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TANGENTIAL-EXCEPTIONAL SETS FOR HARDY-SOBOLEV SPACES[1]

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Section 1. Introduction

Let B denote the unit ball in $\mathbb{C}^n$, and S its boundary. Let $d\sigma$ be the normalized Lebesgue measure on S. For $\alpha > 0$ and $0 < p < +\infty$, the Hardy-Sobolev space H_α^p is the space of holomorphic functions f in B so that $R^\alpha f \in H^p(B)$, where if $f = \Sigma_k f_k$ is its homogeneous expansion, $R^\alpha f = \Sigma_k (k+1)^\alpha f_k$. It is well known that for $\alpha p > n$, this space consists of Lipschitz functions.

In recent years there has been a great number of works dealing with the convergence along tangential approach regions of functions in the space of Poisson integrals of Bessel potentials of H^p functions. It turns out that the "tangentiality" of the approach region depends on n, α and p (see [*N-R-S*], [*N-S*] and [*A-N*]), and it flattens as the order of regularity increases. In the unit ball case, and for $p > 1$, it is easy to see, following [*N-R-S*] how these approach regions look.

If $f \in H_\alpha^p$, $n - \alpha p > 0$ and $\zeta \in S$, then

$$|f(z)| \le C\left[|1 - z\bar{\zeta}|^{n/p}(1 - |z|)^{\alpha - n/p} M_p R^\alpha f(\zeta) + M_1 R^\alpha f(\zeta)\right] \quad (1)$$

where M_1 denotes the Hardy-Littlewood maximal function, and where $M_p f = (M_1 |f|^p)^{1/p}$.

In the extreme case $\alpha p = n$, the above pointwise estimate is replaced by

$$|f(z)| \le C\left[|1 - z\bar{\zeta}|^{n/p}\left(\log\frac{1}{1 - |z|}\right)^{1-1/p} M_p R^\alpha f(\zeta) + M_1 R^\alpha f(\zeta)\right]. \quad (2)$$

Now, if we define

$$\Omega(\zeta) = \left\{z \in B; |1 - z\bar{\zeta}| \le (1 - |z|)^{1 - \alpha p/n}\right\},$$

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respectively

$$\mathscr{E}(\zeta) = \left\{ z \in B; |1 - z\bar{\zeta}| < \frac{1}{\left(\log \frac{1}{1 - |z|}\right)^{(p-1)/n}} \right\},$$

and denote by Mf, respectively Pf, the corresponding maximal operators, (1) and (2) say that they are of weak type (p, p). In particular if $f \in H_\alpha^p$, there exists $\lim f(z)$, as z approaches ζ, z in the tangential region, for almost every $\zeta \in S$.

Our purpose is to study the size of the exceptional set where the limit of a function in H_α^p along some intermediate tangential regions fails to exist. For the real case, the first result is in [*A-N*], and problems in the same direction for a class of holomorphic functions has been obtained in [Su] (see [Ci-Do-Su] and [Do] for related results).

Before stating more precisely our results, we need some more definitions. For $0 < \delta \leq +\infty$, and $\omega(t)$ a non-decreasing function in $[0, +\infty]$, vanishing at zero and satisfying $\omega(2t) \leq c\omega(t)$, and for $E \subset S$,

$$H_\delta^\omega(E) = \inf\left\{ \sum_j \omega(\delta_j); E \subset \cup B(\zeta_j, \delta_j), \delta_j \leq \delta \right\},$$

where $B(\zeta_j, \delta_j)$ is a non-isotropic ball. The non-isotropic Hausdorff measure is then defined by

$$H^\omega(E) = \lim_{\delta \to 0} H_\delta^\omega(E).$$

It is well known that H_∞^ω and H^ω have the same zero sets.

In Section 2 we begin with a characterization of the Hardy-Sobolev spaces H_α^p, $p > 1$, obtaining a representation in terms of a "fractional Cauchy-type" transform of functions in $L^p(d\sigma)$. As an immediate corollary we deduce from this representation and the results in [Su], the desired size of the tangential-exceptional set of H_α^p functions. The case $p \leq 1$ follows directly from the methods in [A].

The third section deals with the extreme case $\alpha p = n$. We obtain a necessary condition for $E \subset S$ to be an exceptional set with respect to tangential regions $\mathscr{E}_\mu(\zeta)$ of exponential type. Similar results for the real case were obtained by [Do] (see also [Ci-Do-Su] for the other results in this line).

On the other hand, we also include two examples which give some information about the sharpness of the necessary condition.

As a final remark on notation, we adopt the usual convention writing by the same letter various absolute constants which values may differ in each occurrence.

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Section 2

We begin this section with an integral representation for the Hardy-Sobolev spaces H^p_α, $p > 1$. Such representation can be viewed as a holomorphic nonisotropic version of the classical Calderon's identity between Sobolev and potential spaces. One of the inclusions is established in [A-Co]. Since to our knowledge there is no written proof of the other one, we include it here.

Let $1 < p < +\infty$ and $0 < \alpha < n$. For $f \in L^p(d\sigma)$, define

$$C_\alpha f(z) = \int_S \frac{f(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}}\, d\sigma(\zeta),$$

and

$$C^p_\alpha = C_\alpha L^p(d\sigma), \quad \text{normed by} \quad \|F\|_{C^p_\alpha} = \inf_{F = C_\alpha f} \|f\|_{L^p(d\sigma)}.$$

THEOREM 2.1. *$H^p_\alpha = C^p_\alpha$, with equivalence of norms provided* $1 < p$, $\alpha < n$.

Proof of Theorem 2.1. As we have already said, Lemma 2.2 in [A-Co] gives $\|C_\alpha f\|_{\alpha,p} \leq C\|f\|_{L^p(d\sigma)}$. Hence, we just need to prove that the map $C_\alpha : L^p \to H^p_\alpha$ is onto.

Let P_k be a homogeneous polynomial of degree k. Using Lemma 2.1 in [A-Co] it is easy to check (see page 433 in same work)

$$C_\alpha P_k(z) = \frac{\Gamma(n)\Gamma(n - \alpha + k)}{\Gamma(n - \alpha)\Gamma(n + k)} P_k(z) \cdot \tag{1}$$

Suppose first that α is an integer. Then $\Gamma(n + k)/\Gamma(n - \alpha + k)$ is a polynomial in k of degree α, and it can be written as

$$\frac{\Gamma(n + k)}{\Gamma(n - \alpha + k)} = a_\alpha(k + 1)^\alpha + a_{\alpha-1}(k + 1)^{\alpha-1} + \cdots + a_0.$$

Since $R^\alpha P_k(z) = (k + 1)^\alpha P_k(z)$, the above formula together with (1), shows that

$$C_\alpha\big(a_\alpha R^\alpha + a_{\alpha-1}R^{\alpha-1} + \cdots + a_0 \,\mathrm{Id}\big)P_k = \frac{\Gamma(n)}{\Gamma(n - \alpha)} P_k. \tag{2}$$

For $f \in H^p_\alpha$, define

$$Tf = \frac{\Gamma(n-\alpha)}{\Gamma(n)}\left(a_\alpha R^\alpha f + a_{\alpha-1}R^{\alpha-1}f + \cdots + a_0 f\right).$$

Then, T is a bounded operator from H^p_α to H^p (see [Gr] and [Kr]) and, by (2), for each $f \in H^p_\alpha$, $C_\alpha Tf = f$, that is, C_α has a right inverse.

For general α, the asymptotic development in [T-E] and Stirling's formula, give that there exist $\lambda_i(\alpha, n)$, $i \in \mathbf{N}$, so that for each $r > 0$,

$$\lim_{k\to+\infty}(k+1)^r\left[1 - \frac{\Gamma(n+k-\alpha)}{\Gamma(n+k)}\left(\lambda_0(k+1)^\alpha + \cdots + \lambda_{r-1}(k+1)^{\alpha-r+1}\right)\right] = \lambda_r.$$

Let

$$b_k = 1 - \frac{\Gamma(n+k-\alpha)}{\Gamma(n+k)}\left(\lambda_0(k+1)^\alpha + \cdots + \lambda_{r-1}(k+1)^{\alpha-r+1}\right);$$

the above convergence says that there exists k_0 so that for $k \geq k_0$,

$$|b_k| < \frac{2|\lambda_r| + 1}{(k+1)^r}. \tag{3}$$

Let T be the operator defined by

$$T(P_k) = \begin{cases} \dfrac{\Gamma(n-\alpha)\Gamma(n+k)}{\Gamma(n)\Gamma(n-\alpha+k)}P_k & \text{if } k < k_0, \\ \dfrac{\Gamma(n-\alpha)}{\Gamma(n)}\left((k+1)^\alpha\lambda_0 + \cdots + (k+1)^{\alpha-r+1}\lambda_{r-1}\right)P_k & \text{if } k \geq k_0, \end{cases}$$

where P_k is a homogeneous polynomial of degree k.

Again as a consequence of [Gr] and [Kr], for each $r \in \mathbf{N}$, $T\colon H^p_\alpha \to H^p$, and we will see that provided r and k_0 are chosen big enough ($r = z$ is sufficient), the operator $I - C_\alpha T : H^p_\alpha \to H^p_\alpha$ has norm strictly less than one. This is equivalent to say that if we let $T_1 = R^\alpha(I - C_\alpha T)$, there exists $\varepsilon < 1$ such that

$$\|T_1 f\|_{H^p} \leq \varepsilon\|f\|_{p,\alpha}.$$

Now, if $f \in H^p_\alpha$, let $f = \sum_{k\geq 0} P_k$ be its homogeneous expansion. Then

$$T_1 f = \sum_{k\geq k_0} (k+1)^\alpha b_k P_k .$$

First suppose $1 < p \leq 2$. Integrating on slices (see [Ru, page 15]) we get

$$\begin{aligned} \|T_1 f\|_2^2 &= \int_S \frac{1}{2\pi}\int_0^{2\pi} |T_1 f(e^{i\theta}\zeta)|^2 d\theta\, d\sigma(\zeta) \\ &= \sum_{k\geq k_0} (k+1)^{2\alpha} |b_k|^2 \int_S |P_k(\zeta)|^2 d\sigma(\zeta) \\ &\leq c \sum_{k\geq k_0} \frac{1}{(k+1)^{2r}} \int_S |R^\alpha P_k(\zeta)|^2 d\sigma(\zeta), \end{aligned}$$

where in last inequality we have used (3).

Now applying Theorem 2.1 in [A-B] to each $R^\alpha P_k$, we obtain

$$\|T_1 f\|_2^2 \leq c \sum_{k\geq k_0} \frac{1}{(k+1)^{2r-2}} \|R^\alpha f\|_1^2 \leq \varepsilon'^2 \|R^\alpha f\|_p^2,$$

where $\varepsilon' < 1$, if r and k_0 are chosen big enough.

Finally, since $p \leq 2$, we deduce that there exists $\varepsilon < 1$ so that

$$\|T_1 f\|_p < \varepsilon \|f\|_{p,\alpha}.$$

If $p > 2$, integrating again on slices, and applying Theorem 6.3 in [Du] to each one of the slices we obtain

$$\begin{aligned} \|T_1 f\|_p^p &= \frac{1}{2\pi}\int_S\int_0^{2\pi} |T_1 f(e^{i\theta}\zeta)|^p d\theta\, d\sigma(\zeta) \\ &\leq \sum_{k\geq k_0} \int_S (k+1)^{p-2+\alpha p} |b_k|^p |P_k(\zeta)|^p d\sigma(\zeta) \\ &\leq c \sum_{k\geq k_0} \int_S \frac{1}{(k+1)^{rp-p+2}} |R^\alpha P_k(\zeta)|^p d\sigma(\zeta) \\ &\leq c \sum_{k\geq k_0} \frac{1}{(k+1)^{p(r-2)+2}} \|R^\alpha f\|_1^p < \varepsilon' \|R^\alpha f\|_p^p \end{aligned}$$

provided K_0 and r are big enough.

Hence $\|T_1 f\|_p \leq \varepsilon \|R^\alpha f\|_p = \varepsilon \|f\|_{p,\alpha}$, and $C_\alpha T$ is invertible in H^p_α.

In particular there exists $S: H^p_\alpha \to H^p_\alpha$ with $C_\alpha TS = Id$. Since $T: H^p_\alpha \to H^p$, we are done. ■

Before stating a result concerning the size of the tangential-exceptional sets for H^p_α, we need some definitions. Assume $n - \alpha p > 0$, and let $\zeta \in S$, $\tau \geq 1$ and $\beta > 0$. Define the tangential approach region

$$\Omega_\tau(\zeta) = \Omega_{\tau,\beta}(\zeta) = \{z \in B; |1 - z\bar{\zeta}|^\tau < \beta(1 - |z|)\},$$

and if $f: B \to \mathbf{C}$, let

$$M_\tau f(\zeta) = M_{\tau,\beta} f(\zeta) = \sup_{z \in \Omega_\tau(\zeta)} |f(z)|$$

be the corresponding maximal function. Notice that if $\tau = 1$, $\Omega_1(\zeta) = D(\zeta)$ is the usual admissible region, and we will denote $M_1 f$ by Nf.

THEOREM 2.2. *Suppose $0 < p < +\infty$, $\tau > 1$, $\alpha p < n$ and $m = \tau(n - \alpha p)$. Let ν be a positive Borel measure on S so that*

$$\nu(B(\zeta, \delta)) = O(\delta^m) \quad \textit{for all } \zeta \in S, \delta > 0.$$

Then there exists $C > 0$ such that for each $f \in H^p_\alpha$,

$$\|M_\tau f\|_{L^p(d\nu)} \leq C \|f\|_{p,\alpha}.$$

Proof of Theorem 2.2. The case $p > 1$ follows immediately from theorem 3.8 in [Su], where the same conclusion is proved for C^p_α, and the previous Theorem 2.1.

The remaining case $0 < p \leq 1$ can be shown using the same methods of Theorem 1.1 in [A], where the admissible case, $\tau = 1$ is considered. ■

Theorem 2.2 together with a non-isotropic Frostman type theorem in [Co], and the same kind of construction of holomorphic functions in H^p_α with prescribed "tangential" exceptional sets in [A-Co], lead to the following characterization of such exceptional sets.

COROLLARY 2.1. *Let $E \subset S$ compact, $0 < p < +\infty$, $\alpha p < n$ and $m = \tau(n - \alpha p)$ with $\tau > 1$. Then $E = E(f)$ for some $f \in H^p_\alpha$ if and only if $H^m(E) = 0$, where*

$$E(f) = \{\zeta \in S; \nexists \lim f(z), z \to \zeta \in \Omega_\tau(\zeta)\}.$$

Remark 2.1. Theorem 2.2 and its corollary still remains valid for $F = P[K_\alpha * g]$, $g \in L^p(d\sigma)$, where $\alpha \in \mathbf{N}$ and $0 < \alpha < n$, $0 < n - \alpha p$, and K_α

is the non-isotropic Riesz kernel given by:

$$K_\alpha(z,\zeta) = \frac{1}{|1 - z\bar{\zeta}|^{n-\alpha}} \quad z,\zeta \in S,$$

and where $P(z,\zeta)$ is the Poisson-Szegö kernel. The proof can still be used, since if F is such an $\mathscr{M}$-harmonic function, then $N(R^\alpha F) \in L^p(d\sigma)$ (see [A-Ca]).

Section 3

We have seen in the previous section that the "wideness" of the tangential approach regions for functions in H_α^p, flattens as $n - \alpha p$ goes to zero. On the other hand, if $f \in H_\alpha^p$, and $n - \alpha p < 0$, f is a continuous function up to the boundary. So one would expect that in the limit case $n = \alpha p$ (as it happens in the real case), the convergence of a function in H_α^p exists within a much wider region.

Let $1 < p < +\infty$, $\mu \geq 1$ and define for $\zeta \in$ S and $C > 0$

$$\mathscr{E}_\mu(\zeta) = \mathscr{E}_{\mu,p}^C(\zeta) = \left\{ z;\, |1 - z\bar{\zeta}| < \frac{C}{\left(\log\frac{1}{1-|z|}\right)^{(p-1)\mu/n}} \right\},$$

and for f defined on B, let $P_\mu f$ be the corresponding maximal function.

THEOREM 3.1. *Let ν be a positive Borel measure on S satisfying*

$$\nu(B(\zeta,\delta)) = O\left(\delta^{n/\tau}\left(\log\frac{1}{\delta}\right)^{1-q}\right), \quad \textit{where } q > p, \mu > 1.$$

Then there exists $C > 0$ so that if $f \in H_\alpha^p$, $\alpha p = n$,

$$\int_S |P_\mu f(\zeta)|^p d\nu(\zeta) \leq C\|f\|_{p,\alpha}^p.$$

Proof of Theorem 3.1. For $f \in H_\alpha^p$ we write

$$f(z) = c(\alpha)\int_0^1 \left(\log\frac{1}{t}\right)^{\alpha-1} g(tz)\, dt,$$

where $g = R^\alpha f$. Thus the theorem will be proved once we show

$$\int_S |P_\mu f(\zeta)|^p d\nu(\zeta) \leq C\|Ng\|^p_{L^p(d\sigma)}.$$

Breaking the integral defining f in two pieces, from 0 to 1/2 and from 1/2 to 1, it is enough to show that if $g \in H^p$, and

$$G(z) = \int_{1/2}^1 (1-t)^{\alpha-1} g(tz)\, dt,$$

then

$$\int_S |P_\mu G(\zeta)|^p d\nu(\zeta) \leq C\|Ng\|^p_{L^p(d\sigma)}.$$

By Hölder's inequality

$$\begin{aligned} |G(z)| \leq & \left\{\int_{1/2}^1 (1-t)^{-1}\left(\log\frac{1}{1-t}\right)^{(1-p)p'/p}\left(\log\log\frac{1}{1-t}\right)^{(1-q)p'/p} dt\right\}^{1/p'} \\ & \cdot\left\{\int_{1/2}^1 (1-t)^{n-1}\left(\log\frac{1}{1-t}\right)^{p-1} \right. \\ & \left. \times\left(\log\log\frac{1}{1-t}\right)^{q-1} |g(tz)|^p\, dt\right\}^{1/p}. \end{aligned}$$

Since $q > p > 1$, the first integral converges. Since $N|g|^p \in L^1(d\sigma)$, and

$$\|N|g|^p\|_{L^1(d\sigma)} = \|N|g|\,\|^p_{L^p(d\sigma)},$$

we may apply Lemma 2.1 in [A-N], and write

$$|g(z)|^p \leq \sum_{k\geq 1} \lambda_k a_k(z), + z \in B,$$

where

$$\sum_{k\geq 1} \lambda_k \leq C\|Ng\|^p_{L^p(d\sigma)},$$

and each a_k is a non-negative α-atom satisfying:

(a) there exists $\zeta_k \in S, \delta_k > 0$ so that

$$\operatorname{supp} a_k \subset T(B(\zeta_k, \delta_k)),$$

where the tent $T(B(\zeta_k, \delta_k)) = B \setminus \cup D(\eta)$, and the union is over all $\eta \in S \setminus B(\zeta_k, \delta_k)$;

(b) $a_k(z) \le \delta_k^{-n}$, for all $z \in B$.

Thus we only need to prove that

$$\int_S \sup_{z \in \mathcal{E}_\mu(\zeta)} \int_0^1 (1-t)^{n-1}\left(\log\frac{1}{1-t}\right)^{p-1}\left(\log\log\frac{1}{1-t}\right)^{q-1} a(tz)\,dt\,d\nu(\zeta) \quad (1)$$

is bounded independently of the atom a, which is supported in $T(B(\zeta_0, \delta))$.

Since a supported in $T(B(\zeta_0, \delta))$, the inner integral is in fact, from $1 - c\delta$ to 1 for some positive constant c.

First, suppose

$$\delta < \frac{C}{\left(\log\frac{1}{\delta}\right)^{(p-1)\mu/n}}.$$

Then $a(tz)$ is zero unless $\zeta \in \tilde{B}$, where

$$\tilde{B} = B\left(\zeta_0, \frac{C}{\left(\log\frac{1}{\delta}\right)^{(p-1)\mu/n}}\right).$$

Consequently, the inner integral above (1) is bounded by

$$\int_{1-c\delta}^1 (1-t)^{n-1}\left(\log\frac{1}{1-t}\right)^{p-1}\left(\log\log\frac{1}{1-t}\right)^{q-1}\delta^{-n}\chi_{\tilde{B}}(\zeta)\,dt$$
$$\le C\left(\log\frac{1}{\delta}\right)^{p-1}\left(\log\log\frac{1}{\delta}\right)^{q-1}\chi_{\tilde{B}}(\zeta).$$

Integrating with respect to ν and using the hypothesis, we deduce that (1) is bounded by

$$C\left(\log\log\frac{1}{\delta}\right)^{q-1}\left(\log\left(\frac{\left(\log\frac{1}{\delta}\right)^{(p-1)\mu/n}}{C}\right)\right)^{1-q} \le C.$$

Finally, if

$$\delta \geq \frac{C}{\left(\log \frac{1}{\delta}\right)^{(p-1)\mu/n}},$$

δ is bounded from below, and $a(tz)$ is zero unless $\zeta \in B(\zeta_0, \delta)$. Hence (1) is bounded by

$$C\left(\log \frac{1}{\delta}\right)^{p-1}\left(\log\log \frac{1}{\delta}\right)^{q-1} \delta^{n/\tau}\left(\log \frac{1}{\delta}\right)^{1-q},$$

which is also bounded.

Remark 3.1. Theorem 3.1 can be used to prove the existence of limits of H^p_α function within "exponential" tangential regions $\mathscr{E}_\mu$, along varieties. For instance, if Γ is a smooth curve and ν is the arc-length measure on it, it is well known that if Γ is transverse, $\nu(B(\zeta, \delta)) = O(\delta)$, whereas if it is complex-tangential $\nu(B(\zeta, \delta)) = O(\delta^{1/2})$. Thus if Γ is transverse (respectively, complex-tangential), $f \in H^p_\alpha$, $\alpha p = n$, and $\mu > n$ (respectively, $\mu > 2n$), $\lim f(z)$ exists as z approaches ζ, $z \in \mathscr{E}_\mu(\zeta)$ for almost every $\zeta \in \Gamma$ (with respect to arc-length). Note that in the transverse case, the tangential region is wider than in the complex-tangential case.

THEOREM 3.2. *Let $1 < p, \mu > 1$, $\alpha p = n$. Then there exists $C > 0$ so that*

$$H^{n/\mu}_\infty\left(\{P_\mu f(\zeta) > t\}\right) \leq C\frac{\|f\|^p_{p,\alpha}}{t^p},$$

for each $f \in H^p_\alpha$.

To prove Theorem 3.2 we need the following lemma.

LEMMA 3.1. *Let $1 < p$, $\alpha p = n$. There exists $C > 0$ so that for each $g \in L^p(d\sigma)$, $z_0 \in S$ and $z \in \mathscr{E}_\mu(z_0)$,*

$$\left|\int_S \frac{g(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}}\, d\sigma(\zeta)\right| \leq CT_{n-n/\mu}g(z_0),$$

where

$$T_{n-n/\mu}g(z_0) = \sup_{z_0 \in Q}\left(\frac{1}{|Q|^{1/\mu}}\int_Q |g(\zeta)|^p\, d\sigma(\zeta)\right)^{1/p},$$

(here Q denotes non-isotropic balls in S).

Proof of Lemma 3.1. Let $z_0 \in S$ and $z \in \mathscr{E}_\mu(z_0)$ and define $Q = \{\zeta; |1 - \zeta\bar{z}_0| < 4|1 - z\bar{z}_0|\}$. Then

$$\left|\int_S \frac{g(\zeta)}{(1 - z\bar{\zeta})^{n-\alpha}}\, d\sigma(\zeta)\right| \le \int_Q \frac{|g(\zeta)|}{|1 - z\bar{\zeta}|^{n-\alpha}}\, d\sigma(\zeta) + \int_{Q^c} \frac{|g(\zeta)|}{|1 - z\bar{\zeta}|^{n-\alpha}}\, d\sigma(\zeta)$$
$$= \mathrm{I} + \mathrm{II},$$

and we will estimate both integrals separately.

In I we apply Hölder's inequality and we get

$$\mathrm{I}\left(\int_Q |g(\zeta)|^p\, d\sigma(\zeta)\right)^{1/p}\left(\int_Q \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^{(n-\alpha)p'}}\right)^{1/p'}$$
$$\le C\left(\int_Q |g(\zeta)|^p\, d\sigma(\zeta)\right)^{1/p}\left(\log\frac{1}{1 - |z|}\right)^{1/p'}.$$

Now, since $z \in \mathscr{E}_\mu(z_0)$, we have

$$\left(\log\frac{1}{1 - |z|}\right)^{1/p'} \le |1 - z\bar{z}_0|^{-n/\mu p}.$$

Thus

$$\mathrm{I} \le C|1 - z\bar{z}_0|^{-n/\mu p}\left(\int_Q |g(\zeta)|^p d\sigma(\zeta)\right)^{1/p} \le CT_{n-n/\mu}g(z_0).$$

In II, let $\delta = 4|1 - z\bar{z}_0|$. Then by Hölder's inequality

$$\mathrm{II} \le \sum_{z^k\delta < M} \frac{1}{(2^k\delta)^{n-n/p}} \cdot \left(\int_{|1-\zeta\bar{z}_0| < 2^k\delta} |g(\zeta)|^p\, d\sigma(\zeta)\right)^{1/p} \cdot (2^k\delta)^{n/p}$$
$$\le CT_{n-n/\mu}g(z_0).$$
■

Proof of Theorem 3.2. Let $f \in H^p_\alpha$. By Theorem 2.1, $f = C_\alpha g$ with $g \in L^p(d\sigma)$. By Lemma 3.1,

$$P_\mu f(\zeta) \le CT_{n-n/\mu}g(\zeta),$$

for any $\zeta \in S$, and since by Lemma 1 in [Do],

$$H^{n/\mu}_\infty\left(\{T_{n-n/\mu}g > t\}\right) \le C\frac{\|g\|_p^p}{t^p},$$

we obtain the desired conclusion. ■

COROLLARY 3.1. *Let* $1 < p$, $\alpha p = n, \mu > 1$. *Then for each* $f \in H_\alpha^p$ *the limit* $f(z)$ *exists as* z *approaches* $\zeta \in S$, z *in* $\mathscr{E}_\mu(\zeta)$, *except for a set* E *with* $H^{n/\mu}(E) = 0$.

On the other direction we begin with the following construction

PROPOSITION 3.1. *Let* $0 < m < n/\mu$, *and* $E \subset S$ *compact so that* $H^m(E) = 0$. *Then there exists* $p > 1$ *and* α *with* $\alpha p = n$, *and* $f \in H_\alpha^p$ *so that for each* $\zeta \in E$, *the maximal function* $P_\mu f(\zeta) = +\infty$. *In particular,*

$$E \subset \{\zeta \in S;\ \nexists \lim f(z), z \to \zeta, z \in \mathscr{E}_\mu(\zeta)\}.$$

Proof of Proposition 3.1. Let $1 < p$ and $\alpha = n/p$ so that $n - \alpha < 1$. For each $z \in B$, let g_z be the function on S defined by

$$g_z(\zeta) = \frac{1}{|1 - z\bar{\zeta}|^\alpha} \frac{1}{\left(\log\dfrac{1}{|1 - z\bar{\zeta}|}\right)^r}, \quad \text{where } r > 1.$$

We will first see that $g_z \in L^p(d\sigma)$ and

$$\|g_z\|_{L^p(d\sigma)} \leq \frac{C}{\left(\log\dfrac{1}{1 - |z|}\right)^{r - 1/p}}.$$

Indeed, let $z_0 = z/|z|$ and $B_0 = B(z_0, 4(1 - |z|))$. Then

$$\int_S |g_z(\zeta)|^p \, d\sigma(\zeta) = \int_{B_0} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n \left(\log\dfrac{1}{|1 - z\bar{\zeta}|}\right)^{rp}} + \int_{B_0^c} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n \left(\log\dfrac{1}{|1 - z\bar{\zeta}|}\right)^{rp}} = \mathrm{I} + \mathrm{II}.$$

In I, $|1 - z\bar{\zeta}| \simeq 1 - |z|$. Hence

$$\mathrm{I} \leq \frac{C}{\left(\log\dfrac{1}{1 - |z|}\right)^{rp}} \int_{B_0} \frac{d\sigma(\zeta)}{|1 - z\bar{\zeta}|^n} \leq C \frac{1}{\left(\log\dfrac{1}{1 - |z|}\right)^{rp - 1}}.$$

In II, let

$$B_k = \{\zeta \in S; 4 \cdot 2^k(1-|z|) \leq |1-\zeta\bar{z}_0| \leq 4 \cdot 2^{k+1}(1-|z|)\}.$$

Then

$$\mathrm{II} \simeq C \sum_{k<C\log\frac{1}{1-|z|}} \int_{B_{k+1}\setminus B_k} \frac{d\sigma(\zeta)}{|1-z\bar{\zeta}|^n \left(\log\frac{1}{|1-z\bar{\zeta}|}\right)^{rp}}$$

$$\leq C \sum_{k<C\log\frac{1}{1-|z|}} \frac{1}{\left(\log\frac{1}{2^k(1-|z|)}\right)^{rp}} \simeq C \frac{1}{\left(\log\frac{1}{1-|z|}\right)^{rp-1}}.$$

Next, define for $z \in B$, the holomorphic function defined by $f_z(\omega) = C_\alpha g_z(\omega)$. By Theorem 2.1,

$$f_z \in H^p_\alpha \quad \text{and} \quad \|f_z\|_{p,\alpha} \leq \frac{C}{\left(\log\frac{1}{1-|z|}\right)^{r-1/p}}.$$

Since $n-\alpha<1$, $\mathrm{Re} f_z(\omega) \simeq K_\alpha * g_z(\omega) \geq 0$ and

$$\mathrm{Re} f_z(z) \geq C\int \frac{1}{|1-z\bar{\zeta}|^n} \frac{1}{\left(\log\frac{1}{|1-z\bar{\zeta}|}\right)^r} d\sigma(\zeta) \geq \frac{C}{\left(\log\frac{1}{1-|z|}\right)^{r-1}}$$

(where in last inequality we have used (1) for $p=1$). Thus if we define

$$h_z(\omega) = \left(\log\frac{1}{1-|z|}\right)^{r-1/p} f_z(\omega),$$

by the above estimates we have

(i) $\|h_z\|_{p,\alpha} \leq C$,

(ii) $\mathrm{Re}\, h_z \geq 0$ and

$$\mathrm{Re}\, h_z(z) \geq C\left(\log\frac{1}{1-|z|}\right)^{1-1/p}.$$

Let $E \subset S$ be compact with $H^m(E) = 0$. For each $k \in \mathbf{N}$ let $\{B(\zeta_{jk}, \delta_{jk})\}_j$ be a disjoint family of non-isotropic balls with $E \subseteq \cup B(\zeta_{jk}, C_1\delta_{jk})$ and $\sum_j \delta^m_{jk} < 1/2^k, C_1 > 0$ an absolute constant.

Define $z_{jk} \in B$ so that $\zeta_{jk} = z_{jk}/|z_{jk}|$ and $1 - |z_{jk}| = \varepsilon_{jk}$, where

$$\delta_{jk} = \frac{C}{\left(\log \frac{1}{\varepsilon_{jk}}\right)^{(p-1)\mu/n}}.$$

Let $F_k = \sum_j \delta_{jk}^m h_{jk}$ and $F = \sum_k F_k$, where $h_{jk} = h_{z_{jk}}$. By (i) and the election of δ_{jk}, $F \in H_\alpha^p$, and by (ii),

$$\begin{aligned}\operatorname{Re} F(z_{jk}) \geq \operatorname{Re} F_k(z_{jk}) &\geq C\delta_{jk}^m \left(\log \frac{1}{\varepsilon_{jk}}\right)^{1-1/p} \\ &\geq C\delta_{jk}^{m-n/\mu p} \to +\infty \quad \text{as } k \to +\infty.\end{aligned}$$

provided we choose $1 < p$ with $m - n/\mu p < 0$.

Next take $\zeta \in E$ and let $k \in \mathbf{N}$ be fixed. There exists $j \in \mathbf{N}$ so that $\zeta \in B(\zeta_{jk}, C_1\delta_{jk})$ and since $\varepsilon_{jk} < \delta_{jk}$,

$$\begin{aligned}|1 - z_{jk}\bar{\zeta}| &\leq 2\big((1 - |z_{jk}|) + |1 - \zeta_{jk}\bar{\zeta}|\big) \leq 2(\varepsilon_{jk} + C_1\delta_{jk}) \\ &\leq \frac{C}{\left(\log \frac{1}{\varepsilon_{jk}}\right)^{(p-1)\mu/n}} = \frac{C}{\left(\log \frac{1}{1 - |z_{jk}|}\right)^{(p-1)\mu/n}}\end{aligned}$$

we have $z_{jk} \in \mathscr{E}_\mu(\zeta)$. Hence $P_\mu F(\zeta) = +\infty$ for every $\zeta \in E$. ■

Finally, for $p = 2$ we can give one more example.

PROPOSITION 3.2. *Let $E \subset S$ be a compact set with* $\operatorname{diam}(E) < 1$ *so that $H^m(E) = 0$, $m < n$. Then for every $\mu > 1$ with $(n+m)/2 < n/\mu$, there is $f \in H_\alpha^2$, $\alpha = n/2$, so that*

$$E = \{\zeta \in S;\ \nexists \lim f(z), z \to \zeta, z \in \mathscr{E}_\mu(\zeta)\},$$

where

$$\mathscr{E}_\mu(\zeta) = \left\{z; |1 - z\bar{\zeta}| < \frac{C}{\left(\log \frac{1}{1 - |z|}\right)^{\mu/n}}\right\}.$$

The proof is based in the following lemma.

LEMMA 3.2. *Let $m < n, \mu > 1$ be as in Proposition 3.2 and let $\eta < 1$. There exists $\delta < 0$ so that for any finite disjoint collection $\{B(\zeta_j, \delta_j)\}$ of pairwise disjoint non-isotropic balls, with $\delta_j < \delta, \zeta_j \in S$ and $|1 - \zeta_j\bar{\zeta}_j| < \eta$ for every j, k, then there exists $F \in H^\infty(B)$ satisfying*:

(i) $\operatorname{Re} F(z) \geq 0$ *for z closed enough to* $\cup\{\zeta_j\}$, $\operatorname{Re} F(z_j) \geq C$ *if* $z_j/|z_j| = \zeta_j, 1 - |z_j| = \varepsilon_j$ *with* $\varepsilon_j = e^{-(c/\delta_j)^{n/\mu}}$.

(ii) $\|F\|^2_{2,\alpha} \leq C\Sigma_j \delta_j^m$, $\alpha = n/2$.

Proof of Lemma 3.2. If h is the holomorphic function on D given by

$$h(w) = \frac{1}{w}\log\frac{1}{1-w},$$

for each $z \in B$ we define the holomorphic function on B given by

$$f_z(\omega) = h(\bar{z}\omega) = \frac{1}{\bar{z}\omega}\log\frac{1}{1-\bar{z}\omega}.$$

Taking real parts we obtain

$$\operatorname{Re} f_z(\omega) = \operatorname{Re}\frac{1}{\bar{z}\omega}\log\frac{1}{|1-\bar{z}\omega|} - \operatorname{Im}\frac{1}{\bar{z}\omega}\operatorname{Arg}\frac{1}{1-\bar{z}\omega}.$$

Since

$$\operatorname{Im}\frac{1}{\bar{z}\omega}\operatorname{Arg}\frac{1}{1-\bar{z}\omega} \leq 0$$

for each z, ω in B, in order to see that $\operatorname{Re} f_z(\omega) \geq 0$, we just need to prove that the product of real parts is positive. If z and ω are chosen sufficiently close to $\cup\{\zeta_j\}$ this is deduced from the hypothesis on the $\zeta_j's$.

Defining

$$F(z) = \sum_j \left(\log\frac{1}{\varepsilon_j}\right)^{-1} f_{z_j}(z), \qquad z \in B,$$

we deduce (i) from the above, and from the fact that $f_z(z) \geq C \log 1/(1 - |z|)$. In order to prove (ii), we will compute the norm by duality.

If we define the polynomial in R of degree n given by

$$Q = Q(R) = (R + (n-2)\mathrm{Id})\ldots(R + \mathrm{Id})R^2,$$

it is then immediate to check that

$$Qf_z(\omega) = \frac{C(n)}{(1 - \bar{z}\omega)^n}.$$

Associated to Q we define the operators $\tilde{Q}_\alpha$ and $\tilde{Q}_{-\alpha}$, $\alpha = n/2$, by

$$\tilde{Q}_\alpha P_k = \left((k + (n-1))\dots(k+2)(k+1)^2\right)^{1/2} P_k,$$
$$\tilde{Q}_{-\alpha} P_k = \left((k + (n-1))\dots(k+2)(k+1)^2\right)^{-1/2} P_k,$$

where P_k is a homogeneous polynomial of degree k. Then, if g is a holomorphic function in a neighbourhood of $\overline{B}$,

$$\left|\int_S \overline{\tilde{Q}_\alpha F(\omega)} g(\omega)\, d\sigma(\omega)\right| = \left|\int_S \overline{QF(\omega)} \cdot \tilde{Q}_{-\alpha} g(\omega)\, d\sigma(\omega)\right|$$
$$= C(n)\left|\sum_j \int_S \left(\log\frac{1}{\varepsilon_j}\right)^{-1} \times \frac{1}{\left(1 - z_j\bar{\omega}\right)^n} \tilde{Q}_{-\alpha} g(\omega)\, d\sigma(\omega)\right|.$$

Using Cauchy's integral formula and Schwarz's inequality, the last expression is bounded by

$$C\left(\sum_j \left(\frac{1}{\log\frac{1}{\varepsilon_j}}\right)^\lambda\right)^{1/2} \left(\sum_j \left(\log\frac{1}{\varepsilon_j}\right)^{\lambda-2} |\tilde{Q}_\alpha g(z_j)|^2\right)^{1/2}, \tag{2}$$

where $\lambda < 1$ is to be chosen.

Since $\log 1/\varepsilon_j = \delta_j^{-n/\mu}$, if we choose λ so that $\lambda n/\mu = m$ we have (2) bounded by

$$C\left(\sum_j \delta_j^m\right)^{1/2} \cdot \left(\sum_j \delta_j^{(2n/\mu)-m} \cdot |\tilde{Q}_{-\alpha} g(z_j)|^2\right)^{1/2} \tag{3}$$

Next, for each $\zeta \in B(\zeta_j, c\delta_j)$, $z_j \in \mathscr{E}_\mu(\zeta)$. Taking infimums, we deduce that

$$|\tilde{Q}_{-\alpha} g(z_j)| \le \inf_{\zeta \in B(\zeta_j, \delta_j)} P_\mu \tilde{Q}_{-\alpha} g(\zeta).$$

Since $2n/\mu - m > n$ and the balls are disjoint, we get

$$C\Big(\sum_j \delta_j^m\Big)^{1/2}\Big(\int_S |P_\mu \tilde{Q}_{-\alpha} g(\zeta)|^2\, d\sigma(\zeta)\Big)^{1/2}$$
$$\leq C\Big(\sum_j \delta_j^m\Big)^{1/2} \|R^\alpha \tilde{Q}_{-\alpha} g\|_{L^2(d\sigma)} \simeq C\Big(\sum_j \delta_j^m\Big)^{1/2} \|g\|_{L^2(d\sigma)},$$

where in the first estimate we have applied Theorem 3.1, and the last estimate is deduced from the fact that $R^\alpha \tilde{Q}_{-\alpha}$ is a multiplier in $L^2(d\sigma)$.

Finally, we deduce from the above that

$$\|\tilde{Q}_\alpha F\|_2 \leq C \sum \delta_j^m,$$

and since $\|\tilde{Q}_\alpha F\|_2 \simeq \|R^\alpha F\|_2$, we have proved (ii) ■

Proof of Proposition 3.2. With fixed $k \in \mathbf{N}$, let $\{B(\zeta_{jk}, \delta_{jk})\}$, be a family of non-isotropic disjoint balls satisfying $E \subset \bigcup_j B(\zeta_{jk}, c_1\delta_{jk})$ and let $m_k \nearrow +\infty$ so that $m_k(\sum_j \delta_{jk}^m)^{1/2} \leq 1/2^k$. Let z_{jk} and F_k be as in Lemma 3.2, and define $F = \sum_k m_k F_k$. Then, (ii) of Lemma 3.2, together with the election of the m_k's give that $F \in H_\alpha^2$.

On the other hand, by (i) of last lemma,

$$\mathrm{Re} F(z_{jk}) \geq \mathrm{Re}\, m_k F_k(z_{jk}) \geq C m_k.$$

Proceeding in the same way as in Proposition 3.1, we deduce from the above, that for each $\zeta \in E$ and for each $k \in \mathbf{N}$, there exists $z_{jk} \in \mathscr{E}_\mu(\zeta)$, with $\mathrm{Re} F(z_{jk}) \geq C m_k$. Hence $P_\mu F(\zeta) = +\infty$ for each $\zeta \in E$.

Finally, from the definition of the E_k's we deduce easily that if $\omega \in K \subset \bar{B} \setminus E$, K compact, there exists $C = C(K) > 0$ with $|F_k(\omega)| \leq C \sum_j \delta_{jk}^{n/\mu}$, and since $m < n/\mu$, the series defining F converges uniformly over the compact sets of $\bar{B} \setminus E$. ■

Remark 3.2. The same methods used in Proposition 3.2 actually show the following statement. Suppose F is an s-set ($s \leq n$); i.e., suppose there exists a positive measure ν on S and $c_1, c_2 > 0$ so that

$$c_1 \delta^s \leq \nu(B(\zeta, \delta)) \leq c_2 \delta^s$$

for each $\zeta \in F$, $\delta > 0$. Assume $E \subset F$ is compact with diam $E < 1$ and $H^m(E) = 0$, $m < s$. Then E is an exceptional set (in the previous sense) with respect to $\mathscr{E}_\mu$ for each $\mu > 1$ so that

$$\frac{s+m}{2} < \frac{n}{\mu} < s.$$

Added in proof. We have recently proved that the estimate in Theorem 3.1 is true if $\nu(B(\zeta,\delta) = O(\delta^{n/\mu})$ and that the condition $H^{n/\mu}(E) = 0$ in fact characterizes the exceptional sets in Corollary 3.1, provided $1 < p \leq 2$.

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