Taylor and Bi-local Piecewise Approximations with Neuro-Fuzzy Systems

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Abstract: A fuzzy neuron (linear combiner of fuzzy systems) with piecewise polynomial characteristic function is defined and analyzed. The linear fuzzy neuron uses a pre-specified type of fuzzy logic systems with complementary pairs of input membership functions. Taylor local approximations are built using the described fuzzy neuron. Moreover, local Taylor approximations are obtained using single fuzzy systems. Hence, the linear combiner fuzzy neurons are universal local approximators implementing truncated Taylor series. Moreover, they represent continuous piecewise approximators. Bi-local approximation with fuzzy logic systems are also introduced and demonstrated.

Keywords: local approximation, TS system, fuzzy neuron, universal local approximator, approximation algorithm.

1. Introduction

Methods of local approximations based on truncated Taylor series are well understood; several methods to compute bounds for the approximation errors are available (Christensen and Christensen, 2006), (Powell, 1981), (Shahriari, 2006). The use of Taylor approximations is convenient, among others, because methods to determine the approximation accuracy are well established. Taylor approximations are widely used in solving differential equations (Kloeden, Platen, 1995), (Tachev, 2009), control applications (Hedjar et al., 2005), circuit modeling (Jridi and Alfalou, 2009), data processing, image generation, signal prediction (Hedjar et al., 2005), and economic modeling (Judd, 1998).

Another class of widely used and well understood approximation methods is based on piecewise polynomials, see for example (Powel, Chapter 18, p. 212). The use of piecewise polynomials approximations is convenient, because methods to determine the approximation accuracy are well established. Proving that some kind of fuzzy systemsare identical with the class of polynomials on a specified interval is important because polynomials constitute a core family of functions that stand at the basis of numerous mathematical constructs and applications.

In addition, because Taylor approximations and piecewise polynomials approximations are so widely used, showing that some kind of fuzzy system may implement such approximating functions is both of intellectual and practical interest.

It is well known that fuzzy logic systems (FLS) endowed with defuzzification perform a mapping from the input $\mathbf{R}^n$ spaceto the output space, the latter being typically the real line $\mathbf{R}$. When the fuzzy system has a single input taking values in $\mathbf{R}$, the mapping performedis $\mathbf{R} \to \mathbf{R}$. The input-output functionsare also named characteristic functions.

Recall that a Sugeno fuzzy system, also named TSK (standing for Takagi-Sugeno-Kang) or T-S fuzzy system, of order zero, is defined by the input-output function

$$f(x) = \sum_{i=1}^{n} \left(s_i \cdot \mu_i(x) \right) / \sum_{i=1}^{n} \mu_i(x),$$

where $s_i \in \mathbf{R}$ are called singletons and $\mu_i : \mathbf{R} \to [0,1]$ are the corresponding membership functions. In case of TSK systems with several inputs, each fuzzified independently, with max-type inference and weighted sum defuzzification, the output is

$$f(x) = \sum_{i=1}^{n} \left(s_i \cdot MAX_h \; \mu_{i,h}(x_h) \right) / \\ / \sum_{i=1}^{n} MAX_h \; \mu_{i,h}(x_h),$$

while for product-type inference the characteristic function is (Tanaka& Wang, 2001), (Yu and Li, 2004),

$$f(x) = \sum_{i=1}^{n} \left(s_i \cdot \prod_h \mu_{i,h}(x_h) \right) / \\ / \sum_{i=1}^{n} \prod_h \mu_{ih}(x_h)$$

with the membership functions related to the i^{th} rule denoted by $\mu_{ih}, i = 1..n$, the index h denoting the h^{th} input variable. TSK fuzzy systems of higher orders are defined in a similar way, but instead of conclusions represented by constants, they have polynomial conclusions. The rules describing a TSK system of order n have the form *If input x is A, then output is* $P_n(x)$, where $P_n(x)$ is a polynomial of order n.

Bikdash (Bikdash, 1999), (Bikdash et al., 2001) have introduced and applied a modified Sugeno (TSK) systems, named Interpretable Sugeno Approximators (ISA), which use outputs in the form $u^k = b_0^k + b_1^k(x - r_1^k) + b_2^k(x - r_2^k) + \cdots + b_n^k(x - r_n^k)$, "where r^k is the rule center, i.e., centers of the membershipfunctions of all inputs tested by the k^{th} rule, […] and the coefficients b are interpreted as Taylor series coefficients."

One of the first papers to use neuro-fuzzy systems in relation to Taylor approximation was (Yu and Li, 2004). However, that approach endeavored only to approximate a Taylor approximating function, not to implement the Taylor approximant. Herera et al. (Herrera et al., 2004)(Herrera et al., 2005a,b) analyzed TSK systems with second order polynomial consequent and connected them to Taylor approximations.(Wang, Li, Niemann, & Tanaka, 2000) provided a detailed analysis of the TSK systems with linear consequent and their approximation power, for the general case. (Sonbol and Fadali 2002) and (Sonbol and Fadali, 2006) also provided an interesting approach for Taylor-like approximation with TSK systems, which contains some of the ideas presented in this paper, while (Fadali, 2002) furthers the analysis for general polynomial approximations.

We show that a constructive approach is possible that allows building a direct, true Taylor approximation using either a simpler neuro-fuzzy system or a modified TSK system. Also, bi-local approximations are introduced, as well as neuro-fuzzy systems for implementing them. A specific choice of the input membership functions allows us the direct implementation of the polynomial approximations, with zero-order TS systems, thus simplifying computations in the implementation of the approximators.

For brevity, we discuss only the case of single input single output (SISO) FLS. Also, to keep the paper focused, we deal here only with Taylor mono- or bi-local approximations, but the methods presented herein are readily extensible to Tchebychev and to other polynomial approximations. These will be dealt with elsewhere.

The flow of the paper is as follows. In the second section we introduce the linear combiner fuzzy neuron (LCFN) and the piecewise polynomial approximations using LCFNs. The piecewise Taylor approximant is derived as a particular case of the piecewise polynomial approximants. The notions of bi-local approximation and piecewise approximant are defined in Section 3 in relation to representations of the approximant by fuzzy logic systems. The fourth section is devoted to a specific class of fuzzy systems able to implement Taylor approximants and bi-local approximants. The general outline of an algorithm for building Taylor-like approximations with fuzzy systems is presented in the fifth section. The final section is devoted to a general discussion and to conclusions.

A preliminary, partial version of this paper was presented in (Teodorescu, 2010).

2. Polynomial Piecewise Approximations with Linear Combiner Fuzzy Neurons

Our aim is to develop a local approximation of a function $f : R \to R,\ f \in C^{\infty}$ around a point x_0, using fuzzy systems. For brevity, we chose $x_0 = 0$, $f \in C^{\infty}[-1,1]$. The Taylor series of f in 0 is denoted by $f(x) = \sum_{n=0}^{\infty} a_n x^n$, $a_n = f^{(n)}(0)/n!$.

The piecewise approximation by the neuro-fuzzy system around $x_0 = 0$ will have two parts, one for $x \le 0$ and the other for $x \ge 0$, both having the same value in $x = 0$. First, we show how to obtain a fuzzy logic system (FLS) function with the characteristic of the form $g(x) = a + bx^n$. Next, such systems are linearly combined in what we name a linear fuzzy neuron, to obtain a characteristic function of the form $h(x) = \sum_{n=0}^{N} \alpha_n x^n$, used in the approximation. The coefficients α_n are then equaled to the corresponding coefficients of the Taylor decomposition of f by solving a system of equations in the weights of the linear combiner and the values of the singletons.

Lemma 1. A Sugeno 0-order FLS can implement the local function $g(x) = a + bx^n$ on $[0,1]$.

Recall that a 0-order Sugeno FLS is defined by the function

$$g(x) = \sum_{k=1}^{P} s_k \cdot \mu_k(x) / \sum_{k=1}^{P} \mu_k(x),$$

where $s_k \in R$ are the values of the singletons associated with the membership functions μ_k and p is the number of membership functions.

The key idea for building the proposed approximators is to choose the input membership functions in complementary pairs, (μ_{k1}, μ_{k2}), such that for every x in the definition domain $\mu_{k1}(x) + \mu_{k2}(x) = 1$, see Fig. 1.Moreover, μ_{k1}, μ_{k2} are expressed by the same power of x. The last condition insures that the denominator is a constant (not depending on x), while the nominator is a polynomial in x. For example, choose $\mu_1(x) = x^i$ on $[0,1]$, $\mu_2(x) = 1 - x^i$ on $[0,1]$. Then, $\mu_1(x) + \mu_2(x) = 1$ and

$g(x) = s_1 \cdot x^i + s_2 \cdot (1 - x^i)$; thus, we obtain

$$g(x) = s_2 + x^i \cdot (s_1 - s_2) = a + bx^i,$$

where $s_1, s_2 \in \mathbf{R}$ are the singletons associated to the membership functions μ_1, μ_2 by rules as

Rule 1: If input is $\tilde{A}_1$, output is s_1.

Rule 2: If input is $\tilde{A}_2$, output is s_2.

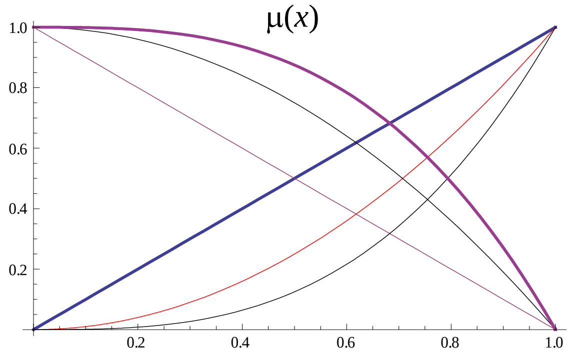

Figure 1. Plot of the complementary membership functions $x, 1 - x, x^2, 1 - x^2$, and $x^3, 1 - x^3$, used as input membership functions in the TS systems

Above, the fuzzy set $\tilde{A}_1$ has the membership function denoted by μ_1, while the fuzzy set $\tilde{A}_2$ has the membership function μ_2.

Consequently, a fuzzy logic system as above implements a natural power ($x^i, i \in \mathbf{N}$) of the input, up to a constant. Polynomial Taylor expressions can be obtained by summing several such fuzzy systems.

Subsequently, we use the notation T_N to denote the N^{th} order Taylor approximation of order *n* of a given function in $x = 0$,

$$T_N(x) = \sum_{n=0}^{N} \left(f^{(n)}(0)/n!\right) \cdot x^n,$$

T_{N,x_0} to denote the approximation in $x_0 \neq 0$, and R_{N+1} to denote the remainder of the Taylor series in $x = 0$ for the N^{th} order approximation. Summarizing the above discussion:

Proposition 1. A linear combiner fuzzy neuron defined by $f(x) = \sum_{i=0}^{N} w_i \cdot f_i(x)$, where f_i represents the characteristic function of the i^{th} elementary fuzzy system in the neuron, implements the Taylor approximation of f in 0, for $x \geq 0$, if the singletons and the weights satisfy the conditions (1),

$$\begin{array}{ll} w_i \cdot \left(s_{1i} - s_{2i}\right) = a_i & i > 0 \\ \sum_{i=0}^{N} w_i \cdot s_{2i} = a_0 & \end{array}, \qquad (1)$$

or, in matrix form, $\mathbf{W}^T \cdot (\mathbf{S}_1 - \mathbf{S}_2) = \mathbf{A}$, $\mathbf{W}^T \cdot \mathbf{S}_2 = a_0$, with unknown column vectors $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{W}$.

Proof. In a basic description, the linear combiner computes the weighted sums from a set of fuzzy systems, each described by a couple of rules as

Rule1_i: If input is $\tilde{A}_{1i}$, output is s_{1i}.

Rule2_i: If input is $\tilde{A}_{2i}$, output is s_{2i}.

where $\tilde{A}_{1i}$ corresponds to the membership function $\mu_{1i}(x) = x^i$ and $\tilde{A}_{1i}$ corresponds to the membership function $\mu_{2i}(x) = 1 - x^i$.

The vectors $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{W}$ may be seen as vectors of parameters of the fuzzy neuron. Notice that the sum $\mathbf{W}^T \cdot \mathbf{S}_2 = a_0$ may be interpreted as a bias and can be cancelled or modified with an external bias to the neuron. The true unknowns are $\mathbf{S} = \mathbf{S}_1 - \mathbf{S}_2$ and $\mathbf{W}$, and the system of equations reduces to $\mathbf{W}^T \cdot \mathbf{S} = \mathbf{A}$, $\mathbf{W}^T \cdot \mathbf{S}_2 = a_0$.

Recall that the linear combiner has the equation $y = \sum_{j=1}^{m} w_j \cdot x_j$, where m is the number of inputs x_j and w_j are called weights. Using N FLS as in *Lemma* 1, with appropriately defined membership functions, $i = 0 \ldots N$, and weights applied to the outputs of the systems, we obtain:

$$y = \sum_{i=0}^{N} w_i \cdot \left[s_{2,\,i} + x^i \cdot \left(s_{1,\,i} - s_{2,\,i} \right) \right],$$

$$y = \mathbf{W} \cdot \left(S_2 + \mathbf{X} \cdot \left(S_1 - S_2 \right) \right) = \mathbf{A}^T \cdot \mathbf{X}$$

or

$$g(x) = \sum_{i=0}^{N} \left(w_i \cdot s_{2,i} + x^i \cdot w_i \cdot \left(s_{1i} - s_{2i} \right) \right) =$$

$$= \sum_{i=0}^{N} a_i \cdot x^i .$$

Elementary computations lead to the conditions (1), proving the proposition.

The system (1) is underdetermined, with linear independent equations, and has solutions because by choosing a set of values for the unknowns w_i and allowing for an external bias $\mathbf{W} \cdot \mathbf{S}_2$, the system becomes linear with exactly N independent equations and N unknowns, $u_i = s_{1i} - s_{2i}$. The function $g(x)$ so obtained is the N^{th} order Taylor approximation of f in $x_0 = 0$, $f(x) = \sum_{n=0}^{\infty} a_n x^n$.

Remark. By translation and scaling, the interval $[0,1]$ can be replaced with any interval to fit the desired partition of the definition domain, according to the choice of membership functions, provided that the radius of convergence of the Taylor series remains larger than the scaled interval. When scaling, the convergence radius of the Taylor series has to be observed.

Example. Approximate with two different linear combiner fuzzy neurons the function $f(x) = \sin x$ around $x = 0$, for $x \geq 0$, using a cubic approximant.

The corresponding Taylor series is

$$f(x) = \sum_{0}^{\infty} \frac{-(-1)^k}{(2k+1)!} \cdot x^{2k+1}, \; a_{2k+1} = \frac{(-1)^k}{(2k+1)!},$$

$a_{2k} = 0$. Choose the interval $[0,1]$ and $N = 3$.

Then, $w_1 \cdot (s_{11} - s_{21}) = a_1 = 1$, $w_3 \cdot (s_{13} - s_{23}) = a_3 = -1/6$, and $w_1 \cdot s_{21} + w_3 \cdot s_{23} + w_0 (s_{1,0} - s_{2,0}) = 0$.

Choosing $w_0 = 0$, $w_1 = 1$, we obtain $s_{11} = 1$, $s_{21} = 0$, $s_{13} = 1$, $s_{23} = s_{21} = 0$, $w_3 = -1/6$, which is the required truncated Taylor series. This case is rather trivial and against the current view in the fuzzy systems domain, as each fuzzy system is described by only one membership function on the interval $[0,1]$, thus, it is described by a single rule. The solution is not unique. Choosing $w_0 = 0$, $w_1 = 3$, we obtain $s_{11} = 1$, $s_{21} = 0$, $s_{13} = 1$, $s_{23} = s_{21} = 0$, $w_3 = -1/6$,

As this elementary example shows, the use of fuzzy systems brings nothing new to the solution of the problem of approximation with Taylor piecewise polynomials, except the demonstration of the fact that fuzzy systems are powerful enough to implement such approximations. Hence, the advantage of using fuzzy systems implementations of the piecewise Taylor approximants is justifiable at the algorithmic and technological level only.

The use of piecewise Taylor approximant is counterintuitive from the standpoint of fuzzy systems practice, because the characteristic function will not behave in general as an interpolation function, as usually with fuzzy systems. That is, the characteristic function will pass through the approximation points but will not take the approximated function values in the points where the membership functions have value 1. To remedy this drawback, a modification of the Taylor approximants is needed, as discussed in the next section.

3. Bi-local Approximants

Next we assume for brevity that the functions f and g are real valued functions of a single real variable belonging to C^{∞}.

Definition 1. An approximant g of f is named bi-local of order (m, n) in $\{x_1, x_2\}$ if the value of the first m derivatives of g in x_1 equal the corresponding derivatives of f in x_1 and if the first n derivatives of g in x_2 equal the corresponding derivatives of f in the same point,

$$g^{(k)}(x_1) = f^{(k)}(x_1), \quad k = 1 \dots m$$

and (2)

$$g^{(h)}(x_2) = f^{(h)}(x_2), \quad h = 1 \dots n$$

The existence of bi-local approximants is guaranteed by the following construction. Let g be a polynomial of order $m+n-1$, $g(x) = \sum_{k=0}^{m+n-1} \lambda_k \cdot x^k$. Choose the first m coefficients to represent the Taylor coefficients of f in x_1, thus ensuring that $g^{(k)}(x_1) = f^{(k)}(x_1)$, $k = 1..m$. Rewrite g as $g(x) = \sum_{k=0}^{m+n} \gamma_k \cdot (x - x_2)^k$, with the first m coefficients predetermined by the Taylor approximation in x_1. Taking the derivative up to the order n and imposing the conditions that the derivatives have the required values in x_2, $g^{(h)}(x_2) = f^{(h)}(x_2)$, $h = 1..n$, a system of independent linear equations is formed. Therefore the bi-local approximation exists. We denote the bi-local approximant of f by ${}^{m+n}T_{x_1,x_2,f}(x_1)$. If g is the bi-local approximant as above, then the Taylor approximant of order m of f in x_1 is

$$T_{x_1,f}(x_1) = \sum_{i=0}^{m} a_i \cdot x_1^i = \sum_{i=0}^{m} \lambda_i \cdot x_1^i =$$
$$= {}^{m,n}T_{x_1,x_2,f}(x_1),$$

moreover

$$T_{x_1,f}(x_2) = \sum_{i=0}^{n} b_i \cdot x_2^i = \sum_{i=0}^{m+n} \lambda_i \cdot x_2^i =$$
$$= {}^{m,n}T_{x_1,x_2,f}(x_2).$$

A set of similar conditions are produced by the conditions on the values of the derivatives in the two points:

$${}^{m,n}T^{(r)}_{x_1,x_2,f}(x_1) = T^{(r)}_{x_1,f}(x_1) \quad r \le m$$
$$f^{(r)}(x_1) = r!\lambda_i = {}^{m,n}T^{(r)}_{x_1,x_2,f}(x_1),$$

moreover

$$T_{x_1,f}(x_2) = \sum_{i=0}^{n} b_i \cdot x_2^i = \sum_{i=0}^{m+n} \lambda_i \cdot x_2^i =$$
$$= {}^{m,n}T_{x_1,x_2,f}(x_2).$$

Bi-local approximations are useful because in general the intuitive fuzzy logic system concept assumes that at the limits of the intervals defining the supports of the membership functions, the characteristic function of the fuzzy logic system takes the desired values, moreover its derivative is continuous. In other words, fuzzy logic systems typically implement bi-local piecewise approximations of order (1,1) with continuous derivatives. Having the derivatives taking specified values may be useful in some applications. In that case, the characteristic function of the fuzzy logic system should implement a bi-local approximation of order (2,2) of the desired behavior (function).

Example. Find a bi-local approximation of order (3,2) of the function $f(x) = e^{-x^2} \cdot \sin(\pi \cdot x)$, used by Herrera (Herrera et al., 2005a,b) and by other authors to demonstrate the approximation capabilities of fuzzy logic approximants. The approximating points are $x_1 = 0$ and $x_2 = 1$. Using Mathematica™, we find the Taylor expansions as $T_{0,3}(x) = \pi x + (-\pi - \pi^3/6)x^3$, respectively:

$$T_{1,3}(x) = -\pi \cdot (x-1)/e + (2\pi(x-1)^2/e + $$
$$+((-6\pi + \pi^3)(x-1)^3/(6e),$$

$$f(1) = T_{1,3}(1) = 0, \; f'(1) = T'_{1,3}(1) = -\pi/e.$$

The graph of the approximated function is shown in Fig. 2. The required bi-local approximant has the form

$$g(x) = T_{0,3}(x) + \lambda_4 \cdot x^4 + \lambda_5 \cdot x^5, \quad \text{or}$$
$$g(x) = \pi x + (-\pi - \pi^3/6)x^3 + \lambda_4 \cdot x^4 + \lambda_5 \cdot x^5,$$

with conditions $g(1) = 0$ and $g'(1) = -\pi/e$. These conditions produce

$$g(1) = \pi + (-\pi - \pi^3/6) + \lambda_4 + \lambda_5 = 0 \quad \text{and}$$
$$g'(1) = \pi + 3(-\pi - \pi^3/6) + 4\lambda_4 + 5\lambda_5 = -\pi/e$$

with solutions $\lambda_4 = -2\pi + \pi^3/3 + \pi/e$ and $\lambda_5 = 2\pi - \pi^3/6 - \pi/e$.

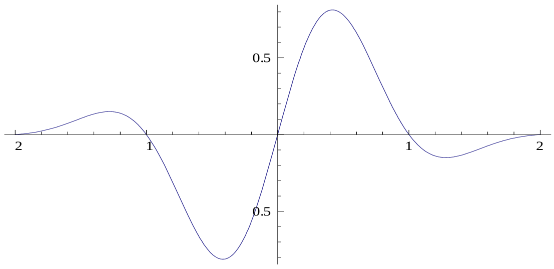

Figure 2.The approximated function

$f(x) = e^{-x^2} \cdot \sin(\pi \cdot x)$

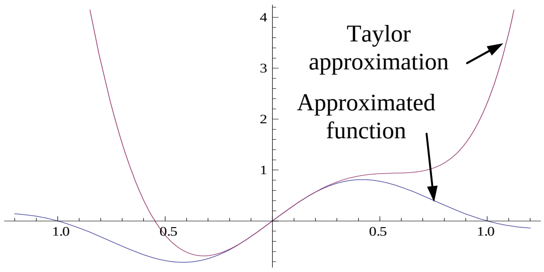

Figure 3.Cubic Taylor approximation in 0 of the function $f(x) = e^{-x^2} \cdot \sin(\pi \cdot x)$

ATaylor approximant is plot in Figure. 3. Figures 3 to 6 compare the results.

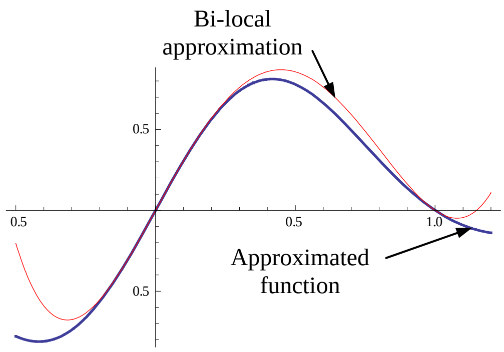

Figure 4.Bi-local approximation in 0 and 1 of the function $f(x) = e^{-x^2} \cdot \sin(\pi \cdot x)$

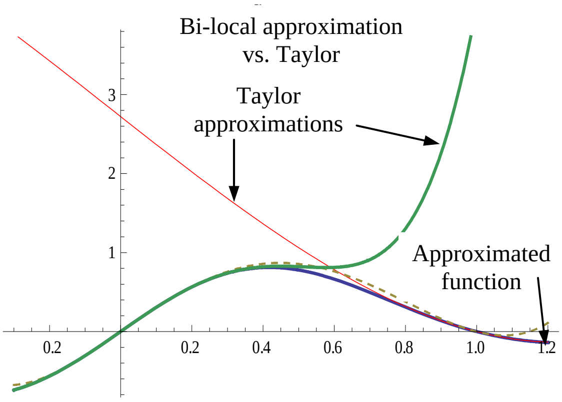

Figure 5.Two Taylor order 3 approximations in 0 and in 1, and the bi-local approximation of order (3,2) in {0, 1}.

The advantage of bilocal approximations with respect to piecewise Taylor approximations is easy to grasp: the latter is discontinuous. An example is shown in Fig. 6, where Taylor approximations on the partition $[-2,-1.5] \cup [-1.5,-1] \cup [-1,-0.5] \cup [-0.5,0]$ are compared with the graph of the function. The glitches (and lack of continuity) in -1.5 and especially in -1, -0.5 and 0 are apparent.

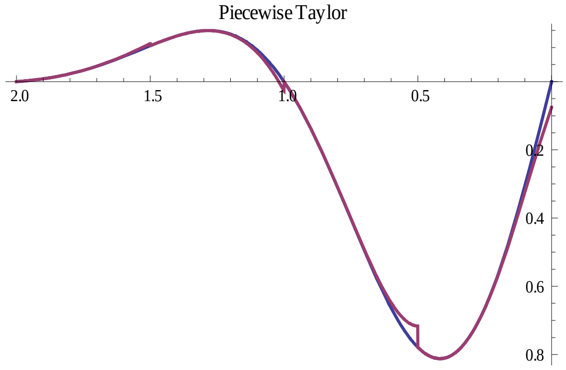

Figure 6.Approximation by piecewise Taylor polynomials for the interval [-2, 0], for $f(x) = e^{-x^2} \cdot \sin(\pi \cdot x)$

Proposition 2. The neuro-fuzzy system can implement bi-local Taylor approximations at both ends of the interval [0,1] simultaneously, provided that conditions (2) are satisfied.

The proof is direct.

Imposing the condition for neuro-fuzzy

$$y(x) = \sum_{i=1}^{N} w_i \cdot \frac{s_{1i} x^i + s_{2i}\left(1 - x^i\right)}{N}$$

$$= \sum_{i=1}^{N} w_i \cdot \frac{s_{21} + x^i\left(s_{1i} - s_{2i}\right)}{N}$$

$$= \sum_{i=1}^{N} w_i \cdot \left[\frac{s_{2i}}{N} + x^i \cdot \frac{s_{1i} - s_{2i}}{N}\right] = \sum_i a^i x^i$$

we obtain

$$\sum_{i=1}^{N} w_i \cdot \frac{s_{2i}}{N} = a_0 \text{ or } \sum w_i \cdot s_{21} = N \cdot a_0$$

$$w_i \cdot \frac{s_{1i} - s_{2i}}{N}$$

The next proposition provides a bound of the error of the bi-local approximation.

Proposition 3. The maximal error ε_{max} of the bi-local approximant $T^{m,n}_{x_1,x_2,f}(x_1)$ in the interval $[x_1, x_2]$ is

$$\varepsilon_{\max} = \max_x \left| T^{m,n}_{x_1,x_2,f}(x_1) - f(x) \right| = \\ = \left| T^{(m)}_{x_0,f}(x_0) - f(x_0) \right|,$$

where x_0 is the solution of $T^{(m)}_{x_1,f}(x) = T^{(n)}_{x_2,f}(x)$.

The proof is elementary.

4. Taylor Approximations with Single Fuzzy Logic Systems

Exact Taylor approximations can be obtained with a single fuzzy logic system as well, allowing a slight, somewhat counter-intuitive modification of the system, namely allowing several memberships functions with the same support. While usually membership functions have overlapping but not identical support, there is no reason to forbid the case when several membership functions have the same support. Then, define on the [0,1] interval the membership functions $\mu_{\tilde{A}_1}(x) = \mu_1(x) = x$, $\mu_{\tilde{A}_2}(x) = \mu_2(x) = 1 - x$, $\mu_{\tilde{A}_3}(x) = \mu_3(x) = x^3$, $\mu_{\tilde{A}_4}(x) = \mu_4(x) = 1 = x^3$, … and define the TSK system by the rules

If x *is* $\tilde{A}_1$ *then* $y = s_{11}$,

If x *is* $\tilde{A}_2$ *then* $y = s_{21}$,

If x *is* $\tilde{A}_3$ *then* $y = s_{21}$,

If x *is* $\tilde{A}_4$ *then* $y = s_{22}, \ldots$

The output is computed, as usually, as

$$y(x) = \sum \left[s_{1i}\mu_{1i} + s_{2i}\mu_{2i} \right] / \sum \left[\mu_{1i}(x) + \mu_{2i}(x) \right].$$

We name such a system *modified TSK system*.

*Proposition*4. A single-input single output modified TSK system implements a Taylor approximation of order N provided that the conditions (3) are satisfied,

$$\sum_{i=1}^{N} s_{2i} = a_0 \cdot N$$

and (3)

$$s_{1i} - s_{2i} = a_i \cdot N \quad i = 1 \ldots N.$$

Consider the output of the fuzzy logic system,

$$y(x) = \frac{\sum \left[s_{1i}\mu_{1i} + s_{2i}\mu_{2i} \right]}{\sum \left[\mu_{1i}(x) + \mu_{2i}(x) \right]},$$

where the membership functions are $\mu_{1i}(x) = x^i$, $\mu_{2i}(x) = 1 - x^i$. The denominator is then $\sum_{i=1}^{N} \left[\mu_{1i}(x) + \mu_{2i}(x) \right] = N$ and the output becomes $y(x) = \frac{1}{N}\left[s_{1i} x^i + s_{2i}\left(1 - x^i\right) \right]$.

Equaling the output to the desired Taylor polynomial, $\frac{1}{N} \cdot \sum_{i=1}^{N} s_{2i} + \frac{1}{N} \cdot \sum x^i \left(s_{1i} - s_{2i} \right) = \sum_{i=1}^{N} a^i x^i$ the conditions required are

$\sum_{i=1}^{N} s_{2i} = a_0 \cdot N$ and

$s_{1i} - s_{2i} = a_i \cdot N \quad i = 1 \ldots N$.

Example 3. Consider the case of Taylor approximation in $x = 0$ of the function $\sin x$, with a single FLS with membership functions defined on $[0,1]$. The relevant conditions are

$$\begin{cases} s_{11} - s_{21} = 1 & s_{11} = 1 + s_{21} \\ s_{21} + s_{23} = 0 & s_{21} = -s_{23} \\ s_{21} + s_{23} = -1/6 & s_{13} + s_{21} = -1/6 \end{cases}.$$

Choose $s_{21} = 0$, $s_{11} = 1$. Then, the system has the remaining parameters $s_{23} = 0, s_{13} = -1/6$ .Other solutions are possible, for example choose $s_{11} = 0.5$, $s_{21} = -0.5$ and obtain $s_{23} = 0.5, s_{13} = -1/2 - 1/6 = -2/3$. The input membership functions and the output singletons are sketched in Figure 7.

Next, assume that we need a local approximation in 1. The relevant conditions are:

$$s_{11}x + s_{21}(1 - x) + s_{12}x^2 + s_{22}\left(1 - x^2\right) + \\ + s_{13}x^3 + s_{23}\left(1 - x^3\right) + \ldots$$

$$\sum_{i=1}^{N} s_{2i} = \sigma = a_0, \; s_{1i} - s_{2i} = a_i \cdot \sigma.$$

Hence, $s_{1i} = s_{2i} + a_i \cdot \sigma = s_{2i} + a_i \cdot a_0$, $i = 1 \ldots N$. The values s_{i2} can be pre-fixed. To cover the case $x = 1$, we need appropriate input membership functions shifted from 0 to 1, as in Figure 8.

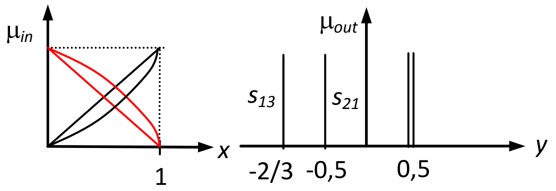

Figure 7.Input membership functions and singletons for the example

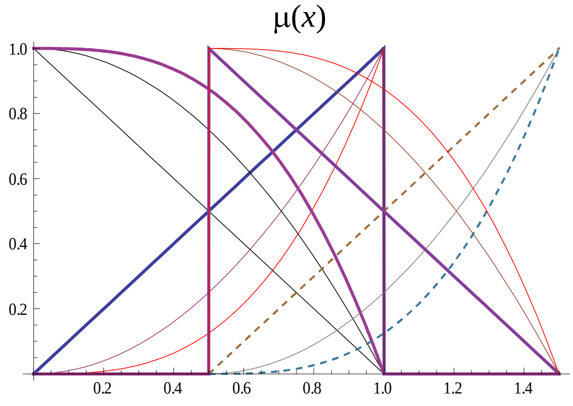

Figure 8.Plot of the functions $x, 1-x, x^2, 1-x^2$, and $x^3, 1-x^3$, and of their translated versions, used as input membership functions

We can improve the fuzzy system implementation asking that the function f is, at the same time, bi-local Taylor approximated in $x=0$ and in $x=1$. Denote the Taylor approximation of order N in $x=1$ by:

$$f(x)=\sum_{i=0}^{N}\frac{f^{(i)}(1)}{i!}\cdot(x-1)^{i}=\sum_{i=0}^{N}b_i\cdot x^{i}.$$

Then, using $\sum_{i=1}^{N}s_{21}=a_0$, $\sum_{i=1}^{N}s_{1i}=b_0$, $s_{1i}=s_{2i}+a_i\cdot a_0$, $s_{1i}=s_{2i}+b_i\cdot b_0$, which implies $a_i=b_i\cdot b_0/a_0$, that is the FLS approximation is valid only for functions having the Taylor approximations of order N in 0 and in 1 identical, up to a multiplicative constant, in the coefficients.

5. Greedy-like Algorithm for Building Approximations with Bi-local Approximants

The algorithm for building the corresponding neuro-fuzzy system approximation consists in the following steps:

Input: approximation interval $[c_0, c_f]$, approximated function, maximal local error.

Output: interval partition, approximating function (fuzzy system)

Initialize $h=0$. Define step Δ for interval length increment. Then,

a. determine the desired partition of the interval domain;
b. determine the Taylor approximants in the border points of the partition intervals;
c. write the systems of equations (1) for the left and right neuro-fuzzy approximating polynomials;
d. pre-determine an appropriate subset of unknowns and solve the system of equations.

The details of the algorithm are as in the examples shown in the previous sections.

6. Discussion and Conclusions

The key ideas in this paper were i) the use of paired complementary membership functions that guarantee constant denominators of the characteristic functions of the TSK fuzzy systems; ii) the use of a linear combiner fuzzy neuron to split the problem of approximation into approximation by natural powers of the variable; iii) the use of an external bias to simplify computations and the choice of weights; iv) the definition of the notion of bi-local polynomial approximation and its use in relation to the Taylor approximation, and v) the universal local approximation capabilities of the systems described.

The theoretical usefulness of Taylor-like approximations consists in showing that the neuro-fuzzy systems described are universal local approximators in Taylor series sense. The linear combiner fuzzy neurons based on order zero Sugeno fuzzy systems are piecewise polynomial continuous approximators in the sense of (Powell, 1981, pp. 28-29).

The results show that linear combiner fuzzy neurons are universal local approximators similar to truncated Taylor series, moreover are implementing continuous piecewise approximators. While the theoretical issues have been discussed only for the [0,1] interval, scaling and translation insures that any other interval can be used, provided that the interval is smaller than the radius of convergence of the Taylor series.

The main computational issues related to the methods described reside in the complexity of solving systems of linear equations and of finding the partition of the universe of discourse that ensures a specified approximation error.

The two methods described in the precedent sections – one based on single fuzzy logic systems and the other based on neuro-fuzzy systems – may be applied to other types of families of polynomials (series), like Tchebychev polynomials.

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