j = 1, \ldots, k)$$

and

$$\tilde\varrho_{(\lambda)} = \sum_{i=1}^{k} \lambda^i \tilde\varrho_j \quad \text{for } \lambda \in \Lambda\,,$$

then we have

$$\partial\bar{\partial}\tilde{\varrho}_{(\lambda)}(x_0) = \sum_{j=1}^{k} \lambda^j\big(\chi''(0)\,\partial\varrho_j(x_0)\big)\wedge\bar{\partial}\varrho_j(x_0) + \big(\chi'(0)\,\partial\bar{\partial}\varrho_j(x_0)\big)\,.$$

If we choose χ to be convex at 0, taking $\chi''(0)$ positive and large, we can obtain that for every $\lambda \in \Lambda$ the Levi form

$$\langle i\,\partial\bar{\partial}\tilde{\varrho}_{(\lambda)}(x_0),\, v\wedge Jv\rangle \tag{31}$$

has

$$\big(p(x_0, \lambda) - d_\lambda + \delta_\lambda\big) + (d_\lambda - \delta_\lambda) = p(x_0, \lambda)$$

positive eigenvalues on $T_{x_0}X$.

If we choose χ to be concave at 0, taking $\chi''(0)$ negative and large, we can obtain that for every $\lambda \in \Lambda$ the Levi form (31) has at least $q(x_0, \lambda)$ negative eigenvalues on $T_{x_0}X$.

The notation of p-pseudoconvexity and q-pseudoconcavity introduced in this way is a pseudo-conformal invariant of Ω at x_0, as it does not depend on the choice of the functions $\varrho_1, \dots, \varrho_k$.

C) The Vanishing Theorems.

With the notations of subsections A) and B): let

$$G = \bigcup_{j=1}^{k} \{x \in X:\, \varrho_j(x) < 0\}\,.$$

Then we have the following

PROPOSITION 11. – *Assume that Ω be p_0-pseudoconvex and q_0-pseudoconcave at x_0 and that locally Ω can be described as the intersection of k open sets with smooth boundaries. Then*

$$H^j(x_0, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar{\partial}_*) = 0 \quad \textit{for } 1\leqslant j\leqslant q_0 - k \textit{ and for } n - p_0 \leqslant j \leqslant n\,, \quad \textit{for } p = 0, \dots, n\,.$$

Moreover with G defined as above, we have:

$$H^j(x_0, \check{\mathfrak{D}}'_G(A^{p,*}), \bar{\partial}_*) = 0 \quad \textit{for } 1\leqslant j < q_0 \textit{ and for } n - p_0 + k - 1 \leqslant j \leqslant n\,, \quad \textit{for } p = 0, \dots, n\,.$$

PROOF. – In view of the remarks in subsection B), repeating the arguments given at the end of the proof of lemma 2 and at the beginning of the proof of lemma 3, we reduce to the statements of lemmas 2, 3, 4.

PROPOSITION 12. – *Under the same assumptions of the previous proposition: if* $q_0 \geqslant 1$, *then the natural map induced by the restriction*:

$$H^0(x_0, \mathfrak{D}'(A^{p,*}), \bar{\partial}_*) \to H^0(x_0, \check{\mathfrak{D}}'_G(A^{p,*}), \bar{\partial}_*)$$

is onto. If $q_0 \geqslant k$, *then the natural map*

$$H^0(x_0, \mathfrak{D}'(A^{p,*}), \bar{\partial}_*) \to H^0(x_0, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar{\partial}_*)$$

is onto.

REMARK. – From the remark at the end of § 2 it follows that, under the assumptions of Proposition 11, we can find a fundamental system of open neighborhoods $\{U_m\}$ of x_0 in X such that

$$H^j(U_m, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar{\partial}_*) = 0 \quad \text{for } j \geqslant n - p_0$$
$$H^j(U_m, \check{\mathfrak{D}}'_G(A^{p,*}), \bar{\partial}_*) = 0 \quad \text{for } j \geqslant n - p_0 + k - 1 .$$

5. – The Poincaré lemma for the tangential Cauchy-Riemann complex on distributions.

Let X be a complex manifold of dimension n and let S be a generic real submanifold of X of real codimension k. As we are interested in local results, we will assume that S is defined by global equations on X, i.e. that for smooth real valued functions $\varrho_1, \dots, \varrho_k : X \to \mathbb{R}$ we have

$$S = \{x \in X : \varrho_1(x) = \dots = \varrho_k(x) = 0\}$$

and that

$$\partial\varrho_1(x) \wedge \dots \wedge \partial\varrho_k(x) \neq 0 , \quad \forall x \in X .$$

The Levi form of S is defined in the following way: let HS denote the analytic tangent bundle of S and let HS^0 denote the annihilator of HS in T^*S. Given any $v \in \Gamma(HS)_x$ and $\lambda \in H_x S^0$ we consider

$$\mathfrak{L}(x, \lambda, v_x) = \lambda([v, Jv]_x) .$$

It turns out indeed that $\mathfrak{L}(x, \lambda, v_x)$ only depends on the value $v_x \in H_x S$ of v at x and that therefore for every fixed $\lambda \in H_x S^0$ the map

$$v_x \to \mathfrak{L}(x, \lambda, v_x) \tag{32}$$

defines a hermitean quadratic form on H_xS, that we call the Levi form of S at x in the direction λ. We denote by

$$e_S(x, \lambda) = (p, q)$$

the function that associates to each $\lambda \in HS^0$ the pair (p, q) where p is the number of positive and q the number of negative eigenvalues of the form (32) on H_xS.

We set

$$E(S, x) = \{e_S(x, \lambda) \colon \lambda \in HS^0 - \{0\}\}\,.$$

If

$$q_0 = \inf\{q \colon (p, q) \in E(S, x)\}$$

we say that *S is q-concave at x if $q \leq q_0$.*

To give an interpretation of (32), let us consider the vector bundle N_S^*X on S of differential forms in X vanishing on TS: we have the inclusion $N_S^*X \subset T^*X|\,S$.

Then we observe that by the genericity assumption the natural projection

$$T^*X|\,S \to T^*S$$

induces an isomorphism

$$J^*N_S^*X \overset{\alpha}{\underset{\sim}{\to}} HS^0\,,$$

where $J^* \colon T^*X \to T^*X$ is the dual of the complex structure.

If $\lambda \in H_xS^0$ and $\omega \in \Gamma(N_S^*X)_x$ is such that $\lambda = \alpha(J^*\omega)$, we have, for $v_x \in H_xS$:

$$\mathfrak{L}(x, \lambda, v_x) = \langle J^*\omega, [v, Jv]_x\rangle = \tfrac{1}{2}\langle dJ^*\omega, v_x \wedge Jv_x\rangle\,.$$

Because the differentials $d\varrho_1, \ldots, d\varrho_k$ generate $\Gamma(N_S^*X)_x$ we have

$$\omega = \alpha^1\, d\varrho_1 + \ldots + \alpha^k\, d\varrho_k \qquad \text{with } \alpha^1, \ldots, \alpha^k \in \Gamma(\mathbb{R})_x\,,$$

and because $\partial\varrho_j(x)$ and $\bar{\partial}\varrho_j(x)$ vanish on v_x and Jv_k for $v_x \in H_xS$ we obtain:

$$\mathfrak{L}(x, \lambda, v_x) = \tfrac{1}{2}\langle dJ^*\omega, v_x \wedge Jv_x\rangle = \tfrac{1}{2}\sum_{j=1}^{k}\big(\alpha^j\langle d(\bar{\partial}\varrho_j - \partial\varrho_j), v_x\wedge Jv_x\rangle + \\ + \langle\partial\alpha^j \wedge \bar{\partial}\varrho_j, v_x \wedge Jv_x\rangle - \langle\bar{\partial}\alpha^j \wedge \partial\varrho_j, v_x\wedge Jv_x\rangle\big) = \sum_{j=1}^{k}\alpha^j(x)\langle\partial\bar{\partial}\varrho_j(x), v_x \wedge Jv_x\rangle\,.$$

One reconnizes the usual Levi from (31), if we choose as coordinates on H_xS^0 the coefficients $\lambda^1, \ldots, \lambda^k$ of the expression of $J^{*-1}\alpha^{-1}(\lambda) \in (N_S^*X)_x$ in the basis $d\varrho_1(x), \ldots$ $\ldots, d\varrho_k(x)$.

We have the following

THEOREM A. – (*Poincaré Lemma for the Tangential Cauchy-Riemann complex on Distribution.*) *If S is q_0-concave at x_0, then*

$$H^j(x_0, \mathfrak{D}'(Q_S^{p,*}), \bar\partial_{S*}) \cong H^j(x_0, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) = 0 \quad \textit{for } p = 0, \dots, n, \quad 1 \leqslant j < q_0 \quad \textit{and} \quad j > n-k-q_0 .$$

If $k = 1$ (S is a hypersurface) and $(p_0, q_0) \in E(S, x_0)$ with $0 \leqslant p_0 \leqslant q_0 \leqslant n-1$, then

$$H^j(x_0, \mathfrak{D}'(Q_S^{p,*}), \bar\partial_{S*}) \cong H^j(x_0, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) = 0 \quad \textit{for } p = 0, \dots, n, \quad 1 \leqslant j < p_0,\ n-q_0-1 < j < q_0 \quad \textit{and} \quad j > n-p_0-1 .$$

PROOF. – By the Corollary to Proposition 10, as x_0 has in X a fundamental system of open Stein neighborhoods, we have

$$H^j(x_0, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) \cong H^{j+k-1}(x_0, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \quad \text{for } j \geqslant 0 . \tag{33}$$

Therefore it is sufficient to prove the vanishing theorem for the right hand side of (33).

(*A*) We consider first the case $k = 1$. Then, if

$$\Omega^+ = \{x\colon \varrho_1(x) > 0\} \quad \text{and} \quad \Omega^- = \{x\colon \varrho_1(x) < 0\},$$

we have

$$H^q(x_0, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \cong H^q(x_0, \check{\mathfrak{D}}'_{\Omega+}(A^{p,*}), \bar\partial_*) \oplus H^q(x_0, \check{\mathfrak{D}}'_{\Omega-}(A^{p,*}), \bar\partial_*) \quad \text{for all } 0 \leqslant p,\ q \leqslant n .$$

By Proposition 11, assuming as we can that

$$i\langle \partial\bar\partial\varrho_1, v \wedge Jv\rangle, \quad v \in H_x S$$

has p_0-positive and q_0-negative eigenvalues, by Proposition 11 we have

$$H^q(x_0, \check{\mathfrak{D}}'_{\Omega+}(A^{p,*}), \bar\partial_*) = 0 \quad \text{if } 1 \leqslant q < p_0 \quad \text{and} \quad n-q_0 \leqslant q \leqslant n,\ p = 0, \dots, n$$

and

$$H^q(x_0, \check{\mathfrak{D}}'_{\Omega-}(A^{p,*}), \bar\partial_*) = 0 \quad \text{if } 1 \leqslant q < q_0 \quad \text{and} \quad n-p_0 \leqslant q \leqslant n,\ p = 0, \dots, n .$$

The statement follows.

(*B*) We assume that $q_0 > 1$ and we prove that

$$H^j(x_0, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_S) = 0 \quad \text{for } p = 0, \dots, n \quad \text{and} \quad 1 \leqslant j < q_0 .$$

Let us set, for a real N,

$$\begin{aligned} \varphi_0 &= -(\varrho_1 + \dots + \varrho_k) - N(\varrho_1^2 + \dots + \varrho_k^2) \\ \varphi_1 &= \varrho_1 - N(\varrho_1^2 + \dots + \varrho_k^2) \\ &\dots\dots\dots\dots\dots\dots \\ \varphi_k &= \varrho_k - N(\varrho_1^2 + \dots + \varrho_k^2) . \end{aligned} \tag{34}$$

If we substitute X by a neighborhood of x_0 in X, the system of open sets

$$\mathfrak{U} = \{\Omega_0, \Omega_1, \dots, \Omega_k\} \tag{35}$$

where

$$\Omega_j = \{x \in X : \varphi_j(x) < 0\} \tag{36}$$

is co-regular and

$$\cup\, \mathfrak{U} = X - S , \qquad \cap\, \mathfrak{U} = \emptyset .$$

Then by Corollary 1 to Proposition 5 we have, for a fixed $q \geqslant k$,

$$H^q(x_0, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) = 0$$

if

$$H^{q-r}\big(x_0, C^r(\mathfrak{U}, \check{\mathfrak{D}}'_*(A^{p,*})), \bar\partial_*\big) \cong \bigoplus_{0 \leqslant h_0 < \dots < h_r \leqslant k} H^{q-r}(x_0, \check{\mathfrak{D}}'_{\Omega_{h_0} \cap \dots \cap \Omega_{h_r}}(A^{p,*}), \bar\partial_*) = 0$$
$$\text{for } r = 0, 1, \dots, k-1 .$$

(indeed we note that $Z^{k-1}(\mathfrak{U}, -) = C^{k-2}(\mathfrak{U}, -)$ and $Z^k(\mathfrak{U}, -) = 0$).

Taking $N > 0$ sufficiently large, all intersections $\Omega_{h_0} \cap \dots \cap \Omega_{h_r}$ $(r = 0, \dots, k-1,$ $0 \leqslant h_0 < h_1 < \dots < h_r \leqslant k)$ are $(q_0 + k - 1)$-concave at x_0. Thus by Proposition 11

$$H^{q-j}(x_0, \hat{\mathfrak{D}}'_{\Omega_{h_1} \cap \dots \cap \Omega_{h_j}}(A^{p,*}), \bar\partial_*) = 0 \quad \text{for } 1 \leqslant q - r \leqslant (q_0 + k - 1) - (r + 1) .$$

If we set $q = j + k - 1$, we obtain the inequality

$$1 \leqslant j + k - (r + 1) \leqslant q_0 + k - r - 2 \quad \text{for } r = 0, 1, \dots, k - 1$$

that is satisfied for $1 \leqslant j \leqslant q_0 - 1$.

(*C*) The proof that

$$H^j(x_0, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) = 0 \quad \text{if } p = 0, \dots, n \quad \text{and} \quad j > n - q_0 - k$$

is obtained repeating the argument of (*B*), only taking this time N large and negative in the definition of the φ_j's.

6. – Extension of Cauchy-Riemann distributions.

We use the notations introduced in § 5. Choosing in (34) an $N > 0$ and sufficiently large, then every linear combination of $\varphi_0, \dots, \varphi_k$ has at $x_0 \in S$ a Levi form with at least k negative eigenvalues on $T_{x_0}X$. Then, with $\mathcal{U} = \{\Omega_0, \Omega_1, \dots, \Omega_k\}$ defined by (35) and (36) we have by Proposition 11:

$$H^q\big(x_0, C^j(\mathcal{U}, \check{\mathfrak{D}}'_*(A^{p,*})), \bar\partial_*\big) = 0 \quad \text{for } 0 \leqslant j \leqslant k-2 \quad \text{and} \quad 1 \leqslant q \leqslant k - j - 1 .$$

Then by Corollary 4 to Proposition 5 we obtain a surjective map:

$$\begin{array}{ccc} H^0\big(x_0, Z^{k-1}(\mathcal{U}, \check{\mathfrak{D}}'_*(A^{p,*})), \bar\partial_*\big) & \to & H^{k-1}(x_0, \check{\mathfrak{D}}'_{X-S}(A^{p,*}), \bar\partial_*) \\ & {\scriptstyle \text{Res}}\searrow & \downarrow \cong \\ & & H^0(x_0, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) \end{array}$$

We notice that

$$Z^{k-1}(\mathcal{U}, \check{\mathfrak{D}}'_*(A^{p,q})) = C^{k-1}(\mathcal{U}, \check{\mathfrak{D}}'_*(A^{p,q})) \cong \bigoplus_{0 \leqslant i_1 < \dots < i_k \leqslant k} \check{\mathfrak{D}}'_{\Omega_{i_1} \cap \dots \cap \Omega_{i_k}}(A^{p,q}) . \tag{37}$$

The inclusion maps of the direct summands in the right hand side of (37) give by composition the restriction maps:

$$\text{Res}_{i_1 \dots i_k} \colon H^0(x_0, \check{\mathfrak{D}}'_{\Omega_{i_1} \cap \dots \cap \Omega_{i_k}}(A^{p,*}), \bar\partial_*) \to H^0(x_0, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*) .$$

If S is q_0-pseudoconcave at x_0, with $q_0 \geqslant 1$, then by the theorem of Hartogs we have

$$H^0(x_0, \check{\mathfrak{D}}'_{\Omega_{i_1} \cap \dots \cap \Omega_{i_k}}(A^{p,*}), \bar\partial_*) = H^0(x_0, \check{\mathfrak{D}}'(A^{p,*}), \bar\partial_*) = \Omega^p_{x_0}$$

where Ω^p denotes the sheaf of germs of holomorphic p-forms on X. Then we have:

THEOREM B. – *If S is q_0-pseudoconcave at x_0 with $q_0 \geqslant 1$, then every germ of Cauchy-Riemann p-distribution on S at x_0 (i.e. every element of $H^0(x_0, {}^0\mathfrak{I}_S^{p,k+*}, \bar\partial_*)$) is the restriction to S of a germ of holomorphic p-form at x_0.*

With the notations of § 5: let $L \subset H_{x_0}S^0$ be the set of $\lambda \in H_{x_0}S$ such that

$$e(x_0, \lambda) \in \mathbb{N} \times \{0\},$$

i.e. such that the Levi form

$$v_x \to \mathfrak{L}(x_0, \lambda, v_x)$$

is non-negative on $H_{x_0}S$.

The set L is a closed convex cone in $H_{x_0}S^0$.

We choose on L the coordinates adjusted to the normal bundle. Then, if L is not a half space in $H_{x_0}S^0$, the open set

$$\Omega = \text{interior of} \bigcap_{\lambda \in L} \{x \in X\colon \varrho_{(\lambda)}(x) < 0\}$$

is non-empty.

In this case we can find $\lambda_1, \ldots, \lambda_k \in \mathbb{R}^k$ such that

$$L \subset \{\alpha^1 \lambda_1 + \ldots + \alpha^k \lambda_k\colon \alpha^j > 0,\ \forall j = 1, \ldots, k\}.$$

Let

$$W = \{x \in X\colon \varrho_{(\lambda_1)}(x) < 0, \ldots, \varrho_{(\lambda_k)}(x) < 0\}.$$

Let L' be any proper convex open cone containing $L - \{0\}$ and let

$$\Omega' = \text{interior of} \bigcap_{\lambda \in L'} \{x \in X\colon \varrho_{(\lambda)}(x) < 0\}.$$

PROPOSITION 13. – *The restriction map*

$$H^0(x_0, \check{\mathfrak{D}}'_{\Omega'}(A^{p,*}), \bar\partial_*) \to H^0(x_0, \check{\mathfrak{D}}'_W(A^{p,*}), \bar\partial_*) \to$$
$$\to H^0(x_0, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) \cong H^0(x_0, \mathfrak{D}'(Q_S^{p,*}), \bar\partial_{S*})$$

is onto.

PROOF. – By unique continuation the statement is true if we show it is true when $\Omega' = W$. We can also assume that $\lambda_1, \ldots, \lambda_k$ is the canonical base of $\mathbb{R}^k$.

Let the φ_j's be defined as in (34) with $N > 0$ and large. Then for $1 \leqslant j_1 < \ldots < j_{k-1} \leqslant k$ we have by Proposition 12 isomorphisms

$$H^0(x_0, \mathfrak{D}'(A^{p,*}), \bar\partial_*) \xrightarrow{\sim} H^0(x_0, \check{\mathfrak{D}}'_{\Omega_0 \cap \Omega_{j_1} \cap \ldots \cap \Omega_{j_{k-1}}}(A^{p,*}), \bar\partial_*).$$

Then, by the remark at the end of subsection E of § 1, and (37), we obtain the result.

7. – Finiteness theorem for global cohomology groups.

A) Tangential Cauchy-Riemann Complexes on Compact Submanifolds.

Let X be a complex manifold of dimension n and let S be a generic orientable real submanifold of X or real codimension k. Then we have:

THEOREM C. – *If $q \geqslant 1$ and S is $(q+1)$-pseudo concave at every point and compact, then*

$$H^q(S, {}^0\mathfrak{I}_S^{p,k+*}, \bar{\partial}_*) \cong H^q(S, \Gamma(Q_S^{p,*}), \bar{\partial}_{S*}) \cong H^q(S, \Omega^p | S) \tag{38}$$

are all finite dimensional.

PROOF. – By Theorems A and B above we have an exact sequence of sheaves:

$$0 \to \Omega^p | S \to {}^0\mathfrak{I}_S^{p,k} \xrightarrow{\bar{\partial}} {}^0\mathfrak{I}_S^{p,k+1} \xrightarrow{\bar{\partial}} {}^0\mathfrak{I}_S^{p,k+2} \to \dots \to {}^0\mathfrak{I}_S^{p,k+q+1} . \tag{39}$$

On the other hand by Proposition 11 and 12 in [14] we have also an exact sequence of sheaves:

$$0 \to \Omega^p | S \to \mathfrak{Q}_S^{p,0} \xrightarrow{\bar{\partial}_S} \mathfrak{Q}_S^{p,1} \xrightarrow{\bar{\partial}_S} \mathfrak{Q}_S^{p,2} \to \dots \to \mathfrak{Q}_S^{p,q+1} . \tag{40}$$

Therefore the isomorphisms (38) follow because (39) and (40) are free resolutions of the same sheaf. We can use then Čech cohomology to compute $H^q(S, \Omega^p | S)$.

By an argument of functional analysis (cf. [2], p. 383, Lemma 3), we deduce the following: « *We can find a finite open covering $\mathfrak{U}$ of S in X such that*:

(i) $H^q(\mathfrak{U}|S, \Omega^p|S) \to H^q(S, \Omega^p | S)$ *is onto*;

(ii) $H^q(\mathfrak{U}, \Omega^p) \to H^q(\mathfrak{U}|S, \Omega^p|S)$ *is onto* ».

But S has in X a fundamental system of relatively compact $(q+1)$-pseudoconcave open reighborhoods and thus we obtain from (i) and (ii) a surjective map

$$H^q(U, \Omega^p) \to H^q(S, \Omega^p | S)$$

for some relatively compact $(q+1)$-pseudoconcave open neighborhood of S in X. Because $H^q(U, \Omega^p)$ is finite dimensional (cf. [3]), the statement follows.

B) Duality on Compact Submanifolds.

The complexes deduced from (38) and (40):

$$(S, {}^0\mathfrak{I}_S^{p,k+*}, \bar{\partial}_*) = \{0 \to {}^0\mathfrak{I}_S^{p,k}(S) \xrightarrow{\bar{\partial}} {}^0\mathfrak{I}_S^{p,k+1}(S) \to \dots \to {}^0\mathfrak{I}_S^{p,n}(S) \to 0\} \tag{39$'$}$$

and

$$(40)' \qquad (S, Q_S^{p,*}, \bar\partial_{S*}) = \{0 \to Q_S^{n-p,0}(S) \xrightarrow{\bar\partial_S} Q_S^{n-p,1}(S) \to \ldots \to Q_S^{n-p,n-k}(S) \to 0\}$$

are dual of each other when S is compact, because the map

$${}^0\mathfrak{J}_S^{p,k+j}(S) \xrightarrow{\bar\partial} {}^0\mathfrak{J}_S^{p,k+j+1}(S)$$

is the dual map of

$$Q_S^{n-p,n-k-j-1}(S) \xrightarrow{\bar\partial_S} Q_S^{n-p,n-k-j}(S)\,.$$

If S is q_0-pseudoconcave at every point, then the two complexes (39)′ and (40)′ have finite dimensional cohomology groups at the places j with $1 \leqslant j \leqslant q_0 - 1$. Then an application of the duality lemma yields the following

PROPOSITION 14. – *If S is compact and q_0-pseudoconcave at every point, then the cohomology groups*

$$H^q(S, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) \cong H^q(S, \Gamma(Q_S^{n-p,*}), \bar\partial_{S*})$$

for $p = 0, \ldots, n$ and $n - k - q_0 + 1 < q < n - k$ are isomorphic and finite dimensional. Moreover

$$H^{n-k}(S, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*) \cong {}^0\mathfrak{J}_S^{p,n}(S)/\{T\colon \langle T, f\rangle = 0,\ \forall f \in \Omega^{n-p}|\, S(S)\}$$

and

$$H^{n-k}(S, \Gamma(Q_S^{p,*}), \bar\partial_{S*}) \cong Q_S^{p,n-k}(S)/\{f\colon \int_S f \wedge \varphi = 0,\ \forall \varphi \in \Omega^{n-p}|\, S(S)\}\,.$$

for $p = 0, \ldots, n$.

REMARK. – Also the groups $H^{n-k-q_0+1}(S, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*)$ and $H^{n-k-q_0+1}(S, \Gamma(Q_S^{p,*}), \bar\partial_{S*})$ are finite dimensional for $p = 0, \ldots, n$. This result can be proved for instance by the method of sub-elliptic estimates (cf. [16]). It implies that also the groups $H^{q_0}(S, {}^0\mathfrak{J}_S^{p,k+*}, \bar\partial_*)$ and $H^{q_0}(S, \Gamma(Q_S^{p,*}), \bar\partial_{S*})$ are separated.

The case of a compact hypersurface is discussed in [9].

C) Domains with piece-wise smooth boundaries.

Let Ω be an open domain in the complex manifold X and let F denote its closure. The Taylor series of real valued smooth functions defined on a neighborhood of a point $x_0 \in F$ and non-negative on F form a convex cone C in the space of formal power series at x_0.

We say that Ω is piece-wise smooth at $x_0 \in \partial\Omega \subset F$ if there are k functions $\varrho_1, \ldots, \varrho_k$ defined in a neighborhood U of x_0 such that

(i) the Taylor series of $\varrho_1, \ldots, \varrho_k$ at x_0 generate C and form an extremal subset of C;

(ii) for every $j = 1, \ldots, k$, we have $d\varrho_j(x_0) \neq 0$.

We say in this case that x_0 is a k-edge of Ω. If every point of $\partial\Omega$ is a k-edge for some finite k, we say that Ω is piece-wise smooth. We say also that $\varrho_1, \ldots, \varrho_k$ define Ω near x_0. Clearly the function that associates to each point of the boundary of a piece-wise smooth open set the « order of edge » is upper semicontinuous. Let us denote by $W_F(E)$ the sheaf of Whitney sections on F of the vector bundle E (cf. [14]). We have the following global version of the results established in § 4 and in ([14], § 5).

PROPOSITION 15. – *Let Ω be a piece-wise smooth relatively compact open subset of X.*

(*a*) *If Ω is $(q+k)$-pseudoconcave at every k-edge point $x \in \partial\Omega$ $(k = 1, 2, \ldots)$, and $q \geqslant 1$ then*

$$H^q(X, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar{\partial}_*) \quad \textit{and} \quad H^q(X, W_F(A^{p,*}), \bar{\partial}_*)$$

are finite dimensional for $0 \leqslant p \leqslant n$.

(*b*) *If Ω is p_0-pseudoconvex at every point of $\partial\Omega$, then*

$$H^j(X, \check{\mathfrak{D}}'_\Omega(A^{p,*}), \bar{\partial}_*) \quad \textit{and} \quad H^j(X, W_F(A^{p,*}), \bar{\partial}_*)$$

are finite dimensional for $j > n - p_0$, $p = 0, \ldots, n$.

PROOF. – The proof of point (*a*) can be done by reducing to Čech cohomology as in Theorem C and therefore is omitted. The proof of statement (*b*) can be obtained by an adaptation of the bump lemma in [3].

It is convenient to fix a finite family of smooth functions $R = \{\varrho_i : i \in I\}$ such that

$$\Omega = \{x \in X : \varrho_i(x) < 0, \forall i \in I\}$$

$$F = \{x \in X : \varrho_i(x) \leqslant 0, \forall i \in I\}$$

and for every $x_0 \in \partial\Omega$ there is a subfamily of R defining Ω near x_0. This is always possible for piecewise smooth relatively compact open sets.

When proving the local vanishing theorem, we showed that the local cohomology groups at points $x_0 \in \partial\Omega$ can be factored through groups of some domains, each of which pseudo-conformally equivalent to a domain of $\mathbf{C}^n$ cutted into strictly convex domains by a family of parallel complex p-planes. This property is preserved under small deformations of Ω: if $x_0 \in \partial\Omega$, we can find an open neighborhood V of x_0 in X such that, for every real valued smooth function φ on X we can find $\varepsilon > 0$ such that, if

$$\Omega_t = \{x\colon \varrho_i(x) + t\varphi(x) < 0,\ \forall i \in I\}\,,$$

then for $|t| < \varepsilon$ the maps

$$H^j(X, W_{\bar\Omega_t}(A^{p,*}), \bar\partial_*) \to H^j(V, W_{\bar\Omega_t}(A^{p,*}), \bar\partial_*) \tag{41}$$

and

$$H^j(X, \check{\mathfrak{D}}'_{\Omega_t}(A^{p,*}), \bar\partial_*) \to H^j(V, \check{\mathfrak{D}}'_{\Omega_t}(A^{p,*}), \bar\partial_*) \tag{42}$$

have zero image for $j > n - p_0$, $p = 0, \ldots, n$.

We choose a finite open covering $\mathfrak{V}$ of $\partial\Omega$ by open sets V in X such that the property above is satisfied and a partition of unity $\{\varphi_V\colon V \in \mathfrak{V}\}$ by smooth real valued nonnegative functions with compact support subordinated to $\mathfrak{V}$.

If

$$\varepsilon\colon \mathfrak{V} \ni V \to \varepsilon_V \in \mathbb{R}\,, \qquad \varepsilon_V \geqslant 0,\ \forall V \in \mathfrak{V}$$

is a given function, we set

$$\Omega_\varepsilon = \{x \in X\colon \varrho_i(x) + \sum_{\mathfrak{V}} \varepsilon_V \varphi_V(x) < 0\}\,.$$

By the remarks made above we deduce that there is $\varepsilon_0 > 0$ such that, if

$$\varepsilon_V < \varepsilon_0\,, \qquad \forall V \in \mathfrak{V}\,,$$

then

$$H^j(X, \check{\mathfrak{D}}'_{\Omega_\varepsilon}(A^{p,*}), \bar\partial_*) \to H^j(V, \check{\mathfrak{D}}'_{\Omega_\varepsilon}(A^{p,*}), \bar\partial_*)$$

and

$$H^j(X, W_{\bar\Omega_\varepsilon}(A^{p,*}), \bar\partial_*) \to H^j(V, W_{\bar\Omega_\varepsilon}(A^{p,*}), \bar\partial_*)$$

have zero image for all $V \in \mathfrak{V}$ and all $j > n - p_0$, $p = 0, \ldots, n$.

Let $V_0 \in \mathfrak{V}$ and let $\varepsilon, \varepsilon'$ be such that

$$\varepsilon_V = \varepsilon'_V \geqslant 0\,, \qquad \forall V \in \mathfrak{V} - \{V_0\}$$

$$\varepsilon_{V_0} < \varepsilon'_{V_0}\,.$$

Then we have (assuming $\varepsilon_0 > 0$ sufficiently small) exact sequences:

$$\ldots \to H^j(X, W_{\bar{\Omega}_{\varepsilon'}}(A^{p,*}), \bar{\partial}_*) \to H^j(X, W_{\bar{\Omega}_\varepsilon}(A^{p,*}), \bar{\partial}_*) \oplus$$
$$\oplus H^j(V_0, W_{\bar{\Omega}_{\varepsilon'}}(A^{p,*}), \bar{\partial}_*) \to H^j(V_0, W_{\bar{\Omega}_\varepsilon}(A^{p,*}), \bar{\partial}_*) \to \ldots$$

and

$$\ldots \to H^j(X, \check{\mathfrak{D}}'_{\Omega_{\varepsilon'}}(A^{p,*}), \bar{\partial}_*) \to H^j(X, \check{\mathfrak{D}}'_{\Omega_\varepsilon}(A^{p,*}), \bar{\partial}_*) \oplus$$
$$\oplus H^j(V_0, \check{\mathfrak{D}}'_{\Omega_{\varepsilon'}}(A^{p,*}), \bar{\partial}_*) \to H^j(V_0, \check{\mathfrak{D}}'_{\Omega_\varepsilon}(A^{p,*}), \bar{\partial}_*) \to \ldots$$

from which we deduce that

$$H^j(X, W_{\bar{\Omega}_{\varepsilon'}}(A^{p,*}), \bar{\partial}_*) \to H^j(X, W_{\bar{\Omega}_\varepsilon}(A^{p,*}), \bar{\partial}_*)$$

and

$$H^j(X, \check{\mathfrak{D}}'_{\Omega_{\varepsilon'}}(A^{p,*}), \bar{\partial}_*) \to H^j(X, \check{\mathfrak{D}}'_{\Omega_\varepsilon}(A^{p,*}), \bar{\partial}_*)$$

are onto for $j > n - p_0$ and $p = 0, \ldots, n$.

By recurrence we obtain, for some ε with $\varepsilon_V > 0$, $\forall V \in \mathfrak{V}$, surjective maps

$$H^j(X, W_{\bar{\Omega}_\varepsilon}(A^{p,*}), \bar{\partial}_*) \to H^j(X, W_F(A^{p,*}), \bar{\partial}_*)$$

and

$$H^j(X, \check{\mathfrak{D}}'_{\Omega_\varepsilon}(A^{p,*}), \bar{\partial}_*) \to H^j(X, \check{\mathfrak{D}}'_{\Omega}(A^{p,*}), \bar{\partial}_*)$$

for $j > n - p_0$ and $p = 0, \ldots, n$.

Because these maps factor through the group $H^j(U, \Omega^p)$ for any open set U with $F \subset U \subset \Omega_\varepsilon$, and F has in X a fundamental system of p_0-pseudoconvex relatively compact open neighborhoods, the statement follows.

REFERENCES

[1] A. ANDREOTTI, *Complexes of Partial Differential Operators*, Yale Univ. Press Chicago, 1975.

[2] A. ANDREOTTI - G. FREDRICKS - M. NACINOVICH, *On the Absence of Poincaré Lemma in Tangential Cauchy-Riemann Complexes*, Ann. Scuola Norm. Sup. Pisa cl. Sci., (4) **8** (1981), pp. 365-404.

[3] A. ANDREOTTI - H. GRAUERT, *Théorèmes de Finitude pour la cohomologie des espaces complexes*, Bull. Soc. Math. France, **90** (1962), pp. 193-259.

[4] A. ANDREOTTI - C. D. HILL, *E. E. Levi convexity and the Hans Levy Problem, I and II*, Ann. Scuola Norm. Sup. Pisa cl. Sci., (3) **26** (1972), pp. 325-363 and 747-806.

[5] A. ANDREOTTI - C. D. HILL - S. ŁOJASIEWICZ - B. MACKICHAN, *Complexes of Differential Operators. The Mayer-Vietoris Sequence*, Invent. Math., **35** (1976), pp. 43-86.
[6] A. ANDREOTTI - M. NACINOVICH, *Noncharacteristic Hypersurfaces for Complexes of Differential Operators*, Ann. Mat. Pura Appl., (4) **125** (1980), pp. 18-83.
[7] M. S. BAOUENDI - C. H. CHANG - F. TREVES, *Microlocal Hypo-Analyticity and Extension of CR functions*, J. Diff. Geometry, **18** (1983), pp. 331-391.
[8] A. BOGGESS - J. POLKING, *Holomorphic Extensions of C.R. functions*, Duke Math. J., **49** (1982), pp. 757-784.
[9] G. P. FOLLAND - J. J. KOHN, *The Neumann Problem for the Cauchy-Riemann Complex*, Ann. of Math. Studies 75, Princeton Univ. Press, 1972.
[10] G. M. HENKIN, *Analytic Representation of C.R. functions on submanifolds of codimension 2 in C^n*, Analytic Functions, Kozubnik, 1979, Lecture Notes in Math., Springer, Berlin, **789** (1980), pp. 169-191.
[11] G. M. HENKIN, *Integral Representation of Differential forms on Cauchy-Riemann Manifolds and the theory of C.R. functions* (in russian) in Uspiehi Math. Nauk., **39** (1984), pp. 39-106.
[12] M. KASHIWARA - P. SCHAPIRA, *Application of the microlocal Theory of sheaves to the study of $\mathcal{O}_X$*, Proc. 1984 Symp. on « Pseudodifferential Operators with applications to Partial Differential Equations » to be published by A.M.S.
[13] A. MARTINEAU, *Distribution et valeurs au bord de fonctions holomorphes*, Strasbourg RCP, **25** (1966).
[14] M. NACINOVICH, *Poincaré Lemma for Tangential Cauchy-Riemann Complexes*, Math. Annalen, **268** (1984), pp. 449-471.
[15] M. NACINOVICH, *On Boundary Hilbert Differential Complexes*, Annales Polonici Mathematici, XLVI, (1985), pp. 213-235.
[16] M. NACINOVICH, *On Weighted Estimates for some systems of partial differential equations*, Rend. Sem. Mat. Univ. Padova, **69** (1983), pp. 221-232.
[17] R. RANGE - YUM-TOM SIU, *Uniform Estimates for the $\bar\partial$-equation on Domains with Piecewise-smooth Pseudoconvex Boundaries*, Math. Ann., **206** (1973), pp. 325-354.
[18] L. SCHWARTZ, *Théorie des Distributions*, 2 nd. ed. Hermann, Paris, 1966.
[19] S. TAJIMA, *Analyse Microlocal sur les variétés de Cauchy-Riemann et problèmes de prolongement des solutions holomorphes des equations aux derivées partielles*, Publi. R.I.M.S. Kyoto Univ., **18** (1982), pp. 911-945.