TAU FUNCTIONS, PRYM-TYURIN CLASSES AND LOCI OF DEGENERATE DIFFERENTIALS

DMITRI KOROTKIN, ADRIEN SAUVAGET, AND PETER ZOGRAF

ABSTRACT. We study the rational Picard group of the projectivized moduli space $P\overline{\mathfrak{M}}_g^{(n)}$ of holomorphic n-differentials on complex genus g stable curves. We define $n-1$ natural classes in this Picard group that we call *Prym-Tyurin* classes. We express these classes as linear combinations of boundary divisors and the divisor of n-differentials with a double zero. We give two different proofs of this result, using two alternative approaches: an analytic approach that involves the Bergman tau function and its vanishing divisor and an algebro-geometric approach that involves cohomological computations on the universal curve.

CONTENTS

1. Introduction

1.1. **Moduli space of n-differentials.** Let g and n be positive integers with $g \geq 2$. Let $\mathcal{M}_g$ (respectively $\overline{\mathcal{M}}_g$) be the moduli space of smooth (respectively stable nodal) complex curves. Denote by $D_0 \subset \overline{\mathcal{M}}_g$ the closure of the locus of stable curves with one nonseparating node. Further, denote by $D_i \subset \overline{\mathcal{M}}_g$, $1 \leq i \leq [g/2]$, the closure of the locus of curves with a separating node and two irreducible components of genera i and $g-i$. Finally, denote by $\pi : \mathcal{C}_g \to \mathcal{M}_g$ or $\pi : \overline{\mathcal{C}}_g \to \overline{\mathcal{M}}_g$ the universal curve and by $\omega = \omega_{\overline{\mathcal{C}}_g/\overline{\mathcal{M}}_g}$ the relative dualizing sheaf.

Let
$$\Omega_g^{(n)} = R^0 \pi_* \omega^{\otimes n}$$
be the direct image of the nth tensor power of ω. Using the Riemann-Roch formula and Serre's duality, one can easily check that $h^1(C, \omega_C^{\otimes n}) = 0$ if $n \geq 2$ and $h^1(C, \omega_C) = 1$ for any stable curve C. Thus $\Omega_g^{(n)}$ is a vector bundle for any $n \geq 1$. For $n = 1$ we write Ω_g instead of $\Omega_g^{(1)}$ and call it the *Hodge bundle*. The Riemann-Roch formula implies:

$$\mathrm{rk}\, \Omega_g^{(n)} = \begin{cases} g & \text{if } n = 1; \\ (2n-1)(g-1) & \text{otherwise.} \end{cases}$$

We define the following elements in the rational Picard group of $\overline{\mathcal{M}}_g$:

- the Chern class $\lambda_n \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ of the determinant line bundle of $\Omega_g^{(n)}$;
- the Poincaré dual classes $\delta_0, \ldots, \delta_{[g/2]} \in \mathrm{Pic}(\overline{\mathcal{M}}_g)$ of the boundary divisors $D_0, D_1, \ldots, D_{[g/2]} \subset \overline{\mathcal{M}}_g$.

For $k = 1$ we write λ instead of λ_1 and call this class the *Hodge class*.

Definition 1.1. The total space of the vector bundle $\Omega_g^{(n)}$ is denoted by $\overline{\mathfrak{M}}_g^{(n)}$ and is called the *space of n-differentials*.

The points of $\overline{\mathfrak{M}}_g^{(n)}$ correspond to equivalence classes of pairs (C, w), where C is a stable genus g algebraic curve, and w is an n-differential on C. We recall that an n-differential w on C is a meromorphic n-differential on each irreducible component of the normalization of C such that

- w can only have poles at the preimages of the nodes;
- these poles are of order at most n;
- at every node the n-residues of w at the poles satisfy

$$\mathrm{res}_{p_1}(w) = (-1)^n \mathrm{res}_{p_2}(w)$$

where p_1 and p_2 are the two preimages of the node.

We denote by $\nu : \overline{\mathfrak{M}}_g^{(n)} \to \overline{\mathcal{M}}_g$ the forgetful map and we will use the same notation for its restriction $\nu : \mathfrak{M}_g^{(n)} \to \mathcal{M}_g$ to the locus of smooth curves. We also denote by $\tilde{\nu} : P\overline{\mathfrak{M}}_g^{(n)} \to \overline{\mathcal{M}}_g$ the projectivized space of n-differentials.

In this paper we study the Picard group of $P\overline{\mathfrak{M}}_g^{(n)}$. We will work over rational numbers, so that Pic will always denote the *rational* Picard group.

By abuse of notation we will denote by λ, λ_n and δ_i both the elements of $\mathrm{Pic}(\overline{\mathcal{M}}_g)$ and their pull-backs in $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$. In addition, we introduce the first Chern class $\psi \in \mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ of the tautological line bundle $L \to P\overline{\mathfrak{M}}_g^{(n)}$.

The following lemma is standard (cf., e.g, [13], Lemma 1).

Lemma 1.2. *The classes $\lambda, \psi, \delta_0, \ldots, \delta_{[g/2]}$ form a basis of $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$.*

The goal of this paper is to define the Prym-Tyurin classes in $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ and express them in the above basis.

1.2. **Stratification of** $P\mathfrak{M}_g^{(n)}$**.** The space of n-differentials is naturally stratified according to the multiplicities of the differential's zeros.

Let $\mathbf{k} = (k_1, \ldots, k_m)$ be a partition of $n(2g-2)$. We denote by $\mathfrak{M}_g^{(n)}[\mathbf{k}] \subset \mathfrak{M}_g^{(n)}$ the locus of pairs (C, w) such that the n-differential w has m pairwise distinct zeros of orders exactly k_i. This locus is $\mathbb{C}^*$-invariant, thus we can also define its projectivization $P\mathfrak{M}_g^{(n)}[\mathbf{k}] \subset P\mathfrak{M}_g^{(n)}$. The space $P\mathfrak{M}_g^{(n)}$ is the disjoint union of the strata $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ for all partitions $\mathbf{k}$ of $n(2g-2)$. The following properties of the strata were proved in [15] and [19].

- Each stratum $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ is smooth.
- If at least one k_i is not divisible by n then either $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ is empty or it has pure dimension $2g - 3 + m$.
- If all k_i's are divisible by n then $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ has at least one irreducible component of dimension $2g-2+m$. A differential (C, w) lies in a component like that if and only if w is the nth power of a holomorphic differential. The stratum $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ may also have irreducible components of dimension $2g - 3 + m$ composed of n-differentials that are not nth powers.

We denote by $P\overline{\mathfrak{M}}_g^{(n)}[\mathbf{k}]$ the closure of $P\mathfrak{M}_g^{(n)}[\mathbf{k}]$ in $P\overline{\mathfrak{M}}_g^{(n)}$. In particular we have $P\overline{\mathfrak{M}}_g^{(n)}[\mathbf{1}] = P\overline{\mathfrak{M}}_g^{(n)}$, where $\mathbf{1}$ stands for the partition $(1, 1, \ldots, 1)$.

Definition 1.3. Let $g, n \geq 2$. The *divisor of degenerate n-differentials* is defined as
$$D_{\mathrm{deg}} = \begin{cases} P\overline{\mathfrak{M}}_g^{(n)}[2, 1, \ldots, 1], & \text{if} \quad (g, n) \neq (2, 2), \\ P\overline{\mathfrak{M}}_g^{(n)}[2, 1, 1] + 2 \cdot P\overline{\mathfrak{M}}_g^{(n)}[2, 2], & \text{if} \quad g = n = 2. \end{cases}$$
We denote by δ_{deg} the cohomology class that is Poincaré dual of D_{deg}.

Remark 1.4. Heuristically, D_{deg} is the divisor of n-differentials with a double zero, and for $(g, n) \neq (2, 2)$ it is just the closure of $P\mathfrak{M}_g^{(n)}[2, 1, \ldots, 1]$ in $P\overline{\mathfrak{M}}_g^{(n)}$. In the case $g = n = 2$, however, D_{deg} has a special component consisting of squares of holomorphic differentials. This is because in genus 2 each quadratic differential is invariant with respect to the hyperelliptic involution. The four simple zeroes of w are pairwise equivalent under the hyperelliptic involution, and when two non-equivalent zeroes coalesce, the other two ones also coalesce, giving a differential with two double zeroes. Since every such differential has two square roots that differ by a sign, the divisor $P\overline{\mathfrak{M}}_g^{(n)}[2, 2]$ comes with a factor of 2. (Note that when two equivalent zeroes coalesce, the differential in the limit has one double zero at a Weierstrass point and two simple zeroes.)

1.3. **First definition of Prym-Tyurin classes.** Let (C, w) be a point in the projectivized moduli space $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$. One can define a canonical cyclic ramified covering $f : \widehat{C} \to C$ of degree n, where
$$\widehat{C} = \{(x, v) | x \in C, \ v \in T_x^* C, \ v^n = w\}.$$

This covering is completely ramified over the zeros of w. The curve $\widehat{C}$ is smooth of genus $\widehat{g} = n^2(g-1) + 1$. It comes with a *canonical holomorphic differential v* given by $v(x, v) = v$. This differential v on $\widehat{C}$ satisfies $v^n = f^* w$.

The action of $\mathbb{Z}/n\mathbb{Z}$ on the covering is given by $\rho^k : (x, v) \mapsto (x, \rho^k v)$, where $\rho = e^{\frac{2\pi\sqrt{-1}}{n}}$. We denote by $\sigma : \widehat{C} \to \widehat{C}$ the automorphism of $\widehat{C}$ corresponding to

$k = 1$. Now consider the natural map

$$\hat{\nu} : P\mathfrak{M}_g^{(n)}[\mathbf{1}] \to \mathcal{M}_{\widehat{g}},$$

$$(C, w) \mapsto \widehat{C}$$

($\widehat{C}$ remains the same when we multiply w by a non-zero constant). We consider the pull-back of the Hodge bundle $\Omega_{\widehat{g}}$ by the map $\hat{\nu}$. The automorphism σ induces an endomorphism σ^* of the vector bundle $\hat{\nu}^*\Omega_{\widehat{g}}$ given by: $((C, w), u) \mapsto ((C, w), \sigma^*u)$, where u is an element of $H^0(\widehat{C}, \omega_{\widehat{C}})$. The endomorphism σ^* satisfies $(\sigma^*)^n = \mathrm{Id}$. Hence we have a decomposition

$$(1.1) \qquad\qquad \hat{\nu}^*\Omega_{\widehat{g}} = \bigoplus_{k=0}^{n-1} \Lambda^{(k)},$$

where $\Lambda^{(k)}$ is the eigenbundle of $\hat{\nu}^*\Omega_{\widehat{g}}$ corresponding to the eigenvalue $\rho^k = e^{\frac{2\pi\sqrt{-1}k}{n}}$.

Remark 1.5. The space $\Lambda^{(k)}$ is a vector bundle because the dimension of the fiber of $\Lambda^{(k)}$ is upper-continuous for all k and $\hat{\nu}^*\Omega_{\widehat{g}}$ is a vector bundle thus the rank of each $\Lambda^{(k)}$ is constant.

Definition 1.6. The vector bundles $\Lambda^{(k)}$ are called the *Prym-Tyurin vector bundles*. The *Prym-Tyurin class* $\lambda_{PT}^{(k)}$ is the first Chern class $c_1(\Lambda^{(k)}) \in \mathrm{Pic}(P\mathfrak{M}_g^{(n)}[\mathbf{1}])$.

For $n = 2$ the study of vector bundles of this type was initiated by Prym [17] and for $n > 2$ by A. N. Tyurin [20].

Remark 1.7. By abuse of notation we denote in the same way the *determinant line bundle* $\lambda_{PT}^{(k)} = \det \Lambda^{(k)}$ and its class in the Picard group.

We will see in Section 5 that the map $\hat{\nu} : \mathfrak{M}_g^{(n)} \to \mathcal{M}_{\widehat{g}}$, used to define the Prym-Tyurin vector bundles, admits no natural extension to $P\overline{\mathfrak{M}}_g^{(n)}$. Nonetheless, in the next section we extend the definition of the Prym-Tyurin class to $P\overline{\mathfrak{M}}_g^{(n)}$ by a construction involving the space of admissible covers and an intermediate bigger stack.

1.4. **Admissible coverings and Prym-Tyurin bundles.** Let $N = 2n(g-1)$ be the degree of $\omega^{\otimes n}$. We denote by Hur_g^n the moduli space whose geometric points are isomorphism classes of pairs $(f : \widehat{C} \to C, \sigma)$ where:

- C and $\widehat{C}$ are smooth curves;
- f is a cyclic ramified covering of degree n which is totally ramified over N distinct points of C;
- σ is an automorphism of $\widehat{C}$ that commutes with f.

We denote by $\overline{\mathrm{Hur}}_g^n$ the compactification of this space by admissible coverings (see [7]). The space of admissible coverings has two forgetful maps (source and target of the covering):

$$\begin{array}{ccc} & \overline{\mathrm{Hur}}_g^n & \\ {\scriptstyle \text{target}} \swarrow & & \searrow {\scriptstyle \text{source}} \\ \overline{\mathcal{M}}_{g,N}/S_N & & \overline{\mathcal{M}}_{\widehat{g}}. \end{array}$$

We consider the pull-back source$^*\Omega_{\widehat{g}}$ of the Hodge bundle under the source map. This vector bundle is endowed with the automorphism

$$\sigma^* : \left((\widehat{C} \to C, \sigma), u\right) \quad \mapsto \quad \left((\widehat{C} \to C, \sigma), \sigma^*u\right).$$

Thus, as in (1.1), we have the decomposition

$$\text{source}^*\Omega_{\widehat{g}} = \bigoplus_{k=0}^{n-1} \Lambda^{(k)},\tag{1.2}$$

where $\Lambda^{(k)}$ is the eigenbundle corresponding to the eigenvalue $\rho^k = e^{\frac{2\pi\sqrt{-1}k}{n}}$.

In the previous section we have constructed an embedding $i : P\mathfrak{M}_g^{(n)}[\mathbf{1}] \hookrightarrow \overline{\text{Hur}}_g^n$ and, by construction, the pull-back $i^*\Lambda^{(k)}$ is, indeed, isomorphic to the Prym-Tyurin bundle $\Lambda^{(k)}$ over $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ as defined in the previous section.

We see that the Prym-Tyurin vector bundles, and therefore the Prym-Tyurin classes, have a natural extension to the compactification of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ by admissible covers. We would like, however, to extend the Prym-Tyurin classes to a different compactification of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$, namely, to $P\overline{\mathfrak{M}}_g^{(n)}$. To do that we will construct a bigger space with a projection to both $\overline{\text{Hur}}_g^n$ and $P\overline{\mathfrak{M}}_g^{(n)}$ and use the push-forward of a pull-back.

1.5. Space of admissible differentials.

Definition 1.8. Let

$$I : \mathfrak{M}_g^{(n)}[\mathbf{1}] \hookrightarrow \overline{\mathfrak{M}}_g^{(n)} \times_{\overline{\mathcal{M}}_g} \overline{\text{Hur}}_g^n$$

be the product of the two natural embeddings. The *moduli space of admissible n differentials* $X(g,n)$, is the Zariski closure of the image of I in $\overline{\mathfrak{M}}_g^{(n)} \times \overline{\text{Hur}}_g^n$.

Remark 1.9. The space of admissible n-differentials is a compactification of the stratum $\mathfrak{M}_g^{(n)}[\mathbf{1}]$. In [2], the authors introduced and described another compactification, the *incidence variety compactification*. The incidence variety is obtained by replacing the space $\overline{\text{Hur}}_g^n$ by the space $\overline{\mathcal{M}}_{g,N}/S_N$ in Definition 1.8. We denote the incidence variety by $X_{g,n}^{\text{inc}}$.

There is a birational and finite map from $X_{g,n}$ to $X_{g,n}^{\text{inc}}$, however this map is not an isomorphism (see Example 4.3 of [2]). We will make use of $X_{g,n}^{\text{inc}}$ once in this text (proof of Lemma 2.3).

The space $X(g,n)$ has two natural morphisms: $\text{adm} : X(g,n) \to \overline{\text{Hur}}_g^n$ and $\text{diff} : X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$. We summarize the notation on the following diagram.

$$
\begin{array}{ccc}
\bigoplus \Lambda^{(k)} & \xrightarrow{\text{source}^*} & \Omega_{\widehat{g}} \\
\downarrow & & \downarrow \\
X(g,n) \xrightarrow{\text{adm}} & \overline{\text{Hur}}_g^n \xrightarrow{\text{source}} & \mathcal{M}_{\widehat{g}} \\
\downarrow{\scriptstyle \text{diff}} & & \\
P\overline{\mathfrak{M}}_g^{(n)} & &
\end{array}
$$

By construction the forgetful map $\text{diff} : X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$ is birational and its restriction to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ is an isomorphism onto its image. In general the space $X(g,n)$ is not normal; thus the push-forward of classes in the Picard group under diff is ill-defined. However we will prove the following proposition in Section 2.

Proposition 1.10. *There exist two smooth open-dense substacks $j : V \hookrightarrow X(g, n)$ and $j' : U \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}$ fitting into the commutative diagram*

$$
\begin{array}{ccc}
V & \xrightarrow{\ j\ } & X(g, n) \\
\downarrow & & \downarrow{\scriptstyle \mathrm{diff}} \\
U & \xrightarrow[\ j'\ ]{} & P\overline{\mathfrak{M}}_g^{(n)}
\end{array}
$$

such that

- *the map $V \to U$ is an isomorphism on the underlying coarse spaces;*
- *the complement of U is of codimension at least 2 in $P\overline{\mathfrak{M}}_g^{(n)}$.*

The existence of U and V as above allows one to define the induced push-forward diff_* in the Picard groups. Indeed, the first property ensures that $\mathrm{Pic}(U)$ and $\mathrm{Pic}(V)$ are isomorphic. Similarly, the second propery ensures that j' induces an isomorphism between $\mathrm{Pic}(U)$ and $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$. Thus, given an element in $\mathrm{Pic}(X_{g,n})$, one defines its push-forward to $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ by first taking its pull-back by j, then the push-forward from V to U, and finally the push-forward by j'.

Definition 1.11. The *Prym-Tyurin class* $\lambda_{PT}^{(k)}$ on $P\overline{\mathfrak{M}}_g^{(n)}$ is defined by the formula $\lambda_{PT}^{(k)} = \mathrm{diff}_* c_1(\Lambda^{(k)})$.

1.6. **Statement of the results.** The main result obtained in this paper is the expression for the Prym-Tyurin classes and the class δ_{deg} of Definition 1.3 in the $(\lambda, \psi, \delta_i)$ basis.

Theorem 1.12. *In the rational Picard group of $P\overline{\mathfrak{M}}_g^{(n)}$ we have*

$$
(1.3) \qquad \delta_{\mathrm{deg}} = 12n(n+1)\lambda - 2(g-1)(2n+1)\psi - n(n+1) \sum_{i=0}^{[g/2]} \delta_i;
$$

$$
(1.4) \quad \lambda_{PT}^{(n-k)} = (6k^2 + 6k + 1)\lambda - \frac{g-1}{n} k(2k+1)\psi - \frac{1}{2} k(k+1) \sum_{i=0}^{[g/2]} \delta_i + c_k \, \delta_{\mathrm{deg}},
$$

where

$$
(1.5) \qquad\qquad c_k = \begin{cases} \frac{2k-n}{2n}, & \text{if } (n-1)/2 < k < n, \\ 0, & \text{otherwise.} \end{cases} \quad .
$$

1.7. **Strategy of the proof.** Formula (1.3) of Theorem 1.12 is proved in two distinct ways.

- In Section 3 we introduce the Bergman tau function on the moduli space $\mathfrak{M}_g^{(n)}$. We study its transformation property and its asymptotic behavior at the boundary divisors D_{deg} and D_i, $0 \leq i \leq [g/2]$. We explicitly compute the vanishing order of the Bergmann tau function along these divisors. We use these results to express the divisor δ_{deg} in the $(\lambda, \delta_i, \psi)$ basis of the Picard group. This first proof is a further development of the ideas introduced in [13] and [14].
- In Section 4 we give an alternative proof of Formula (1.3) based on algebro-geometric computations as introduced in [18] and [22] in the context of abelian differentials. We consider the moduli space of n-differentials on genus g curves with one marked point. This space carries a vector bundle of 2-jets of an n-differential at the marked point. The Euler class of this vector

bundle has a natural expression involving the locus A_2 of n-differentials with a double zero at the marked point. The locus A_2 pushes forward to D_{deg} under the forgetful map that forgets the marked point. This allows one to compute the cohomology class δ_{deg} that is Poincaré dual to the divisor class of D_{deg}.

To prove Formulas (1.4) and (1.5) we combine (1.3) with the following two facts.

- First, the well-known Mumford formula [16] expressing the first Chern class of the vector bundle of k-differentials on $\overline{\mathcal{M}}_g$ via the Hodge class:

$$(1.6) \qquad \lambda_k = (6k^2 - 6k + 1)\lambda - \frac{k(k-1)}{2} \sum_{i=0}^{[g/2]} \delta_i \ .$$

- Second, the fact that the morphism

$$(1.7) \qquad \begin{array}{rcl} \Phi_k : \Lambda^{(k)} \otimes T^{\otimes n-k} & \to & H^0(C, \omega_C^{n-k+1}) \\ (q, v^{n-k}) & \mapsto & qv^{n-k} \end{array}$$

is actually an isomorphism of vector bundles outside D_{deg}.

The second fact allows one to compute the rank of the Prym-Tyurin vector bundle $\Lambda^{(k)}$:

$$(1.8) \qquad \operatorname{rk} \Lambda^{(k)} = \operatorname{rk} \Omega^{(n-k+1)} = (2n - 2k + 1)(g - 1) \ , \qquad k = 1, \ldots, n - 1 \ .$$

It also implies that

$$(1.9) \qquad \lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi + \text{const} \cdot \delta_{\text{deg}} \ .$$

In Section 5, we study the asymptotic of the determinant of Φ_k along the divisor D_{deg} to obtain Expressions (1.4) and (1.5).

Remark 1.13. Presence of an additional contribution proportional to δ_{deg} in (1.4) for $k > (n-1)/2$ was first suggested by the third author in [21] using an idea of [22].

Plan of the paper. In Section 2 we prove Proposition 1.10 and thus complete the definition of the Prym-Tyurin classes. In Sections 3 and 4 we prove Formula (1.3) of Theorem 1.12 using the two different approaches described above. In Section 5 we discuss the relationship between Prym-Tyurin vector bundles and vector bundles of k-differentials and derive a relationship between corresponding determinant line bundles. This allows us to express the Prym-Tyurin classes in the $(\lambda, \delta_i, \psi)$ basis of the Picard group and complete the proof of Theorem 1.12.

2. Space of admissible n-differentials

In this Section we justify the definition of the Prym-Tyurin classes by proving the following extended version of Proposition 1.10.

Proposition 2.1. *There exist two smooth open-dense substacks $j : V \hookrightarrow X(g, n)$ and $j' : U \hookrightarrow P\overline{\mathfrak{M}}_g^{(n)}$ fitting into the commutative diagram*

$$\begin{array}{ccc} V & \xrightarrow{\ j\ } & X(g, n) \\ \downarrow & & \downarrow{\scriptstyle \text{diff}} \\ U & \xrightarrow{\ j'\ } & P\overline{\mathfrak{M}}_g^{(n)} \end{array}$$

such that

- *V and U contain the image of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ under the embeddings into $X(g, n)$ and $P\overline{\mathfrak{M}}_g^{(n)}$;*

- *the complement of U is of codimension at least 2 in $P\overline{\mathfrak{M}}_g^{(n)}$;*
- *the map $V \to U$ is an isomorphism on the underlying coarse spaces;*
- *there exists a line bundle $T \to V$ such that T is a sub-vector bundle of $\Lambda^{(1)}$, $T^{\otimes n} \simeq L$ and the restriction of T to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ coincides with the sub-vector bundle of $\hat{\nu}^* \Omega_{\widehat{g}}$ spanned by v.*

2.1. Distinguished local coordinates on a cyclic covering.

Consider a curve C endowed with an n-differential w with simple zeros. Let $f : \widehat{C} \to C$ be the associated covering and $v = w^{1/n}$ be the canonical abelian differential on $\widehat{C}$. Denote by $x_i \in C$, $i = 1, \ldots, N = 2n(g-1)$, the branch points of $\widehat{C} \to C$, which coincide with the zeros of w. Denote by $\hat{x}_i$ the unique preimage of x_i in $\widehat{C}$. Here we describe a specific parametrization of the covering curve $\widehat{C}$ near the branching points of $f : \widehat{C} \to C$. It is easy to see that all zeros of the holomorphic 1-form v are situated at the ramification points $\hat{x}_i$ and have multiplicity n. In other words,

$$(2.1) \qquad (v) = n\hat{x}_1 + \cdots + n\hat{x}_N.$$

Introduce a local parameter ζ_i in a neighborhood of $x_i \in C$ and a local parameter ξ_i in a neighborhood of $\hat{x}_i \in \widehat{C}$ such that

$$(2.2) \qquad w = \zeta_i (d\zeta_i)^n, \qquad \xi_i(\hat{x})^{n+1} = \int_{\hat{x}_i}^{\hat{x}} v.$$

Both parameters are defined up to an $(n+1)$st root of unity and we make one choice in such a way that

$$\zeta_i = \left(\frac{n+1}{n} \right)^{n/(n+1)} \xi_i^n.$$

The local parameters ξ_i on $\widehat{C}$ and ζ_i on C given by (2.2) are called *distinguished*.

Since $f^* v = \rho\, v$, the local parameter $\xi_i(x)$ transforms under the action of f as $\xi_i(f(x)) = \rho \xi_i(x)$.

2.2. Extension of the Prym-Tyurin bundles to codimension 1 loci.

By construction, $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ is an open dense substack of $X(g,n)$. Thus diff $: X(g,n) \to P\overline{\mathfrak{M}}_g^{(n)}$ is birational and its restriction to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ is an isomorphism onto its image.

By abuse of notation we denote by D_i, $0 \leq i \leq [g/2]$, the preimage in $P\overline{\mathfrak{M}}_g^{(n)}$ of the boundary divisor $D_i \subset \overline{\mathcal{M}}_g$. Then the complement of $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ in $P\overline{\mathfrak{M}}_g^{(n)}$ is the union of the divisors D_i and $D_{\deg}$.

For each of these divisors we define a dense open locus $\widetilde{D} \subset D$ as follows.

- $\widetilde{D}_0$ is the locus of (C, w) such that the curve C has exactly one non-separating node and the differential w has poles of order n at the node and N simple zeros;
- $\widetilde{D}_i$ for $i \geq 1$ is the locus of (C, w) such that the curve C has exactly one separating node and the differential w has poles of order n at the node and N simple zeros.
- $\widetilde{D}_{\deg}$ if $(g, n) \neq (2, 2)$ is the locus of (C, w) such that the curve C is a smooth curve and the differential w has one zero of order exactly 2 and its other zeros are simple;
- $\widetilde{D}_{\deg}$ if $(g, n) = (2, 2)$ is the disjoint union of the locus $\widetilde{D}_{\deg}(2, 1, 1)$ described above and of the locus $\widetilde{D}_{\deg}(2, 2)$ of pairs (C, w) where C is smooth and w is a square of a holomorphic differential with simple zeros.

We define U as the union of $P\mathfrak{M}_g^{(n)}[\mathbf{1}] \cup \widetilde{D}_{\deg} \cup_i \widetilde{D}_i \subset P\overline{\mathfrak{M}}_g^{(n)}$. We define V as $\mathrm{diff}^{-1}(U) \subset X(g,n)$. We will prove that U and V satisfy the properties of Proposition 2.1.

Property 1 is satisfied by construction.

Lemma 2.2 (Property 2). *The stack U is an open substack of $P\overline{\mathfrak{M}}_g^{(n)}$ and its complement is of codimension at least 2.*

Proof. The complement of U is a union of closed substacks: the strata of curves with at least two nodes and the strata $P\overline{\mathfrak{M}}_g^{(n)}[\mathbf{k}]$ for all $\mathbf{k}$ except $(1,\ldots,1)$ and $(2,1\ldots,1)$. Thus U is an open substack in $P\overline{\mathfrak{M}}_g^{(n)}$ and by dimension count its complement is of codimension at least 2. $\square$

Lemma 2.3 (Property 3). *The restriction of $\mathrm{diff} : V \to U$ induces an isomorphism on the underlying schemes. Moreover the map of stacks $\mathrm{diff} : V \to U$ is of degree one over $U \setminus D_{\deg}$ and of degree $1/2$ over $D_{\deg}$.*

Proof. The underlying scheme of U is smooth and the map $\mathrm{diff} : V \to U$ is birational. Thus the map diff of schemes is an isomorphism if and only if it is finite. We consider the incidence variety $X_{g,n}^{\mathrm{inc}}$ compactification defined in Remark 1.9. We have two birational map: $\mathrm{diff}' X_{g,n}^{\mathrm{inc}} \to X_{g,n}$ and $\epsilon : X_{g,n}^{\mathrm{inc}} \to \overline{\mathfrak{M}}_g^{(n)}$ such that $\mathrm{diff} = \mathrm{diff}' \circ \epsilon$. The map diff' is obtained by forgetting the admissible covering (but not the markings) and ϵ is obtained by forgetting the markings. As we have already stated in Remark 1.9, the map diff' is finite because $\overline{\mathrm{Hur}}_{g,n} \to \overline{\mathcal{M}}_{g,N}/S_N$ is finite. Therefore we need to check that the map ϵ restricted to $\epsilon^{-1}(U)$ is finite.

The restriction $\epsilon : \epsilon^{-1}(U) \to U$ is a bijection. Indeed, if (C,w) be a n-differential in $U \setminus D_{\deg}$, then the preimage of (C,w) under ϵ is the n-differential w with the marked simple zeros. Now if (C,w) is a n-differential in $\tilde{D}_{\deg}$ then the preimage of (C,w) is the point (C',w',x_i) where C' is the curve with two components: one component isomorphic to C and one rational component attached to C at the double zero; the differential w' is then given by w on the main component and vanishes identically on the rational component; finally the marked points are the simple zeros on the main component and two marked points on the rational component.

Therefore $\epsilon : \epsilon^{-1}(U) \to U$ is finite and $\mathrm{diff} : V \to U$ is birational and finite thus an isomorphism of the underlying schemes. Moreover the restriction of the map ϵ to $U \setminus D_{\deg}$ is obviously an isomorphism of stacks. The degree of diff along $D_{\deg}$ will be computed in the next paragraphs. $\square$

Lemma 2.4 (Property 4). *There exists a line bundle $T \to V$ such that $T^{\otimes n} \simeq L$, T is a sub-vector bundle of $\Lambda^{(1)}$, and the restriction of T to $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ coincides with the sub-vector bundle of $\hat{\nu}^*\Omega_{\widehat{g}}$ spanned by v.*

The proof requires a detailed analysis of the inverse morphism $\mathrm{diff}^{-1} : U \to V$ and is contained in the next two subsections.

2.2.1. *Nodal curves.* Let $0 \leq i \leq [g/2]$ and let (C,w) be a point in $\widetilde{D}_i$. The n-fold covering associated to (C,w) is given by $\widehat{C} = \{(x,v) \in T_C^*/v^n = w\}$ and the canonical differential v is still defined by $v(x,v) = v$. We can describe the topology of $\widehat{C}$ and the singularities (zeros and poles) of v:

- If $i = 0$, then the curve $\widehat{C}$ is an irreducible curve with n self-intersections. The differential v has zeros of order n at the marked points and poles of order 1 at the nodes.

- If $i \geq 1$ then the $\widehat{C}$ has two irreducible components intersecting at n distinct nodes. The differential v has also zeros of order n at the marked points and poles of 1 at the nodes.

The canonical differential v is well-defined on $U \setminus D_{\mathrm{deg}}$, thus the line bundle T can be extended to $\mathrm{diff}^{-1}(U \setminus D_{\mathrm{deg}})$.

2.2.2. *Degenerate differentials.* Here we describe the local structure of the stacks $X(g,n)$ and $\overline{\mathfrak{M}}_g^{(n)}$ close to D_{deg}. This allows us to explicit the isomorphism of Lemma 2.3 and to describe the fiber of the canonical line bundle along $\widetilde{D}_{\mathrm{deg}}$. Let (C_0, w_0) be a point in $\widetilde{D}_{\mathrm{deg}}$ and let W be a neighborhood of (C_0, w_0) in D_{deg}. We will give a local parametrization of $X(g,n)$ and $\overline{\mathfrak{M}}_g^{(n)}$ around the point (C_0, w_0).

- *Parameters of $\overline{\mathfrak{M}}_g^{(n)}$.* A neighborhood of (C_0, w_0) is given by $W \times \Delta$ where Δ is a disk of $\mathbb{C}$ centered at zero. A point (u,a) in $W \times \Delta$ parametrizes an n-differentials (C,w) such that

$$w = (\zeta^2 + a)d\zeta^n.$$

  where the parameter ζ of the curve C is uniquely determined by the choice of a. The parameter a is a tranverse local parameter of D_{deg} in $\overline{\mathfrak{M}}_g^{(n)}$.

- *Parameters of $X(g,n)$.* A neighborhood of (C_0, w_0) in $X_{g,n}$ is parametrized by $W \times \Delta'/(\mathbb{Z}/2\mathbb{Z})$ where Δ' is a disk of $\mathbb{C}$ centered at zero. Indeed, suppose first that the two colliding zeros of a differential (C,w) are labeled x_1 and x_2. Let ζ be a local parameter of C such that the positions of x_1 and x_2 are given by ζ_1 and ζ_2, respectively. To fix ζ uniquely we can define it by exact relation:

$$w(x) = (\zeta(x) - \zeta_1)(\zeta(x) - \zeta_2)(d\zeta(x))^n , \qquad x \in C.$$

  The parameter $(\zeta_1 - \zeta_2)/\{\pm 1\}$ is a local transverse parameter to D_{deg} in $X(g,n)$ (See Lemma 3.6 for a proof).

With these two local parametrizations, the map $\mathrm{diff} : X(g,n) \to \overline{\mathfrak{M}}_g^{(n)}$ is given by

$$\begin{array}{ccc} W \times \Delta'/(\mathbb{Z}/2\mathbb{Z}) & \to & W \times \Delta \\ (u, \zeta_1 - \zeta_2) & \mapsto & (u, (\zeta_1 - \zeta_2)^2). \end{array}$$

This map is indeed an isomorphism of the underlying schemes. However it is of degree $1/2$ along $D_{deg} \subset X(g,n)$ once we consider the stack structures of $X(g,n)$ and $\overline{\mathfrak{M}}_g^{(n)}$. This finishes the proof of Lemma 2.3.

Finally we describe the extension of the canonical line bundle T to D_{deg}. Let (C,w) be a family of differentials with simple zeros which tends to $(C_0, w_0) \in D_{\mathrm{deg}} \subset X(g,n)$. Once again we label the two coalescing zeros x_1 and x_2 and we use the local parameter of the curve ζ with $\zeta(x_i) = \zeta_i$ and $w(x) = (\zeta(x) - \zeta_1)(\zeta(x) - \zeta_2)(d\zeta(x))^n$

As (C,w) tends to a pair (C_0, w_0) the zeros x_1 and x_2 tend to the double zero x_0 of w_0. The limit curve C_0 is a nodal curve with two components: a Riemann sphere C_1 which gets naturally equipped with the *meromorphic n-differential $w_1(\zeta)$* given by the formula

$$(2.3) \qquad w_1(\zeta) = (\zeta - \zeta_1)(\zeta - \zeta_2)(d\zeta)^n , \qquad \zeta \in C_1 ,$$

which is holomorphic outside of $\zeta = \infty$ and has two simple zeros at ζ_1 and ζ_2. The Riemann surface C_2 is equipped with the holomorphic n-differential w_0. The nodal

point on C_0 is formed by identifying the point $\zeta = \infty$ on C_1 with the point x_0 on C_2.

The limit n-differential (C_0, w_0) determines a canonical n-sheeted covering $\widehat{C}_0 \to C_0$. The curve $\widehat{C}_0$ consists of two components $\widehat{C}_1$ and $\widehat{C}_2$. The canonical covering $\widehat{C}_1$ of C_1 is given by equation

$$(2.4) \qquad\qquad v_1^n = w_1(\zeta)$$

which, if we write $v_1 = y d\zeta$, is the curve

$$(2.5) \qquad\qquad y^n = (\zeta - \zeta_1)(\zeta - \zeta_2)$$

of genus $\widehat{g}_1 = [(n-1)/2]$. The canonical covering $\widehat{C}_2$ of C_2 is defined by the equation

$$(2.6) \qquad\qquad v_2^n = w_2 \ ;$$

its genus equals $\widehat{g}_2 = \widehat{g} - [n/2]$. Therefore, for odd n we have $\widehat{g} = \widehat{g}_1 + \widehat{g}_2$ while for even n we have $\widehat{g} = \widehat{g}_1 + \widehat{g}_2 + 1$.

The difference between the case of even n and the case of odd n is due to the fact that for odd n the coverings $\widehat{C}_1$ and $\widehat{C}_2$ intersect at only one nodal point while for even n the nodal point on C_0 has two pre-images on $\widehat{C}_0$ i.e. for even n $\widehat{C}_1$ and $\widehat{C}_2$ intersect at two nodal points, $x_0^{(1)}$ and $x_0^{(2)}$ (see Figure 1 below).
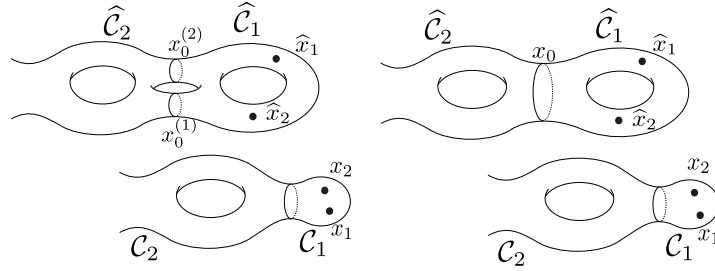

FIGURE 1. Examples of degeneracy of canonical n-coverings (before colliding the zeros): on the left $n = 4$ and on the right $n = 3$.

With this description of the limit covering, we define the limit canonical differential v_0 as follows: it is given by $v_0 = v_2$ on the component $\widehat{C}_2$ and vanishes identicaly on the rational component $\widehat{C}_1$. It satisfies $v_0^n = f^* w_0$. Thus the canonical line bundle T can be extended to the open set V. This completes the proof of Lemma 2.4.

3. Bergman tau function and Hodge class on $P\overline{\mathfrak{M}}_g^{(n)}$

Tau functions play an important role in the theory of integrable systems providing canonical generators for commuting flows on the phase space [1]. In some cases tau functions carry interesting algebro-geometric information, like the isomonodromic tau function of the Riemann-Hilbert problem that is relevant in the theory of Frobenius manifolds [4].

The Bergman tau function introduced in [11] allowed to express the Hodge class on the space of admissible covers of the projective line as an explicit linear combination of the boundary divisors [12]. Then in [9] this result was proven by pure algebro-geometric methods (namely, by means of the Grothendieck-Riemann-Roch theorem) and later in [8] used to answer a question of Harris-Mumford [7] about the classes of Hurwitz divisors in the moduli space $\overline{\mathcal{M}}_g$ of stable complex algebraic curves of even genus g.

A version of the Bergman tau function for the moduli space of holomorphic abelian differentials on algebraic curves [10] allowed to get new relations in the rational Picard group of this space and was applied to the Kontsevich-Zorich theory of Teichmüller flow [13], see also [5]. In [14], the Bergman tau function was used to express the Prym class on the moduli space of holomorphic quadratic differentials in terms of the standard generators. Here we continue with developing these ideas further for the moduli space of holomorphic n-differentials.

3.1. **Bergman tau function on strata of n-differentials.** We begin with defining the Bergman tau function for each stratum $\mathfrak{M}_g^{(n)}[\mathbf{k}]$ where $\mathbf{k} = (k_1, \ldots, k_m)$ is a partition of $N = 2n(g-1)$. We introduce the following notation (see [6] for precise definitions):

- $v_1, \ldots, v_g$ – the normalized basis of holomorphic abelian differentials with respect to a given Torelli marking (or cut system) on C;
- Ω – the corresponding period matrix;
- $\Theta(z, \Omega)$ – the theta function associated with Ω;
- $W(x)$ – the Wronskian determinant of differentials $v_1, \ldots, v_g$;
- $\tilde{C}$ – the fundamental polygon corresponding to the chosen cut system on C;
- $E(x, y)$ – the prime form on $C \times C$;
- $\mathcal{A}_x$ – the Abel map corresponding to the initial point x;
- K^x – the vector of Riemann constants.

The distinguished local parameters on C in a neighborhood of the points x_i (zeroes of the n-differential w) are given by

$$(3.1) \qquad \zeta_i(x) = \left(\int_{x_i}^x v \right)^{n/(k_i+n)}$$

where k_i is the order of x_i (in terms of these parameters $w \sim \zeta_i^{k_i}(d\zeta_i)^n$ near $x_i \in C$, and $v = w^{1/n} \sim \zeta_i^{k_i/n} d\zeta_i$ near $\hat{x}_i = f^{-1}(x_i) \in \widehat{C}$). Then for the prime form $E(x, y)$ on $C \times C$ we have

$$E(x, y) = \frac{E(\zeta(x), \zeta(y))}{\sqrt{d\zeta(x)}\sqrt{d\zeta(y)}},$$

and we put

$$E(\zeta, x_k) = \lim_{y \to x_k} E(\zeta(x), \zeta(y)) \sqrt{\frac{d\zeta_k}{d\zeta}(y)},$$

$$E(x_k, x_l) = \lim_{\substack{x \to x_k \\ y \to x_l}} E(\zeta(x), \zeta(y)) \sqrt{\frac{d\zeta_k}{d\zeta}(x)} \sqrt{\frac{d\zeta_l}{d\zeta}(y)}.$$

We define two vectors $Z, Z' \in \frac{1}{n}\mathbb{Z}^g$ by the condition

$$(3.2) \qquad \frac{1}{n}\mathcal{A}_x((w)) + 2K^x = \Omega Z + Z'.$$

Definition 3.1. The Bergman tau function on the space $\mathfrak{M}_g^{(n)}[\mathbf{k}]$ is given by

$$(3.3) \quad \tau(C, w) =$$

$$c(x)^{2/3} e^{-\frac{\pi}{6}\langle \Omega Z, Z \rangle - \frac{2\pi\sqrt{-1}}{3}\langle Z, K^x \rangle} \left(\frac{w(x)}{\prod_{i=1}^m E^{k_i}(x, x_i)} \right)^{(g-1)/3n} \prod_{i<j} E(x_i, x_j)^{\frac{k_i k_j}{6n^2}},$$

where

$$c(x) = \frac{1}{W(x)} \left(\sum_{i=1}^g v_i(x) \frac{\partial}{\partial z_i} \right)^g \theta(z; \Omega) \Big|_{z=K^x}.$$

Proposition 3.2. *Under the change of Torelli marking on C*

$$\left(\begin{array}{c} \tilde{\boldsymbol{b}} \\ \tilde{\boldsymbol{a}} \end{array} \right) = \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \left(\begin{array}{c} \boldsymbol{b} \\ \boldsymbol{a} \end{array} \right), \quad \left(\begin{array}{cc} A & B \\ C & D \end{array} \right) \in Sp(2g, \mathbb{Z}),$$

the tau function (3.3) transforms as follows:

$$(3.4) \qquad \frac{\tau(C, w, \{\tilde{a}_i, \tilde{b}_i\})}{\tau(C, w, \{a_i, b_i\})} = \epsilon \det(C\Omega + D)$$

where ϵ is a root of unity of degree $48d$ with $d = \text{l.c.m.}(k_1 + n, \ldots, k_m + n)$.

The proof can be obtained by using standard transformation properties of all factors in (3.3) under the change of Torelli cut system on C (cf. [6]). The root of unity appears due to an ambiguity in the definition of the distinguished local parameters (3.1), which translates into an ambiguity in the definition of $E(x, x_i)$ and $E(x_i, x_j)$. The appearance of the term $\det(C\Omega + D)$ can also be seen from variational formulas for $\tau(C, w)$ discussed below, similarly to [10, 13, 14].

Proposition 3.3. *The tau function has the following quasi-homogeneity property:*

$$(3.5) \qquad \tau(C, \delta w) = \delta^\kappa \tau(C, w)$$

with

$$(3.6) \qquad \kappa = \frac{1}{12n^2} \sum_{i=1}^{m} \frac{k_i(k_i + 2n)}{k_i + n}.$$

This proposition follows from the explicit formula (3.3), but can also be derived by applying the Riemann bilinear identity to variational formulas for τ as in was done in [12] in the context of Hurwitz spaces.

Combining Propositions 3.2 and 3.3, we arrive at the following

Theorem 3.4. *On the stratum $\mathfrak{M}_g^{(n)}[\mathbf{1}] \subset \overline{\mathfrak{M}}_g^{(n)}$ of n-differentials with simple zeroes, the power $\tau^{48n(n+1)}$ of the tau function $\tau = \tau(C, w)$ is a nowhere vanishing holomorphic section of the line bundle $\lambda^{48n(n+1)} \otimes L^{-8(g-1)(2n+1)} \longrightarrow \mathfrak{M}_g^{(n)}[\mathbf{1}]$.*

In order to find the divisor of the section $\tau^{48n(n+1)}$ on $\overline{\mathfrak{M}}_g^{(n)}$, we will compute the asymptotics of τ at the boundary divisors D_{deg} and D_j, $j = 0, 1, \ldots, [g/2]$. For that we need to study the tau function more carefully.

3.2. **Homological coordinates and variational formulas for the tau function.** The tau function $\tau(C, w)$ satisfies a system of linear differential equations on the space $\mathfrak{M}_g^{(n)}$ similar to the tau functions on Hurwitz spaces, or spaces of abelian or quadratic differentials [10, 12, 13, 14]. Here we assume that all zeros of w are simple i.e. that all $k_i = 1$ in (3.3).

The homology group $H_1(\widehat{C}, \mathbb{C})$ can be decomposed into the eigenspaces of the automorphism σ_*:

$$H_1(\widehat{C}, \mathbb{C}) = \bigoplus_{i=0}^{n-1} \mathcal{H}_k,$$

where $\dim \mathcal{H}_0 = 2g$ and in the case of simple zeros the dimensions of $\mathcal{H}_k$ for $k = 1, \ldots, n-1$ are independent of k and

$$(3.7) \qquad \dim \mathcal{H}_k = (2n + 2)(g - 1), \quad k = 1, \ldots, n-1$$

The dimensions (3.7) can be computed as the dimensions of the dual spaces $\mathcal{H}^k$ in cohomology of $\widehat{C}$, where $\mathcal{H}^k$ is the subspace of $H^1(\widehat{C}, \mathbb{R})$ corresponding to eigenvalue ρ^k. The space $\mathcal{H}^k$ can be decomposed as $\Omega^{(k)} \oplus \overline{\Omega}^{(n-k)}$ (since $\rho^k = \overline{\rho}^{n-k}$) and, using (1.8), we get $\dim \mathcal{H}^k = (2n + 2)(g - 1)$.

For any two classes $s_1 \in \mathcal{H}_l$ and $s_2 \in \mathcal{H}_k$ we have $s_1 \circ s_2 = 0$ unless $k + l = n$. The spaces $\mathcal{H}_k$ and $\mathcal{H}_{n-k}$ are, therefore, dual to each other with respect to the standard intersection pairing (the space $\mathcal{H}_0$ can be identified with $H_1(C)$, and, therefore, it is self-dual). On the other hand, for any $q \in \Omega^{(k)}$ and $s \in \mathcal{H}_l$ we have $\int_s q = 0$ unless $k = l$. In particular, since $v = w^{1/n} \in \Omega^{(1)}$, it can have non-trivial periods only over the cycles representing homology classes in $\mathcal{H}_1$. Actually, $\mathcal{H}_1$ can be naturally identified with the tangent space to the moduli space $\mathfrak{M}_g^{(n)}$. Choosing a basis $\{s_i\}_{i=1}^{(2n+2)(g-1)}$ in $\mathcal{H}_1$ we introduce *homological coordinates* $\mathcal{P}_i$ on $\mathfrak{M}_g^{(n)}$ by the formula

$$(3.8) \qquad \mathcal{P}_i = \int_{s_i} v$$

(see also Corollary 2.3 of [2]).

Choose a Torelli marking on C and define the associated canonical bimeromorphic differential $B(x,y) = d_x d_y \log E(x,y)$ on C (here $E(x,y)$ is the prime form). The bidifferential B is symmetric with a second order pole with biresidue 1 on the diagonal $x = y$ and vanishing a-periods with respect to both arguments.

The bidifferential $B(x,y)$ has the following local behaviour near the diagonal $x = y$:

$$(3.9) \qquad B(x,y) = \left(\frac{1}{(\zeta(x) - \zeta(y))^2} + \frac{1}{6} S_B(\zeta(x)) + \dots \right) d\zeta(x) d\zeta(y);$$

here $\zeta(x)$ is a local parameter, and $S_B(x)$ is the so-called Bergman projective connection.

To study the tau function on $\mathfrak{M}_g^{(n)}$ we will need some variational formulas for $B(x,y)$. For a basis $\{s_i\}_{i=1}^{(2n+2)(g-1)}$ in $\mathcal{H}_1$ consider the dual basis $\{s_j^*\}_{j=1}^{(2n+2)(g-1)}$ in $\mathcal{H}_{n-1}$, so that $s_j^* \circ s_i = \delta_{ij}$.

Consider a fundamental polygon $\tilde{C}$ of C (that is, dissect C along the cuts representing the Torelli marking). Choose a system Γ of non-intersecting cuts that lie within $\tilde{C}$ and connect the first zero x_1 with other zeros of w. Pick a connected component of $f^{-1}(\tilde{C} \setminus \Gamma) \subset \hat{C}$ and identify it with $\tilde{C} \setminus \Gamma$. On $\tilde{C} \setminus \Gamma$ introduce the coordinate

$$(3.10) \qquad z(x) = \int_{x_1}^x v,$$

where the path connecting x with x_1 entirely lies in $\tilde{C} \setminus \Gamma$.

Theorem 3.5. *The following variational formula holds for* $i = 1, \dots, (2n+2)(g-1)$:

$$(3.11) \qquad \frac{\partial}{\partial \mathcal{P}_i} B(z(x), z(y)) = \frac{1}{2\pi\sqrt{-1}n} \int_{s_i^*} \frac{B(z(x), \cdot) B(\cdot, z(y))}{v}$$

Formula (3.11) can be derived from the variational formulas for the stratum $\mathcal{H}_{\hat{g}}(n, \dots, n)$ of the moduli space of holomorphic 1-differentials of genus $\hat{g} = n^2(g-1) + 1$, see Theorem 3 of [10], in a way similar to Lemma 5 of [14] and Proposition 3.2 of [3].

Consider the differential operator $S_v = \frac{v''}{v} - \frac{3}{2}\left(\frac{v'}{v}\right)^2$ (that is, the Schwarzian derivative of the abelian integral $\int^x v$ with respect to the coordinate z on C). For the holomorphic 1-differential v, S_v is a meromorphic projective connection on C, so that the difference $S_B - S_v$ is a meromorphic quadratic differential.

The tau function $\tau = \tau(C, w)$ satisfies the following system of differential equations with respect to the homological coordinates $\mathcal{P}_i$ on $P\mathfrak{M}_g^{(n)}$:

$$(3.12) \qquad \frac{d}{d\mathcal{P}_i} \log \tau = -\frac{1}{12\pi\sqrt{-1}n} \int_{s_i^*} \frac{S_B - S_v}{v}$$

(notice that the differential $(S_B - S_v)/v$ gets multiplied by ρ^{-1} under the action of f^*; thus its integral over $s_i^* \in \mathcal{H}_{n-1}$ is non-trivial). The compatibility of the system (3.12) follows from the symmetry of the expression

$$\frac{\partial}{\partial \mathcal{P}_j} \int_{s_i^*} \frac{S_B - S_v}{v} = \frac{1}{12\pi\sqrt{-1}n} \int_{s_i^*} \int_{s_j^*} \frac{B(z(x), z(y))B(z(y), z(x))}{v(z(x))v(z(y))}$$

under the interchange of i and j.

3.3. Asymptotics of tau function near the boundary of $\overline{\mathfrak{M}}_g^{(n)}$.

Here we compute the asymptotics of τ near the boundary $\overline{\mathfrak{M}}_g^{(n)} \setminus \mathfrak{M}_g^{(n)}[\mathbf{1}]$ of $\overline{\mathfrak{M}}_g^{(n)}$ that consist of the following divisors:

- D_{deg}, the divisor of n-differentials with multiple zeroes,
- D_0, the divisor of n-differentials on irreducible nodal curves, and
- D_j, $j = 1, \ldots, [g/2]$, the divisors of n-differentials on reducible nodal curves.

3.3.1. *Coalescing simple zeros of w: divisor D_{deg}.*

Lemma 3.6. *Let x_1 and x_2 be two zeros of w coalescing at D_{deg}. Then a transversal local coordinate on $\overline{\mathfrak{M}}_g^{(n)}$ in a tubular neighbourhood of D_{deg} is given by*

$$(3.13) \qquad t_{\text{deg}} = \left(\int_{x_1}^{x_2} v \right)^{2n/(n+2)} .$$

Proof. It is parallel to the proof of Lemma 8 of [14]. Denote by ζ a local coordinate in a small disk U containing the coalescing zeros $x_{1,2}$ and no other zeros. Then we can write in U

$$(3.14) \qquad w(\zeta) = (\zeta - \zeta(x_1))(\zeta - \zeta(x_2))(d\zeta)^n,$$

so that

$$\int_{x_1}^{x_2} v = \int_{\zeta(x_1)}^{\zeta(x_2)} ((\zeta - \zeta(x_1))(\zeta - \zeta(x_2)))^{1/n} \, d\zeta = \text{const} \cdot (\zeta(x_1) - \zeta(x_2))^{(n+2)/n} .$$

and the parameter t_{deg} defined by (3.13) looks like Since $(\zeta(x_1) - \zeta(x_2))^2$ is independent of labeling of zeroes, t_{deg} is a coordinate transversal to D_{deg}. $\square$

Lemma 3.7. *The tau function $\tau(C, w)$ has the following asymptotics near D_{deg}:*

$$(3.15) \qquad \tau(C, w) = t_{\text{deg}}^{\frac{1}{12n(n+1)}} \tau(C_0, w_0)(1 + o(1))$$

where $(C_0, w_0) \in D_{\text{deg}}$.

Proof. The asymptotics (3.15) can be derived by computing the asymptotics of all factors in the explicit formula (3.3). Alternatively, using the system of equations (3.12) we see that in the limit $t_{\text{deg}} \to 0$ the tau function $\tau(C, w)$ behaves like $t_{\text{deg}}^p \tau(\mathbb{C}_0, w_0)$ for some power p, where w_0 is a differential with one zero of order two and all other zeroes simple. To find p explicitly, we look at the transformation properties of τ, τ_0 and t_{deg} under the rescaling $w \mapsto \delta w$. The homogeneity degrees κ of τ and κ_0 of τ_0 are given by the formula (3.6), so that

$$\kappa - \kappa_0 = \frac{1}{6n(n+1)(n+2)}$$

On the other hand, the local parameter t_{deg} has homogeneity degree $2/(n+2)$, which gives $p = \frac{1}{12n(n+1)}$.   □

3.3.2. *Asymptotics of τ near D_0.* Take two loops a and b on C intersecting transversally at one point; their homology class we will also denote by $a, b \in H_1(C, \mathbb{Z})$. Let us pinch a to a point, then C degenerates to a nodal curve C_0 that we represent by a smooth curve of genus $g-1$ with two points (say, x_0 and y_0) identified (we assume that all zeros of w remain far from the node). The holomorphic n-differential w on C degenerates to a meromorphic n-differential w_0 on C_0 with poles of degree n at x_0 and y_0 such that the corresponding n-residues differ by $(-1)^n$.

The canonical covering $\widehat{C}$ of C degenerates to a nodal curve $\widehat{C}_0$ with n nodes that can be thought of as n pairs of points $x_0^{(m)} = \sigma^m(x_0)$ and $y_0^{(m)} = \sigma^m(y_0)$, $m = 0, \dots, n-1$, on the normalization of $\widehat{C}_0$ that are pairwise identified (here σ is the covering automorphism of $\widehat{C}_0$). The differential v on $\widehat{C}$ degenerates to a meromorphic differential v_0 on $\widehat{C}_0$ with simple poles at the preimages of nodal points with residues at $x_0^{(m)}$ and $y_0^{(m)}$ that differ by a sign.

Choose one of n simple loops on in the preimage $f^{-1}(a) \subset \widehat{C}$. Let us assume that this loop pinches to the first node on $\widehat{C}_0$.

Consider the classes $\alpha, \beta \in \mathcal{H}_1$ given by

$$(3.16) \qquad \alpha = \sum_{m=0}^{n-1} \rho^{-m} \sigma_*^{-m} a , \quad \beta = \sum_{m=0}^{n-1} \rho^{-m} \sigma_*^{-m} b$$

(where σ is the covering automorphism of $\widehat{C}$), and introduce the homological coordinates $\mathcal{P}_\alpha = \int_\alpha v$ and $\mathcal{P}_\beta = \int_\beta v$ associated with α and β.

A local coordinate on $\mathfrak{M}_g^{(n)}$ transversal to D_0 in a tubular neighborhood can be chosen as

$$t_0 = e^{2\pi i \mathcal{P}_\beta / \mathcal{P}_\alpha}$$

(notice that $\operatorname{Im}(\mathcal{P}_\beta / \mathcal{P}_\alpha) > 0$ near D_0).

We may assume that $\mathcal{P}_\alpha$ remains constant under the degeneration of C to C_0. Let $\omega_{x,y}$ be the abelian differential of the 3rd kind on $\widehat{C}_0$ with simple poles at points x and y of residues $+1$ and -1 respectively, normalized with respect to some canonical basis (a_i, b_i) in $H_1(\widehat{C}_0, \mathbb{Z})$. Since $v_0 \in \mathcal{H}^1$, it can be written as

$$v_0 = \frac{\mathcal{P}_\alpha}{2\pi\sqrt{-1}} \sum_{j=0}^{n-1} \rho^m \omega_{\sigma^m(x_0), \sigma^m(y_0)} + \text{holomorphic terms} .$$

In the limit $t_0 \to 0$ the bidifferential $B(x,y)$ on $C \times C$ tends to the meromorphic bidifferential $B_0(x,y)$ on $C_0 \times C_0$ with the same properties.

To find the asymptotics of the tau fuction τ as $t_0 \to 0$ (i.e. $\mathcal{P}_\beta \to \infty$), consider the equation

$$(3.17) \qquad \frac{\partial \log \tau}{\partial \mathcal{P}_\beta} = -\frac{1}{12\pi\sqrt{-1}n} \int_{\beta^*} \frac{S_B - S_v}{v} \xrightarrow{t_0 \to 0} -\frac{1}{6n} \operatorname{res}_{x_0} \frac{S_{B_0} - S_{v_0}}{v_0} ,$$

where $\beta^* = \frac{1}{n} \sum_{m=0}^{n-1} \rho^m \sigma_*^m a \in \mathcal{H}_{n-1}$ is the class dual to $\beta \in \mathcal{H}_1$.

To compute the residue, choose a local coordinate ζ near x_0 on C_0 such that $S_{B_0} = 0$. Then we have $v_0 = \frac{\mathcal{P}_\alpha}{2\pi\sqrt{-1}n} \frac{d\zeta}{\zeta} + O(1)$ and

$$\left\{ \int v_0, \zeta \right\} = \left(\frac{v_0'}{v_0} \right)' - \frac{1}{2} \left(\frac{v_0'}{v_0} \right)^2 = \frac{1}{2\zeta^2} + O(1)$$

as $t_0 \to 0$, so that $S_{v_0}/v_0 = \frac{\pi\sqrt{-1}n}{\mathcal{P}_\alpha}\frac{d\zeta}{\zeta} + O(1)$. Therefore, (3.17) implies

$$(3.18) \qquad \left.\frac{\partial \log \tau}{\partial \mathcal{P}_\beta}\right|_{t_0=0} = \frac{\pi\sqrt{-1}}{6\mathcal{P}_\alpha}$$

and $\tau \sim e^{\pi\sqrt{-1}\mathcal{P}_\beta/6\mathcal{P}_\alpha}$, i.e.

$$(3.19) \qquad \tau = t_0^{1/12}(\text{const} + o(1))$$

as $t \to 0$.

3.3.3. *Asymptotics of τ near D_j.* Contracting a homologically trivial simple loop γ on C we get a reducible nodal curve C_0 that splits into two irreducible components C_1 and C_2 of genera $g_1 = j$ and $g_2 = g - j$ respectively, $j = 1, \ldots, [g/2]$. Denote by $x_0 \in C_1$ and $y_0 \in C_2$ the intersection point of C_1 and C_2 (the node of C_0). The holomorphic n-differential w on C degenerates to a pair of meromorphic n-differentials w_1 and w_2 on C_1 and C_2 respectively, with poles of order n at $x_0 \in C_1$ and $y_0 \in C_2$ whose n-residues differ by $(-1)^n$ (we assume that under generation the zeroes of w stay away from the vanishing cycle γ).

Denote by $f_i : \widehat{C}_i \to C_i$ the canonical n-fold covering ($i = 1, 2$), and let $x_0^{(1)}, \ldots, x_0^{(n-1)}$ (resp. $y_0^{(1)}, \ldots, y_0^{(n-1)}$) denote the preimages of the node in $\widehat{C}_1$ (resp. $\widehat{C}_2$) that are cyclically ordered relative to the covering maps $\sigma_i : \widehat{C}_i \to \widehat{C}_i$. The canonical cover $\widehat{C}_0$ of the nodal curve C_0 is obtained from $\widehat{C}_1$ and $\widehat{C}_2$ by identifying $x_0^{(m)}$ with $y_0^{(m)}$ for each $m = 0, \ldots, n-1$.

Define the 1-form v_i on $\widehat{C}_i$ by $v_i^n = f_i^* w_i$, ($i = 1, 2$). The (meromorphic) 1-forms v_1 and v_2 have first order poles at the n preimages of the node whose residues differ by a sign, i. e. $\text{res}|_{x_0^{(m)}} v_1 = -\text{res}|_{y_0^{(m)}} v_2$. Moreover, applying m times the covering map σ_0, we see that $\text{res}|_{x_0^{(m)}} v_1 = \rho^{-m}\text{res}|_{x_0^{(0)}} v_1$ (resp. $\text{res}|_{y_0^{(m)}} v_2 = \rho^{-m}\text{res}|_{y_0^{(0)}} v_2$).

The preimage $f^{-1}(\gamma) \subset \widehat{C}$ of the loop γ on C is the disjoint union of n loops γ_m, $m = 0, \ldots, n-1$ (we enumerate them in such a way that $\gamma_{m+1} = \sigma(\gamma_m)$, assuming that $\gamma_n = \gamma_0$). Note that the union of γ_m, $m = 0, \ldots, n-1$, is homologically trivial on $\widehat{C}$. Consider also a simple loop η_0 on $\widehat{C}$ such that $\gamma_0 \circ \eta_0 = 1$, $\gamma_1 \circ \eta_0 = -1$, and $\gamma_k \circ \eta_0 = 0$ for $k = 2, \ldots, n-1$, where $\circ$ denotes the intersection pairing of 1-cycles on $\widehat{C}$.

Introduce the loops $\eta_m = \sigma^m(\eta_0)$, $m = 1, \ldots, n-1$, and consider the cycles

$$(3.20) \qquad \alpha = \sum_{m=1}^{n-1}(\rho^{-m} - 1)\gamma_m , \quad \beta = \frac{1}{\rho - 1}\sum_{m=1}^{n-1}(1 - \rho^{-m})\eta_m$$

(for homology classes in $H_1(\widehat{C}, \mathbb{Z})$ represented by γ_m and η_m we use the same notation); clearly, $\alpha, \beta \in \mathcal{H}_1$. The class $\beta^* \in \mathcal{H}_{n-1}$ dual to β is given by

$$(3.21) \qquad \tilde{\alpha} = \frac{1}{n}\sum_{m=1}^{n-1}(\rho^m - 1)\gamma_m .$$

Introduce the following homological coordinates:

$$(3.22) \qquad \mathcal{P}_\alpha = \int_\alpha v = n\int_{\gamma_0} v , \quad \mathcal{P}_\beta = \int_\beta v = \frac{n}{1 - \rho}\int_{\eta_0} v .$$

Without loss of generality we may assume that while C degenerates to C_0 all homological coordinates except $\mathcal{P}_\beta$ remain finite.

Lemma 3.8. *A local parameter tranversal to $D_j \subset \overline{\mathfrak{M}}_g^{(n)}$ can be chosen as*

$$(3.23) \qquad t_j = e^{2\pi\sqrt{-1}\mathcal{P}_\beta/\mathcal{P}_\alpha} .$$

Proof. We can realize the loops γ_m and η_m by simple closed geodesics in hyperbolic metric on $\widehat{C}$. Denote by T_i the maximal hyperbolic cylinder with waist γ_i (the collar of γ_i). Put $\eta_0^{(i)} = \eta_0 \cap T_i$, $i = 0, 1$. Then $\frac{1}{n}\mathcal{P}_\beta = \frac{1}{1-\rho}\int_{\eta_0} v \sim \int_{\eta_0^{(0)}} v$ when C approaches C_0 is the "complex heght" of the cylinder T_0 while $\frac{1}{n}\mathcal{P}_\alpha = \int_{\gamma_0} v$ is its "complex waist". Therefore, (3.23) gives a local coordinate transversal to D_j. $\square$

To find the asymptotics of τ when $t_j \to 0$, consider

$$(3.24) \qquad \frac{\partial \log \tau}{\partial \mathcal{P}_\beta} = -\frac{1}{12\pi\sqrt{-1}n} \int_{\beta^*} \frac{S_B - S_v}{v}$$
$$= -\frac{1}{12\pi\sqrt{-1}n} \int_{\gamma_0} \frac{S_B - S_v}{v} \longrightarrow -\frac{1}{6n}\mathrm{res}\big|_{x_0^{(0)}} \frac{S_{B_0} - S_{v_0}}{v_0} \ .$$

Pick a coordinate ζ near $x_0^{(0)}$ such that $S_{B_0} = 0$, then $v_0 = \frac{\mathcal{P}_\alpha}{2\pi\sqrt{-1}n}\frac{d\zeta}{\zeta} + O(1)$ and $\frac{S_{v_0}}{v_0} = \frac{\pi\sqrt{-1}n}{\mathcal{P}_\alpha}\frac{d\zeta}{\zeta} + O(1)$ as $t_j \to 0$. Therefore, (3.24) implies

$$\frac{\partial \log \tau}{\partial \mathcal{P}_\beta}\Big|_{t_j=0} = \frac{\pi\sqrt{-1}}{6\mathcal{P}_\alpha} \ .$$

Thus, $\tau \sim e^{\pi\sqrt{-1}\mathcal{P}_\beta/6\mathcal{P}_\alpha}$ as $t_j \to 0$, and

$$(3.25) \qquad \tau = t_j^{1/12}(const + o(1)) \ .$$

3.4. **Hodge class on $P\overline{\mathfrak{M}}_g^{(n)}$.** A straightforward combination of Theorem 3.4 with asymptotic formulas (3.15), (3.19) and (3.25) yields

Theorem 3.9. (Formula (1.3) of Theorem 1.12) *The Hodge class λ on $\overline{\mathfrak{M}}_g^{(n)}$ is a linear combination of the tautological class ψ and the classes of boundary divisors D_{deg} and D_j, $j = 0, 1, \ldots, [g/2]$, as follows:*

$$(3.26) \qquad \lambda = \frac{(g-1)(2n+1)}{6n(n+1)}\psi + \frac{1}{12n(n+1)}\delta_{\mathrm{deg}} + \frac{1}{12}\sum_{j=0}^{[g/2]}\delta_j \ .$$

4. An alternative computation of δ_{deg}

An alternative proof of Theorem 3.9 was given in [21] using an approach proposed by D. Zvonkine [22] and developed in [18].

Let $g, n \geq 2$. In order to compute the class δ_{deg} in $\mathrm{Pic}(P\overline{\mathfrak{M}}_g^{(n)})$ we begin with marking one point on C, *i.e.* we study the space $P\overline{\mathfrak{M}}_{g,1}^{(n)}$. In $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ we define the loci

$$Z_i = \{(C, x_1, w) \mid C \text{ is smooth and } x_1 \text{ is a zero of } w \text{ of order at least } i\}.$$

We denote by $\overline{Z}_i$ the closure of Z_i. This is a closed substack of $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ of pure codimension i for $1 \leq i < N = (2g-2)n$, while for $i = N$ it has two components of codimensions $N-1$ and N respectively, cf Section 1.2.

Let $\pi : P\overline{\mathfrak{M}}_{g,1}^{(n)} \to P\overline{\mathfrak{M}}_g^{(n)}$ be the forgetful map of the marked point. Then it is easy to see that the image of $\overline{Z}_2$ under π is the divisor D_{deg}. This statement takes into account the multiplicities of the components of D_{deg}. Indeed, if $(g, n) \neq (2, 2)$ then the restriction of π to Z_2 is of degree 1 and if $(g, n) = (2, 2)$ then π is of degree one onto $P\overline{\mathfrak{M}}_g^{(n)}(2, 1, 1)$ and two onto $P\overline{\mathfrak{M}}_g^{(n)}(2, 2)$. Therefore $\pi_*[\overline{Z}_2] = \delta_{\mathrm{deg}}$. Thus to find an expression of δ_{deg} it suffices to compute the class $[\overline{Z}_2] \in A^2(P\overline{\mathfrak{M}}_{g,1}^{(n)})$.

Computation of $[\overline{Z}_2]$. Let $\mathcal{L}_1 \to \overline{\mathcal{M}}_{g,1}$ be the line bundle whose fiber is the cotangent line to the curve C at x_1, and put $\psi_1 = c_1(\mathcal{L}_1)$. Consider the following line bundle over $P\overline{\mathfrak{M}}_{g,1}^{(n)}$:

$$\mathcal{O}(1) \otimes \mathcal{L}_1^{\otimes n} \simeq \mathrm{Hom}\left(L, \mathcal{L}_1^{\otimes n}\right).$$

This line bundle has a natural section

$$s_1 : (C, w) \mapsto w(x_1).$$

In other words, s_1 is the evaluation of w at the marked point. The class of the vanishing locus of s_1 is given by the first Chern class of the line bundle:

$$\{s_1 = 0\} \;=\; c_1\left(\mathcal{O}(1) \otimes \mathcal{L}_1^{\otimes n}\right) \;=\; -\psi + n\psi_1.$$

It is easy to see that $\overline{Z}_1$ is a component of the vanishing locus $\{s_1 = 0\}$. In the next section we will show that the vanishing locus has no other components and that the vanishing order of s_1 along $\overline{Z}_1$ is equal to 1. Thus we have $[\overline{Z}_1] = -\psi + n\psi_1$.

Now we restrict to $\overline{Z}_1$ and study the line bundle $\mathcal{O}(1) \otimes \mathcal{L}_1^{\otimes n+1}$. This line bundle has a natural section

$$s_2 : (C, w) \mapsto w'(x_1).$$

In other words, assuming that w vanishes at x_1, the section s_2 assigns to w its derivative at x_1. It is easy to see that $\overline{Z}_2$ is a component of the vanishing locus $\{s_2 = 0\}$. In the next section we will show that the vanishing locus has no other components and that the vanishing order of s_2 along $\overline{Z}_2$ is equal to 1. Thus we have

$$[\overline{Z}_2] = (-\psi + (n+1)\psi_1)[\overline{Z}_1] = (-\psi + n\psi_1)(-\psi + (n+1)\psi_1).$$

Recall that δ_{deg} is the push-forward by π of this expression. To compute this push-forward we use

- ψ is a pull-back under π;
- $\pi_* \psi_1 = 2g - 2$;
- $\pi_* \psi_1^2 = \kappa_1 = 12\lambda_1 - \sum_{i=1}^{[g/2]} \delta_i$.

The last equality is the well-known Mumford's formula.

Applying these equalities we get

$$\begin{aligned}
\delta_{\mathrm{deg}} &= \pi_*\left((-\psi + n\psi_1)(-\psi + (n+1)\psi_1)\right) \\
&= n(n+1)\pi_*(\psi_1^2) - (2n+1)\pi_*(\psi_1)\psi \\
&= 12n(n+1)\lambda_1 - n(n+1)\sum_{i=1}^{[g/2]} \delta_i - (2n+1)(2g-2)\psi.
\end{aligned}$$

This coincides with the expression of Theorem 3.9.

In order to complete the proof of Theorem 3.9, it remains to prove that the vanishing locus of s_1 (respectively s_2) is exactly $\overline{Z}_1$ (respectively $\overline{Z}_2$) and that the vanishing order of s_1 and s_2 is 1.

Vanishing loci of s_1 and s_2. Let W be an irreducible divisor of $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ in the vanishing locus of s_1. Let k be the number of nodes of the curve represented by a generic point of W. The vanishing locus of s_1 is of codimension 1 in $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ thus $k = 0$ or 1. We investigate both cases.

- Let $k = 0$. Then a dense subset of W is contained in Z_1 and thus W is a component of $\overline{Z}_1$.

- Let $k = 1$. Then the divisor W is contained in $\tilde{\nu}^{-1}(D)$ for some irreducible boundary divisor D of the moduli space of stable curves with one marked point (we recall that $\tilde{\nu} : P\overline{\mathfrak{M}}_{g,1}^{(n)} \to \overline{\mathcal{M}}_{g,1}$ is the forgetful map). Since the D is irreducible, and $\tilde{\nu}$ is the projectivization of a vector bundle, we necessarily have $W = \tilde{\nu}^{-1}(D)$. However there exists an n-differential in $\tilde{\nu}^{-1}(D)$ which is not identically zero on the component of marked point. Therefore, there exists a point in $\tilde{\nu}^{-1}(D)$ which is not in the vanishing locus of s_1. Thus the case $k = 1$ does not occur.

To study the vanishing locus of s_2 we follow the same strategy. First we can check by dimension count that no irreducible component of Z_1 is in the zero locus of s_2. Now let W be an irreducible divisor in the vanishing locus of s_2 and let k be the number of nodes of the curve represented by a generic point of W. We have now 3 cases to study: $k = 0, 1$ and 2.

- Let $k = 0$. Then a dense subset of W is contained in Z_2 and thus W is a component of $\overline{Z}_2$.
- Let $k = 2$. Then $W = \tilde{\nu}^{-1}(D)$ where D is a codimension 2 boundary stratum of $\overline{\mathcal{M}}_{g,1}$. As above $\tilde{\nu}^{-1}(D)$ is not contained in the vanishing locus of s_2. Thus the case $k = 2$ cannot occur.
- Let $k = 1$. Then W is a co-dimension 1 locus inside $\tilde{\nu}^{-1}(D)$ for a boundary divisor D of $\overline{\mathcal{M}}_{g,1}$. The generic curve has two components of genera g' and $g - g'$ with $1 \le g' \le g - 1$. We assume that the marked point is carried by the component of genus g'. The rank of the bundle of n-differentials with a pole of order at most n at the node is $n(2g' - 2 + 1) > 1$. Thus the divisor D is not contained in the locus of differentials that vanish identically on this component. Thus D is not contained in the vanishing locus of s_2.

We conclude that $\{s_i = 0\} = \overline{Z}_i$ for $i = 1$ and 2.

Vanishing order of s_1. Let $y_0 = (C_0, w_0, x_0)$ be a point in Z_1. We recall that $P\overline{\mathfrak{M}}_{g,1}^{(n)} \to P\overline{\mathfrak{M}}_g^{(n)}$ is isomorphic the universal curve. Thus a neighborhood of y_0 in Z_1 is given by $U \times \Delta$ where U is a neighborhood of (C_0, w_0) in $P\mathfrak{M}_g^{(n)}[\mathbf{1}]$ and $\zeta \in \Delta$ is the distinguished parameter around x_0 in C_0 (cf Section 2.1). Let (C, w, x) be an n-differential in $U \times \Delta$. In coordinates $(u, \zeta) \in U \times \Delta$, the differential w is given by $w = \zeta d\zeta^n$, the locus Z_1 is $\{\zeta = 0\}$ and the section s_1 is given by $s_1(u, \zeta) = \zeta$. Therefore the vanishing order of s_1 along Z_1 is 1.

Vanishing order of s_2. Let $y_0 = (C_0, w_0, x_0)$ be a point in Z_2. A neighborhood of y_0 in $P\overline{\mathfrak{M}}_{g,1}^{(n)}$ is now given by $U \times \Delta \times \Delta'$ where U is a neighborhood of y_0 in Z_2 and Δ and Δ' are disks of the complex plane centered at 0 and parametrized by ζ and a such that:

$$w = (\zeta^2 + a)d\zeta^n.$$

With the parameters $(u, \zeta, a) \in U \times \Delta \times \Delta'$, the locus Z_1 is defined by $\zeta^2 + a = 0$. Moreover with these parameters, the section s_2 is given by $s_2(u, \zeta, a) = a$. Thus the vanishing order of s_2 along Z_2 is again 1.

5. Prym-Tyurin differentials on $\widehat{C}$ and holomorphic n-differentials on C

Here we relate Prym-Tyurin vector bundles to vector bundles of holomorphic k-differentials on the base Riemann surface C. We use this relation to finish the proof of Theorem 1.12.

We also prove that the Prym-Tyurin bundle is not a pullback from $P\overline{\mathfrak{M}}_g^{(n)}$ in general.

5.1. **Prym-Tyurin bundles and n-differentials.** Consider two vector bundles $\Lambda^{(k)}$ and $\tilde{\nu}^*\Omega_g^{(n-k+1)}$ over $X(g,n)$. The fiber of $\Lambda^{(k)}$ is the kth eigenspace in the space of abelian differentials on $\widehat{C}$. The fiber of Ω_{n-k+1} is the space of $(n-k+1)$-differentials on C. There are natural morphisms:

(5.1)
$$\begin{array}{rcll} \Phi_0: & \Lambda^{(0)} & \to & \tilde{\nu}^*\Omega_g\,, \\ \Phi_k: & \Lambda^{(k)} \otimes T^{\otimes(n-k)} & \to & \tilde{\nu}^*\Omega_g^{(n-k+1)} \quad \text{for } 1 \le k \le n. \end{array}$$

Indeed, let (C,w) be a point in $U \setminus D_{\deg}$ and let q be a differential in the fiber of $\Lambda^{(k)}$. The $n-k+1$ differential qv^{n-k} is invariant under the action of the automorphism group of the covering, thus qv^{n-k} is the pull-back of $n-k+1$ differential on C. For $k=0$ the differential q is already invariant under the action of the automorphism group of the covering, so there is no need to multiply it by a power of v.

Lemma 5.1. *The map Φ_k is an isomorphism over $V \setminus D_{\deg}$.*

Proof. The maps Φ_k for $0 \le k \le n-1$ are injective because v does not vanish identically on any component of the nodal curve $\widehat{C}$. They are bijective because the sum of ranks of the Prym-Tyurin bundles $\Lambda^{(k)}$ for $0 \le k \le n-1$ is equal to the sum of ranks of the vector bundles Ω_{n-k+1}. Indeed,

$$\sum_{k=0}^{n-1} \operatorname{rk}\Lambda^{(k)} = \operatorname{rk}\Omega_{\widehat{g}} = n^2(g-1)+1$$

and

$$\operatorname{rk}\Omega_g + \sum_{k=1}^{n-1} \operatorname{rk}\Omega_g^{(n-k+1)} = g + \sum_{k=1}^{n-1}(2n-2k+1)(g-1) = n^2(g-1)+1.$$

$\square$

Corollary 5.2. *The rank of the Prym-Tyurin bundle is g for $k=0$ and $(2n-2k+1)(g-1)$ for $1 \le k \le n-1$.*

Corollary 5.3. *In $\operatorname{Pic}(P\overline{\mathfrak{M}}_g^{(n)} \setminus D_{\deg})$ we have*

(5.2)
$$\lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi$$

for $1 \le k \le n-1$.

Proof. On the locus where Φ_k is an isomorphism we have

$$\begin{aligned} \lambda_{PT}^{(k)} &= c_1(\Lambda^{(k)}) \\ &= c_1(\tilde{\nu}^*\Omega_k \otimes T^{\otimes-(n-k)}) \\ &= \lambda_{n-k+1} - (g-1)(n-k)(2n-2k+1)c_1(T) \\ &= \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi, \end{aligned}$$

where the last equality is due to $T^{\otimes n} = L$.

The locus where Φ_k is an isomorphism coincides with $P\overline{\mathfrak{M}}_g^{(n)} \setminus D_{\deg}$ up to codimension 2 substacks that are immaterial for the first Chern class computations. $\square$

To study the extension of the formula (5.2) to $P\overline{\mathfrak{M}}_g^{(n)}$ we need to study the behavior of Φ_k along the boundary divisor $D_{\deg}$. The determinant of Φ_k is a global section of

$$\det(\Lambda^{(k)} \otimes T^{n-k})^{-1} \otimes \det(\tilde{\nu}^*\Omega_g^{(n-k+1)}).$$

Thus the difference between $\lambda_{PT}^{(k)}$ and $\lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi$ is an effective divisor defined as the vanishing locus of $\det\Phi_k$.

In Section 2.2, we have described a parametrization of the cyclic coverings along $D_{\deg}$. We use here this parametrization to prove the following Lemma.

Lemma 5.4. *If $[(n-1)/2]+1 \leq k \leq n-1$ or $k = 0$, then the morphism Φ_k is an isomorphism of vector bundles over $V \subset X(g,n)$. Otherwise, $\det(\Phi_k)$ vanishes along $D_{\deg}$ with order $(1 - \frac{2k}{n})$.*

This lemma implies the following

Corollary 5.5. *The following relations between Prym-Tyurin class $\lambda_{PT}^{(k)}$, the class $\psi = c_1(L)$ and the pullback of class λ_{n-k+1} from $\overline{\mathcal{M}}_g$ to $P\overline{\mathfrak{M}}_g^{(n)}$ holds:*

$$(5.3) \qquad \lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi + \left(\frac{1}{2} - \frac{k}{n}\right)\delta_{\deg},$$
$$1 \leq k \leq [(n-1)/2],$$

$$(5.4) \qquad \lambda_{PT}^{(k)} = \lambda_{n-k+1} - \frac{g-1}{n}(n-k)(2n-2k+1)\psi,$$
$$[(n-1)/2]+1 \leq k \leq n-1.$$

This corollary together with Theorem 3.9 and Formula (1.6) completes the proof of Theorem 1.12.

Remark 5.6. Note the difference of a factor 2 between the vanishing order of $\det \Phi_k$ and the contribution of $\delta_{\deg}$ in $\lambda_{PT}^{(k)}$. This is due to the fact that $V \to U$ is of degree $1/2$ along $D_{\deg}$.

The proof of Lemma 5.4 will occupy the two following sections. We consider separately the cases of even n and odd n.

5.2. **Odd $n = 2m+1$.**

5.2.1. *Kernel and cokernel of Φ_k.* Let (C_0, w_0) be a point in $D_{\deg}$ and $f : \widehat{C}_0 \to C_0$ be the associated admissible covering. We recall that $\widehat{C}_0$ is a curve with two components intersecting in one point. The two components $\widehat{C}_1$ and $\widehat{C}_2$ are of genera $\widehat{g}_1 = m$ and $\widehat{g}_2 = \widehat{g} - m$ (see Section 2.1). We have denoted by w_1 the meromorphic n-differential on C_1 given by $(\zeta - \zeta_1)(\zeta - \zeta_2)d\zeta^n$. The curve $\widehat{C}_1$ is parametrized by $y^n = (\zeta - \zeta_1)(\zeta - \zeta_2)$ and the nth root of $f^* w_1$ is given by $v_1 = yd\zeta$. The covering $\widehat{C}_2 \to C_2$ is defined by $v_2^{\otimes n} = w_0$. Finally, let v be the canonical differential satisfying $v^{\otimes n} = f^* w_0$: it is determined by $v = v_2$ on the component $\widehat{C}_2$ and vanishes identically on the component $\widehat{C}_1$.

Denote the fiber of kth Prym-Tyurin vector bundle $\Lambda^{(k)}$ over the point $(C_0, w_0) \in D_{\deg}$ by $\Omega_0^{(k)}$. We can decompose

$$\Omega_0^{(k)} = \Omega_1^{(k)} \oplus \Omega_2^{(k)}$$

where $\Omega_1^{(k)}$ is the space of holomorphic differentials on $\widehat{C}_1$ which get multiplied by ρ^k under the action of the automorphism $(y, \zeta) \to (\rho^k y, \zeta)$; the space $\Omega_2^{(k)}$ is the analogous space of holomorphic differentials on $\widehat{C}_2$.

The canonical differential v vanishes identically on $\widehat{C}_1$. Thus the kernel of Φ_k contains space $\Omega_1^{(k)} \otimes T^{\otimes n-k}$. On another hand, the restriction of the morphism Φ_k to the linear subspace $\Omega_2^{(k)} \otimes T^{\otimes n-k}$ is injective. Indeed the differential v does not vanish identically on $\widehat{C}_2$. Thus

$$\ker \Phi_k \simeq \Lambda_1^{(k)} \otimes T^{(n-k)}.$$

over a generic locus of $D_{\deg}$.

We have $\dim \Omega_1^{(k)} = 1$ for $k = 1, \ldots, m$. These one-dimensional spaces are generated by the holomorphic differentials on $\widehat{C}_1$ given by $q_1^{(k)} = \frac{d\zeta}{y^{2m+1-k}}$. For $k = m+1 \ldots 2m+1$ the eigenspaces $\Omega_1^{(k)}$ are trivial. Therefore,

$$\dim \Omega_2^{(k)} = \dim \Omega_0^{(k)} - 1 \,, \qquad k = 1, \ldots, m \,;$$

$$\dim \Omega_2^{(k)} = \dim \Omega_0^{(k)} \,, \qquad k = m+1, \ldots, 2m.$$

We can also describe the images of Φ_k. For $k = m+1, \ldots, 2m$, the morphism Φ_k is an isomorphism from $\Omega_2^{(k)} \otimes T^{\otimes n-k}$ to $H^0(C_0, \omega_{C_0}^{n+k-1})$. However, for $k = 1, \ldots, m$ the image of $\Omega_2^{(k)} \otimes T^{\otimes n-k}$ is the space of holomorphic $n-k+1$ differentials vanishing at x_0.

We have proved the following

Lemma 5.7. *The kernel of Φ_k over D_{deg} is the vector bundle $\Omega_1^{(k)} \otimes T^{\otimes n-k}$. This kernel is trivial for $k > m$ and $k = 0$. If $1 \leq k \leq m$, the image of Φ_k is the vector bundle whose fibers are the $H^0(C, \omega^{n+k-1}(-x_0))$ where x_0 is the unique zero of order 2.*

Therefore the first part of Lemma 5.4 and Formula (5.4) are valid for odd n.

5.2.2. *Study of Φ_k for $k = 1 \ldots m$.* We fix $1 \leq k \leq m$. We have seen that the kernel and cokernel of Φ_k are of dimension 1. Let (C_0, w_0) be a generic point in $D_{\deg}$. Let W be an open neighborhood of (C_0, w_0) in $X(g, n)$ with a non-vanishing section $q_0^{(k)}$ of $\Lambda^{(k)}|_W$ such that $q_0^{(k)}|_{D_{\deg}}$ spans $\ker \Phi_k|_{W \cap D_{\deg}}$. The section Φ_k is of co-rank 1 along $D_{\deg}$, thus the vanishing order of $\det \Phi_k$ is equal to the vanishing order of $\Phi_k(q_0^{(k)} \otimes v^{\otimes n-k})$ along $D_{\deg}$. Therefore, we will construct such a local section $q_0^{(k)}$ of $\Lambda^{(k)}$ and study the asymptotic behavior of $q_0^{(k)} \otimes v^{\otimes n-k}$ along $D_{\deg}$.

Let $\tilde{q}_0^{(k)}$ be a non-vanishing section of $\tilde{\nu}^*(\Omega^{(n-k+1)})$ over W such that: for all $(C, w) \in D_{\deg}$, $q_0^{(k)}(C, w)$ is a differential that does not vanish at the double zero of w. Up to a choice of a smaller W, such a section exists. We label the coalescing zeros by x_1 and x_2. We chose the parameter of the curve ζ such that position of x_1 and x_2 are ζ_1 and ζ_2 and $w = (\zeta - \zeta_1)(\zeta - \zeta_2)(d\zeta)^n$ (see Section 2.2). We define

$$q_0^{(k)} = (\zeta_1 - \zeta_2)^{-1+2k/n} \cdot \frac{f^*(\tilde{q}_0^{(k)})}{v^k}.$$

We recall that root $(\zeta_1 - \zeta_2)^{2/n}$ is well defined, it is the integral of v between $\hat{x}_1$ and $\hat{x}_2$ (see Lemma 3.6).

Over $W \setminus D_{\deg}$, the differential $q_0^{(k)}$ is obviously a non-vanishing section of $\hat{\nu}^*(\Lambda^{(k)})$. Beside, along $D_{\deg}$ the differential $q_0^{(k)}$ vanishes identically on $\widehat{C}_2$ because of the factor $(\zeta_1 - \zeta_2)^{-1+2k/n}$. To compute the limit of $q_0^{(k)}$ on $\widehat{C}_1$, we remark that $q_0^{(k)}$ is equivalent to

(5.5) $$(\zeta_1 - \zeta_2)^{1-2k/n}[(\zeta(x) - \zeta_1)(\zeta(x) - \zeta_2)]^{k/n-1} d\zeta \,.$$

in coordinate ζ. The differential (5.5) is invariant under simultaneous rescaling $\zeta \to \epsilon\zeta$, $\zeta_i \to \epsilon\zeta_i$, $i = 1, 2$. Therefore, as $x_{1,2} \to x_0$, the differential $q_0^{(k)}$ tends to the holomorphic differential

$$q_1^{(k)} = y^{-n+k} dx = (\zeta_1 - \zeta_2)^{1-2k/n}[(\zeta(x) - \zeta_1)(\zeta(x) - \zeta_2)]^{k/n-1} d\zeta$$

on the curve $\widehat{C}_1$ (the generator of $\Omega_1^{(k)}$).

The image of $q_0^{(k)} \otimes v^{\otimes n-k}$ under Φ_k is $(\zeta_1 - \zeta_2)^{1-2k/n} \tilde{q}_0^{(k)}$ by construction. Thus the determinant of Φ_k is equivalent to a constant times $(\zeta_1 - \zeta_2)^{1-2k/n}$. The

parameter $(\zeta_1 - \zeta_2)^2$ being a transverse parameter to $D_{\deg}$, we conclude that the vanishing order of Φ_k along $D_{\deg}$ is $1 - \frac{2k}{n}$. This finishes the proof of Lemma 5.4 for odd n. $\square$

5.3. **Even** $n = 2m$.
The proof of Lemma 5.4 is essentially identical to the odd case. Let (C_0, w_0) be a generic point in $D_{\deg}$. Now we have $\widehat{g}_1 = m - 1$ and $\widehat{g}_2 = \widehat{g} - m$ and the two components intersect in two points. Let Ω_0^k be the PT bundle over (C_0, w_0) and let $\Omega_i^{(k)}$ be the subspace of Ω_0^k of holomorphic differentials supported on $\widehat{C}_i$. We still have the decomposition:

$$\Omega_0^{(k)} = \Omega_1^{(k)} \oplus \Omega_2^{(k)}$$

except for $k = m$. The kernel Φ_k is $\Omega_1^{(k)} \otimes T^{\otimes n-k}$. This kernel is trivial for $k = m+1, \ldots, 2m-1$. For $k = 1, \ldots, m-1$, the map Φ_k has co-rank one and the generator of $\Omega_1^{(k)}$ is the differential $q_1^{(k)} = y^{k-2m} d\zeta$ on the canonical covering $\widehat{C}_1$ $y^{2m} = (\zeta - \zeta_1)(\zeta - \zeta_2)$ (see Section 2.2 for definition of the parameters).

For $k = m$, the space $\Omega_0^{(k)}$ contains a 1-dimension subspace of differentials of a third kind. These are differentials with simple poles at the nodal points $x_0^{(1)}$ and $x_0^{(2)}$ (both on $\widehat{C}_1$ and $\widehat{C}_2$) and opposite residues. The image of such differential under Φ_m is a holomorphic m-differential. Beside $\Omega_1^{(m)}$ is trivial. Therefore the morphism Φ_k is also bijective for $k = m$.

Similarly to the case of odd n, for $k = 1, \ldots, m-1$, let $\tilde{q}_0^{(k)}$ be a section of $\tilde{\nu}^*(\Omega^{(n-k+1)})$ over a neighborhood of (C_0, w_0) that does not vanish at the double zero of w_0. We define $q_0^{(k)} = (\zeta_1 - \zeta_2)^{-1+2k/n} \cdot \frac{f^*(\tilde{q}_0^{(k)})}{v^k}$. and we study the asymptotic behavior of $\Phi_k(q_0^{(k)} \otimes v^{n-k})$ along $D_{\deg}$.

As in the odd case, the differential $q_0^{(k)}$ is a non vanishing section of $\Lambda^{(k)}$ and $\Phi_k(q_0^{(k)} \otimes v^{n-k}) = (\zeta_1 - \zeta_2)^{-1+2k/n} \cdot \tilde{q}_0^{(k)}$ by construction. Therefore the vanishing order of $\det(\Phi_k)$ is given by $1 - \frac{2k}{n}$ for $k = 1, \ldots, m-1$. $\square$

5.4. **Obstruction to the extension of the Prym bundles to** $\overline{\mathfrak{M}}_g^{(n)}$**.**
In order to define the Prym-Tyurin classes we have constructed the space of admissible differentials $X(g, n)$ (see 1.5). Indeed, the Prym-Tyurin bundles are naturally defined over $X(g, n)$. The following theorem explains the necessity of the introduction of the space $X(g, n)$.

Theorem 5.8. *Let $g > 2$ and $k > 0$. There exists no vector bundle $\tilde{\Lambda} \to P\overline{\mathfrak{M}}_g^{(n)}$ such that $\Lambda^{(k)} = \mathrm{diff}^* \tilde{\Lambda}^{(k)}$, where $\mathrm{diff} : X(g, n) \to P\overline{\mathfrak{M}}_g^{(n)}$ is the forgetful map.*

Proof. Suppose that there exists $\tilde{\Lambda} \to P\overline{\mathfrak{M}}_g^{(n)}$ such that $\Lambda^{(k)} = \mathrm{diff}^* \tilde{\Lambda}^{(k)}$. Then in particular $\lambda_{PT}^k = c_1(\tilde{\Lambda}^{(k)})$ and thus $c_1(\Lambda^k) = \mathrm{diff}^* \lambda_{PT}^k$. We will prove that this equality does not hold for $g > 2$ and $k > 0$.

Let $m > 2$ and μ be the partition of $n(2g - 2)$ given by $(m, 1, \ldots, 1)$. We denote by $\mathfrak{M}_g^{(n)}[m]$ the locus $\mathfrak{M}_g^{(n)}[\mu] \subset \overline{\mathfrak{M}}_g^{(n)}$. We suppose that $\gcd(m, n) = 1$. We denote by D_m the preimage of $\mathfrak{M}_g^{(n)}[m]$ in $X(g, n)$ under diff. The locus D_m is a divisor whose generic points are elements $(C, w, x_i, f : \widehat{C} \to C) \in X(g, n)$ such that:

- the curve C is a nodal curve with two components: C_2 isomorphic to C with $n(2g-2) - m$ marked points attached in one node to a rational component C_1 at the zero of order m;
- the n-differential w is identically zero on C_1 and has profile μ on C_1;

- the covering curve $\widehat{C} \to C$ has two component $\widehat{C}_2$ and $\widehat{C}_1$. The component $\widehat{C}_2$ determined by the w as in Section 2.2 and $\widehat{C}_1 \to C_1$ is the unique n-sheeted ramified covering maximally ramified at the marked points and the node.

Moreover the canonical root v of f^*w vanishes identically on $\widehat{C}_1$ and has a zero of order $m + n - 1$ at the preimage of the zero of order m. Thus the morphism $\Phi_k : \Lambda^{(k)} \otimes T^{\otimes(n-k)} \to \tilde{\nu}^* \Omega_g^{(n-k+1)}$ has a non-empty co-kernel along D_m for m large enough. Therefore the line bundle

$$\det(\Lambda^{(k)}) \otimes \mathrm{diff}^* \left(\det \tilde{\Lambda}^{(k)\vee} \right)$$

has a global section which vanishes along divisors contained in $X(g, n) \setminus V$. Thus $\Lambda^{(k)} \neq \mathrm{diff}^* \tilde{\Lambda}^{(k)}$. $\qquad \square$

References

[1] Babelon O., Bernard D., Talon M., *Introduction to classical integrable systems*, Cambridge University Press (2004).

[2] Bainbridge, M., Chen, D., Gendron, Q., Grushevsky, S., Moeller, M., *Strata of k-differentials*, arXiv:1610.09238 (2016).

[3] Bertola, M., Korotkin, D., Norton, C., *Symplectic geometry of the moduli space of projective structures in homological coordinates*, arXiv:1506.07918 (2015).

[4] Dubrovin, B. *Geometry of 2D topological field theories*, in: "Integrable systems and quantum groups", Proceedings, Montecatini Terme 1993, Lecture Notes in Math. 1620, Springer, 120-348 (1996).

[5] Eskin, A., Kontsevich, M., Zorich, A., *Sum of Lyapunov exponents of the Hodge bundle with respect to the Teichmuller geodesic flow*, Publ. Math. Inst. Hautes Études Sci. 120, 207-333 (2014).

[6] Fay, J., *Kernel functions, analytic torsion, and moduli spaces,* Memoirs of the AMS, Vol. 96, No. 464 (1992).

[7] Harris, J., Mumford, D., *On the Kodaira dimension of the moduli space of curves*, Invent. Math. 67, 23-86 (1982).

[8] van der Geer, G., Kouvidakis, A., *The Hodge bundle on Hurwitz spaces*, Pure Appl. Math. Quart. 7, 1297-1308 (2011).

[9] van der Geer, G., Kouvidakis, A., *The class of a Hurwitz divisor on the moduli of curves of even genus*, Asian J. Math. 16, No. 4, 787-806 (2012).

[10] Kokotov, A., Korotkin, D., *Tau-functions on spaces of Abelian differentials and higher genus generalization of Ray-Singer formula*, J. Diff. Geom. 82, 35-100 (2009).

[11] Kokotov, A., Korotkin, D., Isomonodromic tau function of Hurwitz Frobenius manifolds and its applications, IMRN, 2006, 1-34 (2006).

[12] Kokotov, A., Korotkin, D., Zograf, P., *Isomonodromic tau function on the space of admissible covers*, Adv. Math., 227, No.1, 586-600 (2011).

[13] Korotkin, D., Zograf, P., *Tau function and moduli of differentials*, Math. Res. Lett. 18, No.3 (2011).

[14] Korotkin, D., Zograf, P., *Tau function and the Prym class*, Contemporary Mathematics, 593, 241–261 (2013).

[15] Lanneau, E., *Connected components of the strata of the moduli spaces of quadratic differentials*, Ann. Sci. Éc. Norm. Supér. (4), 41(1), 1–56 (2008).

[16] Mumford, D., *Stability of projective varieties*, L'Ens. Math., 23, 39-110 (1977).

[17] Prym, F. *Ableitung. Einer allgemeinen Thetaformel,* Acta Math. 3, no. 1, 216–239 (1883).

[18] Sauvaget, A., *Cohomology classes of strata of abelian differentials*, arXiv:1701.07867 (2017).

[19] Schmitt, J., *Dimension theory of the moduli space of twisted k-differentials*, arXiv:1607.08429 (2016).

[20] Tyurin A.N., *Vector bundles. Collected works* Vol. I, ed. by F.Bogomolov, A.Gorodentsev, V.Pidstrigach, M.Reid and N.Tyurin. Universitätsverlag Göttingen, Göttingen, 330 pp. (2008).

[21] Zograf, P., *Moduli of differentials and Prym-Tyurin classes*, unpublished (2013).

[22] Zvonkine, D., *Strata in the projectivization of the Hodge bundle*, unpublished (2013).

Department of Mathematics and Statistics, Concordia University, 1455 de Maisonneuve W., Montreal, Québec, Canada H3G 1M8
  E-mail address: `korotkin@mathstat.concordia.ca`

IMJ-PRG, Université Pierre et Marie Curie, 4 place Jussieu, 75005 Paris, France
  E-mail address: `adrien.sauvaget@imj-prg.fr`

Steklov Mathematical Institute, Fontanka 27, Saint Petersburg 191023 Russia, and Chebyshev Laboratory, Saint Petersburg State University, 14-th Line V.O. 29, Saint Petersburg 199178 Russia
  E-mail address: `zograf@pdmi.ras.ru`