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Taut contact hyperbolas on three-manifolds

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Dedicated to my grandson Davide and my grand-daughter Elisa

Abstract
In this paper, we introduce the notion of taut contact hyperbola on three-manifolds. It is the hyperbolic analogue of the taut contact circle notion introduced by Geiges and Gonzalo (Invent. Math., 121: 147–209, 1995), (J. Differ. Geom., 46: 236–286, 1997). Then, we characterize and study this notion, exhibiting several examples, and emphasizing differences and analogies between taut contact hyperbolas and taut contact circles. Moreover, we show that taut contact hyperbolas are related to some classic notions existing in the literature. In particular, it is related to the notion of conformally Anosov flow, to the critical point condition for the Chern–Hamilton energy functional and to the generalized Finsler structures introduced by R. Bryant. Moreover, taut contact hyperbolas are related to the bi-contact metric structures introduced in D. Perrone (Ann. Global Anal. Geom., 52: 213–235, 2017).

Keywords Taut contact hyperbolas · Taut contact circles · Bi-contact metric structures · Three-manifolds · 3D Lie groups · Generalized Finsler structures · Conformally Anosov flow · Chern–Hamilton energy functional

Mathematics Subject Classification 53C15 · 53D10 · 53C25

1 Introduction

Following [13, 15], a pair (η_1, η_2) of contact 1-forms on a three-manifold M is called a **contact circle** if the linear combination $\eta_a = a_1\eta_1 + a_2\eta_2$ is also a contact form for every $a = (a_1, a_2) \in \mathbb{S}^1$, the unit circle in $\mathbb{R}^2$. If in addition the volume forms $\eta_a \wedge (d\eta_a)$ are equal for every $a \in \mathbb{S}^1$, then (η_1, η_2) is said to be a **taut contact circle**.

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In the paper [24], we studied taut contact circles on a three-manifold from point of view of the Riemannian geometry and introduced the notion of bi-contact metric structure (η_1, η_2, g), that is, (η_1, η_2) is a pair of contact 1-forms and g is a Riemannian metric associated to both the contact forms η_1, η_2 such that the corresponding Reeb vector fields are orthogonal.

The main purpose of this paper is to start the study of the hyperbolic analogue, in dimension three, of taut contact circles study. The unit circle $\mathbb{S}^1$ of the Euclidean plane has its counterpart in the pseudo-Euclidean plane, that is, in the Minkowski plane, in the four arms of the unit equilateral hyperbolas $\mathbb{H}^1_r : x^2 - y^2 = r, r = \pm 1$. Then, we call **contact hyperbola** a pair of contact 1-forms (η_1, η_2) such that, for every $a = (a_1, a_2) \in \mathbb{H}^1_r$, the linear combination $\eta_a = a_1\eta_1 + a_2\eta_2$ is also a contact form. If in addition the volume forms $\eta_a \wedge (d\eta_a) = r\eta_1 \wedge (d\eta_1)$ for every $a \in \mathbb{H}^1_r$, then (η_1, η_2) is said to be a **taut contact hyperbola**. In particular, if (η_1, η_2, g) is a bi-contact metric structure on a three-manifold, then (η_1, η_2) is either a taut contact circle or a taut contact hyperbola.

We note that the notion of taut contact hyperbola introduced in this paper is very natural because is related to some classic notions existing in the literature: the notion of conformally Anosov flow introduced by Mitsumatsu [21] and Eliashberg-Thurston [12], the critical point condition for the Chern–Hamilton energy functional [9, 26] and the generalized Finsler structures introduced by Bryant [5, 6], are related to the taut contact hyperbolas. So we believe that this study is worthy of subsequent insights.

The present paper, where in particular we emphasize differences and analogies between taut contact hyperbolas and taut contact circles, is organized in the following way.

In Sect. 2, we collect some basic facts about contact Riemannian geometry.

In Sect. 3, we introduce the notion of taut contact hyperbola on a three-manifold. In particular, in the compact case, a taut contact hyperbola defines a conformally Anosov flow in the sense of Mitsumatsu [21] and Eliashberg-Thurston [12]. Then, we study left invariant taut contact hyperbolas on $3D$ Lie groups. The Lie groups $\widetilde{SL}(2, R)$ and Sol^3 are the only unimodular Lie groups which admit left invariant taut contact hyperbolas; thus, we determine the left invariant taut contact hyperbolas on these Lie groups. Then, we study taut contact hyperbolas on non-unimodular Lie groups, in particular there are non-unimodular Lie groups with the Milnor's invariant $\mathcal{D} = 0$ which admit a taut contact hyperbola with the corresponding Reeb vector fields dependent.

Section 4 contains some characterization of taut contact hyperbolas (cf. Theorem 4.2), and a remark about a difference between a taut contact circle and a taut contact hyperbola in terms of symplectic structures.

In Sect. 5, in analogy with the notion of taut contact 2-sphere, we introduce the notion of taut contact 2-hyperboloid, in particular we get that the Lie group $\widetilde{SL}(2, R)$ is the only simply connected three-manifold which admits a taut contact 2-hyperboloid (η_1, η_2, η_3) with the corresponding Reeb vector fields (ξ_1, ξ_2, ξ_3) that constitute the frame dual of the coframe (η_1, η_2, η_3).

In Sect. 6, we show that the critical point condition for the Chern–Hamilton energy functional ([9, 26]) is a sufficient condition for the existence of a taut contact hyperbola on a non-Sasakian contact metric three-manifold (M, η, g) (cf. Theorem 6.1). In particular, in the compact case, $(M, \ker \eta)$ is universally tight. Thus, we exhibit an example related to this Theorem.

In Sect. 7 we study the geometry of a three-manifold M determined by the existence of a bi-contact metric structure. We characterize the existence of a bi-contact metric structure (η_1, η_2, g) on M by the condition that (η_1, η_2) is a $(-\varepsilon)$-Cartan structure (cf. Theorem 7.1). In particular, there are a 1-form η_3 and a function κ, that we call the Webster function (cf.

Remark 7.4), uniquely determined by this structure. Then, (cf. Theorem 7.5) η_3 is Killing (resp. a contact form) if and only if (η_1, η_2) is taut contact circle (resp. the Webster function $\kappa \neq 0$ everywhere). Besides, we study the geometry of M when the 1-form η_3 is a contact form and in particular when the Webster function $\kappa = \pm 1$ (cf. Theorem 7.5 and Corollary 7.6).

Finally, in Sect. 8, we see how bi-contact metric structures are related to the generalized Finsler structures (introduced by Bryant [5, 6]) and construct an explicit example of bi-contact metric structures (η_1, η_2, g) where (η_1, η_2) is a taut contact hyperbola with the Webster function κ non-constant (in particular, this example gives a positive answer to a question posed in [24]).

2 Riemannian geometry of contact manifolds

In this section, we collect some basic facts about contact Riemannian geometry and refer to the two monographs [3, 4] for more information. All manifolds are supposed to be connected and smooth. Moreover, in what follows, for a Riemannian manifold (M, g), we shall denote by ∇ the Levi-Civita connection of the Riemannian metric g, by R the corresponding Riemannian curvature tensor and by Ric the Ricci tensor.

A *contact manifold* is a $(2n+1)$-dimensional manifold M equipped with a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M. It has an underlying *almost contact structure* (ξ, η, φ) where ξ is a global vector field (called the *Reeb vector field*, or the *characteristic vector field*) and φ is a global tensor of type (1,1) such that $\eta(\xi) = 1\,,\ \varphi\xi = 0\,,\ \varphi^2 = -I + \eta \otimes \xi$. A Riemannian metric g can be found such that

$$\eta = g(\xi, \cdot)\,, \quad d\eta = g(\cdot, \varphi\cdot)\,.$$

In such a case, g is called an *associated metric*, and we refer to (M, η, g), or $(M, \xi, \eta, \varphi, g)$, as a contact metric (or contact Riemannian) manifold. The tensor $h = \frac{1}{2}\mathcal{L}_\xi\varphi$, where $\mathcal{L}$ denotes the Lie derivative, plays a fundamental role in contact Riemannian geometry, it is symmetric and satisfies: $h\varphi = -\varphi h, \quad h\xi = 0$ and

$$\nabla\xi = -\varphi - \varphi h. \tag{2.1}$$

In particular, the Reeb vector field ξ is a geodesic vector field: $\nabla_\xi\xi = 0$.

More in general, given an almost contact structure (ξ, η, φ), a Riemannian metric g can be found such that $g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y)$, and in this case (ξ, η, φ, g) is called *almost contact metric structure*. An almost contact structure (ξ, η, φ) is said to be *normal* if the almost complex structure J on $M \times \mathbb{R}$ defined by $J(X, f\mathrm{d}/\mathrm{d}t) = (\phi X - f\xi, \eta(X)\mathrm{d}/\mathrm{d}t)$ is integrable, where f is a real-valued function. A contact metric manifold is said to be a *K-contact* manifold if the Reeb vector field ξ is a Killing vector field with respect to the associated metric g. Since the torsion $\tau = \mathcal{L}_\xi g$ satisfies $\tau = 2g(\cdot, h\varphi\cdot)$ and $Ric(\xi, \xi) = 2n - trh^2$, a contact metric manifold M is K-contact if and only if the tensor $h = 0$ or, equivalently, $Ric(\xi, \xi) = 2n$. A contact metric manifold is said to be a *Sasakian* manifold if the almost contact structure (η, ξ, φ) is normal. Any Sasakian manifold is K-contact and the converse also holds in dimension three. A contact metric manifold (M, η, g) is said to be an *H-contact manifold* if Reeb vector field ξ is a harmonic vector field, that is, ξ satisfies the critical point condition for the energy functional defined on the space of all unit vector fields; moreover, a contact metric manifold (M, η, g) is H-contact if and only if ξ is an eigenvector of the Ricci operator Q, that is, $Q\xi = (2n - \mathrm{tr}h^2)\xi$ [23]. Sasakian manifolds and K-contact manifolds are H-contact manifolds, but the converse, in general, is not true.

Recently, we have considered a Riemannian metric g as an associated metric for two contact forms. More precisely, we have

Definition 2.1 ([24]) Let M be a three-manifold. A *bi-contact metric structure* on M is a triple (η_1, η_2, g) where (η_1, η_2) is a pair of contact forms and g is an associated metric for both the contact forms η_1, η_2, such that the corresponding Reeb vector fields satisfy $g(\xi_1, \xi_2) = 0$, equivalently the corresponding almost contact structures $(\xi_i, \eta_i, \varphi_i)$, $i = 1, 2$, satisfy the condition:

$$\varphi_1\varphi_2 + \varepsilon\eta_1 \otimes \xi_2 = -(\varphi_2\varphi_1 + \varepsilon\eta_2 \otimes \xi_1), \quad \varepsilon = \pm 1, \tag{2.2}$$

where ε is defined by $\varphi_2\xi_1 = \varepsilon\,\varphi_1\xi_2$.

Then, in [24] we gave a complete classification of simply connected three-manifolds which admit a bi-H-contact metric structure (η_1, η_2, g), i.e., (η_1, g) and $(\eta_2, g)$are both H-contact.

We note that in the classical definition of contact metric 3-structure (see, for example, [4] Chapter 13 and [3] Chapter 14) we have three contact metric structures $(\xi_i, \eta_i, \varphi_i, g)$, $i = 1, 2, 3$, such that:

$$\varphi_i\varphi_j - \eta_j \otimes \xi_i = \varphi_k = -(\varphi_j\varphi_i - \eta_i \otimes \xi_j) \tag{2.3}$$

for any cyclic permutation (i, j, k) of (1, 2, 3). A *contact metric 3-structure* is called Sasakian 3-structure if the three contact metric structures are Sasakian. The condition (2.3) implies, in particular, the orthogonality of the three Reeb vector fields with respect to g. So, if $(\eta_1, \eta_2, \eta_3, g)$ *is a contact metric* 3*-structure then* (η_i, η_j, g) *are bi-contact metric structures for any* $i, j = 1, 2, 3, i \neq j$. *However, this is only a necessary condition for a contact metric* 3*-structure. In fact, the Lie group* $\widetilde{SL}(2, \mathbb{R})$ *admits three bi-contact metric structures which do not define a contact metric* 3*-structure (cf. Remark* 7.7).

3 Taut contact hyperbolas: first properties and examples

3.1 Definitions and first properties

We begin this Subsection recalling the definitions of contact circle, contact sphere and taut contact circle introduced by H. Geiges and J. Gonzalo on a manifold of dimension three (see, for example, [13, 15]). In all this section, by M we will denote always a three-manifold.

Let (η_1, η_2) be a pair of contact 1-forms on M. The pair (η_1, η_2) is called a **contact circle** if for every $a = (a_1, a_2) \in \mathbb{S}^1$, the unit circle in $\mathbb{R}^2$, the linear combination $\eta_a = a_1\eta_1 + a_2\eta_2$ is also a contact form. A contact circle (η_1, η_2) is said to be a **taut contact circle** if the volume forms $\eta_a \wedge (d\eta_a)$ are equal for every $a \in \mathbb{S}^1$. Equivalently, a pair of contact forms (η_1, η_2) is a taut contact circle if and only if

$$\eta_1 \wedge d\eta_1 = \eta_2 \wedge d\eta_2 \quad \text{and} \quad \eta_1 \wedge d\eta_2 = -\eta_2 \wedge d\eta_1. \tag{3.1}$$

In the case of closed three-manifolds, taut contact circles exist only on compact left quotients of the Lie groups: $\mathbb{S}^3 = SU(2)$, $\widetilde{SL}(2,R)$, $\widetilde{E}(2)$ (cf. [13], Theorem 1.2).

The unit circle $\mathbb{S}^1$ of the Euclidean plane has its counterpart, in the pseudo-Euclidean plane, that is, in the Minkowski plane, in the four arms of the unit equilateral hyperbolas

$$\mathbb{H}^1_r : x^2 - y^2 = r, \quad r = \pm 1.$$

Indeed, the equilateral hyperbolas have many of the properties of circles in the Euclidean plane (cf., for example, [8]). So we give the following definitions.

Definition 3.1 A pair (η_1, η_2) of contact forms on M is called a **contact hyperbola** if for every $a = (a_1, a_2) \in \mathbb{H}^1_r$, the linear combination $\eta_a = a_1\eta_1 + a_2\eta_2$ is also a contact form.

This definition implies that any non-trivial linear combination $\eta_a = a_1\eta_1 + a_2\eta_2$ with constant coefficients (a_1, a_2), $a_1^2 - a_2^2 \neq 0$, is again a contact form.

Definition 3.2 A contact hyperbola (η_1, η_2) is said to be a **taut contact hyperbola** if the volume forms $\eta_a \wedge (d\eta_a)$ satisfy

$$\eta_a \wedge (d\eta_a) = r\,\eta_1 \wedge (d\eta_1) \; for\, all \;\; a \in \mathbb{H}^1_r. \tag{3.2}$$

(equivalently, $\eta_a \wedge (d\eta_a) = -r\,\eta_2 \wedge (d\eta_2)$).

In [21], Mitsumatsu introduced a bi-contact structure (η_1, η_2) on a three-manifold, that is, η_1 and η_2 are mutually transverse contact 1-forms which induce opposite orientations. Anosov flow naturally induces a bi-contact structure whose intersection as a pair of plane fields is tangent to the flow. In general, the intersection of a bi-contact structure does not define an Anosov flow. In fact, he showed that if (η_1, η_2) is a bi-contact structure on a compact three-manifold, then the vector field directing the intersection of the two contact subbundles is a **conformally Anosov flow** (that they called projectively Anosov flow) which is a generalization of an Anosov flow. Eliashberg and Thurston [12] studied bi-contact structures and conformally Anosov flows from the viewpoint of confoliation theory.

The next proposition shows that the notion of taut contact hyperbola is related to that of conformally Anosov flow.

Proposition 3.3 *Let (η_1, η_2) be a pair of contact forms on M. Then, (η_1, η_2) is a taut contact hyperbola if and only if*

$$\eta_2 \wedge d\eta_2 = -\eta_1 \wedge d\eta_1 \quad \text{and} \qquad \eta_1 \wedge d\eta_2 = -\eta_2 \wedge d\eta_1. \tag{3.3}$$

In particular, when M is compact, a taut contact hyperbola defines a conformally Anosov flow. But, a pair of contact forms that defines a conformally Anosov flow, in general, does not define a taut contact hyperbola.

Proof Let (η_1, η_2) be a taut contact hyperbola. If we take $a = (0, 1) \in H^1_r$, $r = -1$, then $\eta_a = \eta_2$ and from (3.2) we get the condition $\eta_2 \wedge d\eta_2 = -\eta_1 \wedge d\eta_1$. Moreover, for all $a \in \mathbb{H}^1_r$, $r = \pm 1$, (3.2) and the above condition imply

$$r\eta_1 \wedge d\eta_1 = \eta_a \wedge d\eta_a = (a_1^2 - a_2^2)\eta_1 \wedge d\eta_1 + a_1 a_2(\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1),$$

and thus we get $\eta_1 \wedge d\eta_2 = -\eta_2 \wedge d\eta_1$.

Vice versa, if we assume (3.3), from

$$\eta_a \wedge d\eta_a = a_1^2\eta_1 \wedge d\eta_1 + a_2^2\eta_2 \wedge d\eta_2 + a_1a_2(\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1),$$

we obtain (3.2). In particular, a taut contact hyperbola is a bi-contact structure (in the sense of Mitsumatsu) and thus, in the compact case, it defines a conformally Anosov flow. The last part is a consequence of Remark 3.10. □

Remark 3.4 By using notations of complex/hyperbolic numbers, the conditions (3.1) and (3.3) can be write, respectively, in the simple forms

$$\eta^c \wedge d\eta^c = 0, \textit{ where } \eta^c = \eta_1 + i\eta_2,\ i^2 = -1, \quad \eta^h \wedge d\eta^h = 0, \text{ where } \eta^h = \eta_1 + j\eta_2, j^2 = 1, j \neq \pm 1.$$

Since the sphere $\mathbb{S}^3$ does not admit a conformally Anosov flow (cf. [21] , p.1420), from Proposition 3.3, follows that the sphere $\mathbb{S}^3$ does not admit a taut contact hyperbola (cf. also [24], Corollary 3.7).

Corollary 3.5 *The torus $\mathbb{T}^3$ admits a taut contact hyperbola (and so a conformally Anosov flow).*

Proof Consider on $\mathbb{R}^3$ the volume form $\Omega = dx \wedge dy \wedge dz$ and the contact 1-forms

$$\eta_1 = \cos z dx - \sin z dy \quad \text{and} \quad \eta_2 = \cos z dx + \sin z dy.$$

Then,

$$\eta_1 \wedge d\eta_1 = \Omega = -\eta_2 \wedge d\eta_2 \quad \text{and} \quad \eta_2 \wedge d\eta_1 = (\cos^2 z - \sin^2 z)\Omega = -\eta_1 \wedge d\eta_2.$$

Thus, by Proposition 3.3, (η_1, η_2) defines a taut contact hyperbola on $\mathbb{R}^3$. On the other hand, η_1 and η_2 are invariant under translation by 2π; therefore, (η_1, η_2) defines a taut contact hyperbola on the torus $\mathbb{T}^3$. □

Remark 3.6 The torus $\mathbb{T}^3$ has many conformally Anosov flows, while it has no Anosov flows because its fundamental group does not grow exponentially.

Now, recall that a taut contact circle with $\eta_1 \wedge d\eta_2 = \eta_2 \wedge d\eta_1 = 0$ is said to be a **Cartan structure** (see [13]). On the other hand, if (3.3) holds with $\eta_1 \wedge d\eta_2 = \eta_2 \wedge d\eta_1 = 0$, we have a taut contact hyperbola like-Cartan structure. So it is natural to give the following

Definition 3.7 A pair of contact 1-forms (η_1, η_2) is said to be a $(-\varepsilon)$-**Cartan structure** on the three-manifold M if $\eta_2 \wedge d\eta_2 = -\varepsilon\eta_1 \wedge d\eta_1$ and $\eta_1 \wedge d\eta_2 = 0 = \eta_2 \wedge d\eta_1$, $\varepsilon = \pm 1$.

Of course, a 1-Cartan structure is a Cartan structure and (-1)-Cartan structure is a taut contact hyperbola with $\eta_1 \wedge d\eta_2 = 0$.

3.2 Taut contact hyperbolas on 3*D* Lie groups

3.3 Unimodular case

Let G be a simply connected unimodular $3D$ Lie group. Then, G contains a discrete subgroup Γ such that the space of right cosets $\Gamma\backslash G$ is a differentiable manifold. Note that a three-dimensional Lie group G admits a discrete subgroup Γ such that $\Gamma\backslash G$ is compact if and only if G is unimodular ([20]). Moreover, each left-invariant tensor field on G descends to $\Gamma\backslash G$.

In particular, if we have a left-invariant taut contact hyperbola on G, then it descends to $\Gamma\backslash G$ and thus defines a conformally Anosov flow.

Now, we determine the simply connected unimodular $3D$ Lie groups G which admit taut contact hyperbolas. Since G is unimodular, there exist a basis of left invariant vector fields (e_1, e_2, e_3) such that ([20]) :

$$[e_2, e_3] = \lambda_1 e_1, \quad [e_3, e_1] = \lambda_2 e_2, \quad [e_1, e_2] = \lambda_3 e_3, \tag{3.4}$$

where $(\lambda_1, \lambda_2, \lambda_3)$ are constant. If $(\lambda_1, \lambda_2, \lambda_3) = (0, 0, 0)$, then G is the Abelian Lie group $\mathbb{R}^3$. The signature of $(\lambda_1, \lambda_2, \lambda_3) \neq (0, 0, 0)$ can be one of the following type:

- $(+, +, +)$ and in this case G is the three-sphere group $SU(2)$;
- $(+, +, 0)$ and in this case $G = \widetilde{E}(2)$ (the universal cover of the group of orientation-preserving isometries of the Euclidean plane);
- $(+, 0, 0)$ and in this case G is the Heisenberg group $\mathcal{H}^3 = Nil^3$;
- $(+, -, -)$ and in this case $G = \widetilde{SL}(2, R)$;
- $(+, -, 0)$ and in this case $G = Sol^3$ (also known as the group $E(1, 1)$ of orientation-preserving isometries of the Minkowski plane).

Denote by η_i the dual 1-forms : $\eta_i(e_j) = \delta_{ij}$. Then

$$d\eta_1 = -\lambda_1 \eta_2 \wedge \eta_3, \quad d\eta_2 = -\lambda_2 \eta_3 \wedge \eta_1, \quad d\eta_3 = -\lambda_1 \eta_1 \wedge \eta_2,$$

and thus

$$\eta_i \wedge d\eta_i = -\lambda_i \eta_1 \wedge \eta_2 \wedge \eta_3 \quad \text{and} \quad \eta_i \wedge d\eta_j = 0 \quad \text{for any } i \neq j.$$

In general, if (η_1, η_2, η_3) is a coframe on a three-manifold, the non-trivial 1-forms $\eta_a = a_1\eta_1 + a_2\eta_2 + a_3\eta_3$, $\eta_b = b_1\eta_1 + b_2\eta_2 + b_3\eta_3$ with constant coefficients $(a_i), (b_i)$, satisfy

$$\begin{aligned} \eta_a \wedge d\eta_b = {} & a_1b_1\eta_1 \wedge d\eta_1 + a_2b_2\eta_2 \wedge d\eta_2 + a_3b_3\eta_3 \wedge d\eta_3 + a_1b_2\eta_1 \wedge d\eta_2 + a_2b_1\eta_2 \wedge d\eta_1 \\ & + a_1b_3\eta_1 \wedge d\eta_3 + a_3b_1\eta_3 \wedge d\eta_1 + a_2b_3\eta_2 \wedge d\eta_3 + a_3b_2\eta_3 \wedge d\eta_2. \end{aligned} \tag{3.5}$$

Thus, two arbitrary left invariant 1-forms η_a, η_b on the unimodular Lie group G satisfy

$$\eta_a \wedge d\eta_b = -\mathcal{L}(a, b)\, \eta_1 \wedge \eta_2 \wedge \eta_3, \quad \text{where we put} \quad \mathcal{L}(a, b) = \lambda_1 a_1 b_1 + \lambda_2 a_2 b_2 + \lambda_3 a_3 b_3.$$

Consequently, by Proposition 3.3, the left invariant 1-forms (η_a, η_b) define a taut contact hyperbola if and only if the symmetric bilinear map $\mathcal{L}$ satisfies

$$\mathcal{L}(a,a) = -\mathcal{L}(b,b) \neq 0, \quad \mathcal{L}(a,b) = 0. \tag{3.6}$$

Therefore, the constant $(\lambda_1, \lambda_2, \lambda_3)$ that define the unimodular Lie groups $SU(2)$, $\tilde{E}(2)$, $\mathcal{H}^3$ and $\mathbb{R}^3$ do not satisfy the condition (3.6). Only the constant $(\lambda_1, \lambda_2, \lambda_3)$ that define the unimodular Lie groups Sol^3 and $\widetilde{SL}(2,\mathbb{R})$ satisfy the condition (3.6). Next, we determine the left invariant taut contact hyperbolas on these unimodular Lie groups.

Example 3.8 On the Lie group Sol^3 we can consider a basis of left invariant vector fields (e_1, e_2, e_3) such that

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = -2e_2, \quad [e_1, e_2] = 0. \tag{3.7}$$

In Example 8.2, we will give an explicit presentation of left invariant vector fields satisfying (3.7). The dual 1-forms η_i satisfy

$$d\eta_1 = -2\eta_2 \wedge \eta_3, \quad d\eta_2 = 2\eta_3 \wedge \eta_1, \quad d\eta_3 = 0,$$

and thus

$$\eta_1 \wedge d\eta_1 = -2\eta_1 \wedge \eta_2 \wedge \eta_3 = -\eta_2 \wedge d\eta_2, \qquad \eta_1 \wedge d\eta_2 = 0 = \eta_2 \wedge d\eta_1.$$

Therefore, by (3.6) (η_1, η_2) is a left invariant taut contact hyperbola on the unimodular Lie group Sol^3 and this structure descends to any compact left-quotient.

Moreover, by using (3.6), two arbitrary left invariant 1-forms (η_a, η_b) on the Lie group Sol^3 define a taut contact hyperbola if and only if

$$(a_1^2 - a_2^2) \neq 0 \text{ and } (b_1, b_2) = \pm(a_2, a_1).$$

Example 3.9 On the Lie group $\widetilde{SL}(2,\mathbb{R})$, we can consider a basis of left invariant vector fields (e_1, e_2, e_3) such that

$$[e_2, e_3] = 2e_1, \quad [e_3, e_1] = -2e_2, \quad [e_1, e_2] = -2e_3. \tag{3.8}$$

In Example 8.2, we will give an explicit presentation of left invariant vector fields satisfying (3.8). The dual 1-forms η_i satisfy

$$d\eta_1 = -2\eta_2 \wedge \eta_3, \quad d\eta_2 = 2\eta_3 \wedge \eta_1, \quad d\eta_3 = 2\eta_1 \wedge \eta_2,$$

and thus

$$-\eta_1 \wedge d\eta_1 = 2\eta_1 \wedge \eta_2 \wedge \eta_3 = \eta_2 \wedge d\eta_2 = \eta_3 \wedge d\eta_3, \qquad \eta_1 \wedge d\eta_j = 0 \text{ for any } i \neq j.$$

Therefore, (η_1, η_2) and (η_1, η_3) satisfy (3.6), i.e., they are left invariant taut contact hyperbolas on the unimodular Lie group $\widetilde{SL}(2,\mathbb{R})$ and these structures descend to any compact left-quotient.

Moreover, we can classify all left invariant taut contact hyperbolas on the Lie group $\widetilde{SL}(2,\mathbb{R})$. In this case, the bilinear map $\mathcal{L}$ defines on $\mathbb{R}^3$ the Lorentzian metric

$$g_0(a,b) = \mathcal{L}(a,b) = a_1 b_1 - a_2 b_2 - a_3 b_3, \quad \text{for any } a, b \in \mathbb{R}^3.$$

Then, by (3.6), *two arbitrary left invariant contact* 1*-forms* η_a,η_b *on* $\widetilde{SL}(2,\mathbb{R})$, *define a taut contact hyperbola if and only if* $g_0(b,b) = -g_0(a,a) \neq 0$ *and* $g_0(a,b) = 0$, *i.e., the vectors a and b are two orthogonal vectors, one of which is timelike and so the other is spacelike.*

Remark 3.10 We can consider two left invariant 1-forms η_a, η_b on the Lie group Sol^3 with

$$\mathcal{L}(b,b) = b_1^2 - b_2^2 = -(a_1^2 - a_2^2) = -\mathcal{L}(a,a) \neq 0 \text{ and } \mathcal{L}(a,b) = a_1 b_1 - a_2 b_2 \neq 0,$$

then η_a, η_b satisfy $\eta_b \wedge d\eta_b = -\eta_a \wedge d\eta_a$ and $\eta_a \wedge d\eta_b = \eta_b \wedge d\eta_a \neq 0$. Analogously, we can consider η'_a, η'_b on the Lie group $\widetilde{SL}(2,\mathbb{R})$ with

$$g_0(b,b) = -g_0(a,a) \neq 0 \text{ and } \ g_0(a,b) \neq 0.$$

In both the cases, (η_a, η_b) and (η'_a, η'_b) do not define a taut contact hyperbola, but they define a bi-contact structure in the sense of Mitsumatsu [21], and so define a conformally Anosov on any compact quotient of Sol^3 and $\widetilde{SL}(2,\mathbb{R})$, respectively. Therefore, *the notion of taut contact hyperbola is stronger than the notion of conformally Anosov.*

3.4 Non-unimodular case

Let G be a non-unimodular $3D$ Lie group. Then G admits a basis of left invariant vector fields e_1, e_2, e_3 such that (cf. [20])

$$[e_1,e_2] = \alpha e_2 + \beta e_3, \quad [e_1,e_3] = \gamma e_2 + \delta e_3, \quad [e_2,e_3] = 0, \quad \alpha + \delta \neq 0. \tag{3.9}$$

This Lie group G can be presented as a semi-direct product Lie group $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \begin{pmatrix} \alpha & \gamma \\ \beta & \delta \end{pmatrix}$, $\mathrm{tr}A = \alpha + \delta \neq 0$. Denote by

$$\mathcal{D} = 4\det A/(\mathrm{tr}A)^2$$

the invariant introduced by Milnor [20, p. 321], which, unless A is a multiple of the identity matrix, completely determines the non-unimodular Lie algebra (and so the Lie group) up to isomorphisms.

Let $(\vartheta^1, \vartheta^2, \vartheta^3)$ be the basis of 1-forms dual of (e_1, e_2, e_3). Then,

$$d\vartheta^1 = 0, \quad d\vartheta^2 = -\alpha\vartheta^1 \wedge \vartheta^2 - \gamma\vartheta^1 \wedge \vartheta^3, \quad d\vartheta^3 = -\beta\vartheta^1 \wedge \vartheta^2 - \delta\vartheta^1 \wedge \vartheta^3,$$

and thus

$$\vartheta^2 \wedge d\vartheta^2 = \gamma\,\Omega, \quad \vartheta^3 \wedge d\vartheta^3 = -\beta\,\Omega, \quad \vartheta^2 \wedge d\vartheta^3 = \delta\,\Omega \quad \text{and} \quad \vartheta^3 \wedge d\vartheta^2 = -\alpha\,\Omega,$$

where $\Omega = \vartheta^1 \wedge \vartheta^2 \wedge \vartheta^3$. Then, by using (3.5), two arbitrary left invariant 1-forms $\eta_a = \sum_{i=1}^3 a_i \vartheta^i$ and $\eta_b = \sum_{i=1}^3 b_i \vartheta^i$ satisfy

$$\eta_a \wedge d\eta_b = (\gamma a_2 b_2 + \delta a_2 b_3 - \alpha a_3 b_2 - \beta a_3 b_3)\Omega,$$
$$\eta_a \wedge d\eta_a = (\gamma a_2^2 + (\delta - \alpha) a_2 a_3 - \beta a_3^2)\Omega.$$

Therefore, (η_a, η_b) is a taut contact hyperbola on the non-unimodular Lie group G if, and only if, are satisfied the following:

$$\begin{cases} (\gamma a_2^2 + (\delta - \alpha) a_2 a_3 - \beta a_3^2) = -(\gamma b_2^2 + (\delta - \alpha) b_2 b_3 - \beta b_3^2) \neq 0, \\ 2\gamma a_2 b_2 + (\delta - \alpha)(a_2 b_3 + a_3 b_2) - 2\beta a_3 b_3 = 0. \end{cases} \tag{3.10}$$

In particular,

$$(\vartheta^2, \vartheta^3) \text{ is a taut contact hyperbola} \iff \beta = \gamma \neq 0 \text{ and } \alpha = \delta \neq 0.$$

In this case, the non-unimodular Lie group is defined by the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \alpha \end{pmatrix}$, $\alpha, \beta \neq 0$, and the Reeb vector fields of ϑ^2, ϑ^3 are given by $\xi_2 = e_2 - (\alpha/\beta)e_3$, $\xi_3 = -(\alpha/\beta)e_2 + e_3$. Then, ξ_2, ξ_3 are linearly independent if and only if the Milnor's invariant $\mathcal{D} = 4 \det A/(trA)^2 \neq 0$. Moreover, $\mathcal{D} = 0$ if and only if $\xi_2 = \pm\xi_3$. Thus, we get the following

Proposition 3.11 *The non-unimodular Lie group* $G = \mathbb{R}^2 \rtimes_A \mathbb{R}$, *where* $A = \begin{pmatrix} \alpha & \pm\alpha \\ \pm\alpha & \alpha \end{pmatrix}$, $\alpha \neq 0$, *admits a taut contact hyperbola* (η_1, η_2) *with the corresponding Reeb vector fields satisfying* $\xi_2 = \mp\xi_1$.

Remark 3.12 The result of the Proposition 3.11 gives an interesting difference with respect to the case of a taut contact circle. In fact, the Reeb vector fields of any taut contact circle are linearly independent (cf. Theorem 4.1).

Remark 3.13 Not all non-unimodular Lie groups admit a left invariant taut contact hyperbola. In fact, for $\alpha = \delta \neq 0$ and $\beta\gamma < 0$, the system (3.10) does not admit solution.

Now, we give an explicit example of non-unimodular Lie group satisfying Proposition 3.11.

Example 3.14 Consider the hyperbolic plane $\mathbb{H}^2 = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0\right\}$ equipped with standard Lie group structure. For any $\alpha \neq 0$, the vector fields

$$E_1 = 2\alpha\, x_2 \partial_1, \quad E_2 = 2\alpha\, x_2 \partial_2$$

define a basis of left invariant vector fields on $\mathbb{H}^2$. Now, consider the direct product Lie group

$$\mathcal{G}_{\mathcal{H}} = \mathbb{H}^2 \times \mathbb{R}.$$

Then $(E_1, E_2, E_3 = \partial_t)$ is a basis of left invariant vector fields on $\mathcal{G}_{\mathcal{H}}$. We note that with respect to the basis

$$e_1 = E_2, \quad e_2 = (E_1 + E_3), \quad e_3 = (E_1 - E_3),$$

the Lie algebra of $\mathcal{G}_{\mathcal{H}}$ is defined by $[e_2, e_3] = 0$, $[e_1, e_2] = [e_1, e_3] = \alpha(e_2 + e_3)$. So, $\mathcal{G}_{\mathcal{H}}$ is the non-unimodular Lie group $\mathbb{R}^2 \rtimes_A \mathbb{R}$, where $A = \alpha I_2$, with the Milnor's invariant $\mathcal{D} = 0$.

4 Some characterization of taut contact hyperbolas

We start this section recalling the following characterizations of taut contact circles.

Theorem 4.1 (Zessin [27]) *Let (η_1, η_2) be a contact circle on a three-manifold M, and let ξ_1, ξ_2 be the corresponding Reeb vector fields. Then, ξ_1, ξ_2 are everywhere linearly independent and $d\eta_1(\xi_2, \cdot)$, $d\eta_2(\xi_1, \cdot)$ never vanish. Moreover, the following properties are equivalent*

(i) (η_1, η_2) *is a taut contact circle;*
(ii) $\xi_a = a_1\xi_1 + a_2\xi_2$ *is the Reeb vector field of* $\eta_a = a_1\eta_1 + a_2\eta_2$ *for any* $a \in \mathbb{S}^1$;
(iii) $\eta_2(\xi_1) = -\eta_1(\xi_2)$ *and* $d\eta_1(\xi_2, \cdot) = -d\eta_2(\xi_1, \cdot)$.

In Proposition 3.11, we showed the existence of taut contact hyperbolas (η_1, η_2) with the corresponding Reeb vector fields $(\xi_1, \xi_2 = \pm\xi_1)$. Next, we give some characterization of taut contact hyperbolas with $\xi_2 \neq \pm\xi_1$. More precisely, we show the following theorem.

Theorem 4.2 *Let (η_1, η_2) be a pair of contact forms on a three-manifold M with $\xi_2 \neq \pm\xi_1$ in any point $p \in M$. Then, the following are equivalent:*

(a) (η_1, η_2) *is a taut contact hyperbola* ;
(b) $\eta_1(\xi_2) = \eta_2(\xi_1)$ *and* $d\eta_1(\xi_2, \cdot) = d\eta_2(\xi_1, \cdot) \neq 0$ *in any point* $p \in M$;
(c) $\eta_a = a_1\eta_1 + a_2\eta_2$*is a contact form with Reeb vector field*$\xi_a = r(a_1\xi_1 - a_2\xi_2)$*for any* $a \in \mathbb{H}^1_r$.

In particular, in a such case, ξ_1, ξ_2 are linearly independent.

We first give the following lemmas.

Lemma 4.3 *Let (η_1, η_2) be a contact hyperbola on M. Then,*

ξ_1, ξ_2 are pointwise linearly independent $\Longleftrightarrow$ $d\eta_1(\xi_2, \cdot)$ and $d\eta_2(\xi_1, \cdot)$ are $\neq 0$ in any point.

Moreover,

$\xi_a = r(a_1\xi_1 - a_2\xi_2)$is the Reeb vector field of $\eta_a = a_1\eta_1 + a_2\eta_2$ for all $a \in \mathbb{H}^1_r$ if and only if $\eta_2(\xi_1) = \eta_1(\xi_2)$ and $d\eta_1(\xi_2, \cdot) = d\eta_2(\xi_1, \cdot)$.

Proof ($\Rightarrow$) Suppose that there exists a point $p \in M$ such that $d\eta_2(\xi_1, \cdot) = 0$ at p, and so $d\eta_a(\xi_1, \cdot) = a_1 d\eta_1(\xi_1, \cdot) + a_2 d\eta_2(\xi_1, \cdot) = 0$ at p for any $a \in \mathbb{H}^1_r$. Then, since $(\eta_a \wedge d\eta_a)(\xi_1, \cdot, \cdot) \neq 0$, we get $\eta_a(\xi_1)_p \neq 0$ for any $a \in \mathbb{H}^1_r$, and in particular $\eta_2(\xi_1)_p \neq 0$. Now, we put $\xi_1' = (1/\varrho)\xi_1$, where $\varrho = \eta_2(\xi_1)_p \neq 0$. Then, $\eta_2(\xi_1')_p = 1$ and $(d\eta_2)(\xi_1', \cdot) = 0$ give the contradiction $(\xi_2)_p = (1/\varrho)(\xi_1)_p$, that is, ξ_1, ξ_2 linearly dependent at p.

($\Leftarrow$) If we suppose ξ_1, ξ_2 linearly dependent in some point p, i.e., $(\xi_2)_p = \varrho(\xi_1)_p$ for some constant $\varrho \neq 0$, then we have the contradiction $(d\eta_1)(\xi_2, \cdot)_p = \varrho(d\eta_1)(\xi_1, \cdot)_p = 0$. For the second part, it is sufficient to remark that

$$\eta_a(\xi_a) = 1 + ra_1a_2\big(\eta_2(\xi_1) - \eta_1(\xi_2)\big) \quad \text{and} \quad d\eta_a(\xi_a, \cdot) = ra_1a_2\big(d\eta_2(\xi_1, \cdot) - d\eta_1(\xi_2, \cdot)\big).$$

□

Lemma 4.4 *Let (η_1, η_2) be a contact hyperbola on M with $\xi_2 \neq \pm\xi_1$ in any point $p \in M$. Then, $d\eta_1(\xi_2, \cdot)$ and $d\eta_2(\xi_1, \cdot)$ are $\neq 0$ in any point $p \in M$.*

In particular, if $\xi_2 \neq \pm\xi_1$ in any point $p \in M$, then ξ_1, ξ_2 are linearly independent in any point $p \in M$.

Proof Assume that $\xi_2 \neq \pm\xi_1$ in any point, and suppose that there exists a point $p \in M$ such that $d\eta_2(\xi_1, \cdot)_p = 0$. Then, for all $a \in \mathbb{H}^1_r$

$$d\eta_a(\xi_1, \cdot) = a_1 d\eta_1(\xi_1, \cdot) + a_2 d\eta_2(\xi_1, \cdot) = 0 \ \text{ at } p.$$

Consequently, since η_a is a contact form, i.e., $\eta_a \wedge d\eta_a(\xi_1, \cdot, \cdot) \neq 0$, we obtain

$$\eta_a(\xi_1)_p \neq 0 \ \text{ for all } \ a \in \mathbb{H}^1_r.$$

In particular, taking $a = (0, 1)$, we have that $\lambda := \eta_2(\xi_1)_p \neq 0$.

Now, we show that $\lambda \neq 0$ gives a contradiction. Consider the function $f : \mathbb{H}^1_r \to \mathbb{R}$ defined by

$$f(a) = \eta_a(\xi_1)_p = a_1\eta_1(\xi_1)_p + a_2\eta_2(\xi_1)_p = a_1 + \lambda a_2.$$

Consider the cases: (I) $\lambda > 0$ and (II) $\lambda < 0$.

For the case (I), we distinguish the following subcases:

$$(I_1)\ \lambda > 1, \quad (I_2)\ 0 < \lambda < 1, \quad (I_3)\ \lambda = 1.$$

For the subcase (I_1) consider the function f defined on the connected subset $C_1 = \{a \in \mathbb{R}^2 : a_1^2 - a_2^2 = 1, a_1 > 0\}$. Put $\mu = (\lambda/\sqrt{\lambda^2 - 1}) > 1$ and take

$$a = (a_1, a_2), \bar{a} = (a_1, -a_2) \in C_1, \ \text{with } a_1 > \mu > 1 \text{ and } a_2 = \sqrt{a_1^2 - 1} > 0.$$

Then, $f(a) = a_1 + \lambda a_2 > 0$ and $f(\bar{a}) = a_1 - \lambda a_2 < 0$. Thus, it should exist $b \in C_1$ such that $f(b) = 0$, and this gives a contradiction.

For the subcase (I_2) consider the function f defined on the connected subset $C_2 = \{a \in \mathbb{R}^2 : a_1^2 - a_2^2 = -1, a_2 > 0\}$. Put $\mu = (1/\lambda) > 1$ and take

$$a = (a_1, a_2), \bar{a} = (-a_1, a_2) \in C_2, \ \text{with } a_1 > 0 \text{ and } a_2 > \mu/\sqrt{\mu^2 - 1} > 0.$$

Then, $f(a) = a_1 + \lambda a_2 > 0$ and $f(\bar{a}) = -a_1 + \lambda a_2 < 0$. Thus, it should exist $b \in C_2$ such that $f(b) = 0$, and this gives a contradiction.

For the subcase (I_3), $\eta_2(\xi_1)_p = 1$ and $d\eta_2(\xi_1, \cdot)_p = 0$ give the contradiction $\xi_2 = \xi_1$ at p.

For the case (II), we distinguish the following subcases:

$$(II_1)\ \lambda < -1, \quad (II_2)\ -1 < \lambda < 0, \quad (II_3)\ \lambda = -1.$$

For the subcase (II_1) consider the function f defined on the connected subset $C_3 = \{a \in \mathbb{R}^2 : a_1^2 - a_2^2 = 1, a_1 < 0\}$. Put $\mu = (\lambda/\sqrt{\lambda^2 - 1}) < -1$ and take

$$a = (a_1, a_2), \bar{a} = (a_1, -a_2) \in C_3, \ \text{with } a_1 < \mu < -1 \text{ and } a_2 = \sqrt{a_1^2 - 1} > 0.$$

Then, $f(a) = a_1 + \lambda a_2 < 0$ and $f(\bar{a}) = a_1 - \lambda a_2 > 0$. Thus, it should exist $b \in C_3$ such that $f(b) = 0$, and this gives a contradiction.

For the subcase (II_2) consider the function f defined on the connected subset $C_4 = \{a \in \mathbb{R}^2 : a_1^2 - a_2^2 = -1, a_2 < 0\}$. Put $\mu = (1/\lambda) < -1$ and take

$$a = (a_1, a_2), \bar{a} = (-a_1, a_2) \in C_4, \text{ with } a_1 = \sqrt{a_2^2 - 1} > 0 \text{ and } a_2 < \mu/\sqrt{\mu^2 - 1} < -1.$$

Then, $f(a) = a_1 + \lambda a_2 > 0$ and $f(\bar{a}) = -a_1 + \lambda a_2 < 0$. Thus, it should exist $b \in C_4$ such that $f(b) = 0$, and this gives a contradiction.

For the subcase (II_3), $\eta_2(-\xi_1)_p = 1$ and $d\eta_2(-\xi_1, \cdot)_p = 0$ give the contradiction $\xi_2 = -\xi_1$ at p. We proceed analogously if suppose $d\eta_1(\xi_2, \cdot)_p = 0$. The second part of the Lemma follows from Lemma 4.3. □

Proof of Theorem 4.2 $(a) \Longleftrightarrow (b)$

If we suppose (a) (respectively (b)), by Lemma 4.4 (respectively Lemma 4.3), we have that the Reeb vector fields ξ_1, ξ_2 are linearly independent. Moreover, we have

$$\begin{aligned} \eta_1 \wedge d\eta_1(\xi_1, \xi_2, \cdot) =& d\eta_1(\xi_2, \cdot), \quad \eta_2 \wedge d\eta_2(\xi_1, \xi_2, \cdot) = -d\eta_2(\xi_1, \cdot), \\ \eta_1 \wedge d\eta_2(\xi_1, \xi_2, \cdot) =& -\eta_1(\xi_2) d\eta_2(\xi_1, \cdot), \quad \eta_2 \wedge d\eta_1(\xi_1, \xi_2, \cdot) = \eta_2(\xi_1) d\eta_1(\xi_2, \cdot). \end{aligned}$$

Then, by using Proposition 3.3, it is not difficult to see that $(a) \Leftrightarrow (b)$.

$(c) \Longleftrightarrow (a)$

Suppose (c), that is, $\eta_a = a_1\eta_1 + a_2\eta_2$ is a contact form with Reeb vector field $\xi_a = r(a_1\xi_1 - a_2\xi_2)$ for any $a \in \mathbb{H}^1_r$. Then, (η_1, η_2) is a contact hyperbola and, from Lemma 4.4, ξ_1, ξ_2 are linearly independent and $d\eta_1(\xi_2, \cdot), d\eta_2(\xi_1, \cdot) \neq 0$ in any point $p \in M$. Then, Lemma 4.3 gives $\eta_1(\xi_2) = \eta_2(\xi_1)$ and $d\eta_1(\xi_2, \cdot) = d\eta_2(\xi_1, \cdot) \neq 0$. So, we get (b) and thus (a).

Conversely, suppose (a), that is, (η_1, η_2) is a taut contact hyperbola and thus $\eta_a = a_1\eta_1 + a_2\eta_2$ is a contact form for any $a \in \mathbb{H}^1_r$. Since (a) is equivalent to (b), by Lemma 4.3 we get (c). □

We close this subsection with remarking that the taut contact hyperbolas are related to the symplectic pair.

Remark 4.5 **(taut contact hyperbola/circle and symplectic structures)**

Let (M, η, ξ, φ) be an almost contact manifold. We denote by $C(M) = \mathbb{R}_+ \times M$ the cone on M, for more information about the geometry of the cone $C(M)$ we refer, for example, to [4] Section 6.5. Consider the (1, 1)-tensor J on $C(M)$ defined by

$$JX = \varphi X \text{ for } X \in \ker \eta, \quad J\xi = \zeta, \quad J\zeta = -\xi,$$

where $\zeta = t\frac{\partial}{\partial t}$ is *Liouville (or Euler) vector field*. Then, J is an almost complex structure invariant under the flow of $\zeta : \mathcal{L}_\zeta J = 0$. Moreover, it well-known that η is a contact form on M if and only if the 2-form $\Omega = d(t^2\eta)$ is a symplectic form on $C(M)$.

Now, let (η_1, η_2) be a pair of contact forms on a three-manifold M and $(\eta_i, \xi_i, \varphi_i)$, $i = 1, 2$, underlying almost contact structures. Then the corresponding symplectic forms $\Omega_i = d(t^2\eta_i) = 2tdt \wedge \eta_i + t^2 d\eta_i$, $i = 1, 2$, satisfy:

$$\begin{aligned} \Omega_1 \wedge \Omega_1 &== 4t^3 dt \wedge \eta_1 \wedge d\eta_1, \\ \Omega_2 \wedge \Omega_2 &= 4t^3 dt \wedge \eta_2 \wedge d\eta_2, \\ \Omega_1 \wedge \Omega_2 &= 2t^3 dt \wedge (\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1) = \Omega_2 \wedge \Omega_1, \\ (a_1\Omega_1 + a_2\Omega_2) \wedge (a_1\Omega_1 + a_2\Omega_2) &= 4t^3 dt \wedge \eta_a \wedge d\eta_a. \end{aligned}$$

So, (η_1, η_2) is a contact hyperbola (resp. circle) if and only if $\Omega_a := (a_1\Omega_1 + a_2\Omega_2)$ is a symplectic 2-form on the four-dimensional cone $C(M)$ for any $(a_1, a_2) \in \mathbb{H}^1_r$ (resp. for any $(a_1, a_2) \in \mathbb{S}^1$).

On the other hand, on a four-manifold, following Bande and Kotschick [2]: a *symplectic pair* is defined by two symplectic forms (ω_1, ω_2) that satisfy

$$\omega_1 \wedge \omega_2 = 0 \text{ and } \omega_1 \wedge \omega_1 = -\omega_2 \wedge \omega_2,$$

and following Geiges [14]: (ω_1, ω_2) is said to be a *conformal symplectic couple* if

$$\omega_1 \wedge \omega_2 = 0 \quad \text{and} \quad \omega_1 \wedge \omega_1 = \omega_2 \wedge \omega_2.$$

Therefore (cf. also Section 6 of [24] where we studied the metric cone of a bi-contact metric manifold) we get:

(η_1, η_2) is a taut contact hyperbola (resp. circle) if and only if the corresponding symplectic 2-forms (Ω_1, Ω_2) define a symplectic pair (resp. a conformal symplectic couple) .

Moreover, given a pair of contact forms (η_1, η_2) on M with $\xi_2 \neq \pm\xi_1$ in any point $p \in M$, the corresponding symplectic 2-forms Ω_1, Ω_2 satisfy:

$$\begin{aligned}
\Omega_1(\xi_2, X) &= 2t(dt \wedge \eta_1)(\xi_2, X) + t^2 d\eta_1(\xi_2, X) = t^2(d\eta_1)(\xi_2, X) \text{ for } X \text{ tangent to } M;\\
\Omega_1(\xi_2, \partial t) &= 2t(dt \wedge \eta_1)(\xi_2, \partial t) + t^2 d\eta_1(\xi_2, \partial t) = -t\eta_1(\xi_2);\\
\Omega_2(\xi_1, X) &= 2t(dt \wedge \eta_2)(\xi_1, X) + t^2 d\eta_2(\xi_1, X) = t^2(d\eta_2)(\xi_1, X) \text{ for } X \text{ tangent to } M;\\
\Omega_2(\xi_1, \partial t) &= 2t(dt \wedge \eta_2)(\xi_1, \partial t) + t^2 d\eta_2(\xi_1, \partial t) = -t\eta_2(\xi_1).
\end{aligned}$$

Then, by Theorems 4.2 and 4.1, we get

(η_1, η_2) is a taut contact hyperbola (resp. circle) if and only if the corresponding symplectic 2-forms (Ω_1, Ω_2) satisfy $\Omega_2(\xi_1, \cdot) = \Omega_1(\xi_2, \cdot) \neq 0$ (resp. $\Omega_2(\xi_1, \cdot) = -\Omega_1(\xi_2, \cdot) \neq 0$) in any point.

5 Taut contact 2-hyperboloid

Recall (cf. [13]) that a *contact sphere* on a three-manifold M is a triple of contact 1-forms (η_1, η_2, η_3) such that any linear combination $(a_1\eta_1 + a_2\eta_2 + a_3\eta_3)$, $a \in \mathbb{S}^2$, is a contact form; moreover, it is taut if the volume forms $\eta_a \wedge (d\eta_a)$ on M are equal for every $a \in \mathbb{S}^2$; moreover in this case the 1-forms (η_1, η_2, η_3) parallelize the three-manifold M.

Now, consider the surface

$$H^2_r : a_1^2 - a_2^2 - a_3^2 = r, r \pm 1,$$

that is, H^2_{-1} is an one-sheeted hyperboloid and H^2_1 is a two-sheeted hyperboloid. In analogy with the definition of (taut) contact 2-sphere we give the following definition. We say that a triple of contact 1-forms (η_1, η_2, η_3) on a three-manifold M, is a *contact 2-hyperboloid* if the 1-form

$$\eta_a = a_1\eta_1 + a_2\eta_2 + a_3\eta_3 \textit{ is a contact form for any } a \in H^2_r.$$

This definition implies that any non-trivial linear combination $\eta_a = a_1\eta_1 + a_2\eta_2 + a_3\eta_3$ with constant coefficients (a_1, a_2, a_3), $a_1^2 - a_2^2 - a_3^2 \neq 0$, is again a contact form. We call the triple (η_1, η_2, η_3) a *taut contact 2-hyperboloid* if the volume forms $r(\eta_a \wedge d\eta_a)$ are equal for every $a \in H^2_r$, that is,

$$\eta_a \wedge d\eta_a = r\,\eta_1 \wedge d\eta_1 \quad \text{for every } \ a \in H^2_r.$$

Besides, we note that

$$\begin{aligned}\eta_a \wedge d\eta_a = {}& a_1^2\eta_1 \wedge d\eta_1 + a_2^2\eta_2 \wedge d\eta_2 + a_3^2\eta_3 \wedge d\eta_3 + a_1a_2(\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1)\\ &+ a_1a_3(\eta_1 \wedge d\eta_3 + \eta_3 \wedge d\eta_1) + a_2a_3(\eta_3 \wedge d\eta_2 + \eta_2 \wedge d\eta_3).\end{aligned}$$

Consequently, we get

Proposition 5.1 *A triple of contact* 1*-forms* (η_1, η_2, η_3) *on a three-manifold M is a taut contact 2-hyperboloid if and only if* (η_1, η_2) *and* (η_1, η_3) *are taut contact hyperbola and the other pair* (η_2, η_3) *is a taut contact circle.*

Remark 5.2 We note that a triple of contact 1-forms (η_1, η_2, η_3) on a three-manifold M defines a taut contact 2-sphere if and only if (η_1, η_2), (η_1, η_3) and (η_2, η_3) are taut contact circles.

A difference, with respect to the taut contact 2-spheres, is that: *in general the* 1*-forms* η_1, η_2, η_3 *that define a taut contact* 2*-hyperboloid are not linearly independent.* In fact, we have the following.

Example 5.3 On the torus $\mathbb{T}^3$ the contact 1-forms

$$\eta_1 = (\cos z)dx - (\sin z)dy, \quad \eta_2 = (\cos z)dx + (\sin z)dy \ \text{ and } \ \eta_3 = (\sin z)dx - (\cos z)dy,$$

are linearly dependent and define a taut contact 2-hyperboloid. More precisely: (η_1, η_2), (η_1, η_3) are taut contact hyperbola, and (η_2, η_3) is a taut contact circle.

Now, we give the following

Proposition 5.4 *Let* (η_1, η_2, η_3) *be a coframe of contact* 1*-forms on a three-manifold M. Denote by* (e_1, e_2, e_3) *the frame dual of* (η_1, η_2, η_3)*. Then,* (η_1, η_2, η_3) *is a taut contact 2-hyperboloid if and only if there exist three* 1*-forms* $\beta_1, \beta_2, \beta_3$ *and a nonzero smooth function* λ *such that*

$$\begin{cases} d\eta_1 = \beta_1 \wedge \eta_1 + \lambda\eta_2 \wedge \eta_3, \\ d\eta_2 = \beta_2 \wedge \eta_2 - \lambda\eta_3 \wedge \eta_1, \\ d\eta_3 = \beta_3 \wedge \eta_3 - \lambda\eta_1 \wedge \eta_2, \end{cases} \tag{5.1}$$

where the 1*-forms* β_i *satisfy*

$(*)$ $\beta_i(e_i) = 0$ and $\beta_i(e_j) = \beta_k(e_j)$ for any (i, j, k) permutation of $(1, 2, 3)$.

In particular, $\beta_1 = \beta_2 = \beta_3$ *if and only if* $\beta_i = 0$ *for any* $i = 1, 2, 3$*. If this is the case, then the function* λ *is a constant* $\neq 0$.

Proof Since (η_1, η_2, η_3) is a coframe, we put

$$\begin{cases} d\eta_1 = f_1\eta_1 \wedge \eta_2 + f_2\eta_2 \wedge \eta_3 + f_3\eta_3 \wedge \eta_1, \\ d\eta_2 = g_1\eta_1 \wedge \eta_2 + g_2\eta_2 \wedge \eta_3 + g_3\eta_3 \wedge \eta_1, \\ d\eta_3 = h_1\eta_1 \wedge \eta_2 + h_2\eta_2 \wedge \eta_3 + h_3\eta_3 \wedge \eta_1, \end{cases}$$

where f_i, g_i, h_i are smooth functions. Then, the conditions

$$\eta_1 \wedge d\eta_1 = -\eta_2 \wedge d\eta_2 = -\eta_3 \wedge d\eta_3$$

are equivalent to the conditions $f_2 = -g_3 = -h_1 = \lambda$, where λ is a nowhere zero function. Moreover, the conditions

$$0 = \eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1 = \eta_1 \wedge d\eta_3 + \eta_3 \wedge d\eta_1 = \eta_2 \wedge d\eta_3 + \eta_3 \wedge d\eta_2$$

are equivalent to the conditions $g_2 = -f_3 = -\lambda_3$, $h_2 = -f_1 = \lambda_2, h_3 = -g_1 = -\lambda_1$. Thus, (η_1, η_2, η_3) is a taut contact 2-hyperboloid if and only if

$$\begin{cases} d\eta_1 = \beta_1 \wedge \eta_1 + \lambda\eta_2 \wedge \eta_3, \\ d\eta_2 = \beta_2 \wedge \eta_2 - \lambda\eta_3 \wedge \eta_1, \\ d\eta_3 = \beta_3 \wedge \eta_3 - \lambda\eta_1 \wedge \eta_2, \end{cases}$$

where

$$\beta_1 = (\lambda_2\eta_2 + \lambda_3\eta_3), \quad \beta_2 = (\lambda_1\eta_1 + \lambda_3\eta_3) \text{ and } \beta_3 = (\lambda_1\eta_1 + \lambda_2\eta_2).$$

Consequently, the 1-forms β_i satisfy $(*)$; moreover, $\beta_1 = \beta_2 = \beta_3$ if and only if $\beta_1 = \beta_2 = \beta_3 = 0$. In this case, we have

$$0 = d^2\eta_1 = d\lambda \wedge \eta_2 \wedge \eta_3 + \lambda(d\eta_2) \wedge \eta_3 - \lambda\eta_2 \wedge d\eta_3 = d\lambda \wedge \eta_2 \wedge \eta_3 \Rightarrow e_1(\lambda) = 0.$$

Analogously, $e_2(\lambda) = e_3(\lambda) = 0$, and so λ is a constant. □

Corollary 5.5 *A simply connected three-manifold M admits a taut contact 2-hyperboloid* (η_1, η_2, η_3), *with the corresponding Reeb vector fields* (ξ_1, ξ_2, ξ_3) *that constitute the frame dual of the coframe* (η_1, η_2, η_3), *if and only if M is the Lie group* $\widetilde{SL}(2, \mathbb{R})$.

Proof In the Proposition 5.4, the condition $\beta_1 = \beta_2 = \beta_3$, i.e., $\beta_i = 0$ for any $i = 1, 2, 3$, is equivalent to the condition that the vector fields e_i are the Reeb vector fields of the contact forms η_i, $i = 1, 2, 3$. Then, if (η_1, η_2, η_3) is a taut contact 2-hyperboloid with the corresponding Reeb vector fields (ξ_1, ξ_2, ξ_3) that constitute the frame dual of the coframe (η_1, η_2, η_3), from (5.1) we get that the 1-forms β_i vanish and so the Reeb vector fields satisfy

$$[\xi_1, \xi_2] = \lambda\xi_3, \quad [\xi_2, \xi_3] = -\lambda\xi_1, \quad [\xi_3, \xi_1] = \lambda\xi_2, \ \lambda = \text{const.} \neq 0.$$

Therefore, M admits a Lie group structure isomorphic to $\widetilde{SL}(2, \mathbb{R})$.

Conversely, if we consider the Lie group $\widetilde{SL}(2, \mathbb{R})$, by using the notations of the Example 3.9, we have the 1-forms (η_1, η_2) and (η_1, η_3) are left invariant contact hyperbolas and (η_2, η_3) is a left invariant taut contact circle. Then, by Proposition 5.1, we get that the 1-forms (η_1, η_2, η_3) define a *left invariant taut contact 2-hyperboloid*. Moreover, the corresponding Reeb vector fields (ξ_1, ξ_2, ξ_3) constitute the frame dual of the coframe (η_1, η_2, η_3). □

Remark 5.6 In the compact case, taut contact 2-spheres exist only on left quotients of the three-sphere group $\mathbb{S}^3 = SU(2)$ ([13], Theorem 1.10). In particular, the torus $\mathbb{T}^3$ does not admit a taut contact 2-sphere, however it admits a taut contact 2-hyperboloid.

6 Taut contact hyperbolas and the Chern–Hamilton energy functional

Let (M, η) be an oriented compact contact manifold. Denote by $\mathcal{M}(\eta)$ the set of all Riemannian metrics associated to the contact form η and by $\mathcal{A}(\eta)$ the set of all almost CR structures J for which the Levi form is positive definite. The sets $\mathcal{M}(\eta)$ and $\mathcal{A}(\eta)$ can be identified (cf., for example, Proposition 8 of [25]). Tanno [26] considered the *Dirichlet energy*

$$E(g) = \int_M \|\tau\|^2 dv, \quad \tau = \mathcal{L}_\xi g, \tag{6.1}$$

defined for any $g \in \mathcal{M}(\eta)$. Then, he found the critical point condition ([26], Theorem 5.1)

$$\nabla_\xi \tau = 2\tau\varphi, \text{ equivalently } \nabla_\xi h = -2\varphi h. \tag{6.2}$$

The Dirichlet energy (6.1) was first studied by Chern and Hamilton [9] for compact contact three-manifolds as a functional defined on the set $\mathcal{A}(\eta)$ (there was an error in their calculation of the critical point condition, as was pointed out by Tanno). This functional is known in literature also with the name of *Chern–Hamilton energy functional*. Moreover, since $Ric(\xi, \xi) = 2n - trh^2 = 2n - \|\tau\|^2/4$, the functional (6.1) is equivalent to the functional $L(g) = \int_M Ric(\xi, \xi)dv$ studied in general dimension, for compact regular contact manifold, by Blair ([3], Section 10.3). We note that K-contact metrics and Sasakian metrics are trivial critical metrics, besides we note that the critical point condition (6.2) has a tensorial character, so it holds also in the non-compact case. On the other hand, the sphere $\mathbb{S}^3$ admits a Sasakian structure, therefore: *in general a Sasakian three-manifold fails to admit a taut contact hyperbola*.

Next, we show that the critical point condition (6.2) is a sufficient condition for the existence of a taut contact hyperbola on a non-Sasakian contact metric three-manifold. In fact, we have the following.

Theorem 6.1 *Let $(M, \eta, g, \varphi, \xi)$ be a non-Sasakian contact metric three-manifold, that is, the torsion $\tau \neq 0$ at any point. If the metric g satisfies the critical point condition for the Dirichlet energy functional* (6.1), *then M admits a taut contact hyperbola.*

Proof Let $(M, \eta, g, \varphi, \xi)$ be a non-Sasakian contact metric three-manifold. Let $\{e_1, e_2 = \varphi e_1, \xi\}$ be an orthonormal basis of smooth eigenvectors for h with $h\xi = 0$, $he_1 = \lambda e_1$, $he_2 = -\lambda e_2$, λ being the positive eigenvalue. Since the three eigenvalues $0, \lambda, -\lambda$ of h are everywhere distinct, the corresponding line fields are global and by the orientability the basis can be taken to be global. Let η_1, η_2 be the 1-forms g-dual to e_1 and e_2, respectively, and hence (η_1, η_2, η) is a global basis of 1-forms. Using (2.1), we have

$$\nabla_{e_1}\xi = -\varphi e_1 - \varphi h e_1 = -(1+\lambda)e_2 \quad \text{and} \quad \nabla_{e_2}\xi = (1-\lambda)e_1. \tag{6.3}$$

By straightforward computation and using $\nabla_\xi \xi = 0$, we get

$$\nabla_\xi e_1 = a\, e_2 \qquad \text{and} \qquad \nabla_\xi e_2 = -a\, e_1, \tag{6.4}$$

where $a = g(\nabla_\xi e_1, e_2)$ is a smooth function. Moreover, $(\nabla_\xi h)\xi = 0$ and by using (6.4) we obtain

$$(\nabla_\xi h)e_1 = \nabla_\xi h e_1 - h(\nabla_\xi e_1) = \xi(\lambda)e_1 + 2a\lambda\, e_2$$

and

$$(\nabla_\xi h)e_2 = (\nabla_\xi h)\varphi e_1 = -\varphi(\nabla_\xi h)e_1 = -\xi(\lambda)e_2 + 2a\lambda\, e_1.$$

Thus,

$$(\nabla_\xi h) = -2a\, h\varphi + \big(\xi(\lambda)/\lambda\big)h.$$

Consequently, since g satisfies the critical point condition (6.2), we have $a = -1$ and $\xi(\lambda) = 0$. Thus, (6.4) becomes

$$\nabla_\xi e_1 = -e_2 \qquad \text{and} \qquad \nabla_\xi e_2 = e_1. \tag{6.5}$$

Then, by using (6.3) and (6.5) we have

$$\begin{aligned}
\eta_1 \wedge d\eta_1(\xi, e_1, e_2) &= -(d\eta_1)(\xi, e_2) = \frac{1}{2} g(e_1, \nabla_\xi e_2 - \nabla_{e_2}\xi) = \frac{\lambda}{2} > 0,\\
\eta_2 \wedge d\eta_2(\xi, e_1, e_2) &= (d\eta_2)(\xi, e_1) = -\frac{1}{2} g(e_2, \nabla_\xi e_1 - \nabla_{e_1}\xi) = -\frac{\lambda}{2} < 0,\\
\eta_1 \wedge d\eta_2(\xi, e_1, e_2) &= -(d\eta_2)(\xi, e_2) = \frac{1}{2} g(e_2, \nabla_\xi e_2 - \nabla_{e_2}\xi) = 0,\\
\eta_2 \wedge d\eta_1(\xi, e_1, e_2) &= (d\eta_1)(\xi, e_1) = -\frac{1}{2} g(e_1, \nabla_\xi e_1 - \nabla_{e_1}\xi) = 0,
\end{aligned}$$

Therefore, by using Proposition 3.3, the 1-forms (η_1, η_2) define a taut contact hyperbola.

□

Following Y. Eliashberg [11], a contact manifold (M, η) is called *overtwisted* if there exists an embedded disk D in M such that $T_pD = \ker \eta_p$ for all $p \in \partial D$. It is called *tight* if it is not overtwisted. Moreover, the contact distribution is called *universally tight* if even its lift to the universal cover of M is tight. Recently, S. Hozoori ([18], Theorem 1.4) proved, in the compact case, that a conformally Anosov contact three-manifold is universally tight. On the other hand, by proof of Theorem 6.1 we note that the intersection of the bi-contact structure (η_1, η_2) is given by $\mathbb{R}\xi$ and thus, by [21], $(M, \ker \eta)$ is a conformally Anosov contact three-manifold. Therefore, we have the following

Corollary 6.2 *Let $(M, \eta, g, \varphi, \xi)$ be a compact non-Sasakian contact metric three-manifold. If the metric g is a critical metric for the Dirichlet energy functional, then $(M, \ker \eta)$ is universally tight.*

Next, we exhibit an example of compact non-Sasakian contact metric three-manifold with the contact Riemannian metric g critical for the Dirichlet energy (6.1).

Example 6.3 Let (M, g) be a compact 2-dimensional Riemannian manifold of constant sectional curvature $k < 0$. By Theorem 7 and Corollary 3 of the paper [1], we get that the unit tangent sphere bundle T_1M admits a family of non-Sasakian contact metric structures $(\tilde{\eta}_a, \tilde{G}_a)$, depending on one parameter $a > 0$, satisfying the critical point condition (6.2), where the critical metric $\tilde{G}_a$ is a Riemannian g-natural metric. In particular, for $a = 1/4$ and $k = -1$, $(\tilde{\eta}_a, \tilde{G}_a)$ is the standard (non-Sasakian) contact Riemannian structure on T_1M satisfying the critical point condition (6.2) ([3], Th. 10.13, p.208), where $\tilde{G}_a$ is the classic Sasaki metric $\tilde{G}_S$. In general, for $a > 0$, $\tilde{G}_a$ is a *metric of Kaluza–Klein type*, i.e., horizontal and tangential lifts are mutually orthogonal with respect to $\tilde{G}_a$. To note that the Sasaki metric on T_1M, in general, is not Sasakian in the sense of the contact Riemannian geometry.

7 Geometry of bi-contact metric structures

In this section, we study the geometry of a three-manifold determined by the existence of a bi-contact metric structure. The following theorem, which can be considered as a more complete presentation of Theorem 3.6 of [24], will be very useful for this study.

Theorem 7.1 *Let (η_1, η_2) be a pair of contact forms on a three-manifold M, with Reeb vector fields (ξ_1, ξ_2), $\xi_2 \neq \pm\xi_1$. Then, the following are equivalent.*

(I) (η_1, η_2) *defines a bi-contact metric structure, i.e., there exists a Riemannian metric g for which (η_1, η_2, g) is a bi-contact metric structure.*

(II) (η_1, η_2) *is a $(-\varepsilon)$-Cartan structure, i.e.,*

$$\eta_1 \wedge d\eta_2 = \eta_2 \wedge d\eta_1 = 0, \quad \eta_2 \wedge d\eta_2 = -\varepsilon \eta_1 \wedge d\eta_1, \ \varepsilon = \pm 1.$$

(III) There exists a unique 1-form η_3 such that

$$d\eta_1 = -2\eta_2 \wedge \eta_3, \quad d\eta_2 = -2\varepsilon \eta_1 \wedge \eta_3, \quad d\eta_3 = 2\kappa \eta_2 \wedge \eta_1,$$

where the smooth function $\kappa = (d\eta_3)(\xi_2, \xi_1)$ satisfies $d\kappa \wedge \eta_1 \wedge \eta_2 = 0$.

Proof $(I) \Rightarrow (II)$.

Let (η_1, η_2, g) be a bi-contact metric structure and $(\eta_1, \xi_1, \varphi_1, g)$, $(\eta_2, \xi_2, \varphi_2, g)$ the corresponding contact metric structure with $g(\xi_1, \xi_2) = 0$. Consider the vector field $\xi_3 = \varphi_1 \xi_2 = \varepsilon \varphi_2 \xi_1$, $\varepsilon = \pm 1$ (cf. Definition 2.1). Then (ξ_1, ξ_2, ξ_3) is a global orthonormal basis and $\eta_i(\xi_j) = \delta_{ij}$ for $i = 1, 2$ and $j = 1, 2, 3$. Consequently, $\eta_1 \wedge d\eta_2 = 0 = \eta_2 \wedge d\eta_1$ and

$$(\eta_2 \wedge d\eta_2)(\xi_1, \xi_2, \xi_3) = -\eta_2(\xi_2) d\eta_2(\xi_1, \xi_3) = -g(\xi_1, \varphi_2 \xi_3) = \varepsilon, \ (\eta_1 \wedge d\eta_1)(\xi_1, \xi_2, \xi_3) = \ldots = 1$$

that is, $\eta_2 \wedge d\eta_2 = -\varepsilon \eta_1 \wedge d\eta_1$.

$(II) \Rightarrow (III)$.

By using (II), (3.1) and (3.3), we get that (η_1, η_2) ia taut contact hyperbola (resp. circle) if $\varepsilon = 1$ (resp. $\varepsilon = -1$). From Theorem 4.2 (if $\varepsilon = 1$) and Theorem 4.1 (if $\varepsilon = -1$), we get that ξ_1, ξ_2 are linearly independent, $\eta_1(\xi_2) = \varepsilon \eta_2(\xi_1)$ and $d\eta_1(\xi_2, \cdot) = \varepsilon d\eta_2(\xi_1, \cdot) \neq 0$

everywhere. Then, $\eta_1 \wedge d\eta_2 = 0 = \eta_2 \wedge d\eta_1$ implies $\eta_1(\xi_2) = \eta_2(\xi_1) = 0$. Now, consider the 1-form

$$\eta_3 = d\eta_1(\cdot, \xi_2) = \varepsilon d\eta_2(\cdot, \xi_1) \neq 0 \text{ everywhere.}$$

Then, $\eta_3(\xi_2) = \eta_3(\xi_1) = 0$ and there exists a vector field ξ_3 such that $\eta_3(\xi_3) = 1$, (ξ_1, ξ_2, ξ_3) is a basis and thus $\eta_1 \wedge \eta_2 \wedge \eta_3$ is a volume form and $(\eta_1 \wedge \eta_2, \eta_1 \wedge \eta_3, \eta_2 \wedge \eta_3)$ is a basis of 2-forms. Since

$$d\eta_1(\xi_1, \cdot) = d\eta_2(\xi_2, \cdot) = 0, d\eta_1(\xi_2, \xi_3) = -\eta_3(\xi_3) = -1 \text{ and } d\eta_2(\xi_1, \xi_3) = -\varepsilon\eta_3(\xi_3) = -\varepsilon,$$

we get

$$d\eta_1 = -2\eta_2 \wedge \eta_3, \quad d\eta_2 = -2\varepsilon\eta_1 \wedge \eta_3. \tag{7.1}$$

Finally, by using (7.1), we have

$$0 = d\eta_2 \wedge \eta_3 = \eta_2 \wedge d\eta_3 \quad \text{and} \quad 0 = d\eta_1 \wedge \eta_3 = \eta_1 \wedge d\eta_3$$

and hence $d\eta_3 = 2\kappa\, \eta_2 \wedge \eta_3$ where κ is a smooth function satisfying $d\kappa \wedge \eta_1 \wedge \eta_2 = 0$, that is, $\kappa_3 = 0$ where we put $d\kappa = \sum_i \kappa_i \eta_i$. The 1-form η_3 satisfying (*III*) is unique because from (7.1) one gets that $\eta_3 = -d\eta_1(\xi_2, \cdot) = -\varepsilon d\eta_2(\xi_1, \cdot)$.

(*III*) $\Rightarrow$ (*I*).

Let (ξ_1, ξ_2, ξ_3) be the triple of vector fields dual to the basis (η_1, η_2, η_3) of 1-forms. We note that ξ_1, ξ_2 are necessarily the Reeb vector fields of η_1, η_2, respectively. Moreover by the usual formulae, the dual of the equations of (*III*) are

$$[\xi_1, \xi_2] = 2\kappa\, \xi_3, \quad [\xi_2, \xi_3] = 2\, \xi_1, \quad [\xi_3, \xi_1] = -2\varepsilon\, \xi_2, \tag{7.2}$$

where κ is a smooth function. Now, we consider the Riemannian metric g defined by $g(\xi_i, \xi_j) = \delta_{ij}$. Then $\eta_1 = g(\xi_1, \cdot)$ and $\eta_2 = g(\xi_2, \cdot)$. Moreover, if we define the (1, 1)-tensors φ_1 and φ_2 by

$$\varphi_1\xi_1 = 0, \quad \varphi_1\xi_2 = \xi_3, \quad \varphi_1\xi_3 = -\xi_2, \quad \varphi_2\xi_2 = 0, \quad \varphi_2\xi_1 = \varepsilon\xi_3, \quad \varphi_2\xi_3 = -\varepsilon\xi_1,$$

then $(\eta_1, \xi_1, \varphi_1)$ and $(\eta_2, \xi_2, \varphi_2)$ are almost contact structures. Moreover, by using (7.2), we get $d\eta_1 = g(\cdot, \varphi_1)$ and $d\eta_2 = g(\cdot, \varphi_2)$, that is , (η_1, η_2, g) is a bi-contact metric structure on M. □

From Theorem 7.1 follows that for a bi-contact metric structure (η_1, η_2, g) are uniquely determined the 1-form $\eta_3 = d\eta_1(\cdot, \xi_2)$ and the smooth function $\kappa = (d\eta_3)(\xi_2, \xi_1)$. In particular, we deduce

Corollary 7.2 *If a 3D Lie group G admits a left invariant bi-contact metric structure, then G is unimodular.*

The next example shows that there exist taut contact hyperbolas/circles which do not define $(-\varepsilon)$-Cartan structures.

Example 7.3 Consider on $\mathbb{R}^3$ the 1-forms

$$\eta_1 = (ay + bz)f(x)\, dx + dy \quad \text{and} \quad \eta_2 = (\varepsilon by + az)f(x)\, dx + dz$$

where $f(x)$ is a positive smooth function and $a, b \in \mathbb{R}$, $a, b \neq 0$. Then

$$d\eta_1 = af(x)\,dy \wedge dx + bf(x)\,dz \wedge dx \quad \text{and} \quad d\eta_2 = \varepsilon bf(x)\,dy \wedge dx + af(x)\,dz \wedge dx.$$

Consequently,

$$\eta_1 \wedge d\eta_1 = bf(x)\,dx \wedge dy \wedge dz, \quad \eta_2 \wedge d\eta_2 = -\varepsilon\eta_1 \wedge d\eta_1 \neq 0$$

and

$$\eta_1 \wedge d\eta_2 = -\eta_2 \wedge d\eta_1 = af(x)\,dx \wedge dy \wedge dz \neq 0.$$

So for $\varepsilon = 1$, (η_1, η_2) defines a taut contact hyperbola, and $\varepsilon = -1$, (η_1, η_2) defines a taut contact circle. In both the cases (η_1, η_2) does not define a $(-\varepsilon)$-Cartan structure. Besides, by Remark 4.5, we get that on the four manifold $\mathbb{R}^3 \times \mathbb{R}_+$ the corresponding symplectic 2-forms define a symplectic pair for $\varepsilon = 1$ and a conformal symplectic couple for $\varepsilon = -1$.

Remark 7.4 **An interpretation of the function κ in terms of the Webster scalar curvature**. Consider the *Webster scalar curvature* $\mathcal{W}$ as defined by Chern and Hamilton [9] in their study on contact Riemannian three-manifolds. If (M, η, g) is a contact Riemannian three-manifold, the Webster scalar curvature is given by ([9], p.284)

$$\mathcal{W} = (1/8)(w - Ric(\xi, \xi) + 4),$$

where w is the usual scalar curvature and $Ric(\xi, \xi)$ is the Ricci curvature in the direction of the Reeb vector field ξ. We note that the generalized Tanaka-Webster scalar curvature $\hat{w}$ (cf. [26]) is eight times the Webster scalar curvature $\mathcal{W}$ as defined by Chern and Hamilton. Moreover, a compact simply connected regular Sasakian $(2n+1)$-manifold is a principal $\mathbb{S}^1$ -bundle over a compact Kaehler manifold B of complex dimension n, and the generalized Tanaka-Webster scalar curvature $\hat{w}$ is the scalar curvature of the Kaehler manifold B ([25] p.26). Of course, in dimension three, B is a Riemann surface and hence the Webster scalar curvature $\mathcal{W}$ determines the Gaussian curvature $(4\mathcal{W})$ and the Eulero-Poincaré characteristic of B.

Now, let (η_1, η_2, g) be a bi-contact metric structure. Then, we have the following (cf. [24], p.224):

- if (η_1, η_2) is a taut contact circle, i.e., $\varepsilon = -1$, the Webster scalar curvatures of (η_1, g) and (η_2, g) are given by the same function

$$\mathcal{W} = (\kappa + 1)/2; \tag{7.3}$$

- if (η_1, η_2) is a taut contact hyperbola, i.e., $\varepsilon = +1$, the Webster scalar curvatures of (η_1, g) and (η_2, g) are given, respectively, by the functions

$$\mathcal{W}_1 = (\kappa - 1)/2 \quad \text{and} \quad \mathcal{W}_2 = -(\kappa + 1)/2.$$

So, in both the cases the function κ determines the Webster curvature and it does not depend on the associated metric g. Therefore, we call κ **the Webster function** of the $(-\varepsilon)$ -Cartan structure (η_1, η_2). If (η_1, η_2) is a taut contact circle, i.e., a Cartan structure, the Webster function κ is invariant for an Euclidean rotation of constant angle (cf. [24]).

Now, we suppose that (η_1, η_2) is a taut contact hyperbola and consider a hyperbolic rotation of constant angle of (η_1, η_2), i.e.,

$$(\eta_1' = a_1\eta_1 + a_2\eta_2, \eta_2' = r(a_2\eta_1 + a_1\eta_2), r = \pm 1, \text{ with } a = (a_1, a_2) \in \mathbb{H}_1^1.$$

Since

$$\begin{aligned} \eta_1' \wedge d\eta_1' &= (a_1^2 - a_2^2)\eta_1 \wedge d\eta_1 = \eta_1 \wedge d\eta_1, \quad \eta_2' \wedge d\eta_2' = (a_2^2 - a_1^2)\eta_1 \wedge d\eta_1 = -\eta_1 \wedge d\eta_1, \\ \eta_1' \wedge d\eta_2' &= r(a_1^2 - a_2^2)\eta_1 \wedge d\eta_2 = 0, \quad \eta_2' \wedge d\eta_1' = r(a_2^2 - a_1^2)\eta_1 \wedge d\eta_2 = 0, \end{aligned}$$

then (η_1', η_2') is again a (-1)-Cartan structure. Moreover, the corresponding Reeb vector fields of the contact forms (η_1', η_2') are $\xi_1' = a_1\xi_1 - a_2\xi_2$ and $\xi_2' = -r(a_2\xi_1 - a_1\xi_2)$. Consequently,

$$\eta_3' = -d\eta_1'(\xi_2', \cdot) = r(a_1 d\eta_1 + a_2 d\eta_2)(a_2\xi_1 - a_1\xi_2, \cdot) = r\eta_3 = \ldots = -d\eta_2'(\xi_1', \cdot),$$

and thus $d\eta_3' = rd\eta_3 = 2r\kappa\eta_2 \wedge \eta_1 = \ldots = 2\kappa\eta_2' \wedge \eta_1'$. Therefore, the Webster function of the (-1)-Cartan structure (η_1', η_2') is $\kappa' = \kappa$, i.e., *for a (-1)-Cartan structure, the Webster function κ is invariant for a hyperbolic rotation of constant angle.*

About the 1-form η_3 and the Webster function κ, we have the following

Theorem 7.5 *Let (η_1, η_2, g) be a bi-contact metric structure on the three-manifold M. Then, for the 1-form η_3 hold the following properties.*

(1) η_3 is a Killing 1-form (with respect to g) if and only if (η_1, η_2) is a taut contact circle.
(2) η_3 is a contact form if and only if the Webster function $\kappa \neq 0$ everywhere.
(3) If η_3 is a contact form, then

$$(\eta = -\bar{\varepsilon}\eta_3, g_\kappa = \varepsilon\bar{\varepsilon}\kappa g + (1 - \varepsilon\bar{\varepsilon}\kappa)\eta \otimes \eta), \text{ where } \bar{\varepsilon} = \varepsilon(\mathit{sign}\kappa),$$

is a contact metric structure of the three-manifold M, and it is a Sasakian structure if and only if (η_1, η_2) is a taut contact circle.

Proof Let (η_1, η_2, g) be a bi-contact metric structure on the three-manifold M. Denote by $(\eta_i, \varphi_i, \xi_i, g)$, $i = 1, 2$, the corresponding contact metric structures with $g(\xi_1, \xi_2) = 0$. The 1-form η_3 defined by (III) of Theorem 7.1 is given by $\eta_3 = -d\eta_1(\xi_2, \cdot) \neq 0$ everywhere, and thus

$$\eta_3 = d\eta_1(\cdot, \xi_2) = g(\cdot, \varphi_1\xi_2) = g(\xi_3, \cdot), \quad \text{where} \xi_3 := \varphi_1\xi_2 = \varepsilon\varphi_2\xi_1.$$

Moreover, $g(\xi_3, \xi_3) = \eta_3(\xi_3) = g(\varphi_1\xi_2, \varphi_1\xi_2) = 1$, $g(\xi_1, \xi_3) = -g(\varphi_1\xi_1, \xi_2) = 0$ and $g(\xi_2, \xi_3) = -\varepsilon g(\varphi_2\xi_2, \xi_3) = 0$. So (ξ_1, ξ_2, ξ_3) is an orthonormal basis, dual to the basis of 1-forms (η_1, η_2, η_3). Moreover, by proof of Theorem 7.1, the basis (ξ_1, ξ_2, ξ_3) satisfies (7.2). Consequently, the fundamental tensors $h_1 = (1/2)\mathcal{L}_{\xi_1}\varphi_1$ and $h_2 = (1/2)\mathcal{L}_{\xi_2}\varphi_2$ of the contact metric structures (η_1, g), (η_2, g), respectively, satisfy

$$\begin{cases} h_1\xi_1 = 0, \quad h_1\xi_2 = (\kappa + \varepsilon)\xi_2, \quad h_1\xi_3 = -(\kappa + \varepsilon)\xi_3, \\ h_2\xi_1 = \varepsilon(1 - \kappa)\xi_1, \quad h_2\xi_2 = 0, \quad h_2\xi_3 = -\varepsilon(1 - \kappa)\xi_3. \end{cases} \tag{7.4}$$

Now, recall that a 1-form η on a Riemannian manifold (M, g) is a Killing form if and only if

$$i(X)d\eta = \nabla_X \eta \text{ for any vector field } X \text{ on } M,$$

where ∇ is the Levi-Civita connection of the metric g. Then, in our case, by using the notations introduced before, η_3 is a Killing form if and only if

$$d\eta_3(\xi_i,\xi_j) = (\nabla_{\xi_i}\eta_3)\xi_j \text{ for } i,j = 1,2,3.$$

By (III) of Theorem 7.1, $d\eta_3 = 2\kappa\,\eta_2 \wedge \eta_1$, thus

$$d\eta_3(\xi_i,\xi_j) = 0 \text{ when } i = 3 or j = 3, \text{ and } d\eta_3(\xi_1,\xi_2) = -\kappa.$$

Moreover, we have

$$(\nabla_{\xi_i}\eta_3)\xi_j = -g\big(\xi_3, \nabla_{\xi_i}\xi_j\big).$$

Since, by using (2.1), (7.4), and (7.2), we get

$$\begin{cases} \nabla_{\xi_1}\xi_1 = 0, & \nabla_{\xi_2}\xi_1 = -(1+\kappa+\varepsilon)\xi_3, \quad \nabla_{\xi_3}\xi_1 = (1-\kappa-\varepsilon)\xi_2, \\ \nabla_{\xi_2}\xi_2 = 0, & \nabla_{\xi_1}\xi_2 = -(1-\kappa+\varepsilon)\xi_3, \quad \nabla_{\xi_3}\xi_2 = (\varepsilon-1+\kappa)\xi_1, \\ \nabla_{\xi_1}\xi_3 = (1-\kappa+\varepsilon)\xi_2, & \nabla_{\xi_2}\xi_3 = (1+\kappa+\varepsilon)\xi_1, \quad \nabla_{\xi_3}\xi_3 = 0. \end{cases} \tag{7.5}$$

Then, we obtain

$$(\nabla_{\xi_1}\eta_3)\xi_2 = (1-\kappa+\varepsilon),\ (\nabla_{\xi_2}\eta_3)\xi_1 = (1+\kappa+\varepsilon) \text{ and } (\nabla_{\xi_i}\eta_3)\xi_j = 0 \text{ in the other cases.}$$

Therefore,

$$d\eta_3(\xi_i,\xi_j) = (\nabla_{\xi_i}\eta_3)\xi_j \text{ for } i,j = 1,2,3 \iff \varepsilon + 1 = 0,$$

that is, the property (1). Since $\eta_3 \wedge d\eta_3 = 2\kappa\,\eta_3 \wedge \eta_2 \wedge \eta_1$, we get the property (2).

Now, suppose that η_3 is a contact form, that is, the Webster function $\kappa \neq 0$ everywhere. Then, $\xi_3 = \varphi_1\xi_2$ is the Reeb vector field of the contact form η_3. If we define the tensor φ_3 by

$$\varphi_3\xi_3 = 0, \varphi_3\xi_1 = -\varepsilon\xi_2, \varphi_3\xi_2 = \varepsilon\xi_1,$$

since $\eta_3(\xi_i) = \delta_{3i}$, we have $\varphi_3^2 = -I + \eta_3 \otimes \xi_3$. Consider the tensors

$$\begin{aligned} \eta =& -\bar\varepsilon\eta_3, \quad \varphi = \varphi_3, \quad \xi = -\bar\varepsilon\xi_3, \\ g_\kappa =& \varepsilon\bar\varepsilon\kappa g + (1-\varepsilon\bar\varepsilon\kappa)\eta_3 \otimes \eta_3, \quad \text{where we put } \bar\varepsilon = sign(\varepsilon\kappa). \end{aligned}$$

It is not difficult to see that these tensors satisfy $\varphi^2 = -I + \eta \otimes \xi$, $\eta = g_\kappa(\xi,\cdot)$ and $d\eta = g_\kappa(\cdot,\varphi)$. Therefore, $(\eta,\varphi,\xi,g_\kappa)$ is a contact metric structure on the three manifold M. Moreover, this structure is Sasakian if and only if the almost contact metric structure (η,φ,ξ) is normal.

Recall that in dimension three, any almost CR structure is integrable, then by using Theorem 11 of [25] we get that the almost contact structure (η,φ,ξ) is normal (equivalently, the induced almost CR structure is normal) if and only if ξ is a CR Reeb vector field, that is, the tensor $\mathcal{L}_\xi J$ vanishes, where $J = \varphi_{|\ker\eta}$, $\ker\eta$ =span(ξ_1,ξ_2). By using (7.2),

$$\begin{aligned} -\bar\varepsilon\big(\mathcal{L}_\xi J\big)\xi_1 =& \big(\mathcal{L}_{\xi_3}\varphi_3\big)\xi_1 = [\xi_3,\varphi_3\xi_1] - \varphi_3[\xi_3,\xi_1] = 2(\varepsilon+1)\xi_1, -\bar\varepsilon\big(\mathcal{L}_\xi J\big)\xi_2 = \big(\mathcal{L}_{\xi_3}\varphi_3\big) \\ \xi_2 =& [\xi_3,\varphi_3\xi_2] - \varphi_3[\xi_3,\xi_2] = -2(\varepsilon+1)\xi_2. \end{aligned}$$

Then, (ξ, η, φ) is normal if and only if $\varepsilon + 1 = 0$, that is, (η_1, η_2) is a taut contact circle. □

Next, we examine the property (3) of Theorem 7.5. Consider the contact metric structure $(\eta = -\bar{\varepsilon}\eta_3, g_\kappa = \varepsilon\bar{\varepsilon}\kappa g + (1 - \varepsilon\bar{\varepsilon}\kappa)\eta_3 \otimes \eta_3)$, $\bar{\varepsilon} = sign(\varepsilon\kappa)$, which is Sasakian if and only if $\varepsilon = -1$. We note that the metric $g_\kappa = g$ if and only if the Webster function $\kappa = \pm 1$.

Now, for $\kappa = \pm 1$, the vector fields (ξ_1, ξ_2, ξ_3) are g-orthonormal and satisfy (7.2), for which $\widetilde{M}$ has a Lie group structure isomorphic to $SU(2)$ or $\widetilde{SL}(2, \mathbb{R})$; moreover,

$$(\eta_1, \eta_2, g), \quad (\eta_1, -\bar{\varepsilon}\eta_3, g), \quad (\eta_2, -\bar{\varepsilon}\eta_3, g) \tag{7.6}$$

are three left invariant bi-contact metric structures with

$$\eta_3 \wedge d\eta_3 = -\varepsilon\kappa\, \eta_2 \wedge d\eta_2 = \kappa\, \eta_1 \wedge d\eta_1, \quad \eta_i \wedge d\eta_j = 0, i \neq j.$$

So, we distinguish the following cases.

a) $(\kappa, \varepsilon) = (1, -1)$. In this case $(\eta, \xi, \varphi, g_\kappa) = (\eta_3, \xi_3, \varphi_3, g)$ and $\widetilde{M}$ is the Lie group $SU(2)$. Moreover, the tensors $(\xi_i, \eta_i, \varphi_i)$, $i = 1, 2, 3$, related to the bi-contact metric structures (7.6), satisfy the condition (2.3). Thus, $(\eta_1, \eta_2, \eta_3, g)$ is a contact metric 3-structure. In particular, by Remark 5.2, the triple (η_1, η_2, η_3) defines a taut contact 2-sphere. On the other hand, a contact metric 3-structure is a Sasakian 3-structure (see, for example, [3] p.293) and g is of constant sectional curvature $+1$.

b) $(\kappa, \varepsilon) = (-1, 1), (1, 1), (-1, -1)$. In this case $\widetilde{M}$ is the Lie group $\widetilde{SL}(2, \mathbb{R})$.

b_1) If $(\kappa, \varepsilon) = (-1, 1)$, then $(\eta, \xi, \varphi, g_\kappa) = (\eta_3, \xi_3, \varphi_3, g)$ is not Sasakian and thus $(\eta_1, \eta_2, \eta_3, g)$ is not a contact metric 3-structure. Moreover, since $\kappa + \varepsilon = 0$, from (7.4) we have that (η_1, g) is Sasakian. Besides, by using (7.5) a direct computation gives that the Ricci tensor of g is $Ric = -6g + 8\eta_1 \otimes \eta_1$. Then, if we consider the corresponding Lorentzian-Sasakian structure $(\eta_1, g_L = g - 2\eta_1 \otimes \eta_1)$, from formula (22) of [25], the corresponding Ricci tensor is given by $Ric_L = Ric + 4g - 4\eta_1 \otimes \eta_1 = -2g_L$, and thus g_L is a Lorentzian metric of constant sectional curvature -1.

b_2) If $(\kappa, \varepsilon) = (1, 1)$, also in this case $(\eta, \xi, \varphi, g_\kappa) = (-\eta_3, -\xi_3, \varphi_3, g)$ is not Sasakian and thus $(\eta_1, \eta_2, -\eta_3, g)$ is not a contact metric 3-structure. Moreover, since $\kappa = \varepsilon$, from (7.4) we have that (η_2, g) is Sasakian. Then, as in the case b_1), we get that $(\eta_2, g_L = g - 2\eta_2 \otimes \eta_2)$ is a Lorentzian-Sasakian structure with g_L Lorentzian metric of constant sectional curvature -1.

b_3) If $(\kappa, \varepsilon) = (-1, -1)$, the structures (η_1, g),(η_2, g) are not Sasakian, thus $(\eta_1, \eta_2, -\eta_3, g)$ is not a contact metric 3-structure, but the structure $(\eta, \xi, \varphi, g_\kappa)$ =$(-\eta_3, -\xi_3, \varphi_3, g)$ is Sasakian. Then, as in the case b_1), we get that $(-\eta_3, g_L = g - 2\eta_3 \otimes \eta_3)$ is a Lorentzian-Sasakian structure with g_L Lorentzian metric of constant sectional curvature -1.

Finally, in all the cases b_i), $i = 1, 2, 3$, from Proposition 5.1, follows that the 1-forms η_1, η_2, η_3 define a taut contact 2-hyperboloid.

Summing up, we get

Corollary 7.6 *Let (η_1, η_2, g) be a bi-contact metric structure on the three-manifold M with the Webster function $\kappa \neq 0$ everywhere. Then, the metric $g_\kappa = g$ if and only if $\kappa =$ const.$= \pm 1$.*

In this case, i.e., for κ *=const.=* ± 1,

$$(\eta_1, \eta_2, g), (\eta_1, -\kappa\varepsilon\eta_3, g) \text{ and } (\eta_2, -\kappa\varepsilon\eta_3, g)$$

are three left invariant bi-contact metric structures and $\widetilde{M}$ *is either* $SU(2)$ *or* $\widetilde{SL}(2, \mathbb{R})$.

More precisely, we have the following.

- *If* $(\kappa, \varepsilon) = (1, -1)$, $\widetilde{M}$ *is* $SU(2)$ *and* $(\eta_1, \eta_2, \eta_3, g)$ *is a* 3*-Sasakian structure on it, where* (η_1, η_2, η_3) *is a taut contact* 2*-sphere and* g *is of constant sectional curvature* $+1$.
- *If* $(\kappa, \varepsilon) = (-1, 1), (1, 1), (-1, -1)$, $\widetilde{M}$ *is* $\widetilde{SL}(2, \mathbb{R})$ *and the* 1*-forms* η_1, η_2, η_3 *define a taut contact* 2*-hyperboloid. Besides, for* $(\kappa, \varepsilon) = (-1, -1)$ (*resp.* $(\kappa, \varepsilon) = (-1, 1), (1, 1)$) *the structure* $(-\eta_3, g_L = g - 2\eta_3 \otimes \eta_3)$ (*resp.* $(\eta_1, g_L = g - 2\eta_1 \otimes \eta_1)$, $(\eta_2, g_L = g - 2\eta_2 \otimes \eta_2)$) *is a Lorentzian-Sasakian structure with* g_L *Lorentzian metric of constant sectional curvature* -1.

Remark 7.7 From Corollary 7.6 we get that the Lie group $\widetilde{SL}(2, \mathbb{R})$ admits three left invariant bi-contact metric structures, with the same associated metric, which do not define a contact metric 3-structure.

8 Generalized Finsler structures and bi-contact metric structures

The main purpose of this section is to see how the taut contact hyperbolas are related to generalized Finsler structures, and construct examples of bi-contact metric structures (η_1, η_2, g) with (η_1, η_2) taut contact hyperbola. On the other hand, in [24], we posed the question to find examples (if there exist) of bi-contact metric structures (η_1, η_2, g) on 3-manifolds which are not homogeneous, and thus with the Webster function κ non-constant, where the 1-forms (η_1, η_2) satisfy the conditions that define a taut contact hyperbola. So, by this study we give, in particular, a positive answer to this question (see Example 8.2).

Let M be a three-manifold. Following R. Bryant [5, 6], a coframe $(\omega_1, \omega_2, \omega_3)$ on M is said to be a (I, J, K)-**generalized Finsler structure** if it satisfies the following structure equations

$$\begin{cases} d\omega_1 = -\omega_2 \wedge \omega_3, \\ d\omega_2 = \omega_1 \wedge \omega_3 + I\omega_3 \wedge \omega_2, \\ d\omega_3 = -K\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3, \end{cases} \tag{8.1}$$

where (I, J, K) are smooth functions on M, known as the *main scalar*, the *Landsberg curvature* and the *flag curvature*, respectively.

We note that if $(\omega_1, \omega_2, \omega_3)$ is a (I, J, K)-generalized Finsler structure, then $(\omega_1, -\omega_2, -\omega_3)$ is a $(-I, -J, K)$-generalized Finsler structure. As remarked in [5] and [6], the difference between the notions of Finsler structure and generalized Finsler structure is global in nature, that is, any generalized Finsler structure is locally diffeomorphic to a Finsler structure, hence M can be realized locally as the unit sphere bundle of a Finsler surface (N, F) in such a way that the given coframing is the canonical coframing induced on M by the (local) Finsler structure F.

In the sequel, for a smooth function f on M equipped with a generalized Finsler structure, we put $df = \sum_{i=1}^{3} f_i\, \omega_i$. Computing the exterior derivative of the structure equations (8.1), one gets the so called Bianchi identities (cf. [5], Section 1; [6], Section 2.2)

$$I_1 = J, \qquad J_1 = -K_3 - KI. \tag{8.2}$$

In particular, $I =$ const. implies $J = 0$; $J =$const. and $K =$const.$\neq 0$ imply $I = J = 0$. When $I = J = 0$, the generalized Finsler structure is locally a Riemannian structure.

Denote by Ω the volume form $\omega_1 \wedge \omega_2 \wedge \omega_3$. From (8.1), a simply computation gives

$$\omega_2 \wedge d\omega_2 = \omega_1 \wedge d\omega_1 = -\Omega, \quad \omega_3 \wedge d\omega_3 = K\omega_1 \wedge d\omega_1 = -K\Omega, \quad \omega_1 \wedge d\omega_2 = -I\,\Omega,$$
$$\omega_2 \wedge d\omega_1 = 0, \quad \omega_1 \wedge d\omega_3 = -J\,\Omega, \quad \omega_3 \wedge d\omega_1 = 0, \quad \omega_2 \wedge d\omega_3 = \omega_3 \wedge d\omega_2 = 0.$$

Then, from Theorem 7.1, we get

Proposition 8.1 *Let $(\omega_1, \omega_2, \omega_3)$ be a (I, J, K)-generalized Finsler structure on a three-manifold M. Then*

a) (ω_1, ω_2) *defines a bi-contact metric structure if and only if* $\varepsilon = -1, I = J = 0$.
b) (ω_1, ω_3) *defines a bi-contact metric structure if and only if* $K = -\varepsilon, J = I = 0$.
c) (ω_2, ω_3) *defines a bi-contact metric structure if and only if* $K = -\varepsilon$.

Bryant et al. [6] studied Finsler surfaces of constant flag curvature $K = 0, \pm 1$, with a Killing field.

Next, we discuss separately the cases *a*), *b*), *c*).

- **The case** *a*) : $\varepsilon = -1, I = J = 0$. In this case the triple of 1-forms $(\eta_1, \eta_2, \eta_3) = (1/2)(\omega_1, \omega_2, \omega_3)$ satisfies (*III*) of Theorem 7.1, where (η_1, η_2) is a taut contact circle with the Webster function $\kappa = K$. So, *if the flag curvature K is $\neq 0$ everywhere, from Theorem 7.5 follows that ω_3 is a Killing contact form*. Moreover, in this case, we have a generalized Riemann structure in the sense of [7]. Of course, a bi-contact metric structure with $\varepsilon = -1$ defines a $(0, 0, \kappa)$ generalized Finsler structure.

  A model for this type of structure is implicitly given in [24]. More precisely, consider the space $\mathbb{R}^3(x_1, x_2, t)$, a smooth function $\sigma = \sigma(x_1, x_2)$ and put $\sigma_1 = \partial\sigma/\partial x_1$, $\sigma_2 = \partial\sigma/\partial x_2$, $\sigma_{11} = \partial^2\sigma/\partial x_1^2$ and $\sigma_{22} = \partial^2\sigma/\partial x_2^2$. Then, the 1-forms

$$\omega_1 = e^{\sigma}\big((\cos t)dx_1 + (\sin t)dx_2\big),$$
$$\omega_2 = e^{\sigma}\big(-(\sin t)dx_1 + (\cos t)dx_2\big),$$
$$\omega_3 = -\sigma_2 dx_1 + \sigma_1 dx_2 + dt,$$

define a coframe on $\mathbb{R}^3(x_1, x_2, t)$. Moreover, they satisfy the structure equations (8.1) of a generalized Finsler structure with $I = J = 0$ and $K = -e^{-2\sigma}(\sigma_{11} + \sigma_{22})$. We note that if (N, G) is a Riemannian surface, using isothermal local coordinates (x_1, x_2) on N, the Riemannian metric G is given by $G = e^{2\sigma}(dx_1^2 + dx_2^2)$ and, in terms of these coordinates, the function $K = -e^{-2\sigma}(\sigma_{11} + \sigma_{22})$ is its *Gaussian curvature*.

- In the case *b*): $K = -\varepsilon, I = J = 0$, the triple of 1-forms $(\eta_1, \eta_2, \eta_3) = (1/2)(\omega_1, \omega_3, -\omega_2)$ satisfies (*III*) of Theorem 7.1 with $\kappa = 1$, where (η_1, η_2) is a taut contact hyperbola (resp. circle) if $\varepsilon = 1$ (resp. $\varepsilon = -1$).

- In the case c): $K = -\varepsilon$, the most interesting case for our study, the structure equations become

$$\begin{cases} d\omega_1 = -\omega_2 \wedge \omega_3, \\ d\omega_2 = \omega_1 \wedge \omega_3 + I\omega_3 \wedge \omega_2, \\ d\omega_3 = \varepsilon\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3. \end{cases}$$

Then, the 1-forms

$$\eta_1 = (1/2)\omega_3, \quad \eta_2 = (1/2)\omega_2 \text{ and } \eta_3 = (1/2)(\varepsilon\omega_1 + J\omega_3 - \varepsilon I\omega_2)$$

satisfy:

$$\begin{aligned} d\eta_1 =& (1/2)d\omega_3 = (1/2)(\varepsilon\omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3) = 2\eta_3 \wedge \eta_2, \\ -2\varepsilon\eta_1 \wedge \eta_3 =& -(1/2)\omega_3 \wedge (\omega_1 + J\varepsilon\omega_3 - I\omega_2) = (1/2)d\omega_2 = d\eta_2. \end{aligned}$$

Besides,

$$\begin{aligned} 2d\eta_3 &= (\varepsilon d\omega_1 + dJ \wedge \omega_3 + Jd\omega_3 - \varepsilon dI \wedge \omega_2 - \varepsilon I d\omega_2) \\ &= (J_2 - J^2 + \varepsilon I_3 + \varepsilon I^2 - \varepsilon)\omega_2 \wedge \omega_3 + (J_1 - \varepsilon I)\omega_1 \wedge \omega_3 + (\varepsilon J - \varepsilon I_1)\omega_1 \wedge \omega_2, \end{aligned}$$

and thus, by using (8.2), we have

$$d\eta_3 = 2\kappa\, \eta_2 \wedge \eta_1, \text{ where } \kappa = (J_2 - J^2 + \varepsilon I_3 + \varepsilon I^2 - \varepsilon).$$

Therefore, for $K = -1$ (resp. $K = 1$), we get taut contact hyperbolas (resp. circles) with the Webster function κ, in general, non-constant. If $K = 1$ and $\kappa \neq 0$ everywhere, from Theorem 7.5 follows that η_3 is a Killing contact form.

Next, we give an explicit example of bi-contact metric structure (η_1, η_2, g) where (η_1, η_2) is a taut contact hyperbola with the Webster function κ non-constant.

Example 8.2 Let M be a connected open subset of $\mathbb{R}^3$. On M we consider the following 1-forms

$$\omega_1 = dx + xdy + dz, \quad \omega_2 = -\frac{\cosh z}{f(x)}dx + f(x)(\sinh z)dy, \quad \omega_3 = -\frac{\sinh z}{f(x)}dx + f(x)(\cosh z)dy$$

where $f(x)$ is a positive smooth function defined on M. These forms satisfy:

$$\begin{aligned} d\omega_1 =& dx \wedge dy = \omega_3 \wedge \omega_2, \\ d\omega_2 =& -\frac{\sinh z}{f(x)}dz \wedge dx + f'(x)(\sinh z)dx \wedge dy + f(x)(\cosh z)dz \wedge dy, \\ d\omega_3 =& -\frac{\cosh z}{f(x)}dz \wedge dx + f'(x)(\cosh z)dx \wedge dy + f(x)(\sinh z)dz \wedge dy, \\ \omega_1 \wedge \omega_3 =& -\frac{\sinh z}{f(x)}dz \wedge dx + f(x)\cosh z\,(dz \wedge dy) + \Big(f(x)(\cosh z) + x\frac{\sinh z}{f(x)}\Big)dx \wedge dy, \\ \omega_1 \wedge \omega_2 =& -\frac{\cosh z}{f(x)}dz \wedge dx + f(x)(\sinh z)dz \wedge dy + \Big(f(x)(\sinh z) + x\frac{\cosh z}{f(x)}\Big)dx \wedge dy. \end{aligned}$$

So, one gets

$$d\omega_1 = \omega_3 \wedge \omega_2, \quad d\omega_2 = \omega_1 \wedge \omega_3 + I\omega_3 \wedge \omega_2, \quad d\omega_3 = \omega_1 \wedge \omega_2 - J\omega_2 \wedge \omega_3,$$

where

$$I = (f' - (x/f)) \sinh z - f \cosh z \text{ and } J = (f' - (x/f)) \cosh z - f \sinh z.$$

Therefore, $(\omega_1, \omega_2, \omega_3)$ is a $(I, J, -1)$ generalized Finsler structure on M. Then, by the discussion of the case c) of Proposition 8.1, $(\eta_1, \eta_2) = (1/2)(\omega_3, \omega_2)$ is a taut contact hyperbola. In this case, the 1-form η_3 is given by

$$\eta_3 = (1/2)\big(\omega_1 + J\omega_3 - I\omega_2\big) = \dots = (1/2)\big(dz + f'(x)f(x)dy\big).$$

Since $2d\eta_3 = (f'f)'(x)\, dx \wedge dy = (f'f)'(x)\omega_3 \wedge \omega_2 = -4(f'f)'(x)\eta_2 \wedge \eta_1$, we get

$$d\eta_3 = 2\kappa\, \eta_2 \wedge \eta_1$$

where the Webster κ function is given by

$$\kappa(x) = -(f'f)'(x) = -(1/2)(f^2)''(x).$$

The frame (ξ_1, ξ_2, ξ_3), dual of the coframe (η_1, η_2, η_3), is given by

$$\begin{aligned} \xi_1 &= 2\Big(f(x)(\sinh z)\partial_x + \frac{\cosh z}{f(x)}\partial_y - f'(x)(\cosh z)\partial_z\Big), \\ \xi_2 &= -2\Big(f(x)(\cosh z)\partial_x + \frac{\sinh z}{f(x)}\partial_y - f'(x)(\sinh z)\partial_z\Big), \quad \xi_3 = 2\partial_z, \end{aligned}$$

where ξ_1, ξ_2 are the Reeb vector fields of η_1, η_2.

Then, if g is the Riemannian metric defined by $g(\xi_i, \xi_j) = \delta_{ij}$, (η_1, η_2, g) is a bi-contact metric structure with the Webster scalar curvatures

$$\mathcal{W}_1 = -\big((f^2)''(x) + 2\big)/4 \quad \text{and} \quad \mathcal{W}_2 = \big((f^2)''(x) - 2\big)/4. \tag{8.3}$$

In this example, we have a family of $(I_f, J_f, -1)$ generalized Finsler structures, depending on a function $f(x)$, that define taut contact hyperbolas. In particular, this family contains generalized Finsler structures that define left invariant taut contact hyperbolas on the Lie groups Sol^3 and $\widetilde{SL}(2, R)$. More precisely,

- Consider $M = \mathbb{R}^3$ and the $(I_f, J_f, -1)$ generalized Finsler structure corresponding to the function $f(x) = 1$. Then, the 1-forms $(\eta_1, \eta_2) = (1/2)(\omega_3, \omega_2)$ and η_3 are given by

  $$\eta_1 = (-(\sinh z)dx + (\cosh z)dy)/2, \quad \eta_2 = (-(\cosh z)dx + (\sinh z)dy)/2, \quad \eta_3 = (1/2)dz,$$

  and the dual vector fields

  $$\begin{aligned} \xi_1 &= 2((\sinh z)\partial_x + (\cosh z)\partial_y), \quad \xi_2 = -2((\cosh z)\partial_x + (\sinh z)\partial_y), \\ \xi_3 &= 2\partial_z \end{aligned}$$

  satisfy (3.7), and hence define on $\mathbb{R}^3$ a Lie group structure isomorphic to Sol^3.
- Consider $M = \mathbb{R}^3_+ = \{(x, y, z) \in \mathbb{R}^3 : x > 0\}$ and the $(-1, I_f, J_f)$ generalized Finsler structure corresponding to the function $f(x) = x$. Then, the 1-forms $(\eta_1, \eta_2) = (1/2)(\omega_3, \omega_2)$ and η_3 are given by

$$\eta_1 = \big(-\frac{\sinh z}{x}dx + x(\cosh z)dy\big)/2, \quad \eta_2 = \big(-\frac{\cosh z}{x}dx + x(\sinh z)dy\big)/2,$$
$$\eta_3 = (1/2)(xdy + dz),$$

and the dual vector fields

$$\xi_1 = 2\big(x(\sinh z)\partial_x + \frac{\cosh z}{x}\partial_y - (\cosh z)\partial_z\big), \quad \xi_2 = -2\big(x(\cosh z)\partial_x + \frac{\sinh z}{x}\partial_y - (\sinh z)\partial_z\big),$$
$$\xi_3 = 2\partial_z,$$

satisfy (3.8), and hence define on $\mathbb{R}^3_+$ a Lie group structure isomorphic to $\widetilde{SL}(2,\mathbb{R})$.

Remark 8.3 About the Webster curvature, we recall that every contact structure on a compact orientable three-manifold has a contact form and an associated Riemannian metric whose Webster scalar curvature is either a constant ≤ 0 or is everywhere strictly positive (see the main result of [9]). On the other hand, every compact orientable three-manifold M has a contact structure [19]. Therefore, every compact orientable three-manifold M has a contact Riemannian structure whose Webster scalar curvature is either a constant ≤ 0 or is everywhere strictly positive.

Now, from (8.3), it is not difficult to find a positive function $f(x)$ for which $\mathcal{W}_1$ be a strictly negative function and $\mathcal{W}_2$ be a strictly positive function. On the other hand, on a three-manifold, a contact Riemannian structure determines a non-degenerate CR structure with the same Webster scalar curvature (cf., for example, [25] p.30). So, we get the following

Proposition 8.4 *Any connected open subset of* $\mathbb{R}^3$ *admits a non-degenerate CR structure whose Webster scalar curvature is a strictly negative function and a non-degenerate CR structure whose Webster scalar curvature is a strictly positive function.*

Final remark

Of course, it is an open question to give a classification of three-manifolds which admit a taut contact hyperbola. Recall that the homothety class of a taut contact circle is defined by multiplication by the same positive function and by a rotation of constant angle [13]. Similarly, we can define the homothety class of a taut contact hyperbola. If f is a positive smooth function and (η_1, η_2) a pair of contact forms on a three-manifold M, the contact forms $(\tilde\eta_1 = f\eta_1, \tilde\eta_2 = f\eta_2)$ satisfy

$$\tilde\eta_i \wedge d\tilde\eta_i = f^2\eta_i \wedge d\eta_i,\ (i = 1, 2), \qquad \tilde\eta_1 \wedge d\tilde\eta_2 + \tilde\eta_2 \wedge d\tilde\eta_1 = f^2(\eta_1 \wedge d\eta_2 + \eta_2 \wedge d\eta_1).$$

Then,

- (η_1, η_2) is a taut contact hyperbola if and only if $(\tilde\eta_1, \tilde\eta_2)$ is a taut contact hyperbola.
  Moreover, if (η_1, η_2) is a taut contact hyperbola and (η_1', η_2') is obtained from (η_1, η_2) by a hyperbolic rotations of constant angle, it is not difficult to see that
- (η_1, η_2) is a taut contact hyperbola if and only if (η_1', η_2') is a taut contact hyperbola.

This suggests to define the homothety class of a taut contact hyperbola by multiplication by the same positive function and by a hyperbolic rotation of constant angle. Hence, to classify taut contact hyperbolas is equivalent to classify their homothety classes.

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