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Tchakaloff polynomial meshes

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Abstract. We construct polynomial meshes from Tchakaloff quadrature points for measures on compact domains with certain boundary regularity, via estimates of the Christoffel function and its reciprocal.

1. Introduction. A norming set for $\mathbb{P}_n^d(K)$, the polynomials of degree not exceeding n restricted to a compact set $K \subset \mathbb{R}^d$, is a subset $X_n \subset K$ such that

$$\|p\|_K \leq C_n \|p\|_{X_n}, \quad \forall p \in \mathbb{P}_n^d(K), \tag{1.1}$$

where $\|f\|_Y$ denotes the sup-norm of a bounded function on the compact set Y.

Sequences of finite norming sets such that both C_n and $\mathrm{card}(X_n)$ grow algebraically in n are of primary interest in multivariate polynomial approximation, and are usually called *weakly admissible polynomial meshes*. If $C_n \equiv C$ is a constant independent of n, the mesh is termed *admissible*. Notice that necessarily $\mathrm{card}(X_n) \geq N_n = N_n(K) = \dim(\mathbb{P}_n^d(K))$, since X_n is determining for $\mathbb{P}_n^d(K)$ (i.e., polynomials vanishing there vanish everywhere on K). Admissible meshes with $\mathrm{card}(X_n) = \mathcal{O}(N_n)$ are termed *optimal*.

Observe that $N_n = \mathcal{O}(n^\beta)$ with $\beta \leq d$, in particular $N_n = \binom{n+d}{d} \sim n^d/d!$ on polynomial determining compact sets (i.e., polynomials vanishing there vanish everywhere in $\mathbb{R}^d$), but we can have $\beta < d$ for example on compact algebraic varieties, like the sphere in $\mathbb{R}^d$ where $N_n = \binom{n+d}{d} - \binom{n-2+d}{d}$.

In case $m \in \mathbb{R}$ is *not* an integer, we will let $\mathbb{P}_m^d(K)$ denote the space $\mathbb{P}_n^d(K)$ for n the *largest* integer less than or equal to m, and similarly for N_m etc.

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We recall among their properties that polynomial meshes are invariant under affine transformations, can be extended by algebraic transformations, finite union and product and are stable under small perturbations. They give good discrete models of a compact set for polynomial approximation, since they are nearly optimal for polynomial least squares, contain extremal subsets of Fekete and Leja type for polynomial interpolation, and have been applied in polynomial optimization and in pluripotential numerics; cf., e.g., [4, 6, 9, 16, 20, 21, 23, 24].

2. Tchakaloff points and polynomial meshes. In the present paper we will make use of the notion of Tchakaloff quadrature points to construct polynomial meshes on compact sets with certain boundary regularity (see Corollary 2.5 for the details). It is therefore worth recalling the celebrated *Tchakaloff Theorem* on the existence of positive algebraic quadrature formulas, originally proved in [28] for the Lebesgue measure, and later extended to general measures (cf., e.g., [25]).

THEOREM 2.1. *Let μ be a positive measure with compact support in $\mathbb{R}^d$. Then there are $s \le N_n = \dim(\mathbb{P}_n^d(\mathrm{supp}(\mu))$ and sets $\mathcal{T}_n = \{t_j\}_{j=1}^s \subseteq \mathrm{supp}(\mu)$ of points and $\boldsymbol{w} = \{w_j\}_{j=1}^s$ of positive real numbers such that*

$$\int_{\mathbb{R}^d} p(x)\, d\mu = \sum_{j=1}^{s} w_j p(t_j), \qquad \forall p \in \mathbb{P}_n^d(\mathrm{supp}(\mu)).$$

The set $\mathcal{T}_n$ may be termed a set of *Tchakaloff quadrature points* of degree n for the measure μ.

Assume now that $\mathrm{supp}(\mu)$ is determining for $\mathbb{P}^d(K)$ (the space of d-variate real polynomials restricted to K; for a fixed degree n, we could even assume that $\mathrm{supp}(\mu)$ is determining for $\mathbb{P}_n^d(K)$). We recall two functions that will play a relevant role in our construction. The first is the reproducing kernel for μ in $\mathbb{P}_n^d(K)$, namely

$$K_n(x,y) = K_n^{\mu}(x,y) = \sum_{j=1}^{N_n} p_j(x)p_j(y), \tag{2.1}$$

where $\{p_j\}$ is any μ-orthonormal basis of $\mathbb{P}_n^d(K)$, for example that obtained from the standard monomial basis by applying the Gram–Schmidt orthonormalization process (it can be shown that $K_n(x,y)$ does not depend on the choice of the orthonormal basis). The diagonal of the reproducing kernel, $K_n(x,x)$, has the important property that

$$\|p\|_K \le \sqrt{\max_{x\in K} K_n(x,x)}\, \|p\|_{L^2_\mu(K)}, \qquad \forall p \in \mathbb{P}_n^d(K). \tag{2.2}$$

The second function is the reciprocal of $K_n(x,x)$, usually called the *Christoffel function* of μ, which has the following relevant characterization:

$$\lambda_n(x) = \frac{1}{K_n(x,x)} = \min_{p\in\mathbb{P}_n^d(K),\,p(x)=1} \int_K p^2(x)\,d\mu; \tag{2.3}$$

see, e.g., [12, 13] for the definition and properties of the Christoffel function $\lambda_n(x)$ and its reciprocal $K_n(x,x)$.

We are now ready to state and prove our main result, beginning with the following lemma.

LEMMA 2.2. *Suppose that*

$$X_{n+m} = \{x_1,\dots,x_s\} \subset K$$

are the points of a positive quadrature formula of precision $2(n+m)$ *for the measure* μ, *i.e., there are positive weights* $w_i > 0$, $1 \le i \le s$, *such that for all polynomials* $p(x)$ *with* $\deg(p) \le 2(n+m)$,

$$\int_K p(x)\,d\mu = \sum_{i=1}^s w_i p(x_i).$$

Then for all polynomials $p(x)$ *with* $\deg(p) \le n$ *and* $\xi \in K$,

$$|p(\xi)| \le \sqrt{K_{n+m}(\xi)\lambda_m(\xi)}\,\|p\|_{X_{n+m}}.$$

Proof. Given $\xi \in K$ let

$$q(x) := \frac{K_m(x,\xi)}{K_m(\xi,\xi)}.$$

Then $q(\xi)=1$ and

$$\int_K q^2(x)\,d\mu = \int_K \frac{K_m(x,\xi)K_m(x,\xi)}{K_m^2(\xi,\xi)}\,d\mu = \frac{K_m(\xi,\xi)}{K_m^2(\xi,\xi)} = \frac{1}{K_m(\xi,\xi)} = \lambda_m(\xi).$$

Now set $r(x) := p(x)q(x)$ and note that $\deg(r) \le n+m$. Then

$$p^2(\xi) = r^2(\xi) = \Big(\int_K K_{n+m}(x,\xi)r(x)\,d\mu\Big)^2 \le \int_K K_{n+m}^2(x,\xi)\,d\mu \times \int_K r^2(x)\,d\mu = K_{n+m}(\xi,\xi) \times \int_K r^2(x)\,d\mu.$$

But, as $\deg(r^2) \le 2(n+m)$,

$$\int_K r^2(x)\,d\mu = \sum_{i=1}^s w_i r^2(x_i) = \sum_{i=1}^s w_i p^2(x_i)q^2(x_i) \le \|p\|_{X_{n+m}}^2 \sum_{i=1}^s w_i q^2(x_i) = \|p\|_{X_{n+m}}^2 \int_K q^2(x)\,d\mu = \|p\|_{X_{n+m}}^2 \lambda_m(\xi).$$

Combining these estimates we obtain

$$p^2(\xi) \leq K_{n+m}(\xi,\xi)\lambda_m(\xi)\|p\|^2_{X_{n+m}}. \ \blacksquare$$

PROPOSITION 2.3. *Let $K \subset \mathbb{R}^d$ be a compact set and μ a measure on K whose support is determining for $\mathbb{P}^d(K)$, and fix $\alpha > 1$. Then the set $\mathcal{T}_{2\alpha n}$ of Tchakaloff quadrature points for μ of degree at most $2\alpha n$ is a norming set for $\mathbb{P}^d_n(K)$, with cardinality not exceeding $N_{2\alpha n}$ and norming constant*

$$C_n = \sqrt{\max_{x\in K} K_{\alpha n}(x,x) \cdot \max_{x\in K} \lambda_{(\alpha-1)n}(x)}. \tag{2.4}$$

Proof. Let $m := (\alpha-1)n$ so that $n+m = \alpha n$ and take the set $X_{n+m} = \mathcal{T}_{2\alpha n}$ which gives a positive quadrature formula for degree $2(n+m) = 2\alpha n$ of cardinality at most $N_{2\alpha n}$. Then the result follows immediately from Lemma 2.2. $\blacksquare$

We can now give some corollaries on the construction of Tchakaloff polynomial meshes on compact sets satisfying certain geometric regularity conditions on the boundary. All the asymptotic bounds are intended as $n \to \infty$ for fixed d. The first concerns the sphere, where a similar result has been proved by means of Fekete points (cf. [3, 18]).

COROLLARY 2.4. *Let $K = S^{d-1}$ be the unit sphere in $\mathbb{R}^d$. Then the sequence $\{\mathcal{T}_{2\alpha n}\}$ of Tchakaloff point sets for the standard surface measure $d\sigma$ is an optimal admissible polynomial mesh.*

Proof. The proof is immediate on recalling that

$$N_n = \binom{n+d}{d} - \binom{n-2+d}{d} \sim \frac{2n^{d-1}}{(d-1)!}$$

and the well-known property $K_n(x,x) = N_n/\omega_{d-1}$ for every $x \in S^{d-1}$, where ω_{d-1} is the surface area of the sphere (cf. [26]). Then Theorem 2.1 and Proposition 2.3 give $\mathrm{card}(\mathcal{T}_{2\alpha n}) \leq N_{2\alpha n} = \mathcal{O}(N_n)$ and $C_n = \sqrt{N_{\alpha n}/N_{(\alpha-1)n}} = \mathcal{O}(1)$. $\blacksquare$

The following result, which concerns $d\mu = dx$ (the Lebesgue measure) and fat compact sets with some geometric regularity, is substantially based on the upper and lower bounds for Christoffel functions obtained in [12]; see also [8] for a general discussion of the functional inequalities involved.

We say that K is a *UIBC set* (satisfies the Uniform Interior Ball Condition) if every point of K belongs to a ball of fixed radius contained in K. This is equivalent to saying that there is a ball of fixed radius that can roll along the boundary of K remaining in K. This holds for example on smooth convex bodies, by the celebrated Rolling Ball Theorem (cf. [15]). We stress however that such a property does not require convexity or smoothness, for example inward corners or even inward cusps are allowed (consider e.g. the union of two overlapping disks or a cardioid in $\mathbb{R}^2$).

On the other hand, K is a *UICC set* (satisfies the Uniform Interior Cone Condition) if every point of K belongs to a fixed suitably rotated cone contained in K. This property is fulfilled on domains with a Lipschitz boundary and holds for example for any convex body; for such geometric properties of compact sets we refer the reader, e.g., to [10].

COROLLARY 2.5. *Let $K \subset \mathbb{R}^d$ be a fat compact set (the closure of a bounded open set) and $\{\mathcal{T}_{2\alpha n}\}$ a sequence of Tchakaloff point sets for the Lebesgue measure on K (whose cardinalities do not exceed $N_{2\alpha n} = \mathcal{O}(n^d)$ by Theorem 2.1). Then*

- *if K is a UIBC set, then $\{\mathcal{T}_{2\alpha n}\}$ is a weakly-admissible polynomial mesh with $C_n = \mathcal{O}(\sqrt{n})$;*
- *if K is a UICC set, then $\{\mathcal{T}_{2\alpha n}\}$ is a weakly-admissible polynomial mesh with $C_n = \mathcal{O}(n^{d/2})$.*

Proof. First, we recall that with the Lebesgue measure, on any compact set we have $\max_{x\in K} \lambda_n(x) = \mathcal{O}(n^{-d})$, by [12, Theorem 6.3]. On the other hand, in [12] it is proved that on compact UIBC sets the bound $\max_{x\in K} K_n(x,x) = \mathcal{O}(n^{d+1})$ holds, which by Proposition 2.3 gives $C_n = \sqrt{\mathcal{O}(n)} = \mathcal{O}(\sqrt{n})$, whereas on compact UICC sets we have $\max_{x\in K} K_n(x,x) = \mathcal{O}(n^{2d})$, which gives $C_n = \sqrt{\mathcal{O}(n^d)} = \mathcal{O}(n^{d/2})$. ∎

It is worth recalling that the existence of weakly-admissible Tchakaloff-like polynomial meshes with $C_n = \mathcal{O}(n^{d/2})$ and $\mathcal{O}(n^d)$ cardinality was proved in [29] by a fully discrete version of the Tchakaloff Theorem, on any compact set admitting an optimal admissible polynomial mesh. The existence of optimal polynomial meshes on UICC compact sets is not known in general, and has been proved only in special cases and conjectured for convex bodies (cf. [16]).

On the other hand, in some sense the result for UIBC compact sets is weaker than what is known for example on general smooth compact bodies, where it is proved constructively that an optimal admissible polynomial mesh always exists (cf. [20]). However, the effective computation of such a mesh does not appear to be an easy task, whereas the computation of Tchakaloff quadrature points is much easier, whenever a starting algebraic quadrature formula is known (as happens with the Lebesgue measure for many classes of compact sets). Indeed, the computation of Tchakaloff points of degree n starting from a known quadrature formula with $M > N_{2\alpha n}$ nodes can be treated as a quadratic or even as a linear programming problem with $N_{2\alpha n}$ constraints and M variables, which (taking for example $\alpha = 2$) can be solved effectively by standard algorithms at least in low dimension d

and for moderate degrees n. We refer the reader, e.g., to [22, 27] for the computational aspects concerning Tchakaloff points.

3. The pluripotential equilibrium measure. For any (regular) compact set $K \subset \mathbb{C}$ there is a distinguished measure for which the associated $K_n(x,x)$ (and hence also $\lambda_n(x)$) has a special behaviour. Indeed, it is known [1, Thm. B] that (under certain conditions), for $d\mu$ a probability measure supported on K,

$$\lim_{n\to\infty} \frac{1}{N_n} K_n(x,x)d\mu(x) = d\mu_K(x) \qquad \text{weak-}^*,$$

where $d\mu_K$ is the so-called *pluripotential equilibrium measure* for K (see e.g. the monograph [14] for an introduction to this subject). Of special note is the fact that for $K = [-1,1] \subset \mathbb{C}$, it is known that

$$d\mu_K = \frac{1}{\pi} \frac{1}{\sqrt{1-x^2}} dx,$$

the so-called *Chebyshev measure.*

In particular if $d\mu$ is chosen to be the equilibrium measure $d\mu_K$ itself, then it follows that

$$\lim_{n\to\infty} \frac{1}{N_n} K_n(x,x)d\mu_K(x) = d\mu_K(x) \qquad \text{weak-}^*, \tag{3.1}$$

and hence

$$\lim_{n\to\infty} \frac{1}{N_n} K_n(x,x) = 1 \qquad \text{weak-}^*.$$

In special cases (see e.g. [5]) it is known that the limit (3.1) is actually pointwise (for x in the interior of K). Moreover, when $K \subset \mathbb{R}^d$ is a ball, simplex or cube, the equilibrium measure is explicitly known, and this can be exploited to show that for $d\mu = d\mu_K$ there are constants $C_1, C_2 > 0$ such that

$$C_1 N_n \le K_n(x,x) \le C_2 N_n, \qquad x \in K. \tag{3.2}$$

From (3.2) and Proposition 2.3 we then immediately obtain

Corollary 3.1. *Let $K \subset \mathbb{R}^d$ be either a ball, a simplex or a cube. Then the sequence $\{\mathcal{T}_{2\alpha n}\}$ of Tchakaloff point sets for the equilibrium measure $d\mu_K$ is an optimal admissible polynomial mesh.*

We now proceed to prove the bounds (3.2) for K a ball, simplex or cube.

3.1. The ball. Here $K = B^d = \{x \in \mathbb{R}^d : \|x\|_2 \le 1\}$ and its equilibrium measure is

$$d\mu_K = c_d \frac{1}{\sqrt{1-\|x\|_2}} dx$$

where the constant c_d is chosen so that $d\mu_K$ is a probability measure.

LEMMA 3.2 (cf. [19, Prop. 5.9]). *There are constants $C_1, C_2 > 0$ such that (for the equilibrium measure)*

$$C_1 N_n \le K_n(x,x) \le C_2 N_n, \qquad x \in B^d.$$

Proof. We will make use of the intimate connection between B^d and the sphere $S^d := \{x \in \mathbb{R}^{d+1} : \|x\|_2 = 1\}$, of which B^d is the principal diameter. Indeed, any function $f : B^d \to \mathbb{R}$ can be "lifted" to a function on S^d by setting, for $x \in B^d$ and $z \in [-1,1]$ such that $\|x\|_2^2 + |z|^2 = 1$, $f(x,z) := f(x)$. Further,

$$\int_{B^d} f(x)\, d\mu_K(x) = \frac{1}{2} \int_{S^d} f(x,z)\, d\sigma(x,z)$$

where $d\sigma$ is surface area on S^d, normalized to be a probability measure. Hence, if $\{p_1(x), \ldots, p_{N_n(B^d)}(x)\}$ is an *orthonormal* basis, with respect to the measure $d\mu_K$, for $\mathbb{P}_n^d(B^d)$, it is also an *orthogonal* set, considered as a subset of $\mathbb{P}_n^d(S^d)$, with respect to the measure $d\sigma$ with

$$\|p_j\|_{L^2(S^d)} = \sqrt{2}.$$

It follows that, for $x \in B^d$,

$$K_n(x,x) = \sum_{j=1}^{N_n(B^d)} p_j^2(x) \le 2K_n((x,z),(x,z)) = 2N_n(S^d) \le 4N_n(B^d)$$

where, by an abuse of notation, $K_n((x,z),(x,z))$ denotes the diagonal of the reproducing kernel for S^d. Hence the upper bound in (3.2) follows with (the non-optimal) $C_2 = 4$.

The proof of the lower bound is a bit more delicate. We use the optimality property of K_n,

$$K_n(x,x) = \max_{p \in \mathbb{P}_n^d(K),\, p(x)=1} \frac{1}{\int_K p^2(x)\, d\mu(x)},$$

so that, for any particular $p \in \mathbb{P}_n^d(K)$ with $p(x) = 1$,

$$K_n(x,x) \ge \frac{1}{\int_K p^2(x)\, d\mu_K(x)}. \tag{3.3}$$

Indeed, by [7, Lemma 3], for every $a \in B^d$ and integer $n > 0$, there exists a "peaking" polynomial $p_a(x)$ of degree at most n with the properties that

1. $p_a(a) = 1$,
2. there is a constant c_1 such that $\max_{x \in B^d} |p_a(x)| \le c_1$,
3. there is a constant c_2 such that

$$|p_a(x)| \le \frac{c_2}{n^{d+1}} \operatorname{dist}(a,x)^{-(d+1)}.$$

Here, for $a, b \in B^d$, $\mathrm{dist}(a,b)$ is the geodesic distance on the sphere S^d between the lifted points $\tilde{a} := (a, \sqrt{1-\|a\|_2^2}) \in S^d$ and $\tilde{b} := (b, \sqrt{1-\|b\|_2^2}) \in S^d$. Using this $p_a(x)$ in (3.3) results in

$$K_n(a,a) \geq \frac{1}{\int_{B^d} p_a^2(x)\, d\mu_K(x)} = \frac{2}{\int_{S^d} p_a^2(x)\, d\sigma(x)}.$$

We claim that there is a constant C_2' such that

$$\int_{S^d} p_a^2(x)\, d\sigma(x) \leq C_2' n^{-d},$$

from which the lower bound follows.

Now to see this we split the domain into two parts: where $\mathrm{dist}(\tilde{a}, x) \leq 1/n$ and where $\mathrm{dist}(\tilde{a}, x) > 1/n$. Firstly, we have

$$\int_{\mathrm{dist}(\tilde{a},x)\leq 1/n} p_a^2(x)\, d\sigma(x) \leq c_1^2 \,\mathrm{vol}(\{x \in S^d : \mathrm{dist}(\tilde{a}, x) \leq 1/n\}) \leq c_3 n^{-d} \tag{3.4}$$

for some constant c_3.

The second part is more delicate, yet completely elementary. We use spherical coordinates with "north pole" at $\tilde{a}$ and let $\phi \in [0, \pi]$ denote the angle between $x \in S^d$ and the axis through the pole, so that

$$\mathrm{dist}(\tilde{a}, x) = \phi \quad \text{with} \quad \cos(\phi) = \tilde{a} \cdot x.$$

We claim that there is a constant c_4 such that

$$\int_{\mathrm{dist}(\tilde{a},x)\geq 1/n} p_a^2(x)\, d\sigma(x) \leq c_4 n^{-d}.$$

To see this, first notice that, by property 3 of $p_a(x)$ it is sufficient to prove that

$$\int_{\epsilon \geq \mathrm{dist}(\tilde{a},x)\geq 1/n} p_a^2(x)\, d\sigma(x) \leq c_4 n^{-d}$$

for any fixed $\epsilon > 0$. Then, as $d\sigma(x) = \sin(\phi)^{d-1} d\sigma'(\theta)$, where $d\sigma'(\theta)$ represents the part of $d\sigma(x)$ coming from the angles other than ϕ,

$$\begin{aligned}
\int_{\epsilon \geq \mathrm{dist}(\tilde{a},x)\geq 1/n} p_a^2(x)\, d\sigma(x) &\leq c_4 \frac{1}{n^{2d+2}} \int_{1/n}^{\epsilon} \frac{1}{\phi^{2d+2}} \phi^{d-1}\, d\phi \\
&= c_4 \frac{1}{n^{2d+2}} \int_{1/n}^{\epsilon} \phi^{-(d+3)}\, d\phi \\
&= c_4 \frac{1}{n^{2d+2}} \frac{1}{d+2} \{n^{d+2} - \epsilon^{-(d+2)}\} \leq c_4 n^{-d}
\end{aligned}$$

(where the meaning of c_4 changes slightly on the last line). ∎

3.2. The simplex. Here $K = T^d = \{x \in \mathbb{R}^d : x_i \geq 0, \sum_{i=1}^d x_i \leq 1\}$ and its equilibrium measure is

$$d\mu_K = c_d \frac{1}{\sqrt{x_1 \cdots x_d(1 - \sum_{i=1}^d x_i)}} dx$$

where the constant c_d is chosen so that $d\mu_K$ is a probability measure.

This case is also intimately related to the sphere S^d by setting, for $x \in T^d$,

$$\tilde{x} := \Big(\sqrt{x_1}, \ldots, \sqrt{x_d}, \sqrt{1 - \sum_{i=1}^d x_i}\Big) \in S^d$$

and its inverse, for $\tilde{x} \in S^d \subset \mathbb{R}^{d+1}$,

$$x = (\tilde{x}_1^2, \tilde{x}_2^2, \ldots, \tilde{x}_d^2) \in T^d.$$

We remark that the equilibrium measure for T^d is just the pullback of surface area on S^d under the mapping $x \mapsto \tilde{x}$.

Lemma 3.3. *There are constants $C_1, C_2 > 0$ such that* (*for the equilibrium measure*)

$$C_1 N_n \leq K_n(x,x) \leq C_2 N_n, \qquad x \in T^d.$$

Proof. To see the upper bound just note that if $\{p_1(x), \ldots, p_{N_n(T^d)}(x)\}$ is an *orthonormal* basis for $\mathbb{P}_n^d(T^d)$ with respect to the measure $d\mu_K$, and we define

$$q_j(\tilde{x}) := p_j(x) = p_j(\tilde{x}_1^2, \ldots, \tilde{x}_d^2),$$

then $\{q_1(\tilde{x}), \ldots, q_{N_n(T^d)}(\tilde{x})\}$ is an *orthogonal* set, considered as a subset of $\mathbb{P}_{2n}^{d+1}(S^d)$, with respect to the measure $d\sigma$ with norm a certain dimensional constant. It follows that, for $x \in T^d$,

$$K_n(x,x) = \sum_{j=1}^{N_n(T^d)} p_j^2(x) = \sum_{j=1}^{N_n(T^d)} q_j^2(\tilde{x}) \leq cK_{2n}(\tilde{x},\tilde{x}) = cN_{2n}(S^d) \leq c' N_n(T^d)$$

for some constant c'. Here, by an abuse of notation, $K_{2n}(\tilde{x},\tilde{x})$ denotes the diagonal of the reproducing kernel for S^d.

The proof of the lower bound is very similar to the proof of the case of the ball. Indeed, [7, Lemma 4] provides, for each $a \in T^d$, a "peaking" polynomial $p_a(x)$ with the same properties as for the ball. We suppress the details. ■

3.3. The cube. Here $K = [-1,1]^d$ for which the equilibrium measure is the product measure

$$d\mu_K(x) = \frac{1}{\pi^d} \prod_{i=1}^d \frac{1}{\sqrt{1 - x_i^2}} dx.$$

Lemma 3.4. *There are constants* $C_1, C_2 > 0$ *such that* (*for the equilibrium measure*)

$$C_1 N_n \le K_n(x,x) \le C_2 N_n, \qquad x \in [-1,1]^d.$$

Proof. This follows easily from the fact that

$$\prod_{i=1}^{d} K_{n/d}(x_i) \le K_n(x,x) \le \prod_{i=1}^{d} K_n(x_i)$$

where $K_m(x_i)$ denotes the *univariate* kernel of degree m for the univariate equilibrium measure for $[-1,1]$. ∎

Remark 3.5. It is an interesting problem to determine for which compact sets K the property (3.2) holds. Given the weak convergence indicated in (3.1) we suspect that it holds for quite a general class of K. Indeed, Kroó and Lubinsky [17, Lemma 5.2] show that it holds locally in the interior for quite general K. We would also mention that (3.2) holds trivially for the sphere when one considers it as a compact subset of its complexification, in which case the equilibrium measure is just normalized surface area (cf. Corollary 2.4).

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