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TAUBERIAN CONDITIONS FOR L^1-CONVERGENCE OF FOURIER SERIES

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Dedicated to Jovan Karamata

ABSTRACT. It is proved that Fourier series with asymptotically even coefficients and satisfying $\lim_{\lambda\to 1}\limsup_{n\to\infty}\sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta\hat{f}(j)|^p = 0$, for some $1 < p \leqslant 2$, converge in L^1-norm if and only if $\|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| = o(1)$, where $E_n(t) = \sum_{k=0}^{n} e^{ikt}$. Recent results of Stanojević [1], Bojanic and Stanojević [2], and Goldberg and Stanojević [3] are special cases of some corollaries to the main theorem.

1. Introduction. The space $L^1(T)$ of complex functions integrable on $T = \mathbf{R}/2\pi Z$ does not admit convergence in norm. Consequently, convergence in norm of the partial sums $S_n(f) = S_n(f,t) = \sum_{|j|\leqslant n}\hat{f}(j)e^{ijt}$ to $f \in L^1(T)$ cannot be characterized in terms of Fourier coefficients without additional assumptions about the sequence $\{\hat{f}(n)\}$.

In the case of even coefficients ($\hat{f}(n) = \hat{f}(-n)$ for all integers n) satisfying certain regularity and/or speed conditions, it is well known that

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty, \tag{1.1}$$

is equivalent with

$$\hat{f}(n)\lg n = o(1), \qquad n \to \infty. \tag{1.2}$$

(A survey of classical and recent results of this kind can be found in **[1, 2** and **3]**.)

Most recent results concerning the equivalence between (1.1) and (1.2) are due to Stanojević **[1]**, Bojanic and Stanojević **[2]** and Goldberg and Stanojević **[3]**.

In **[1]** it is proved that if $\{\hat{f}(n)\}$ is even and satisfies

$$\frac{1}{n}\sum_{k=1}^{n} k|\Delta\hat{f}(k)| = o(1), \qquad n \to \infty, \tag{1.3}$$

and

$$n\Delta\hat{f}(n) = O(1), \qquad n \to \infty, \tag{1.4}$$

then (1.1) is equivalent with (1.2).

Goldberg and Stanojević **[3]** proved that if

$$\{(\hat{f}(n) - \hat{f}(-n))\lg n\} \quad \text{is a null-sequence of bounded variation,} \tag{1.5}$$

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and if for some $1 < p \leqslant 2$

$$\frac{1}{n}\sum_{j=n}^{2n} j^p|\Delta\hat{f}(j)|^p = o(1), \qquad n \to \infty, \tag{1.6}$$

then (1.1) if and only if (1.2). An earlier result of Bojanic and Stanojević [2] is a corollary to the Goldberg-Stanojević theorem.

In this paper I shall extend and generalize the Goldberg-Stanojević theorem in two ways. Instead of (1.5), a weaker condition will be assumed, i.e.,

$$\text{(AE)} \quad \begin{aligned} &\frac{1}{n}\sum_{j=1}^{n} |\hat{f}(j) - \hat{f}(-j)|\lg j = o(1), \qquad n \to \infty, \\ &\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))|\lg j = 0, \end{aligned}$$

and (1.6) will be relaxed as follows:

$$\text{(HK)} \qquad \lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta\hat{f}(j)|^p = 0,$$

for some $1 < p \leqslant 2$.

A sequence of complex numbers satisfying (AE) is called *asymptotically even*. Clearly, every even sequence satisfies (AE).

The condition (HK) is a Tauberian condition of Hardy-Karamata [4] kind. Plainly (1.6) implies (HK).

As a consequence of the main theorem it will follow that the condition (1.3) is superfluous, and that (1.4) can be weakened if a certain speed of $\|\sigma_n(f) - f\|$ is assumed, where $\sigma_n(f)$ is the Fefér sum of $S_n(f)$.

2. Main theorem. Fourier series considered throughout this section are the series with asymptotically even coefficients. That is

$$\frac{1}{n}\sum_{j=1}^{n} |\hat{f}(j) - \hat{f}(-j)| \lg j = o(1), \qquad n \to \infty, \tag{2.1.1}$$

$$\lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))|\lg j = 0. \tag{2.1.2}$$

MAIN THEOREM. *Let $S[f] \sim \sum_{|n|<\infty} \hat{f}(n)e^{int}$ be the Fourier series of $f \in L^1(T)$ with asymptotically even coefficients.*

If for some $1 < p \leqslant 2$

$$\text{(HK)} \qquad \lim_{\lambda \to 1} \limsup_{n \to \infty} \sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta\hat{f}(j)|^p = 0,$$

then $\|S_n(f) - f\| = o(1)$, $n \to \infty$, if and only if

$$\|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| = o(1), \qquad n \to \infty, \tag{2.2}$$

where $E_n(t) = \sum_{k=0}^{n} e^{ikt}$.

PROOF. It suffices to show that

$$\limsup_{n\to\infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| = 0. \tag{2.3}$$

Let $\lambda > 1$ and $n > 1$. Then the following identity can be established:

$$\begin{aligned} S_n(f,t) &- f(t) - \left(\hat{f}(n)E_n(t) + \hat{f}(-n)E_{-n}(t)\right) \\ &= \frac{[\lambda n]+1}{[\lambda n]-n}\left[\sigma_{[\lambda n]}(f,t) - f(t)\right] - \frac{n+1}{[\lambda n]-n}\left[\sigma_n(f,t) - f(t)\right] \\ &\quad - \frac{1}{[\lambda n]-n}\sum_{j=n-1}^{[\lambda n]} \hat{f}(j)E_j(t) - \frac{1}{[\lambda n]-n}\sum_{j=n-1}^{[\lambda n]} \hat{f}(-j)E_{-j}(t) \\ &\quad - \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n]+1-j}{[\lambda n]-n}\left[\Delta\hat{f}(j)\right]E_j(t) \\ &\quad - \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n]+1-j}{[\lambda n]-n}\left[\Delta\hat{f}(-j)\right]E_{-j}(t). \end{aligned} \tag{2.4}$$

The Dirichlet kernel can be written as

$$D_j(t) = E_j(t) + E_{-j}(t) - 1.$$

Thus the third and the fourth terms on the right-hand side of (2.4) can be grouped in the following way:

$$\begin{aligned} I_{1n} &= \frac{1}{[\lambda n]-n}\sum_{j=n}^{[\lambda n]} \hat{f}(j)D_j(t) \\ &\quad - \frac{1}{[\lambda n]-n}\sum_{j=n}^{[\lambda n]} \left(\hat{f}(j) - \hat{f}(-j)\right)E_{-j}(t) + \frac{1}{[\lambda n]-n}\sum_{j=n}^{[\lambda n]} \hat{f}(j); \end{aligned} \tag{2.5}$$

and the fifth and the sixth terms as

$$\begin{aligned} I_{2n} &= \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n]+1-j}{[\lambda n] \neq}\left[\Delta\hat{f}(j)\right]D_j(t) \\ &\quad - \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n]+1-j}{[\lambda n]-n}\left[\Delta\left(\hat{f}(j) - \hat{f}(-j)\right)\right]E_{-j}(t) \\ &\quad + \sum_{j=n}^{[\lambda n]-1} \frac{[\lambda n]+1-j}{[\lambda n]-n}\Delta\hat{f}(j). \end{aligned} \tag{2.6}$$

Taking the norm of both sides of (2.5) we obtain

$$
\begin{aligned}
\|I_{1n}\| \leqslant{} & \frac{1}{[\lambda n]-n}\left\|\sum_{j=n}^{[\lambda n]} \hat{f}(j) D_j(t)\right\| \\
& + \frac{1}{[\lambda n]-n} \sum_{j=n}^{[\lambda n]} |\hat{f}(j)-\hat{f}(-j)| \lg j \\
& + \frac{1}{[\lambda n]-n} \sum_{j=n}^{[\lambda n]} |\hat{f}(j)| \qquad (2.7) \\
={} & J_n + \frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)-\hat{f}(-j)| \lg j\right) \\
& + \frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|\right).
\end{aligned}
$$

Applying first the Hölder inequality and then the Hausdorff-Young equality to J_n, we have

$$
J_n \leqslant A_p \frac{[\lambda n]^{1/q}}{[\lambda n]-n}\left(\sum_{j=n}^{[\lambda n]} |\hat{f}(j)|^p\right)^{1/p} = A_p \frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p\right)^{1/p},
$$

where A_p is an absolute constant depending on p, and $1/p + 1/q = 1$. The last term in (2.7) is majorized by

$$
\frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p\right)^{1/p}.
$$

Hence

$$
\begin{aligned}
\|I_{1n}\| \leqslant{} & C_1 \frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)|^p\right)^{1/p} \\
& + C_2 \frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]} \sum_{j=1}^{[\lambda n]} |\hat{f}(j)-\hat{f}(-j)| \lg j\right),
\end{aligned}
$$

where C_1 and C_2 are absolute constants.

In a similar manner we obtain

$$
\|I_{2n}\| \leqslant C_3 \sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j)-\hat{f}(-j))| \lg j + C_4\left(\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p\right)^{1/p},
$$

where C_3 and C_4 are absolute constants.

Combining estimates for both $\|I_{1n}\|$ and $\|I_{2n}\|$ we get

$$\begin{aligned}\big|\,\|S_n(f)-f\|-\|\hat{f}(n)E_n+\hat{f}(-n)E_{-n}\|\,\big| &\leqslant \frac{[\lambda n]}{[\lambda n]-n}\|\sigma_{[\lambda n]}(f)-f\| \\ &+\frac{n+1}{[\lambda n]-n}\|\sigma_n(f)-f\|+C_1\frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]}|\hat{f}(j)|^p\right)^{1/p} \\ &+C_2\frac{[\lambda n]}{[\lambda n]-n}\left(\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]}|\hat{f}(j)-\hat{f}(-j)|\lg j\right) \\ &+C_3\sum_{j=1}^{[\lambda n]}|\Delta(\hat{f}(j)-\hat{f}(-j))|\lg j+C_4\left(\sum_{j=n}^{[\lambda n]}j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p}.\end{aligned} \tag{2.8}$$

Since for $\lambda > 1$ we have $\lambda n/([\lambda n]-n)\sim\lambda/(\lambda-1)$, $n\to\infty$, it follows that

$$\limsup_{n\to\infty}\frac{[\lambda n]}{[\lambda n]-n}C_n=0,$$

for any null-sequence $\{C_n\}$.

After taking the limit superior of both sides of (2.8) we get

$$\begin{aligned}&\limsup_{n\to\infty}\big|\,\|S_n(f)-f\|-\|\hat{f}(n)E_n+\hat{f}(-n)E_{-n}\|\,\big| \\ &\qquad\leqslant C_3\limsup_{n\to\infty}\sum_{j=n}^{[\lambda n]}|\Delta(\hat{f}(j)-\hat{f}(-j))|\lg j \\ &\qquad+C_4\limsup_{n\to\infty}\left(\sum_{j=n}^{[\lambda n]-1}j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p}.\end{aligned} \tag{2.9}$$

For $\|\sigma_n(f)-f\|=o(1)$, $n\to\infty$, $\hat{f}(n)=o(1)$, $n\to\infty$, and

$$\frac{1}{[\lambda n]}\sum_{j=1}^{[\lambda n]}|\hat{f}(j)-\hat{f}(-j)|\lg j=o(1),\qquad n\to\infty,$$

because of (2.1.1).

Taking the limit as $\lambda\to1$ of both sides of (2.9) we obtain

$$\begin{aligned}&\lim_{\lambda\to1}\limsup_{n\to\infty}\big|\,\|S_n(f)-f\|-\|\hat{f}(n)E_n+\hat{f}(-n)E_{-n}\|\,\big| \\ &\qquad\leqslant C_3\lim_{\lambda\to1}\limsup_{n\to\infty}\sum_{j=n}^{[\lambda n]}|\Delta(\hat{f}(j)-\hat{f}(-j))|\lg j \\ &\qquad+C_4\lim_{\lambda\to1}\limsup_{n\to\infty}\left(\sum_{j=n}^{[\lambda n]}j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p}.\end{aligned}$$

Because of (2.1.2) and (HK) we finally have (2.3), i.e.

$$\limsup_{n\to\infty} \left| \|S_n(f) - f\| - \|\hat{f}(n)E_n + \hat{f}(-n)E_{-n}\| \right| = 0.$$

This completes the proof of the main theorem.

3. Corollaries and additional results. By strengthening either (2.1.2) or (HK), or both, one can obtain a number of corollaries that, as a special case, contain the results of Stanojević [**1**], Bojanic and Stanojević [**2**], and Goldberg and Stanojević [**3**].

The class of complex null-sequences $\{c_n\}$ satisfying

$$\frac{1}{n}\sum_{k=1}^{n} k|\Delta c_k| = o(1), \qquad n \to \infty, \tag{3.1}$$

includes as a proper subclass null-sequences of bounded variation.

COROLLARY 3.1. *Let* $S[f] \sim \Sigma_{|n|<\infty} \hat{f}(n)e^{int}$ *be the Fourier series of* $f \in L^1(T)$, *and let* $\{(\hat{f}(n) - \hat{f}(-n))\lg n\}$ *satisfy* (3.1). *If* (HK) *holds then*

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty,$$

if and only if

$$\hat{f}(n)\lg n = o(1), \qquad n \to \infty.$$

PROOF. The condition (2.1.1) is satisfied. It remains to show that (2.1.2) holds. Since for $\lambda > 1$

$$\sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))|\lg j \leqslant \frac{1}{n}\sum_{j=n}^{[\lambda n]} j|\Delta(\hat{f}(j) - \hat{f}(-j))|\lg j$$

$$\leqslant \frac{\lambda}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j|\Delta(\hat{f}(j) - \hat{f}(-j))|\lg j$$

$$\leqslant \frac{\lambda}{[\lambda n]}\sum_{j=1}^{[\lambda n]} j|\Delta[(\hat{f}(j) - \hat{f}(-j))\lg j]| + \frac{\lambda}{[\lambda n]}\sum_{j=1}^{[\lambda n]} |\hat{f}(j) - \hat{f}(-j)|\lg(1 + 1/j)^{-j},$$

it follows that if $\{(\hat{f}(n) - \hat{f}(-n))\lg n\}$ satisfies (3.1) then (2.1.2) holds.

A special case of Corollary 3.1 is the Goldberg-Stanojević theorem. Indeed, let $1 < \lambda \leqslant 2$. Then

$$\sum_{j=n}^{[\lambda n]} j^{p-1}|\Delta\hat{f}(j)|^p \leqslant \frac{2}{n}\sum_{j=n}^{2n} j^p|\Delta\hat{f}(j)|^p.$$

COROLLARY 3.2. *Let* $S[f] \sim \Sigma_{|n|<\infty} \hat{f}(n)e^{int}$ *be the Fourier series of* $f \in L^1(T)$, *and let* (2.1.1) *hold. If* (HK) *holds and if*

$$n[\Delta(\hat{f}(n) - \hat{f}(-n))]\lg n = O(1), \qquad n \to \infty, \tag{3.2}$$

then (1.1) *is equivalent with* (1.2).

PROOF. Due to (3.2) we have $\sum_{j=n}^{[\lambda n]} |\Delta(\hat{f}(j) - \hat{f}(-j))| \lg j \leqslant C \lg \lambda$, where C is an absolute constant. Hence $\{\hat{f}(n)\}$ is an asymptotically even sequence.

Since (3.2) is a summability condition in the sense of Hardy [**6**], from (2.1.1) it follows that

$$(f(n) - \hat{f}(-n)) \lg n = o(1), \qquad n \to \infty. \tag{3.3}$$

But (3.3) implies that (2.2) is equivalent with (1.2), for $\| D_n \| = (4/\pi^2) \lg n + O(1)$, $n \to \infty$.

COROLLARY 3.3. *Let* $S[f] \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}$ *be the Fourier series of* $f \in L^1(T)$ *with even coefficients. If* (HK) *holds then* (1.1) *is equivalent with* (1.2).

A special case of this corollary is the main theorem of Bojanic and Stanojević [**2**].

PROOF. Every even sequence is asymptotically even.

COROLLARY 3.4. *Let* $S[f] \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}$ *be the Fourier series of* $f \in L^1(T)$ *with even coefficients. If* $n \Delta \hat{f}(n) = O(1)$, $n \to \infty$, *then*

$$\|S_n(f) - f\| = o(1), \qquad n \to \infty$$

if and only if

$$\hat{f}(n) \lg n = o(1), \qquad n \to \infty.$$

PROOF. The condition $n \Delta \hat{f}(n) = O(1)$, $n \to \infty$ implies (HK), for

$$\sum_{j=n}^{[\lambda n]} j^{p-1} |\Delta \hat{f}(j)|^p \leqslant C \lg \lambda,$$

where C is an absolute constant.

A special case of Corollary 3.4 is the Stanojević theorem.

In what follows it will be assumed that, for simplicity's sake, $\{\hat{f}(n)\}$ are even sequences.

All classical conditions as well as (HK) imply that

$$n^\alpha \Delta \hat{f}(n) = o(1), \qquad n \to \infty, \text{ for some } 0 < \alpha < 1. \tag{3.4}$$

It seems unlikely that (3.4) would imply that (1.1) $\Leftrightarrow$ (1.2). But a slightly stronger form of (3.4) such as

$$n [\lg n]^{-1/p} \max_{n \leqslant j \leqslant n + [n/\lg n]} |\Delta \hat{f}(j)| = o(1), \qquad n \to \infty, \tag{3.5}$$

for some $1 < p \leqslant 2$ and $1/p + 1/q = 1$, and certain conditions on the speed with which $\| \sigma_n(f) - f \|$ goes to zero as $n \to \infty$ could imply that (1.1) $\Leftrightarrow$ (1.2).

PROPOSITION 3.1. *Let* $S[f] \sim \sum_{|n|<\infty} \hat{f}(n) e^{int}$ *be the Fourier series of* $f \in L^1(T)$ *with even coefficients. If, for some* $1 < p \leqslant 2$, $1/p + 1/q = 1$, (3.5) *holds and*

$$\begin{aligned} &\lg n \|\sigma_n(f) - f\| = o(1), \qquad n \to \infty, \\ &\lg n \left(\frac{1}{n} \sum_{j=n}^{n+[n/\lg n]} |\hat{f}(j)|^p \right)^{1/p} = o(1), \qquad n \to \infty, \end{aligned} \tag{3.6}$$

then (1.1) *if and only if* (1.2).

PROOF. Let $m > n > 1$. Then using the same technique as in the proof of the main theorem one can obtain the inequality

$$
\begin{aligned}
\big|\,\|S_n(f) - f\| - |\hat{f}(n)|\,\|D_n\|\,\big| \leqslant{} & \frac{m+1}{m-n}\|\sigma_m(f) - f\| + \frac{n+1}{m-n}\|\sigma_n(f) - f\| \\
& + C_1\left(\sum_{j=n}^{m} j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p}\left(\frac{m}{n}\right)^{1/q} \qquad (3.7)\\
& + C_2\left(\frac{1}{m-n}\sum_{j=n}^{m}|\hat{f}(j)|^p\right)^{1/p}\left(\frac{m}{m-n}\right)^{1/q},
\end{aligned}
$$

where C_1 and C_2 are absolute constants.

Let $m = n + [n/\lg n]$. Then (3.7) becomes

$$
\begin{aligned}
\big|\,\|S_n(f) - f\| - |\hat{f}(n)|\,\|D_n\|\,\big| \leqslant{} & B_1\left[\lg\left(n + \left[\frac{n}{\lg n}\right]\right)\right]\|\sigma_{n+[n/\lg n]}(f) - f\| \\
& + B_2[\lg n]\|\sigma_n(f) - f\| + B_3\left(\sum_{j=n}^{n+[n/\lg n]} j^{p-1}|\Delta\hat{f}(j)|^p\right)^{1/p} \\
& + B_4\left(\frac{1}{n}\sum_{j=n}^{n+[n/\lg n]}|\hat{f}(j)|^p\right)^{1/p}\lg n
\end{aligned}
$$

where $B_1, \ldots, B_4$ are absolute constants.

Due to (3.5) and (3.6), for sufficiently large n we have

$$
\big|\,\|S_n(f) - f\| - |\hat{f}(n)|\lg n\,\big| = O\left(n[\lg n]^{-1/p}\max_{n \leqslant j \leqslant n+[n/\lg n]}|\Delta\hat{f}(j)|\right) + o(1).
$$

This completes the proof of Proposition 3.1.

If instead of $\{[n/\lg n]\}$ we take a sequence of integers $\{[n/L(n)]\}$ where $L(n)$ is a slowly varying function in the sense of Karamata [**5**], such that $L(n + [n/L(n)]) \geqslant L(n)$ for all n greater than some n_0, we can obtain a generalization of Proposition 3.1.

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