[23] W. B. Johnson, *Factoring compact operators*, Israel J. Math. 9 (1971), 337–345.
[24] H. Kamowitz and S. Sheinberg, *Derivations and automorphisms of* $L^1(0,1)$, Trans. Amer. Math. Soc. 135 (1969), 415–427.
[25] D. Lamb, *The second dual of certain Beurling algebras*, preprint.
[26] T. W. Palmer, *Banach Algebras and the General Theory of *-Algebras, Vol. I, Algebras and Banach Algebras*, Cambridge Univ. Press, 1994.
[27] G. Pisier, *Factorization of Linear Operators and Geometry of Banach Spaces*, CBMS Regional Conf. Ser. in Math. 60, Amer. Math. Soc., 1986.
[28] J. Pym, *Remarks on the second duals of Banach algebras*, J. Nigerian Math. Soc. 2 (1983), 31–33.
[29] C. J. Read, *Discontinuous derivations on the algebra of bounded operators on a Banach space*, J. London Math. Soc. (2) 40 (1989), 305–326.
[30] S. Sakai, *C^*-Algebras and W^*-Algebras*, Springer, New York, 1971.
[31] Yu. V. Selivanov, *Biprojective Banach algebras*, Izv. Akad. Nauk SSSR 43 (1979), 1159–1174 (in Russian); English transl.: Math. USSR-Izv. 15 (1980), 381–399.
[32] M. V. Sheĭnberg, *A characterization of the algebra $C(\Omega)$ in terms of cohomology groups*, Uspekhi Mat. Nauk 32 (5) (1977), 203–204 (in Russian).
[33] D. R. Sherbert, *The structure of ideals and point derivations in Banach algebras of Lipschitz functions*, Trans. Amer. Math. Soc. 111 (1964), 240–212.
[34] S. Watanabe, *A Banach algebra which is an ideal in the second dual algebra*, Sci. Rep. Niigata Univ. Ser. A 11 (1974), 95–101.
[35] M. Wodzicki, *Vanishing of cyclic homology of stable C^*-algebras*, C. R. Acad. Sci. Paris Sér. I Math. 307 (1988), 329–334.
[36] N. J. Young, *The irregularity of multiplication in group algebras*, Quart. J. Math. Oxford Ser. (2) 24 (1973), 59–62.

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Received February 3, 1997
Revised version August 18, 1997
(3834)

STUDIA MATHEMATICA 128 (1) (1998)

Tauberian theorems for vector-valued Fourier and Laplace transforms

by

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Abstract. Let X be a Banach space and $f \in L^1_{\mathrm{loc}}(\mathbb{R}; X)$ be absolutely regular (i.e. integrable when divided by some polynomial). If the distributional Fourier transform of f is locally integrable then f converges to 0 at infinity in some sense to be made precise. From this result we deduce some Tauberian theorems for Fourier and Laplace transforms, which can be improved if the underlying Banach space has the analytic Radon–Nikodym property.

0. Introduction. In the last decade Tauberian theorems for vector-valued Laplace transforms attained much attention because of the intimate relation with the asymptotic behavior of Cauchy problems in Banach spaces ([1]–[6], [20], [21]). A typical Tauberian theorem, essentially Ingham's theorem, says the following: Let $f : \mathbb{R}_+ \to X$ be uniformly continuous (where X is a Banach space) and assume that the Laplace transform has a continuous extension to $\overline{\mathbb{C}}_+$. Then $\lim_{t\to\infty} f(t) = 0$ (cf. [4, Thm. 3.5]).

The proof in [4] uses a tricky contour argument from Korevaar [18], which has been exploited in most of the cited papers. In the first section of this paper we present a new approach to Ingham's theorem via Fourier transforms. Our proofs are not more difficult, and moreover, this approach allows us to relax the Tauberian hypothesis considerably and to go beyond the most recent results even for asymptotically almost periodic functions. To give an example, under suitable ergodic conditions on f it suffices in Ingham's theorem to suppose a continuous extension of the Laplace transform to the imaginary axis minus a closed, countable set. This Tauberian hypothesis arises naturally in Volterra equations ([4]). In particular, our result answers a problem asked in the introduction of [2].

In the second section we derive a Tauberian theorem in a similar way to Section 1, but for a larger class of Laplace transformable functions (than

1991 *Mathematics Subject Classification*: 47A35, 44A10, 34C28.
Key words and phrases: Tauberian theorem, Fourier transform, Laplace transform, asymptotically almost periodic, analytic Radon–Nikodym property, Cauchy problem.

the uniformly continuous ones). The price one has to pay is that one obtains a weaker notion of limit. However, for the abstract Cauchy problem it can be shown that this result leads to stable individual solutions even if the underlying semigroup is not bounded. The semigroup version of this result has recently been obtained in [15].

It is also surprising that under suitable assumptions on the geometry of the underlying Banach space the Tauberian hypothesis can be further relaxed. Indeed, in Section 3 we show the following: If the Banach space has the analytic Radon–Nikodym property, then instead of assuming a continuous extension of the Laplace transform to the imaginary axis it suffices to assume that the Laplace transform of f is bounded on bounded subsets of $\mathbb{C}_+$.

1. Tauberian theorems for bounded or slowly oscillating functions. We start this section with some notations, which we need throughout the paper. By X we denote a Banach space. The spaces $\mathcal{S}(\mathbb{R})$ and $\mathcal{D}(O)$ ($O \subset \mathbb{R}$ open) are the well known spaces of test functions from the theory of distributions, whereas $\mathcal{S}'(\mathbb{R};X) := \mathcal{L}(\mathcal{S}(\mathbb{R});X)$ and $\mathcal{D}'(O;X) := \mathcal{L}(\mathcal{D}(O);X)$ denote the corresponding spaces of (vector-valued) distributions. If $\varphi \in \mathcal{S}(\mathbb{R})$, then we define the Fourier transform $\mathcal{F}\varphi$ by

$$\mathcal{F}\varphi(\eta) := \int_{\mathbb{R}} e^{-i\eta t}\varphi(t)\,dt \qquad (\eta \in \mathbb{R}),$$

and we denote also by $\mathcal{F}$ the resulting Fourier transform on $\mathcal{S}'(\mathbb{R};X)$ or $\mathcal{D}'(\mathbb{R};X)$, where it is defined by duality.

According to the classical theory of distributions we will call a function $f \in L^1_{\mathrm{loc}}(\mathbb{R};X)$ *absolutely regular* if

$$g_k : t \mapsto f(t)(1+t^2)^{-k/2}$$

is integrable for some $k \in \mathbb{N}$. A function f is absolutely regular if and only if $f\varphi$ is integrable for all $\varphi \in \mathcal{S}(\mathbb{R})$ (the proof in [25, Theorem 1] goes through in the vector-valued case). Every absolutely regular function defines a tempered distribution T_f by

$$\langle T_f, \varphi\rangle := \int_{\mathbb{R}} f(s)\varphi(s)\,ds \qquad (\varphi \in \mathcal{S}(\mathbb{R})),$$

and we identify T_f and f.

Throughout the paper we will consider $L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ as a subspace of $L^1_{\mathrm{loc}}(\mathbb{R};X)$ in the canonical way. In the results of this paper we shall mainly be confronted with functions defined on $\mathbb{R}_+$, but for the proofs we need their extensions to $\mathbb{R}$ as well (for example, *all* the convolutions in this paper are understood as convolutions of functions defined on $\mathbb{R}$). For $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ we define the Laplace transform $\widehat{f}$ by

$$\widehat{f}(z) := \int_0^\infty e^{-zt} f(t)\,dt$$

for all $z \in \mathbb{C}$ for which this integral exists.

DEFINITION 1.1. Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ be absolutely regular. We define the *spectrum* $\mathrm{Sp}(f) \subset \mathbb{R}$ by setting for $\eta \in \mathbb{R}$,

$\eta \notin \mathrm{Sp}(f) :\Leftrightarrow$ There exist a neighborhood U of η and $g \in L^1(U;X)$ such that
$$\lim_{\xi\to 0^+} \widehat{f}(\xi + i\cdot)|_U = g \text{ in } \mathcal{D}'(U;X).$$

REMARK 1.2. (a) From the definition it is clear that the spectrum $\mathrm{Sp}(f)$ is always closed.

(b) Let U be a neighborhood of some point $\eta \in \mathbb{R}$. If the functions $\widehat{f}(\xi + i\cdot)|_U$ converge in $L^1(U;X)$ as $\xi \to 0^+$ then evidently $\eta \notin \mathrm{Sp}(f)$. In particular, if the Laplace transform $\widehat{f}$ has an analytic extension to $i\eta$, then $\eta \notin \mathrm{Sp}(f)$.

The following proposition is fundamental for this paper. Although its proof is short, it has deep consequences in Tauberian theory.

PROPOSITION 1.3. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ be absolutely regular. If $\varphi \in \mathcal{D}(\mathbb{R})$ satisfies $(-\operatorname{supp}\varphi) \cap \mathrm{Sp}(f) = \emptyset$, then $f * \mathcal{F}\varphi \in C_0(\mathbb{R};X)$.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ be such that $(-\operatorname{supp}\varphi) \cap \mathrm{Sp}(f) = \emptyset$. It is well known that the convolution of a tempered distribution and a test function in $\mathcal{S}(\mathbb{R})$ is continuous, so it remains to show that $\lim_{|s|\to\infty} \int_{\mathbb{R}} f(s-t)\mathcal{F}\varphi(t)\,dt = 0$ (notice that the integral makes sense). Since $\operatorname{supp}\varphi$ is compact, the assumption implies that there exist $\eta_j \in \operatorname{supp}\varphi$ ($1 \le j \le n$), neighborhoods U_j of η_j and functions $g_j \in L^1(U_j;X)$ so that $\operatorname{supp}\varphi \subset \bigcup_{j=1}^n U_j$ and $\widehat{f}(\xi + i\cdot)|_{U_j} \to g_j$ in $\mathcal{D}'(U_j;X)$ as $\xi \to 0^+$. Choose test functions φ_j such that $\operatorname{supp}\varphi_j \subset U_j$ and $\sum_{j=1}^n \varphi_j = \varphi$. For every j and $s \in \mathbb{R}$ we obtain, by Plancherel's Theorem,

$$\begin{aligned}\int_{\mathbb{R}} f(s-t)\mathcal{F}\varphi_j(t)\,dt &= \lim_{\xi\to0^+}\int_{\mathbb{R}} e^{-\xi t} f(t)\mathcal{F}\varphi_j(s-t)\,dt\\ &= \lim_{\xi\to0^+}\int_{U_j} \widehat{f}(\xi+i\eta)e^{i\eta s}\varphi_j(\eta)\,d\eta\\ &= \int_{U_j} g_j(\eta)e^{i\eta s}\varphi_j(\eta)\,d\eta.\end{aligned}$$

It follows from the Riemann–Lebesgue Lemma that $f * \mathcal{F}\varphi_j \in C_0(\mathbb{R};X)$, so that we have proved the proposition. ∎

REMARK 1.4. Consider Ingham's Tauberian theorem from the introduction. Changing the function f on a bounded interval we can assume that $f(0)=0$ and that the spectrum $\mathrm{Sp}(f)$ is still empty. It is not difficult to see that for a uniformly continuous function f (now defined on $\mathbb{R}$) there exists $M\ge 0$ such that for all $s,t\in\mathbb{R}$ one has

$$\|f(t+s)-f(t)\|\le M(1+s^2)^{1/2}.$$

Taking $\varphi_1\in\mathcal{D}(\mathbb{R})$ such that $2\pi\varphi_1(0)=\int_{\mathbb{R}}\mathcal{F}\varphi_1(s)\,ds=1$, we deduce from the inequality

$$\|f*\mathcal{F}\varphi_1(t)-f(t)\|=\Big\|\int_{\mathbb{R}}(f(t-s)-f(t))\mathcal{F}\varphi_1(s)\,ds\Big\| \le\int_{\mathbb{R}}M(1+s^2)^{1/2}|\mathcal{F}\varphi_1(s)|\,ds$$

and Proposition 1.3 that f is actually bounded. Put $\varphi_n:=\varphi_1(\cdot/n)$. Then $f*\mathcal{F}\varphi_n$ converges uniformly to f and Proposition 1.3 yields $f\in C_0(\mathbb{R}_+;X)$.

For $\eta\in\mathbb{R}$ and $x\in X$ we define the function $e_\eta\otimes x$ by $e_\eta\otimes x(s):=e^{-i\eta s}x$ ($s\in\mathbb{R}$). Then we define the space of *almost periodic functions* by

$$\mathrm{AP}(\mathbb{R};X):=\overline{\mathrm{span}}\{e_\eta\otimes x:\eta\in\mathbb{R},\ x\in X\}$$

(the closure in $\mathrm{BUC}(\mathbb{R};X)$) and $\mathrm{AP}(\mathbb{R}_+;X):=\{f|_{\mathbb{R}_+}:f\in\mathrm{AP}(\mathbb{R};X)\}$. Notice that every almost periodic function on $\mathbb{R}_+$ has a unique almost periodic extension to $\mathbb{R}$.

Let $f\in\mathrm{AP}(\mathbb{R}_+;X)$ and set $f_s(t):=f(s+t)$ $(s,t\ge 0)$. By the Mean Value Theorem for almost periodic functions ([19, Section 2.3]) we know that for all $\eta\in\mathbb{R}$,

(1.1) $$M_\eta(f,s):=\lim_{\xi\to 0+}\xi\widehat{f_s}(\xi+i\eta)\quad\text{exists uniformly in } s\ge 0.$$

One has $M_\eta(f,\cdot)=e_\eta\otimes M_\eta(f,0)$, and moreover, all the means $M_\eta(f,0)$ are zero except, possibly, for η in a countable set E. The Uniqueness Theorem [19, Section 2.3] says that if for $f\in\mathrm{AP}(\mathbb{R};X)$ all the means are zero, then f is zero itself.

The space $\mathrm{AAP}(\mathbb{R}_+;X):=\mathrm{AP}(\mathbb{R}_+;X)\oplus C_0(\mathbb{R}_+;X)$ will be called the space of *asymptotically almost periodic functions*. It is easy to check that (1.1) also holds for all AAP-functions and all $\eta\in\mathbb{R}$.

We will say that a function $f\in L^\infty(\mathbb{R}_+;X)$ is *uniformly ergodic* on a set $E\subset\mathbb{R}$ if (1.1) holds for all $\eta\in E$ (notice that the uniformity is with respect to $s\ge 0$). If $E=\mathbb{R}$, then f is sometimes called *totally (uniformly) ergodic* (see e.g. [2], [3]).

The following main theorem of this section then is a Tauberian theorem.

THEOREM 1.5. *Let $f\in L^\infty(\mathbb{R}_+;X)$ be such that $\mathrm{Sp}(f)$ is countable. If f is uniformly ergodic on $\mathrm{Sp}(f)$, then $(f*\varphi)|_{\mathbb{R}_+}\in\mathrm{AAP}(\mathbb{R}_+;X)$ for all $\varphi\in L^1(\mathbb{R})$.*

Proof. Consider the following bounded operator:

$$\kappa:L^1(\mathbb{R})\to\mathrm{BUC}(\mathbb{R}_+;X)/\mathrm{AAP}(\mathbb{R}_+;X)=:Z,$$
$$\varphi\mapsto(f*\varphi)|_{\mathbb{R}_+}+\mathrm{AAP}(\mathbb{R}_+;X).$$

We will show that $\kappa=0$ by showing that the adjoint operator $\kappa':Z'\to L^\infty(\mathbb{R})$ is zero. Let $z'\in Z'$ and $\varphi\in\mathcal{D}(\mathbb{R})$ be such that $(-\operatorname{supp}\varphi)\cap\mathrm{Sp}(f)=\emptyset$. Proposition 1.3 yields

$$\langle\mathcal{F}(\kappa'z'),\varphi\rangle=\langle\kappa'z',\mathcal{F}\varphi\rangle=\langle z',\kappa(\mathcal{F}\varphi)\rangle =\langle z',(f*\mathcal{F}\varphi)|_{\mathbb{R}_+}+\mathrm{AAP}(\mathbb{R}_+;X)\rangle=0.$$

Hence, the support of the distribution $\mathcal{F}(\kappa'z')$ is contained in $-\mathrm{Sp}(f)$. Let $\psi\in L^1(\mathbb{R})$. Observing that $\mathcal{F}(\kappa'z'*\psi)=\mathcal{F}(\kappa'z')\mathcal{F}\psi$, one can see that the support of $\mathcal{F}(\kappa'z'*\psi)$ is also contained in $-\mathrm{Sp}(f)$. Since $\kappa'z'*\psi$ is bounded and uniformly continuous and since $\mathrm{Sp}(f)$ is closed and countable, $\kappa'z'*\psi\in\mathrm{AP}(\mathbb{R})$ by Loomis's Theorem (see e.g. [17, Ex. 7, p. 169] or [19, Section 6.4, Theorem 4]). Equivalently, we can say that $(\kappa'z'*\psi)^\vee$ is almost periodic and that the support of its Fourier transform is contained in $\mathrm{Sp}(f)$ (here and in the rest of the proof we put $\check g(t):=g(-t)$).

We show that for all $\eta\in\mathrm{Sp}(f)$ the mean $M_\eta((\kappa'z'*\psi)^\vee,0)$ is zero in order to conclude that $\kappa'z'*\psi=0$. Let $\eta\in\mathrm{Sp}(f)$. Then one computes

$$\begin{aligned}M_\eta((\kappa'z'*\psi)^\vee,0)&=\lim_{\xi\to 0+}\xi\int_0^\infty e^{-(\xi+i\eta)t}\kappa'z'*\psi(-t)\,dt\\&=\lim_{\xi\to 0+}\langle(\kappa'z'*\psi)^\vee,1_{\mathbb{R}_+}\xi e^{-(\xi+i\eta)\cdot}\rangle\\&=\lim_{\xi\to 0+}\langle\kappa'z',(1_{\mathbb{R}_-}\xi e^{(\xi+i\eta)\cdot})*\check\psi\rangle\\&=\lim_{\xi\to 0+}\langle z',(f*(1_{\mathbb{R}_-}\xi e^{(\xi+i\eta)\cdot})*\check\psi)|_{\mathbb{R}_+}+\mathrm{AAP}(\mathbb{R}_+;X)\rangle.\end{aligned}$$

The second assumption implies that in $\mathrm{BUC}(\mathbb{R}_+;X)$,

(1.2) $$\lim_{\xi\to 0+}(f*(1_{\mathbb{R}_-}\xi e^{(\xi+i\eta)\cdot}))|_{\mathbb{R}_+}=M_\eta(f,\cdot)|_{\mathbb{R}_+}\in\mathrm{AAP}(\mathbb{R}_+;X).$$

Hence, $M_\eta(\kappa'z'*\psi,0)=0$ for all $\eta\in\mathbb{R}$, so that $\kappa'z'*\psi=0$ for all $\psi\in L^1(\mathbb{R})$ and all $z'\in Z'$. This implies $\kappa'z'=0$ for all $z'\in Z'$, which completes the proof. ∎

We say that a locally integrable function $f:\mathbb{R}_+\to X$ is *slowly oscillating* if

(1.3) $$\lim_{\substack{t\to\infty\\ \delta\to 0}}\|f(t+\delta)-f(t)\|=0.$$

The following lemma shows that slowly oscillating functions are, in a sense, uniformly continuous at infinity.

LEMMA 1.6. *A locally integrable function* $f:\mathbb{R}_+\to X$ *is slowly oscillating if and only if* $f=g_0+g_1$ *where* $g_0\in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$, $\lim_{t\to\infty}g_0(t)=0$ *and* $g_1\in \mathrm{UC}(\mathbb{R}_+;X)$.

Proof. The "if" part is evident. So let f be slowly oscillating. By (1.3) we can find two monotone sequences $(t_n)_{n\ge0}$, $(\delta_n)_{n\ge0}\subset\mathbb{R}_+$ such that $t_n\to\infty$, $\delta_n\to0$ and

$$\|f(t+\delta)-f(t)\|\le 1/n \quad \text{whenever } t\ge t_n \text{ and } 0<\delta\le\delta_n.$$

For every $n\ge0$ choose a partition of the interval $[t_n,t_{n+1}]$ so that every subinterval has length smaller than δ_n. We define the function g_1 to be linear on every subinterval and equal to f at the end points of the subintervals. On the interval $[0,t_0]$ we can extend g_1 continuously. By construction $g_1\in\mathrm{UC}(\mathbb{R}_+;X)$. Putting $g_0:=f-g_1$, we have the desired decomposition of f. ■

For slowly oscillating functions there is a refinement of Theorem 1.5. Notice that the following corollary improves the Tauberian theorems in [3], [2], [5] and [6] in various directions.

COROLLARY 1.7. *Let* $f\in L^\infty(\mathbb{R}_+;X)$ *be slowly oscillating. If the spectrum* $\mathrm{Sp}(f)$ *is countable and if* f *is uniformly ergodic on* $\mathrm{Sp}(f)$, *then there exists* $h\in\mathrm{AP}(\mathbb{R}_+;X)$ *such that* $\lim_{t\to\infty}\|f(t)-h(t)\|=0$.

Proof. By Lemma 1.6 there are $g_0\in L^\infty(\mathbb{R}_+;X)$ and $g_1\in\mathrm{BUC}(\mathbb{R}_+;X)$ such that $\lim_{t\to\infty}g_0(t)=0$, $g_1(0)=0$ and $f=g_0+g_1$. Since $\mathrm{Sp}(f)$ is countable and since f is uniformly ergodic on $\mathrm{Sp}(f)$, Theorem 1.5 implies that $(g_1*\varphi)|_{\mathbb{R}_+}\in\mathrm{AAP}(\mathbb{R}_+;X)$ for all $\varphi\in L^1(\mathbb{R})$.

For every $n\ge1$ we put $\varphi_n:=n1_{[0,1/n]}$. Then $g_1*\varphi_n$ converges to g_1 uniformly on $\mathbb{R}$, so that $g_1\in\mathrm{AAP}(\mathbb{R}_+;X)$. This completes the proof. ■

REMARK 1.8. We remark that in Corollary 1.7 it was not necessary to suppose f to be bounded. In fact, since f is a slowly oscillating function that is uniformly ergodic at 0 or for which $0\notin\mathrm{Sp}(f)$, it is necessarily eventually bounded.

In order to see this, one has to prove first that there exist constants $M>0$ and $t_0\ge0$ such that for all $t\ge t_0$ and all $s\in\mathbb{R}$ with $t+s\ge t_0$ one has

$$\|f(t+s)-f(t)\|\le M(1+s^2)^{1/2}.$$

Shifting the function f, we can assume that $t_0=0$.

In the case $0\notin\mathrm{Sp}(f)$ one can proceed as in Remark 1.4 to show that f is bounded. The case $0\in\mathrm{Sp}(f)$ and f uniformly ergodic at 0 is treated similarly.

2. A Tauberian theorem for functions with absolutely regular Fourier transforms. Whereas in Section 1 we restricted ourselves to bounded, slowly oscillating or absolutely regular functions, we now consider arbitrary locally integrable functions and study their asymptotic behavior.

We will do this with the help of the Fourier transform which is also defined on $\mathcal{D}'(\mathbb{R};X)$. The Fourier transform of a distribution $T\in\mathcal{D}'(\mathbb{R};X)$ is an element of $\mathcal{L}(\mathcal{Z}(\mathbb{R});X)$, where $\mathcal{Z}(\mathbb{R})$ is the image of $\mathcal{D}(\mathbb{R})$ under the Fourier transform. We say that the Fourier transform of a function $f\in L^1_{\mathrm{loc}}(\mathbb{R};X)$ is *absolutely regular* if there exists an absolutely regular function $g\in L^1_{\mathrm{loc}}(\mathbb{R};X)$ such that

$$\int_{\mathbb{R}} f(s)\mathcal{F}\varphi(s)\,ds=\int_{\mathbb{R}} g(s)\varphi(s)\,ds \quad \text{for all } \varphi\in\mathcal{Z}(\mathbb{R}). \tag{2.1}$$

The function g is uniquely determined and we identify g and $\mathcal{F}f$.

PROPOSITION 2.1. *Let* $f\in L^1_{\mathrm{loc}}(\mathbb{R};X)$ *be such that its Fourier transform is absolutely regular. Then* $f*\varphi\in C_0(\mathbb{R};X)$ *for all* $\varphi\in\mathcal{D}(\mathbb{R})$.

Choose $\alpha\ge0$ *such that* $\int_{\mathbb{R}}\|\mathcal{F}f(s)\|(1+s^2)^{-\alpha/2}\,ds<\infty$. *Then the function* $f*\varphi$ *converges to* 0 *as* $|t|\to\infty$, *uniformly for* φ *in each subset* $\mathcal{A}\subset\mathcal{D}(\mathbb{R})$ *for which* $\mathcal{F}\mathcal{A}_\alpha:=\{s\mapsto(1+s^2)^{\alpha/2}\mathcal{F}\varphi(s):\varphi\in\mathcal{A}\}$ *is bounded in* $C_0(\mathbb{R})$ *and uniformly equicontinuous.*

Proof. Let $(S(t))_{t\in\mathbb{R}}$ and $(\widetilde{S}(t))_{t\in\mathbb{R}}$ be the right-shift groups on $L^1(\mathbb{R};X)$ and $C_0(\mathbb{R})$, respectively. We choose $\alpha\ge0$ so that $g(s):=(1+s^2)^{-\alpha/2}\mathcal{F}f(s)$ defines an integrable function and we put $\psi(s):=(1+s^2)^{\alpha/2}\mathcal{F}^{-1}\varphi(s)$. In a similar way to Proposition 1.3 we obtain, using the Riemann–Lebesgue Lemma in a more explicit way (in the first inequality),

$$\begin{aligned}\Big\|\int_{\mathbb{R}} f(t-s)\varphi(s)\,ds\Big\| &=\Big\|\int_{\mathbb{R}} e^{its}\mathcal{F}f(s)\mathcal{F}^{-1}\varphi(s)\,ds\Big\|\le\frac12\Big\|\Big(S\Big(\frac{\pi}{t}\Big)-I\Big)(\mathcal{F}f\mathcal{F}^{-1}\varphi)\Big\|_{L_1}\\ &\le\frac12\Big\|S\Big(\frac{\pi}{t}\Big)g\cdot\Big(\widetilde{S}\Big(\frac{\pi}{t}\Big)\psi-\psi\Big)\Big\|_{L_1}+\frac12\Big\|\Big(S\Big(\frac{\pi}{t}\Big)g-g\Big)\cdot\psi\Big\|_{L_1}\\ &\le\frac12\|g\|_{L_1}\cdot\Big\|\widetilde{S}\Big(\frac{\pi}{t}\Big)\psi-\psi\Big\|_{C_0}+\frac12\Big\|S\Big(\frac{\pi}{t}\Big)g-g\Big\|_{L_1}\cdot\|\psi\|_{C_0}.\end{aligned}$$

Now, let $\mathcal{A}$ be a subset of $\mathcal{D}(\mathbb{R})$ such that $\mathcal{F}\mathcal{A}_\alpha$ is bounded in $C_0(\mathbb{R})$. Then the second term on the right hand side converges to 0 as $|t|\to\infty$, uniformly in $\varphi\in\mathcal{A}$. Moreover, $\mathcal{F}\mathcal{A}_\alpha$ is uniformly equicontinuous if and only if the right-shift group $(\widetilde{S}(t))_{t\in\mathbb{R}}$ is uniformly continuous on $\mathcal{F}\mathcal{A}_\alpha$. Hence, we can conclude. ■

Also in this section we are mainly interested in a Tauberian theorem for Laplace transforms. In order to relate the Fourier transform of a function $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ to its Laplace transform we introduce the space $H_+(\mathbb{C}_+; X)$ of all holomorphic functions $\widehat{f} : \mathbb{C}_+ \to X$ with the following two properties:

(i) There exists $k \in \mathbb{N}$ such that for all $\xi > 0$ there exists $C_\xi \geq 0$ with

$$\|\widehat{f}(z)\| \leq C_\xi (1 + |z|)^k \quad \text{whenever } \operatorname{Re} z \geq \xi.$$

(ii) The functions $\widehat{f}(\xi + i\cdot)$ converge in the sense of tempered distributions as $\xi \to 0^+$.

The following lemma is then a special case of a more general result in the scalar case ([7, Thm. 2.7, Chapter 2]). We give an easy proof of it.

LEMMA 2.2. *Let $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ be Laplace transformable and assume that the Laplace transform $\widehat{f}$ has an analytic extension (also denoted by $\widehat{f}$) to the right half plane $\mathbb{C}_+$ and that $\widehat{f} \in H_+(\mathbb{C}_+; X)$. Then the distributional limit T of the functions $\widehat{f}(\xi + i\cdot)$ as $\xi \to 0^+$ is the Fourier transform of f.*

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$, $\xi > 0$ and $\mu \geq \operatorname{abs}(f)$, where $\operatorname{abs}(f)$ denotes the *abcissa of absolute convergence* of the Laplace integral of f. Then by Plancherel's Theorem we obtain

$$\int_{\mathbb{R}_+} e^{-\xi s} f(s)\varphi(s)\, ds = \int_{\mathbb{R}_+} e^{-(\mu+\xi)s} f(s) e^{\mu s}\varphi(s)\, ds$$
$$= \int_{\mathbb{R}} \widehat{f}(\mu + \xi + i\eta)\mathcal{F}^{-1}(e^{\mu\cdot}\varphi)(\eta)\, d\eta.$$

Note that for every $\varphi \in \mathcal{D}(\mathbb{R})$ the Fourier transform $\mathcal{F}\varphi$ is an entire function. Changing the path of integration and using the assumption that $\widehat{f}$ is polynomially bounded in every strip away from the imaginary axis, we then obtain by the Paley–Wiener and Cauchy Theorems (see [21, Section 4.3] for a similar argument)

$$\int_{\mathbb{R}_+} e^{-\xi s} f(s)\varphi(s)\, ds = \int_{\mathbb{R}} \widehat{f}(\xi + i\eta)\mathcal{F}^{-1}(e^{\mu\cdot}\varphi)(\eta + i\mu)\, d\eta$$
$$= \int_{\mathbb{R}} \widehat{f}(\xi + i\eta)\mathcal{F}^{-1}\varphi(\eta)\, d\eta.$$

Letting $\xi \to 0$ on both sides of the equation, we obtain

$$\int_{\mathbb{R}_+} f(s)\varphi(s)\, ds = \langle T, \mathcal{F}^{-1}\varphi\rangle \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R}). \ \blacksquare$$

Proposition 2.1 and Lemma 2.2 obviously yield the following Tauberian theorem for Laplace transforms, in which we do not state the uniformity of the limit again.

THEOREM 2.3. *Let $f \in L^1_{\text{loc}}(\mathbb{R}_+; X)$ be Laplace transformable and assume that the Laplace transform $\widehat{f}$ has an analytic extension to the right half plane $\mathbb{C}_+$, that $\widehat{f} \in H_+(\mathbb{C}_+; X)$, and that for some $k \in \mathbb{N}$ the functions $\eta \mapsto \widehat{f}(\xi + i\eta)(1 + \eta^2)^{-k/2}$ converge in $L_1(\mathbb{R}; X)$ as $\xi \to 0^+$. Then $f * \varphi \in C_0(\mathbb{R}; X)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.*

The novelty of this theorem is that one only assumes f to be a Laplace transformable function, but no other growth conditions are imposed. Of course, we cannot obtain strong convergence of f, but a convergence that seems to be closely related to the B-convergence discussed in [4] and [10].

Theorem 2.3 has the following consequence for the linear Cauchy problem. Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on the Banach space X. Assume that for some $x \in X$ the local resolvent $R(\cdot, A)x$ has an analytic extension F to the open right half plane $\mathbb{C}_+$ (for example, one could assume that the spectral bound $s(A)$ is less than or equal to 0). Assume, furthermore, that $F \in H_+(\mathbb{C}_+; X)$ and that $F(\xi + i\cdot)$ converges in $L^1_{\text{loc}}(\mathbb{R}; X)$ to some absolutely regular function as $\xi \to 0^+$.

Theorem 2.3 then yields that for all $\varphi \in \mathcal{D}(\mathbb{R}_+)$ one has

$$\lim_{t\to\infty} \int_{\mathbb{R}} T(t + s)x\varphi(s)\, ds = \lim_{t\to\infty} T(t) \int_0^\infty T(s)x\varphi(s)\, ds = 0. \tag{2.2}$$

Therefore we have found stable solutions for initial values of the form

$$\int_0^\infty T(s)x\varphi(s)\, ds \in D(A^\infty).$$

Using the second part of Proposition 2.1, one can obtain an even larger class of initial values with corresponding stable solutions (see [11] for details). This semigroup version of Theorem 2.3 has already been obtained before in [15]. We only remark that in [15] the extension of the function F onto the imaginary axis is slowly increasing. In Theorem 2.3 we only assume an absolutely regular extension, which is a slightly weaker condition.

To end this section we show that the introduction of the space $H_+(\mathbb{C}_+; X)$ was necessary in Theorem 2.3. For this, let $(J(t))_{t\geq 0}$ be the Riemann–Liouville semigroup on $L^2(0,1)$, given by

$$(J(t)f)(x) := \frac{1}{\Gamma(t)} \int_0^x (x - y)^{t-1} f(y)\, dy \quad (f \in L^2(0,1),\ t \geq 0),$$

and let G denote its generator (see [14, Section 23.16]). It is proved in [14, Section 23.16] that G has empty spectrum and that iG generates a C_0-group. Choose $\omega \in \mathbb{R}$ such that for $B := \omega + iG$ the resolvent $R(\cdot, B)$ is bounded on $\{z \in \mathbb{C} : \operatorname{Re} z \leq 1\}$ and let $(S(t))_{t\in\mathbb{R}}$ be the group generated by B. If the assumption $F \in H_+(\mathbb{C}_+; X)$ in Theorem 2.3 were not necessary, we would deduce (using the fact that the analytic extension of $R(\cdot, B)x$ is bounded on $i\mathbb{R}$ for every $x \in X$) that

$$\lim_{t\to\infty} S(t)y = 0 \quad \text{for all } y \in Y := \Big\{ \int_0^\infty S(s)x\varphi(s)\, ds : x \in X,\ \varphi \in \mathcal{D}(\mathbb{R}_+) \Big\}.$$

But then $R(\cdot, B)y$ would be a bounded entire function, which would imply $y = 0$. This is a contradiction, because Y is dense in $L^2(0,1)$.

3. The analytic Radon–Nikodym property. In the Tauberian theorems for Laplace transforms of both Sections 1 and 2 the convergence of the functions $\widehat{f}(\xi + i\cdot)$ in appropriate spaces of locally integrable functions was an important condition. In fact, it was a sufficient condition for the existence of a locally integrable Fourier transform. In this section we show that there are weaker conditions, provided that the underlying Banach space has the analytic Radon–Nikodym property.

For the moment $g : \mathbb{C}_+ \to X$ will be an arbitrary analytic function (not necessarily a Laplace transform). Let $U \subset \mathbb{R}$ be a bounded interval and consider the following condition:

$$\limsup_{\xi\to 0^+} \int_U \|g(\xi + i\eta)\|\, d\eta < \infty. \tag{3.1}$$

PROPOSITION 3.1. *With the above notations the following two assertions are true*:

(i) *The functions $g(\xi + i\cdot)|_U$ converge in $L^1(U; X)$ as $\xi \to 0^+$ if and only if they converge almost everywhere.*

(ii) *If the Banach space X has the analytic Radon–Nikodym property* (see Definition 3.3 below), *then the functions $g(\xi + i\cdot)|_U$ converge in $L^1(U; X)$ as $\xi \to 0^+$.*

In view of Sections 1 and 2 the second statement of Proposition 3.1 is of great interest. It will give us a weak condition for the existence of a locally integrable Fourier transform. This idea in Tauberian theory is worked out in detail in [10] (however, with different proofs); the basic theory of Hardy spaces and the analytic Radon–Nikodym property can be found in [8], [9], [12] and [23].

An analogue of Proposition 3.1 is well known on the unit disc D. Let $f : D \to X$ be an analytic function. We set

$$\|f\|_1 := \sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \|f(re^{i\theta})\|\, d\theta,$$

and we denote by $H_1(D; X)$ the space of all analytic functions $f : D \to X$ with $\|f\|_1 < \infty$. Equipped with the norm $\|\cdot\|_1$, this space becomes a Banach space, called a *vector-valued Hardy space*. The following result from [12] is well known, even in the vector-valued case.

THEOREM 3.2. *Let $f \in H_1(D; X)$. Then the following two assertions are equivalent*:

(i) *There is a measurable function $\widetilde{f} : [0, 2\pi] \to X$ such that*

$$\widetilde{f}(\theta) = \lim_{r\to 1^-} f(re^{i\theta}) \quad \text{for almost all } \theta \in [0, 2\pi].$$

(ii) *There is a function $\widetilde{f} \in L^1([0, 2\pi]; X)$ such that*

$$\lim_{r\to 1^-} \int_0^{2\pi} \|f(re^{i\theta}) - \widetilde{f}(\theta)\|\, d\theta = 0.$$

In the scalar case $X = \mathbb{C}$ each of the statements of Theorem 3.2 holds for all functions $f \in H_1(D; X)$, but for arbitrary Banach spaces this is no longer true. Take for example the Banach space c_0 and the function f defined by $f(z) = (z^n)_{n\ge 0}$. Then $f \in H_1(D; c_0)$ but it does not admit any radial (or nontangential) limit on the unit circle. This example motivates the following definition, first given by Bukhvalov (see e.g. [8]).

DEFINITION 3.3. A Banach space X has the *analytic Radon–Nikodym property* if for every function $f \in H_1(D; X)$ one of the equivalent statements in Theorem 3.2 is true.

One can show that a Banach space X has the analytic Radon–Nikodym property if and only if every function $F : [0, 2\pi] \to X$ of normalized bounded variation, whose Fourier–Stieltjes coefficients $\int_0^{2\pi} e^{-int}\, dF(t)$ vanish for $n < 0$, admits a Radon–Nikodym derivative. Since such a function is absolutely continuous by a theorem of F. and M. Riesz (see [24]), the Radon–Nikodym property implies the analytic Radon–Nikodym property. Hence, every reflexive space and every separable dual space have the analytic Radon–Nikodym property. Moreover, a Banach lattice has the analytic Radon–Nikodym property if and only if it does not contain a subspace isomorphic to c_0 ([9, Theorem 1]). This shows that L^1 has the analytic Radon–Nikodym property, whereas it fails to have the Radon–Nikodym property in general (every finite-dimensional space and the space l^1 have the Radon–Nikodym property, but $L^1[0, 1]$ does not).

Proof of Proposition 3.1. Let $g : \mathbb{C}_+ \to X$ be an analytic function and U a bounded interval. If g satisfies condition (3.1) then Tonelli's Theorem implies

$$\int_0^1 \int_U \|g(\xi + i\eta)\|\, d\eta\, d\xi = \int_U \int_0^1 \|g(\xi + i\eta)\|\, d\xi\, d\eta < \infty.$$

Hence, for almost all $\eta \in U$ the integral $\int_0^1 \|g(\xi+i\eta)\|\,d\xi$ is finite. Let $\eta_1 < \eta_2$ be two such points, and consider the rectangle $\Omega = \{z \in \mathbb{C} : 0 < \operatorname{Re} z < 1 \text{ and } \eta_1 < \operatorname{Im} z < \eta_2\}$. Then there exists a sequence $(\Gamma_n)_{n\ge 0}$ of the boundaries of rectangles in Ω which converge to $\partial\Omega$ in the sense that they eventually surround each compact subdomain in Ω, such that

$$\sup_{n\ge 0} \int_{\Gamma_n} \|g(z)\|\,|dz| < \infty. \tag{3.2}$$

Now, choose a conformal mapping $\varphi : D \to \Omega$. By the theorems of Carathéodory and Riesz–Privalov [23, pp. 18 and 126], φ has a continuous extension to the closed unit disc and the extension is absolutely continuous on the unit circle. Moreover, $g\circ\varphi \in H_1(D;X)$ because of (3.2) and [12, Thm. 10.1] (that theorem is also true in the vector-valued case). Since φ is angle preserving on the boundary, the existence of a radial limit almost everywhere of $g\circ\varphi$ on the unit circle is equivalent to the existence of a limit almost everywhere of the functions $g(\xi+i\cdot)|_{(\eta_1,\eta_2)}$. Since this is true for almost all $\eta_1, \eta_2 \in U$, Theorem 3.2 implies the statement (i).

The second statement is just a consequence of the definition of the analytic Radon–Nikodym property and of part (i). ■

One aim of this section is the following Tauberian theorem, which is a variant of Corollary 1.7. The condition on the spectrum $\operatorname{Sp}(f)$ in Corollary 1.7 is replaced by a considerably weaker condition. Notice that in this theorem the Laplace transform $\widehat{f}$ has to be known in the *open* right half plane $\mathbb{C}_+$ only.

THEOREM 3.4. *Let X be a Banach space with the analytic Radon–Nikodym property and let $f \in L^\infty(\mathbb{R}_+;X)$ be slowly oscillating. Let $O \subset \mathbb{R}$ be open such that $E := \mathbb{R}\setminus O$ is countable. Assume*

(i) *for every $\eta \in O$ there exists $\varepsilon > 0$ such that*

$$\limsup_{\xi\to 0^+} \int_{\eta-\varepsilon}^{\eta+\varepsilon} \|\widehat{f}(\xi+i\eta')\|\,d\eta' < \infty, \quad \textit{and} \tag{3.3}$$

(ii) *f is uniformly ergodic on E.*

Then there exists $h \in \operatorname{AP}(\mathbb{R}_+;X)$ such that $\lim_{t\to\infty}\|f(t)-h(t)\| = 0$.

Proof. One only has to observe that $\operatorname{Sp}(f)$ is contained in E by assumption (i) and Proposition 3.1. The conclusion follows from Corollary 1.7. ■

There is also a variant of Theorem 2.3 which is valid in Banach spaces with the analytic Radon–Nikodym property. It reads as follows.

THEOREM 3.5. *Let X be a Banach space with the analytic Radon–Nikodym property and let $f \in L^1_{\mathrm{loc}}(\mathbb{R}_+;X)$ be Laplace transformable. Assume that $\widehat{f}$ has an analytic extension to $\mathbb{C}_+$, $\widehat{f} \in H_+(\mathbb{C}_+;X)$ and for some $k \in \mathbb{N}$ one has*

$$\limsup_{\xi\to 0^+} \int_{\mathbb{R}} \|\widehat{f}(\xi+i\eta)\|(1+\eta^2)^{-k/2}\,d\eta < \infty.$$

Then $f * \varphi \in C_0(\mathbb{R};X)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.

Proof. Notice that for every $k \in \mathbb{N}$ the factor $(1+\eta^2)^{-k/2}$ is strictly positive on bounded sets. Proposition 3.1 and (3.4) then yield that the functions $\widehat{f}(\xi+i\cdot)$ converge in $L^1_{\mathrm{loc}}(\mathbb{R};X)$ to an absolutely regular function as $\xi \to 0^+$. The assertion now follows from Theorem 2.3. ■

REMARK 3.6. (a) The conclusion of Theorem 3.4 is in general false if X does not have the analytic Radon–Nikodym property. Define the function $f : \mathbb{R}_+ \to c_0$ by

$$f(t) = (n+1-t)e_n + (t-n)e_{n+1} \quad \text{whenever } t \in [n, n+1),$$

where the e_n are the canonical unit vectors in c_0. Then $f \in \operatorname{BUC}(\mathbb{R}_+;c_0)$ and $\widehat{f}$ is uniformly bounded on $\mathbb{C}_+$, but f is *not* asymptotically almost periodic. In fact, since $\widehat{f}$ is uniformly bounded on $\mathbb{C}_+$, all the means of f are 0, but clearly f does not converge to 0.

(b) Since $\mathbb{C}$ has the analytic Radon–Nikodym property, Theorems 3.4 and 3.5 are true in arbitrary Banach spaces if one replaces the limits by weak limits.

(c) Let X be a separable Banach space and let $f \in H_1(D;X')$. Then there exists a weak* measurable function $\widetilde{f} : [0,2\pi] \to X'$ such that for all $n \ge 0$ and all $x \in X$ one has

$$\langle a_n, x\rangle = \int_0^{2\pi} e^{-in\theta}\langle \widetilde{f}(\theta), x\rangle\,d\theta,$$

where the a_n are the Taylor coefficients of f ([8, Theorem 2.3]). By a remark from Pisier [22], the Riemann–Lebesgue Lemma is true in the strong sense for weak* measurable and integrable functions if and only if X does not contain l^1. Hence, if $l^1 \not\subset X$, then $\lim_{n\to\infty}\|a_n\| = 0$. Since the essential part of our Tauberian theorems was only the Riemann–Lebesgue Lemma (see Propositions 1.3 and 2.1), the above theorems might also be true in the duals of separable Banach spaces not containing l^1. Since we do not know any example of such a Banach space without the analytic Radon–Nikodym property, we omitted this study.

Acknowledgements. The author wishes to thank Prof. W. Arendt from the Universität Ulm for a lot of discussions and encouragement for this work.

References

[1] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups*, Trans. Amer. Math. Soc. 306 (1988), 837–852.

[2] —, —, *Asymptotically almost periodic solutions of inhomogeneous Cauchy problems on the half line*, J. London Math. Soc., to appear.

[3] —, —, *Almost periodic solutions of first and second order Cauchy problems*, J. Differential Equations 137 (1997), 363–383.

[4] W. Arendt and J. Prüss, *Vector-valued Tauberian theorems and asymptotic behavior of linear Volterra equations*, SIAM J. Math. Anal. 23 (1992), 412–448.

[5] C. J. K. Batty, J. M. A. M. van Neerven and F. Räbiger, *Local spectra and individual stability of uniformly bounded C_0-semigroups*, Trans. Amer. Math. Soc., to appear.

[6] —, —, —, *Tauberian theorems and stability of solutions of the Cauchy problem*, ibid., to appear.

[7] E. J. Beltrami and M. R. Wohlers, *Distributions and the Boundary Values of Analytic Functions*, Academic Press, New York, 1966.

[8] A. V. Bukhvalov, *Hardy spaces of vector-valued functions*, J. Soviet Math. 16 (1981), 1051–1059 (English transl.).

[9] A. V. Bukhvalov and A. A. Danilevich, *Boundary properties of analytic and harmonic functions with values in Banach space*, Math. Notes 31 (1982), 104–110 (English transl.).

[10] R. Chill, *Taubersche Sätze und Asymptotik des abstrakten Cauchy-Problems*, Diplomarbeit, Universität Tübingen, 1995.

[11] —, *Stability results for individual solutions of the abstract Cauchy problem via Tauberian theorems*, Ulmer Sem. Funktionalanal. Differentialgleichungen 1 (1996), 122–133.

[12] P. L. Duren, *Theory of H^p-Spaces*, Academic Press, New York, 1970.

[13] J. Esterle, E. Strouse et F. Zouakia, *Stabilité asymptotique de certains semi-groupes d'opérateurs et idéaux primaires de $L^1(\mathbb{R}_+)$*, J. Operator Theory 28 (1992), 203–227.

[14] E. Hille and R. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, 1957.

[15] S. Huang and J. M. A. M. van Neerven, *B-convexity, the analytic Radon–Nikodym property and individual stability of C_0-semigroups*, J. Math. Anal. Appl., to appear.

[16] A. E. Ingham, *On Wiener's method in Tauberian theorems*, Proc. London Math. Soc. 38 (1933), 458–480.

[17] Y. Katznelson, *An Introduction to Harmonic Analysis*, Wiley, New York, 1968.

[18] J. Korevaar, *On Newman's quick way to the prime number theorem*, Math. Intelligencer 4 (1982), 108–115.

[19] B. M. Levitan and V. V. Zhikov, *Almost Periodic Functions and Differential Equations*, Cambridge Univ. Press, 1982.

[20] Y. I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces*, Studia Math. 88 (1988), 37–42.

[21] J. M. A. M. van Neerven, *The Asymptotic Behaviour of Semigroups of Linear Operators*, Oper. Theory Adv. Appl. 88, Birkhäuser, 1996.

[22] G. Pisier, *Une propriété de stabilité de la classe des espaces ne contenant pas l^1*, C. R. Acad. Sci. Paris Sér. A 286 (1978), 747–749.

[23] C. Pommerenke, *Boundary Behaviour of Conformal Maps*, Grundlehren Math. Wiss. 299, Springer, Berlin, 1992.

[24] R. Ryan, *The F. and M. Riesz theorem for vector measures*, Indag. Math. 25 (1962), 558–562.

[25] Z. Szmydt, *Characterization of regular tempered distributions*, Ann. Polon. Math. 41 (1983), 255–258.

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Received March 13, 1997 (3856)
Revised version August 21, 1997