Target Tracking With Explicit Control of Filter Lag

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Abstract

This paper presents a method of gain adjustment for an alpha-beta filter when data points are lost or when the tracking interval changes. The steady state position and velocity lags are first derived for a step acceleration input. The standard predictor-corrector form of the filter equations are algebraically rearranged into two uncoupled difference equations; one equation for the smoothed position and one for smoothed velocity. The equations are then solved for the smoothed estimates using the method of undetermined coefficients. The solution is shown to consist of input acceleration, transient terms and steady state lags. The transient terms counteract the effects of the steady state lags until the time determined by the filters lag time. The steady-state lags are used for optimal adjustment of filter gains for aperiodic track conditions. For a varying track update interval, the filter gains which preserve a nominal periodic filter lag are derived. Such gain selection will preserve the nominal lags associated with the constant tracking interval regardless of how the update interval varies. An example demonstrates the improvement in performance from using this approach.

1. Derivation of Acceleration Responses

Kalman filters are at the core of many tracking systems [1], [2]. For the particular class of problems where the noise is nearly stationary or stationary, the Kalman gains converge to fixed values. For the class of problems when this occurs, the filter can be viewed as a constant gain filter which is nothing more than a matrix difference equation. This equation can then be solved provided the model is deterministic. In general, the update equations for a constant gain filter can be written as

$$\eta_{n+1} = F\eta_n + x_{n+1}G, \tag{1}$$

where

$$\begin{aligned} \eta_n &\doteq n^{th}\text{vector state estimate} \\ F, G &\doteq \text{filter update gain matrices,} \\ x_n &\doteq \text{scalar measurement model.} \end{aligned}$$

Alpha-beta filters, which are commonly used in radar tracking systems when it is necessary to track a large number of objects, are a special case of this general class. The tracking equations of an alpha-beta filter assume a target that is moving with constant velocity. For the x-coordinate, these equations are given by;

$$\begin{aligned} x_p(k) &= x_S(k-1) + v_S(k-1)T \\ v_p(k) &= v_S(k-1) \\ x_S(k) &= x_p(k) + \alpha(x_m(k) - x_p(k)) \\ v_S(k) &= v_S(k-1) + \frac{\beta}{T}(x_m(k) - x_p(k)) \end{aligned} \tag{2}$$

where

- $x_S(k) \equiv$ smoothed position at time step k
- $x_p(k) \equiv$ predicted position at time step k
- $x_m(k) \equiv$ measured position at time step k
- $v_S(k) \equiv$ smoothed velocity at time step k
- $v_p(k) \equiv$ predicted velocity at time step k
- $T \equiv$ radar update interval
- $\alpha, \beta \equiv$ filter gains

Similar equations apply for y and z. The filter gains satisfy the following relations ,

$$\begin{aligned} 0 &< \alpha < 1 \\ \beta &= 2(2-\alpha) - 4\sqrt{1-\alpha} \end{aligned}$$

The relation between α and β is known as the Kalata relation and is obtained from steady state Kalman filter theory assuming zero mean white noise in the position and velocity state equations [4]. Therefore, for a given α, the optimal choice for β is given by the above.

From Eq. (2), substitute the predictor terms into the expressions for x_S and v_S, apply the Z-transform and solve for $x_S(z)$, and $v_S(z)$. As in [3], the equations for x_S and v_S in the Z-transform domain are;

$$\begin{aligned} x_S(z) &= \frac{z[(z-1)\alpha+\beta]}{z^2 - z(2-\alpha-\beta) + 1 - \alpha} x_m(z) \\ v_S(z) &= \frac{\beta}{T} \frac{z(z-1)}{z^2 - z(2-\alpha-\beta) + 1 - \alpha} x_m(z) \end{aligned}$$

In the time domain we arrive at the following two uncoupled difference equations in $x_S(k)$ and $v_S(k)$

$$x_S(k+2) - c_1 x_S(k+1) + c_2 x_S(k) =$$

$$\alpha x_m(k+2) - (\alpha-\beta) x_m(k+1)$$

$$v_S(k+2) - c_1 v_S(k+1) + c_2 v_S(k) = \qquad (3)$$

$$\frac{\beta}{T}(x_m(k+2) - x_m(k+1))$$

$$c_1 = 2 - \alpha - \beta \qquad c_2 = 1 - \alpha$$

The method of undetermined coefficients will be used to find solutions for $x_S(k)$ and $v_S(k)$ with input $x_m(k) = \frac{1}{2}ak^2T^2$, where a is constant acceleration and $t = kT$ is the discrete time. T is the sampling interval and $k \geq -2$. For $k \leq 0$, $x_m(k) = 0$. First $x_S(k)$ will be found from Eq. (3). Initial conditions are $x_S(0) = 0$ and $x_S(1) = \alpha x_m(1)$. With auxiliary equation $m^2 - c_1 m + c_2$ the complementary solution is

$$x_c(k) = d_1 r^k \cos(k\theta) + d_2 r^k \sin(k\theta)$$

$$\text{where } r = \sqrt{1-\alpha},\ 0 < \alpha < 1$$
$$\text{and}$$
$$\theta = \arctan\left(\frac{\sqrt{4\beta - (\alpha+\beta)^2}}{2-\alpha-\beta}\right)$$

$d_1$and d_2 are constants that are determined from initial conditions. Substituting $x_m(k)$ into the rhs of Eq. (3), we have the trial solution $A_1k^2 + A_2k + A_3$ from which we get $A_1 = \frac{1}{2}aT^2, A_2 = 0$, and $A_3 = -(\frac{1-\alpha}{\beta})aT^2$. The solution for $x_S(k)$ takes the form

$$x_S(k) = x_c(k) + \tfrac{1}{2}ak^2T^2 - \left(\frac{1-\alpha}{\beta}\right) aT^2 \qquad (4)$$

The term $(1-\alpha)\beta^{-1}$, denoted ℓ_p, is the position lag coefficient and accounts for the lag in the response due to the acceleration input. Equation (4) will be used to find the constants d_2 and d_2. Applying the initial conditions we get $d_1 = aT^2\ell_p$ and $d_2 = \alpha d_1(2r\sin\theta)^{-1}$

The complete solution is

$$x_S(k) = aT^2\ell_p r^k \cos(k\theta) + \frac{aT^2\alpha\ell_p}{2\sin\theta} r^{k-1}\sin(k\theta)$$
$$+\tfrac{1}{2}ak^2T^2 - a\ell_pT^2 \qquad (5)$$

Since $0 < r < 1$, the terms r^k and r^{k-1} represent the exponential damping in the transient response. The steady state solution is

$$x_{SS}(k) = \tfrac{1}{2}ak^2T^2 - a\ell_pT^2$$

Repeating the above steps for the difference equation in $v_S(k)$, we use a trial solution of the form $A_1k + A_2$, with initial conditions $v_S(0) = 0, v_S(1) = 0.5a\beta T$. We have $A_1 = aT$, $A_2 = aT(\frac{\alpha}{\beta} - \frac{1}{2})$. The term $\ell_v = (\frac{\alpha}{\beta} - \frac{1}{2})$ is the velocity lag coefficient. The complementary solution for $v_S(k)$ has the same form as $x_S(k)$. The solution for $v_S(k)$ is

$$v_S(k) = \frac{aT}{\sin\theta}\left(\frac{\beta}{2} - 1 + \frac{\ell_v}{2}(\alpha+\beta)\right) r^{k-1}\sin(k\theta) \qquad (6)$$
$$+aT\ell_v r^k \cos(k\theta) + akT - a\ell_vT$$

After the transients die out, we have the steady state solution for the smoothed velocity $v_s(k)$

$$v_{SS}(k) = akT - a\ell_vT$$

In summary, the steady state position and velocity lags for a constant acceleration input are $L_p = \ell_p aT^2$ and $L_v = \ell_v aT$ where $\ell_p = \frac{1-\alpha}{\beta}$, $\ell_v = (\frac{\alpha}{\beta} - \frac{1}{2})$. The lag coefficients are related by $\ell_p = \frac{\ell_v^2}{2}$, assuming the Kalata [4] relationship ($\beta = 2(2-\alpha) - 4\sqrt{1-\alpha}$) between the coefficients. The terms ℓ_p and ℓ_v grow asymptotically as the gains α and β become small. We can define a velocity lag time, $\tau = l_vT$, since in the steady state the estimated velocity lags the true velocity by τ seconds. From $v_S(k)$, notice that the transient term and the steady state lag act to cancel each other out until the time $t = \tau$ or $k = \ell_v$, The same observation can be made about the smoothed position response, $x_S(k)$. Using $\ell_p = 0.5\ell_v^2$,we see that the steady state portions of the responses, $x_S(k)$ and $v_S(k)$ equals zero at time $k = \ell_v$.

The preceding analysis can be applied to the predicted position and velocity as well. From equation (2), the predicted or one step ahead lags for position and velocity are;

$$\ell_{p+1} = \frac{1}{\beta}$$

$$\ell_{v+1} = \frac{\alpha}{\beta} + \frac{1}{2}$$

We also have $\ell_{p+1} = \dfrac{\ell_{v+1}^2}{2}$

2. Explicit Control of Filter Lag

There are some instances where the tracking interval, T , may vary due to missed data points or some adverse environment. Eq. (2) will be re-written in terms of a varying data interval T_k along with varying filter gains α_k and β_k. The task will be to find the filter gains that will preserve the nominal filter lags, $L_p = \ell_p a T^2$ and $L_v = \ell_v a T$, where T is the nominal update interval, ℓ_p and ℓ_v are the lag coefficients corresponding to the nominal gains α and β. The predicted terms $x_p(k)$ and $v_p(k)$ are substituted into the expressions for $x_S(k)$ and $v_S(k)$;

$$\begin{aligned} x_S(k) &= (1-\alpha_k)x_S(k-1) \\ &\quad +(1-\alpha_k)T_k v_S(k-1) + \alpha_k x_m(k) \\ v_S(k) &= \frac{-\beta_k}{T_k} x_S(k-1) \\ &\quad +(1-\beta_k)v_S(k-1) + \frac{\beta_k}{T_k} x_m(k) \end{aligned}$$

In matrix form, these equations may be written as

$$\underline{\widehat{X}}_k = A_k \underline{\widehat{X}}_{k-1} + x_m(k)\underline{G}_k \qquad (7)$$

where

$$A_k = \begin{pmatrix} 1-\alpha_k & (1-\alpha_k)T_k \\ \frac{-\beta_k}{T_k} & (1-\beta_k) \end{pmatrix}$$

$$\underline{G}_k = \begin{pmatrix} \alpha_k \\ \frac{-\beta_k}{T_k} \end{pmatrix} \text{ and } \underline{\widehat{X}}_k = \begin{pmatrix} x_S(k) \\ v_S(k) \end{pmatrix}$$

Writing $x_m(k) = x(k) + \epsilon(k)$, where $x(k)$ is the true target position and $\epsilon(k)$ is zero mean white measurement noise, Eq. (7) becomes

$$\underline{\widehat{X}}_k = A_k \underline{\widehat{X}}_{k-1} + x(k)\underline{G}_k + \epsilon(k)\underline{G}_k \qquad (8)$$

Define $\underline{X}_k = [x(k) \quad v(k)]^T$,

$$\begin{aligned} x(k) &= x(k-1) + T_k v(k-1) + .5aT_k^2 \\ v(k) &= v(k-1) + aT_k. \end{aligned}$$

The vector $\underline{X}_k$ represents the true target state and 'a' is the acceleration bias. Subtracting $\underline{X}_k$ from both sides of Eq. (8) we have

$$\underline{\widehat{X}}_k - \underline{X}_k = A_k \underline{\widehat{X}}_{k-1} + x(k)\underline{G}_k - \underline{X}_k + \epsilon(k)\underline{G}_k$$

A computation shows that

$$x(k)\underline{G}_k - \underline{X}_k = -A_k \underline{\widehat{X}}_{k-1} + \underline{C}_k,$$

where

$$\underline{C}_k = \begin{bmatrix} .5aT_k^2(1-\alpha_k) \\ aT_k(1-.5\beta_k) \end{bmatrix}$$

Eq. (8) becomes

$$\underline{\widehat{X}}_k - \underline{X}_k = A_k(\underline{\widehat{X}}_{k-1} - \underline{X}_{k-1}) + \underline{C}_k + \epsilon(k)\underline{G}_k$$

Defining $\underline{\widehat{e}}_k = \underline{\widehat{X}}_k - \underline{X}_k$,we have a state equation for the error vector $\underline{\widehat{e}}_k$, i.e.

$$\underline{\widehat{e}}_k = A_k \underline{\widehat{e}}_{k-1} + \underline{C}_k + \epsilon(k)\underline{G}_k \qquad (9)$$

With E($\underline{\widehat{e}}_k$)= $\underline{b}_k$, where $\underline{b}_k$ is defined as the bias vector and E is the expected value operator, we get from taking expected values of Eq. (9);

$$\underline{b}_k = A_k \underline{b}_{k-1} + \underline{C}_k \qquad (10)$$

Due to the presence of the acceleration in the target model, the bias vector $\underline{b}_k$ will be non-zero in the steady state. Assume that Eq. (10) reaches a steady state corresponding to the nominal update interval, T, and nominal filter gains α and β. From the previous section this vector is

$$\underline{b} = \begin{bmatrix} \ell_p a T^2 \\ \ell_v a T \end{bmatrix}$$

$$\ell_p = \frac{1-\alpha}{\beta}, \ \ell_v = (\frac{\alpha}{\beta} - \frac{1}{2})$$

The same expression may also be obtained from Eq. (10) with $\underline{b} = \underline{b}_k$, $A = A_k$;and $\underline{C} = \underline{C}_k$;

$$\underline{b} = (I - A)^{-1}\underline{C}$$

The steady state version of Eq. (10) corresponding to varying update interval T_k and filter gains α_k and β_k is;

$$\underline{b} = A_k\underline{b} + \underline{C}_k \tag{11}$$

In order to find the varying filter gains α_k and β_k which will maintain a constant bias corresponding to time varying updates T_k, Eq. (11) is solved for α_k and β_k in terms of ℓ_p and ℓ_v

$$\alpha_k = \frac{(\ell_v T_k \ + \ .5T_k)T_k}{\ell_p T_k^2 \ + \ (\ell_v T_k \ + \ .5T_k)T_k}$$

$$\beta_k = \frac{T_k^2}{\ell_p T_k^2 \ + \ (\ell_v T_k \ + \ .5T_k)T_k}$$

Setting $\tau = \ell_v T$ and using $\ell_p = \frac{\ell_v^2}{2}$ (which implictily assumes the Kalata relationship), the above formulas may be written in terms of velocity lag time, τ, and varying update interval T_k;

$$\begin{aligned} \alpha_k &= 1 - \frac{\tau^2}{(T_k \ + \ \tau)^2} \\ \beta_k &= 2\frac{{T_k}^2}{(T_k \ + \ \tau)^2} \end{aligned} \tag{12}$$

This choice of gains will maintain the nominal filter lags no matter how the update interval varies. From equation (12), we also have

$$\beta_k = 2(2 - \alpha_k) - 4\sqrt{1 - \alpha_k} \tag{13}$$

Therefore for the case of aperiodic updates, Eq. (12) will preserve the nominal or periodic filter lags and also maintain the optimal relation between the filter gains. The following one dimensional example considers an incoming target initially being tracked at 2 Hz with negligible acceleration and an initial velocity of 50 yds/sec at 250 data miles. The true target trajectory is corrupted with 1 milliradian of angle noise. Least squares filter gains are used to settle the track to steady state. As the target undergoes a 1 g maneuver, the tracking interval changes to 1/2 Hz followed by a transition to a 1/2 g maneuver with a corresponding tracking interval of 1 Hz. Two cases are considered; upon transition to steady state, case 1 uses a constant value of $\alpha = .15$ while case 2 will use the same constant gain until the maneuver and drop in data rate occur. Thereafter case 2 will use the gain from Eq. (12) which maintains the 2 Hz lag; from $\alpha = .15$ to .4419, to .2686. From figures 1 and 2 we have the position and velocity errors.

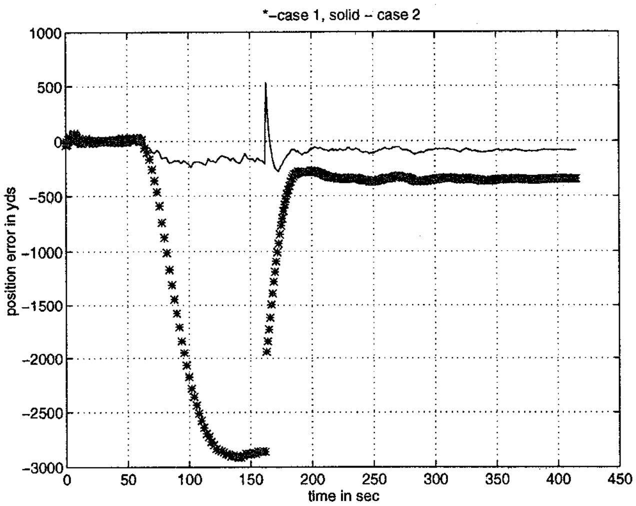

Figure 1. Position Error

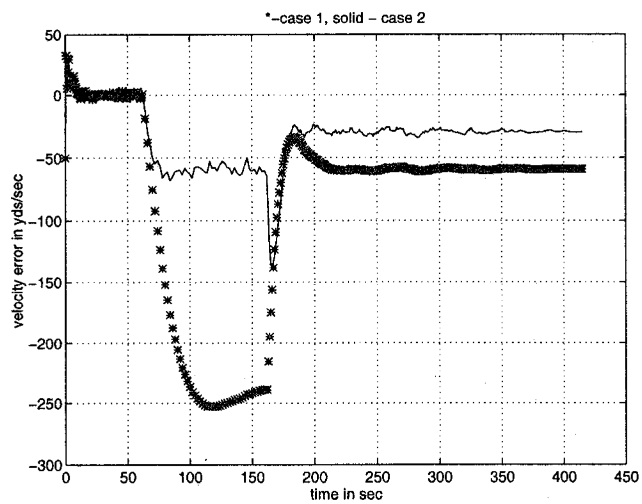

Figure 2. Velocity Error

With the exception of the brief transients due to gain switching, case 2 suffers relatively little degradation in performance. The price payed for maintaining the 2 Hz lag is a tolerable increase in track noise due to the increase in filter gain.

3. Conclusions

For a step acceleration input, we have derived the position and velocity lags for an alpha-beta filter along with the closed form expressions for smoothed position and velocity. For the case of aperiodic tracking, the lag terms were used to derive the filter gains which preserve the optimal Kalata relation and also maintain the periodic filter lag. A simple example demonstrated the improvement in performance for tracking an accelerating target with missed data points.

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