



## UvA-DARE (Digital Academic Repository)

### Teacher interventions aimed at mathematical level raising during collaborative learning

Dekker, R.; Elshout-Mohr, M.

**Publication date**

2004

**Document Version**

Final published version

**Published in**

Educational Studies in Mathematics

[Link to publication](#)

**Citation for published version (APA):**

Dekker, R., & Elshout-Mohr, M. (2004). Teacher interventions aimed at mathematical level raising during collaborative learning. *Educational Studies in Mathematics*, 56(1), 39-65.

**General rights**

It is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), other than for strictly personal, individual use, unless the work is under an open content license (like Creative Commons).

**Disclaimer/Complaints regulations**

If you believe that digital publication of certain material infringes any of your rights or (privacy) interests, please let the Library know, stating your reasons. In case of a legitimate complaint, the Library will make the material inaccessible and/or remove it from the website. Please Ask the Library: <https://uba.uva.nl/en/contact>, or a letter to: Library of the University of Amsterdam, Secretariat, Singel 425, 1012 WP Amsterdam, The Netherlands. You will be contacted as soon as possible.

TEACHER INTERVENTIONS AIMED AT MATHEMATICAL LEVEL  
RAISING DURING COLLABORATIVE LEARNING

**ABSTRACT.** This article addresses the issue of helping students who work collaboratively on mathematical problems with the aim of raising the level of their mathematical understanding and competence. We investigated two kinds of teacher interventions aimed at helping students. The rationale of these interventions was based on a process model for interaction and mathematical level raising. One kind of interventions focused on the interaction between the students, the other – on the mathematical content of the tasks. The effects of the two kinds of interventions were investigated using a pre-test – post-test comparison of students' learning outcomes and analyzing the transcripts of students' verbal utterances and worksheets. Our analyses point to interventions focused on students' interactions as more effective in terms of students' learning outcomes. Theoretical and practical implications of the research are discussed.

**KEY WORDS:** collaborative learning, interaction, mathematical level raising, teacher intervention

1. INTRODUCTION

The potential of working in small groups is widely recognised (Webb, 1989; Yackel, Cobb and Wood, 1991; Pea, 1993; Van Boxtel, Van der Linden and Kanselaar, 1997; Van der Linden and Renshaw, 2004). There are, however, still questions about how to maximise benefits and how to prepare teachers to provide adequate tasks and help (Grugnetti and Jaquet, 1996). In our study we address the issue of how to help students through appropriate verbal interventions.

Brodie (2001) stressed the need of such interventions, because students often find it difficult to communicate with each other and might reinforce, rather than challenge each other's mathematical misconceptions. She also showed that the teacher's role in dealing with these difficulties is problematic. A main problem is that it is practically impossible for a teacher in a classroom situation to keep track of each group's work. Teacher interventions therefore tend to lack precision with the result that they interfere with ongoing thinking and learning processes of the students. Brodie showed how the negative effects of teacher's verbal interventions can outweigh their potential positive effects. The findings were all the more noteworthy



because the teacher was experienced and the content of her help was based on sound principles, namely Lakatos' theory about learning mathematics by a series of conjectures, attempts at proof and refutations.

The possibility that we want to explore here is that an intervention may be more effective if it focuses on interactions between students than if it has to do with the content of the given tasks.

## 2. THEORETICAL FRAMEWORK

Collaborative learning tasks are in general designed as complex, challenging and authentic problems. Such problems motivate students to attempt different strategies and co-construct and justify solutions (Cohen, 1994; Elshout-Mohr and Dekker, 2000; Kramarski, Mevarech and Arami, 2002). Manifold cognitive abilities being needed, deficits may occur in the available cognitive and social abilities. Help can be offered in several ways. While Brodie (2001) studied the effects of *hints* used by the teacher based on her knowledge of appropriate methods to solve the *mathematical* problems at hand, others investigated the effects of promoting the use of more general problem solving strategies. Kramarski et al. (2002), for instance, performed a study in which help was provided in the form of a *general problem solving strategy* consisting of four steps, namely comprehension, connection, strategy selection and reflection. Students were instructed to let their own problem solving process be guided by self-addressed *metacognitive* questions such as: "How is this task different from/ similar to what I have already solved?" and "What strategy/tactic/principle can be used and why?" The students in Kramarski et al.'s study worked in small groups and had to take turns in asking and answering the metacognitive questions. After an initial instruction phase, the teachers' role was restricted to encouraging students to adhere to the procedure. Positive effects of the intervention were observed for both lower and higher achievers.

Help may also be directed at the way students communicate while working on the task at hand. Sfard (2001) argued that communication is at the heart of mathematics education and learning in groups. According to her view teachers should strive for initiating students into a certain well-defined discourse, characterized by symbolic artefacts as its communication tools and by meta-rules that regulate communication. Establishment of clear classroom norms such as advocated by Wood (1999) with regard to mathematical communication can be seen as part of this initiation. An example of a symbolic artefact as a communication tool may be the use of words that refer to mathematical concepts, such as 'rotation' and 'translation'. An example of a meta-rule that may be incorporated in

a classroom norm might be that it is quite right for a student to say “I disagree” and to bring alternative problem solutions or strategies to the fore. Wood (1999) showed positive effects of norm related learning, such as learning to participate in disagreement and argument.

Until now, to our knowledge, there is little empirical research aimed at comparing the effectiveness of teacher interventions focused on mathematical content and product of students’ work, and those that focus on meta-rules for the collaborative process. In the present study such comparison will be made within the framework of our earlier research on interaction and mathematical level raising.

### 3. MATHEMATICAL LEVEL RAISING

Raising students’ mathematical level is a major aim of mathematics education. The term ‘level’ refers here to the theory of Van Hiele (1986), who distinguished three levels of mathematical understanding and competence. The first level is a prescientific perceptual (visual) level dominated by *concrete operations*. The second level is a conceptual (descriptive) level dominated by the use of mathematical *concepts and the mutual relations between these concepts*. The third level is a theoretical level dominated by *formal operations on mathematical concepts and mathematical principles*. When mathematical concepts such as rotation and translation are introduced in the classroom, these concepts are not entirely new to the students. Students are already acquainted with relevant phenomena in daily life, which they may have investigated, for instance, in the concrete context of tiling. It is new for them, however, that rotation and reflection are conceptualized by mathematicians as mathematical transformations, and eventually as elements of a group structure. For the present study the transition between the first and second level is the most relevant. Level raising within this range is characterised by growing competence in discerning *aspects* of transformations (as concrete operations) and *application of descriptive* knowledge, for instance in solving construction and reconstruction problems.

Mathematical level raising processes have been researched by Freudenthal (1978, 1991), Dekker (1991), and Dekker and Elshout (1998) in the context of small heterogeneous learning groups. Dreyfus and his colleagues (Kieran and Dreyfus, 1998; Hershkowitz et al., 2001) have also studied these processes. They refer to level raising as ‘abstraction in context’ and ‘an activity of vertically reorganising previously constructed mathematics into a new mathematical structure’. Although solid evidence is still

scarce, it is plausible that working together in small groups is a facilitating factor.

#### 4. INTERVENTIONS IN THE PROCESS OF MATHEMATICAL LEVEL RAISING

##### 4.1. *The process model*

Dekker and Elshout-Mohr (1998) modelled processes of mathematical level raising and thereby created a framework for helping students during the process. In the model three types of activities are discerned, namely key activities, regulating activities and mental or cognitive activities. For the present study the regulating and key activities are very important. These activities are presented in Figure 1.

| <i>Regulating activities</i>        | <b>Key activities</b>              |
|-------------------------------------|------------------------------------|
| <i>A asks B to show his work</i>    | <b>B shows his own work</b>        |
| <i>A asks B to explain his work</i> | <b>B explains his own work</b>     |
| <i>A criticizes B's work</i>        | <b>B justifies his own work</b>    |
| <i>A rejects B's justification</i>  | <b>B reconstructs his own work</b> |

Figure 1. Regulating and key activities for interaction and mathematical level raising.

There are four regulating activities in the model: asking to see someone's work (*what are you doing? what have you got?*), expressing the wish to understand (*why are you doing that? how did you get that?*), uttering criticism (*but that's wrong, because. . .*) and uttering rejection of justification (*no, it isn't right, because. . .*). These activities are called regulating because they regulate (monitor, control) the activities of the other student(s) (see the second column). When a what-question is asked, students attempt to **show** and verbalise what they are doing. When a how- or why-question is asked, they attempt to **explain** their operations, constructions and mathematical reasoning. When challenged to answer criticism they attempt to **justify** their actions and reasoning. Finally, when the justification is insufficient or unconvincing, they attempt to **reconstruct** their work. These activities are key activities, because they keep the interaction going and at the same time

help the students to make progress in developing their mathematical competences. A detailed description of the model can be found in Dekker and Elshout-Mohr (1998). Here, we just want to add that it is quite common that students who perform regulating activities, such as asking for an explanation or offering criticism, subsequently take an active part in attempts to provide an explanation or to counter the critique. In other words, the model is compatible with collaborative processes such as co-construction and negotiation of meaning.

#### 4.2. *Two kinds of help*

In the present study we compare two kinds of teacher interventions aimed at helping students in accomplishing collaborative tasks. We call one of them 'process help' and the other 'product help'.

##### *Process help*

The process help capitalizes on the regulating and key activities performed by the students themselves (see Figure 1). We reasoned that the process of mathematical level raising would benefit from interventions that stimulate students to perform these activities in a consistent manner. Whenever students feel that they are insufficiently informed about the work of other students, that more explanation is needed, and so on, they should say so. Also students should take any regulating activity of other students seriously. They should perform the corresponding key activities or at least attempt to do so. Basically, the process-help interventions address a specified set of interaction norms. They are not concerned with students' reasoning and products, but with their interactions.

We expect that it is not difficult for the teachers to engage in this type of interventions even if they have to assist several groups of students simultaneously: groups where students refrain from regulating activities or corresponding key activities symptoms are easily recognizable. Students, who do not express themselves when they feel insufficiently informed, or do not understand how and why the work is done and justified, will show disinterest, unrest and unease. The same symptoms are to be expected when regulating activities are disregarded and remain unanswered by relevant key activities.

Theoretically, process help is an interesting variation on the intervention used in the research of Kramarski et al., because the aim of the help is not to improve the *problem solving process* but the *interaction process*. It is also a variation on Wood's studies on the effects of establishing classroom norms, because the norms in our study are derived from the process model for mathematical level raising, rather than from a more general theory

about the learning of mathematics. Therefore the norms are relatively clear and restricted in number, which justifies the expectation that they can be established and sustained effectively, even in a brief experiment.

### *Product help*

The second kind of interventions that we want to investigate is complementary to the process-help interventions in the sense that these interventions are concerned not with the students' interaction but with their mathematical reasoning and products. They aim at providing what we have called *product help*. Teachers engaging in product-help interventions play the role of flexible part-time assistants for the collaborating students. They may perform regulating activities, such as asking the students to explain and justify their work. They may provide hints and scaffolding when key activities become (too) difficult for the students. Product-help interventions are expected to be within the reach of teachers, especially when students' collaborative work is clearly observable on neatly arranged worksheets.

Theoretically, this type of help is interesting because it is somewhat different from the help in Brodie's study. The help is purely supportive and does not have the additional aim to initiate students to a specific problem solving strategy, such as the Lakatosian conjecture – proof – refutation sequence.

We expected that the process-help interventions would be the most effective for mathematical level raising. If implemented efficiently, these interventions are likely to promote students' collaborative work along the lines of the process model. The process-help interventions do not provide hints and scaffolding that risk to interfere with students' own mathematical reasoning. The other side of the coin, however, is that tasks that are the most likely to bring about mathematical level raising, are complex and hard to solve. Therefore some product-help interventions might be more beneficial than disturbing for the solution progress.

## 5. THE EXPERIMENTAL STUDY

### 5.1. *Design*

Two groups of students participated in the experiment in two different teacher-intervention conditions, a product-help condition and a process-help condition. Each student took a pre-test first, and then the students worked on a series of problems in triples under one of the help conditions. Finally the students took a post-test.

### 5.2. *Participants*

The students and their teachers were already familiar with working in small groups, as an alternative to whole class lessons and individual work. The students, aged 16 to 17, were pupils from two classes in a high school. Fifteen students worked in the product-help condition, and 20 students worked in the process-help condition. The numbers were not equal because of an unplanned absence of some of the students. In the product-help condition a school mathematics teacher provided the help. In the process-help condition the interventions were made by one of the researchers. Students were informed beforehand about the kind of help they could expect.

The mathematics teacher was informed in an interview about the kind of help he was supposed to provide. The main instruction for the teacher in the product condition was that hints and scaffolding had to be concerned with mathematical content and strategies. Even in situations, in which teachers would normally provide process help, he was asked to refrain from this kind of help for the sake of the experiment. Being used to collaborative learning as instructional arrangement, the teacher habitually limited himself to hints, avoided direct instructions or lengthy explanations, and gave help only when this was manifestly needed. In sum, it was agreed that the 'product-help teacher' was to offer content help, to refrain from process help, to restrict content help to situations where a need for help was manifest in students' behaviour or work, and to inform students beforehand about the kind of help that they could expect.

The process help was provided by one of the investigators. We are aware that this may have influenced the results, but we preferred this solution over the alternative of training a mathematics teacher in offering a kind of help that s/he was not familiar with and that was not expected of him/her by the students. The process help was based on the assumption that students might need help to regulate key activities by adequate regulating activities in the manner presented in Figure 1. It was agreed that the 'process-help teacher' was to encourage students to engage in active showing, explaining, justifying and reconstructing their work, and in giving comments (questions, critique) that would trigger these key activities in others. Further, she had to refrain from mathematical content help, to restrict help to situations where a need for help was manifest in the students' behaviour, and to inform students beforehand about the kind of help that they could expect.

### 5.3. *Experimental tasks*

A series of four sets of problems concerning the transformations of reflection, rotation, translation, and glide reflection was constructed for the



students to work on. These problems explicitly aimed at raising students' mathematical understanding and competence.

For the first set of problems, each small group of students received the following materials:

- one handout with examples of the four transformations (see Figure 2);
- two cards, one with a symmetry drawing of Escher in colour and one with a fragment of Alhambra tiling;
- one little bag with red and blue triangles;
- one large sheet of paper with some information about Escher and Alhambra and a lot of space to work.

On the large sheet of paper, titled 'Escher', the following problems were formulated:

Here you see a drawing of Escher and a fragment of Alhambra tiling. Which transformations can you discover?

Make your own 'Escher' with the red and blue triangles and try to apply the four transformations.

Make a sketch of your design on this sheet and show which transformations you have applied.

The problems were developed in a previous study (see Pijls, 1996). One of the outcomes of the study was that, for the students, the problems were *realistic* and *meaningful*. Analyzing transformations in a drawing of Escher makes sense both for students who do not think in terms of the four transformations given in Figure 2 and for students who are already familiar with mathematical transformations. The problems were *complex* and *constructive*, allowing several approaches and starting points resulting in visible actions and constructions. The *level raising* aspect was brought about by a conjunction of several factors: the realistic, complex and constructive nature of the problems, the educational setting, the sheet with examples of the four transformations, and the content of subsequent problems. In the handout the transformations were offered as *visual objects*. The activities of collaboratively analyzing the tiling and constructing the design imply that the students work with the transformations, discuss them and discover many properties of them. According to the level theory of Van Hiele (1986) this means that the students start working on *the visual level* on which concrete objects are subject to analysis and gradually proceed towards *the descriptive level* on which the properties of the objects are subject to analysis.

The first set of problems, titled 'Escher', has an open character in the sense that the design can be more or less complicated.

The second set of problems, titled 'Driehoekje leggen' [Laying a little triangle], requires students to discover and reconstruct the transformations

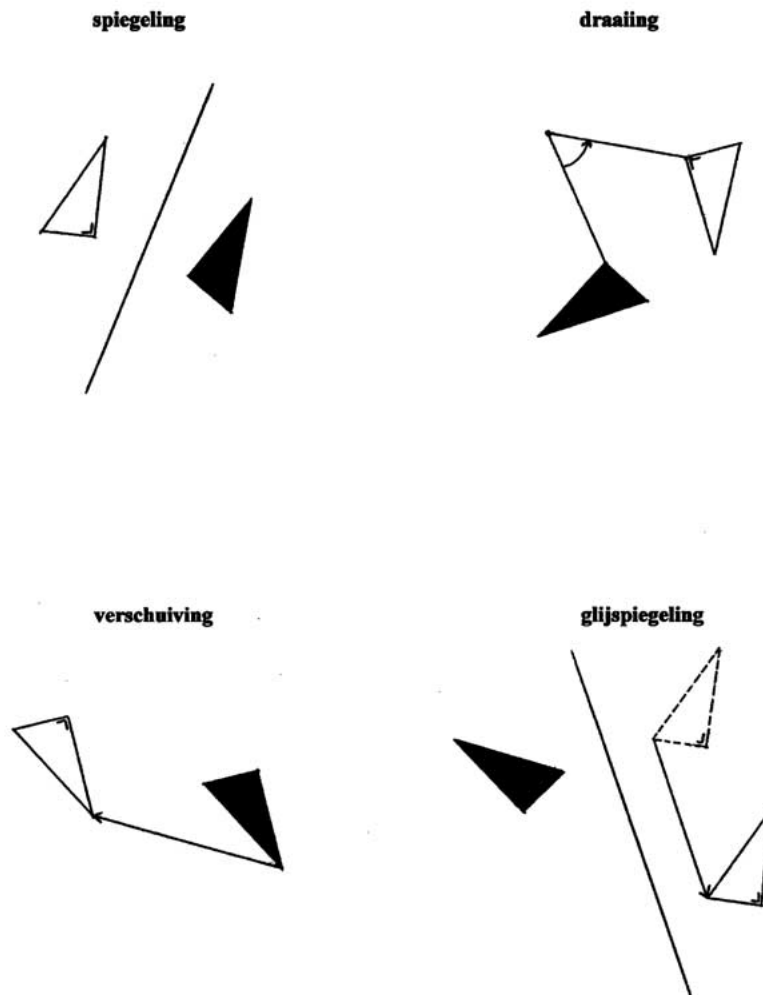
**Meetkundige afbeeldingen**

Figure 2. Sheet with the four transformations [meetkundige afbeeldingen]: reflection [spiegeling], rotation [draaiing], translation [verschuiving] and glide reflection [glijspiegeling].

in a given design, indicating points and angles of rotations, translation arrows, and reflection axes (see Figure 3):

In the design below you can put each time the little black triangle with just one *transformation* on a little white triangle. Can you discover with which one? Reveal the construction of the design by indicating (glide) reflection axes, translation arrows, rotation points and rotation angles as accurately as possible.

These problems are more restrictive in the sense that only one solution is correct, and students are forced to be more explicit in discussing the properties of the transformations.

In the third set of problems students work with mirrors and analyze relations between reflections, rotations and translations and the special role of reflections in these relations. Those problems aim at deeper understanding of the transformations, their properties and their being part of a more complex mathematical concept.

The fourth group of problems aims at reasoning on a higher level. Students have to find out in a game how many transformations are needed to put a blue triangle on a red one of equal shape and size and they have to justify their findings.

#### 5.4. *Pre- and post-test*

A pre- and a post-test were constructed to measure the results of students' learning. The tests consisted of different items, but were parallel in relevant aspects. A student's mathematical-level score was operationalized as the total score of points awarded for the following performances:

- *recognizes* reflection, rotation, translation, and glide reflection (maximum 4 points),
- *constructs* and *reconstructs* a reflection, rotation, translation, and glide reflection (maximum 8 points),
- *applies knowledge about properties of* reflection, rotation, translation and glide reflection (maximum 6 points),
- *recognizes relations between* reflection, rotation, translation, and glide reflection (maximum 2 points),
- *applies knowledge about (relations between)* reflection, rotation, translation, and glide reflection (maximum 5 points).

Maximum total scores were 25 for both pre- and post-test.

Students who functioned at a purely visual level could achieve a maximum score of 4 points. Scores from 5 to 18 points could be achieved by students who had partly to fully mastered knowledge about mathematical transformations on a conceptual (descriptive) level. Scores from 18

### Driehoekje leggen

In het onderstaande ontwerp kun je het zwarte driehoekje met telkens één meetkundige afbeelding op een wit driehoekje leggen. Kun je ontdekken met welke? Onthul de constructie van het ontwerp door (glij)spiegelassen, verschuifpijlen, draaipunten en draaihoeken zo nauwkeurig mogelijk weer te geven.

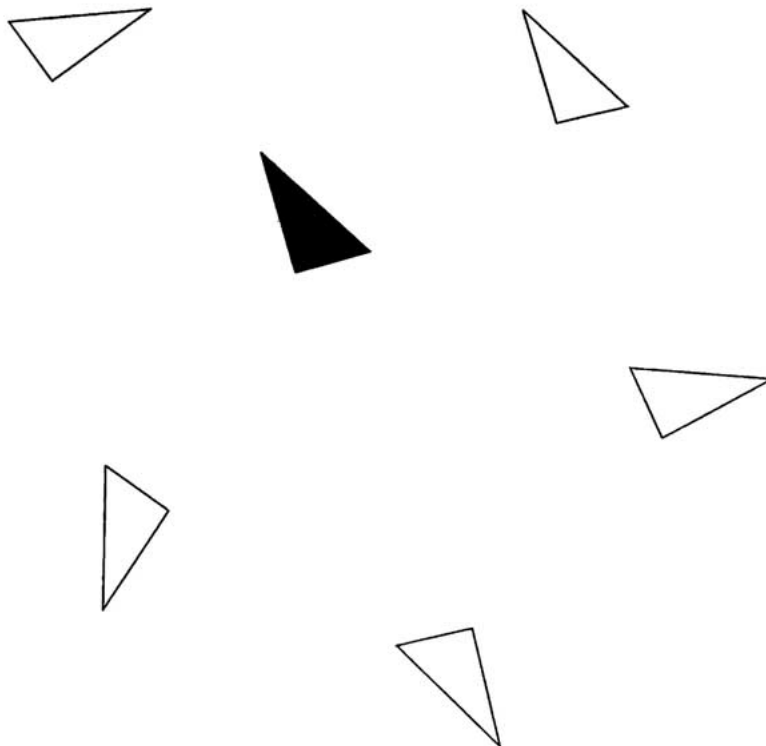


Figure 3. Driehoekje leggen [Laying a little triangle].

TABLE I

Means and standard deviations of pre- and post-test scores

|                        | pre-test                | post-test               |
|------------------------|-------------------------|-------------------------|
| product help<br>(n=15) | mean 12.20<br>s.d. 1.93 | mean 13.13<br>s.d. 2.17 |
| process help<br>(n=20) | mean 11.90<br>s.d. 2.45 | mean 14.45<br>s.d. 2.93 |

to 25 points could be achieved by students who functioned (partly) on the theoretical mathematical level.

### 5.5. *Experimental procedure*

Two classes participated in the experiment. During the pre-test the students could use the handout with the four transformations (Figure 2). The information on the handout allowed them to recall prior knowledge and to demonstrate their level justly. On the basis of the pre-test results students in the two classes were divided into two comparable subgroups, each taken in charge by one of the two instructors. The product-help teacher and the process-help teacher then divided their subgroup into heterogeneous triples and an occasional couple on the basis of the pre-test scores.

During one week the students worked in two sessions of 65 minutes on the four sets of problems on transformations. Worksheets of each group were collected. The students made a post-test shortly after the last session *without* the help of the handout with the four transformations. During the post-test the handout was not needed for recall, and we wanted to prevent students from using it and thus continue learning in uncontrolled ways. Discussions between students and interventions by the teachers were tape-recorded during the lessons. The audiotapes were then transcribed verbatim in preparation for a qualitative analysis.

### 5.6. *Hypotheses*

The first hypothesis was that pre- and post-test differences would be larger in the process-help condition than in the product-help condition. The second hypothesis was that teacher interventions would interfere less with the students' interaction and learning processes in the process-help condition than in the product-help condition.

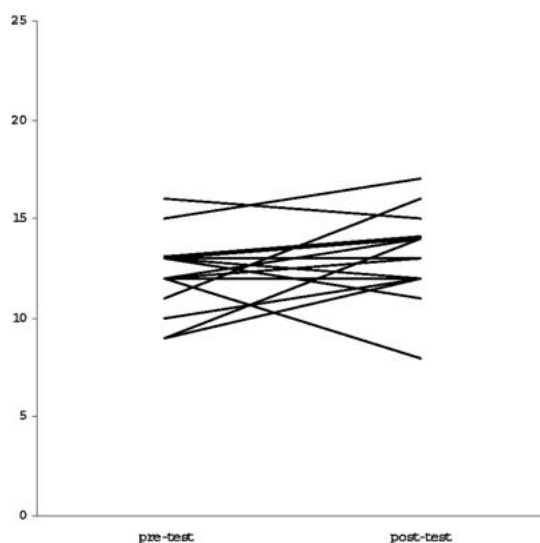


Figure 4a. Graphs of the pre- and post-test scores of the students in the product-help condition.

## 6. COMPARISON OF THE PRE- AND POST-TEST SCORES

The means and standard deviations of the pre- and post-test scores of 35 students in two different conditions are presented in Table I.

Analysis of the pre-test scores showed that differences in pre-test scores between the two groups were not significant. Evidently, the attempts to form experimental groups with comparable pre-test scores were successful. The hypothesis about the post-test scores was that these would be higher in the process-help condition than in the product-help condition. This hypothesis was confirmed ( $p < .05$ ). Students' pre- and post-test scores are presented in a graphical form in Figures 4a and 4b. These figures reveal that students' progress in the product-help condition was more heterogeneous than in the process-help condition. In the product-help condition, for instance, more fallbacks were found than in the process-help condition.

Regarding the question of whether the level raising occurred or not, it should be noticed that students scored well above 4 points in the pre-test. Thus, they must have already had at least some conceptual knowledge about one or more transformations. It remains to be seen in the qualitative analysis for which transformations, if any, students truly raised their level from a purely visual to a more conceptual one.

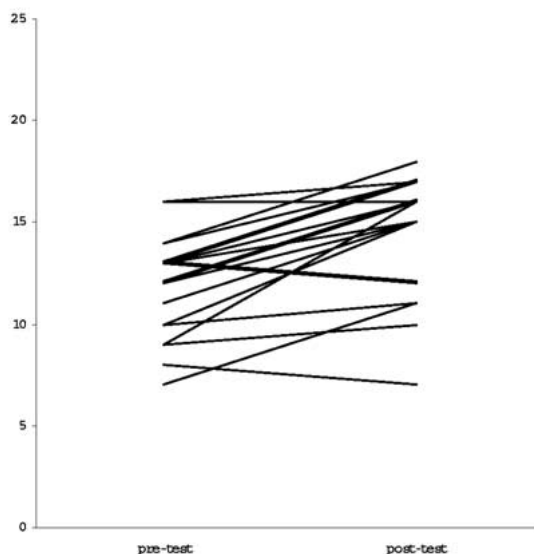


Figure 4b. Graphs of the pre- and post-test scores of the students in the process-help condition.

## 7. ANALYSIS OF THE TRANSCRIBED EXCHANGES

### 7.1. Implementation of the help

Part of the teacher help concerned organisational activities that were the same for both conditions, such as distributing worksheets and asking students to put their names on their work. This kind of help was not analyzed any further. The second category concerned all interventions whereby teachers demonstrated how they had implemented the help conditions. The data showed that both teachers were reticent in offering help and tried not to disturb students' discussions. In sheer numbers of utterances during the series of lessons, the product teacher's utterances (73) exceeded the number of those made by the process teacher (21), but teacher interference was still very limited. We will describe the roles of the teachers in more detail for two triples.

### 7.2. Two triples

Two triples were selected for further investigation. The first triple: Maaïke, Stefan and Rolf worked in the product-help condition. The second triple: Jelmer, Rafik and Thomas worked in the process-help condition. Pre- and post-test scores of the students are presented in Table II.

TABLE II

Pre- and post-test scores of the product and the process triple

|        | pre-test | post-test |
|--------|----------|-----------|
| Maaike | 15       | 17        |
| Stefan | 12       | 8         |
| Rolf   | 9        | 14        |
| Jelmer | 14       | 17        |
| Rafik  | 13       | 17        |
| Thomas | 9        | 16        |

The two triples were representative of the whole group in the following aspects: Pre-test scores in both triples were not different from average; average post-test scores were higher than pre-test scores and average learning gains were higher in the process triple than in the product triple. The fallback of one of the students (Stefan) in the product triple is also representative. Figures 4a and 4b showed more fallbacks in the product condition than in the process condition.

A first point of interest was how the two kinds of help were implemented. A second issue was when, how and why teacher interventions facilitated or hindered students in their collaborative work and learning. While investigating the second issue, we also address the question whether students merely extended higher order mathematical knowledge that they already possessed, or truly transformed lower-level knowledge into higher-level knowledge.

### 7.2.1. *Two episodes*

Two episodes related to the mathematical questions in the second set of problems were analysed in depth. The problems were concerned with the construction and the reconstruction of rotations. The episodes were selected because the protocols and the students' worksheets showed that this subject matter was highly challenging for the students. Moreover, the achieved learning outcomes were striking in the sense that both positive and negative individual learning outcomes were found. Supposedly, these episodes would allow an interesting view on the effects of the different kinds of teacher intervention.

The episodes singled out for analysis concerned the second set of problems, titled 'Driehoekje leggen' [Laying a little triangle] (See Figure 3).



### 7.2.2. *Pre- and post-test performances related with rotations*

A close inspection of students' performance on the pre- and post-tests revealed that all three students in the product triple could recognize a rotation in the pre-test but were not able to reconstruct one. In the post-test Stefan was not able to recognize or to construct a rotation. Maaïke performed well on all items related to rotation, and Rolf still could not construct a rotation, but he was able to recognize one. Pre-test results of the process triple revealed that two of the three could not recognize a rotation and none of them could reconstruct one. In the post-test scores all three recognized and constructed rotations well. From these data it must be concluded that, in respect to rotations, all students, with the exception of Stefan, have successfully raised their mathematical level from the pre-scientific, perceptual level to a higher, conceptual level.

How they proceeded and how teacher interventions were helpful will be discussed in the following. Each episode will be first described in terms of the ongoing collaborative work. Then we submit conjectures about the manner, in which the learning of individual students might have been influenced by the available kind of teacher help. In order to better understand the description of the collaborative work and the fragments from the protocols, the work of each triple is presented in Figures 5a and 5b.

### 7.3. *An episode in the collaborative work of the product triple*

The episode (see 7.2.1) was focused on the reconstruction of the two rotations in the design: a regular rotation which puts the little black triangle upon the white one below to the left, and the half turn (rotation by an angle of 180 degrees), which puts the black triangle on the lowest white one (see Figure 5a). The students roughly proceeded as follows. During the reconstruction of the regular rotation Maaïke (M) was looking for the rotation angle by constructing the angle between two corresponding sides. She criticized her own construction, however, because the angle was not positioned at the rotation point. She wanted to revise her solution by creating something 'more technical', as she expressed it herself. Rolf (R) proposed to connect the right angles with a line and draw the perpendicular bisector of that line with the same length of the line. That would give the rotation point. Stefan (S) agreed. So did Maaïke, but she was hesitant. They applied the same strategy to the reconstruction of the half turn. Maaïke followed again, because she wanted the group to work consistently, but then she started to criticize the strategy.

The transcribed protocol for the episode reads as follows:

Maaite, Stefan, Kolf

### Driehoekje leggen

In het onderstaande ontwerp kun je het zwarte driehoekje met telkens één meetkundige afbeelding op een wit driehoekje leggen. Kun je ontdekken met welke? Onthul de constructie van het ontwerp door (glij)spiegelassen, verschuifpijlen, draaipunten en draaihoeken zo nauwkeurig mogelijk weer te geven.

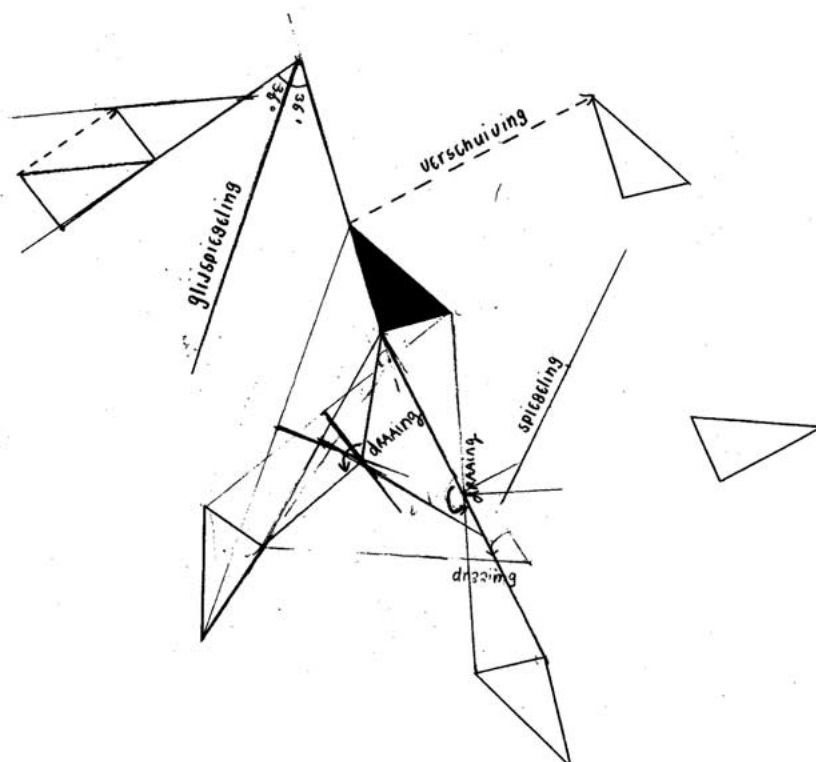


Figure 5a. Product of the product triple, working on 'Driehoekje leggen' [Laying a little triangle].

- 1 M: yes but the angle always comes out the same because if this is the same length as this you get this is always the same size and that's wrong
- 2 R: yes that's exactly the idea
- 3 M: no because then a rotation is always with the same angle
- 4 S: no because it depends how long this is
- 5 M: yes but
- 6 S: if this is longer
- 7 M: you always make this the same length as this according to your rule
- 8 S: yes
- 9 M: and then you always get the same angle  
[...]
- 10 S: no you always get the same angle if these two lie exactly like this
- 11 M: mm
- 12 S: I think if they lie differently then this angle will be
- 13 M: yes but what you can it doesn't make any difference at all between which point you do it? With everything you get something else different. . .

This transcript evidently represents an interactive dialogue of the type described in Figure 1. A variety of regulating and key processes can be identified. Maaïke criticizes the construction of Stefan and Rolf by showing and explaining that if the connecting line is as long as the perpendicular bisector, the rotation angle is constant (1). Rolf justifies the construction by saying that both lines are supposed to be equally long (2). Maaïke indicates again that in that case the angle is the same (3). Stefan criticizes her and states that it depends on the length (which is not true) (4). Maaïke repeats her critique by arguing that the angle is always the same (5, 7, 9). Stefan seems to check this and agrees with Maaïke (10). He is working on the construction (12) and this stimulates Maaïke to consider the other angles as well (13). [Does she want to show that the rule of Stefan and Rolf applied to other angles gives other rotation points? A definite critique of the rule of Stefan and Rolf? Or is she looking for a reconstruction of the construction they already have on paper?]

Maaïke continues to connect other related angles and to indicate the midpoints, although she says that she doesn't know why. Stefan and Rolf follow her. All three are looking for a better solution, which is closer than they realize, because in the 'official' construction of the rotation point one

can find the rotation point as the intersection point of the perpendicular bisectors of the connecting lines of the corresponding angles.

Then the product teacher (CT) comes along and Maaïke addresses him:

- 1 M: this rotation here this is wrong. . .  
[. . .]
- 2 M: how can you calculate a rotation
- 3 CT: well I see that at least you have worked with the angles of the right-angled triangle
- 4 R: yes
- 5 CT: you've done something with that, but you know yourselves that that is not enough because if this was rotated a bit for example and this was also rotated a bit then you would get another rotation, wouldn't you? So you shouldn't only work with the angles of the right-angled triangle but you have to work with two other angles as well
- 6 M: yes you have here those middle points, what do you do with those then
- 7 CT: yes yes you've got you've got it that rotation point lies on the perpendicular bisector of this line segment

In this second part of the episode, we see how Maaïke asks CT for an explanation (2). CT looks at their work and tells them how they may have been thinking, according to him (3). He expresses the same critique as Maaïke had expressed before and he confirms that one has to work with the other angles as well (5). Maaïke shows what they did with that and asks for an explanation (6). CT guesses that they are already aware of the 'official' construction (which is not correct!). His interpretation goes too far (7) and the students don't follow him anymore. Maaïke accepts his suggestions, but she is unsure and continues to ask CT for help in the minutes that follow. In the end CT sees no other way than to tell her the construction. The construction is put on paper correctly and applied well with the half turn too (see Figure 4a). But the subsequent transcript shows that especially Stefan participated less in the collaborative work than before the triple received help. It was interesting to note that the effect was a lasting one over the lessons. In the discussion about rotation in the next lesson, over the third set of problems, Stefan participated with only 3 utterances, while Maaïke contributed 17 utterances and Rolf 15.

#### 7.4. *An episode in the collaborative work of the process triple*

Before we describe the related episode of the process triple, we want to present how the process teacher provided help to the triple before the start of the episode.

##### 7.4.1. *The process teacher*

The process teacher (PT) clarified to the whole group of students her role and expressed her expectations at the beginning of the first lesson:

- PT: What I expect. . . I am not going to help you with the content  
 T: no  
 PT: but I do want you to discuss a lot  
 R: yes  
 PT: to show each other your work, to give each other explanations, that's what makes you learn  
 R: yes  
 PT: to give critic to each other, so that the work improves

The protocol shows that the process teacher explicitly stimulates the key activities of showing and explaining and the regulating activity of criticizing. With the latter she implicitly stimulates the process of justifying and reconstructing. The inbetween remarks of Thomas and Rafik reveal that they will try to apply the norms for collaborative work formulated by the teacher.

After the introduction, the students worked collaboratively and discussed a lot, taking equal turns in the conversation. At some point they criticized each other's work and had trouble reaching an agreement:

- T: yes  
 R: you did it first like this  
 J: stop it now, man  
 T: we had to talk a lot, so  
 R: okay  
 J: we now have to

In this fragment we see that the moment Jelmer wants to stop the discussion, Thomas refers to the expectations of the teacher and the students continue their discussion and talk through their point of disagreement.

Some time later Jelmer and Thomas discussed closely together and Rafik didn't seem to participate. The process teacher noticed this and addressed Rafik:

- PT: Rafik, do think aloud  
R: what I am thinking? O yes  
PT: if you are puzzling over something  
R: no  
PT: speak, say so

Here, the process teacher tries to stimulate Rafik to present his ideas and voice his criticism. And although Rafik denies that he is puzzling over something, the interruption of the process teacher is a stimulus for him to participate in the discussion again and in the follow up he does it by revealing his ideas.

The students continue to work collaboratively. At some point the discussion was intense between Rafik and Jelmer. Thomas seemed to drop out. The process teacher noticed this and addressed him:

- PT: Is it okay what they are doing, Thomas?  
R: do you get it?  
J: you just have to  
T: yes, I do get it  
J: with those lines  
R: this is what you see

The process teacher tries to make Thomas participate in the discussion again by stimulating him to look critically at the other students' work. The teacher's action makes Rafik aware that Thomas was not participating and his guess is that it was because he didn't understand it. Jelmer starts to explain and although Thomas says that he does understand, Rafik starts to explain, too. From then on, Thomas becomes again an active participant in the discussion.

#### 7.4.2. *The episode*

Now we turn to the episode focused on the reconstruction of the rotation and the half turn in the process group (see Figure 5b). In the first phase of the episode, Jelmer, Rafik and Thomas did not manage to reconstruct the regular rotation. Therefore they switched to the reconstruction of the half turn. They found it by connecting the two related right angles and indicated the rotation point in the middle of the connecting line.

Rafik and Thomas then wanted to apply the same construction to the reconstruction of the regular rotation, but Jelmer did not agree:

*Jelmer, Thomas, Rafik*

### Driehoekje leggen

In het onderstaande ontwerp kun je het zwarte driehoekje met telkens één meetkundige afbeelding op een wit driehoekje leggen. Kun je ontdekken met welke?  
 Onthul de constructie van het ontwerp door (glij)spiegels, verschuifpijlen, draaipunten en draaihoeken zo nauwkeurig mogelijk weer te geven.

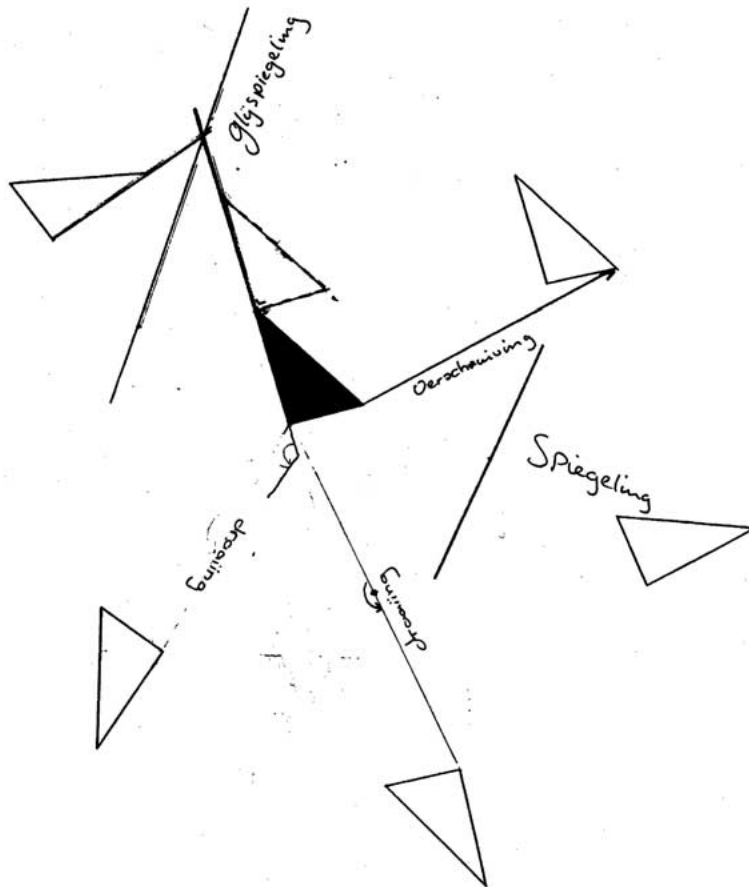


Figure 5b. Product of the process triple, working on 'Driehoekje leggen' [Laying a little triangle].

- 1 J: this isn't right, you know  
2 T: it is  
3 J: it has to be it just has to be it can't be an angle just 180 degrees  
4 T: huh?  
5 J: in that case it won't work out right  
6 R: no this one is wrong see  
7 J: look this line has to be yes it has to go through that thing at exactly the same point say  
8 R: yes that's what I'm saying too  
9 J: yes  
10 T: that's not so with this one either is it?  
11 R: that's what I'm saying the whole time  
12 J: yes this one goes along there exactly  
13 R: yes  
14 J: and this one goes like that along there too so so that's right  
15 R: that's why I was just saying this a moment ago  
16 J: that's not right  
17 R: because this isn't right  
18 J: try something again?  
[...]  
19 T: it's definitely a rotation but

A variety of regulating and key processes can be identified here. Jelmer criticizes the solution of Rafik and Thomas (1) by saying that the rotation angle should not be 180 degrees (3) and by explaining that if it were the case, the connecting line of the two other corresponding angles would cross the rotation point as well (7), a property of the half turn. The line does not do that, so it is not a half turn. Thomas criticizes Jelmer's argument by demonstrating that the half turn they just made, does not have that property either. But Jelmer justifies his reasoning by demonstrating that with the half turn, the other connecting lines do cross the rotation point (12, 14) and he repeats his critique (16). Rafik seems to be convinced and Thomas starts to hesitate (19). In this phase of the episode, the process triple started to reconstruct their prior solution. The students knew how to find the right angle of rotation by constructing the angle between two corresponding sides (the same original construction as Maaïke in the product triple had found), but they could not figure out where the rotation point is laying, and left it at that. They did not even take the time to adequately 'translate' their discussion into a corresponding reconstruction on their work sheet. Therefore, their learning process was not adequately reflected in the quality of



the product (see Figure 5b). In the next lesson this group discussed rotation again, and did that extensively (more extensively than the product triple). Utterances were equally spread over the participants. Jelmer contributed 34 utterances about rotation, Rafik 19 and Thomas 28.

Although no process help was actually provided in this fragment, we wish to argue that the students were still affected by the previous interventions. Students' interactive discourse was in line with the process model for interaction and mathematical level raising and students refrained from seeking the help of the teacher.

## 8. SUMMARY

The pre-test – post-test comparison revealed that students in the process-help condition raised their mathematical level more than students in the product-help condition. Moreover, product help tended to lead to divergence rather than convergence of student's gain scores, while process help had the opposite effect.

The analysis of transcripts showed that the two experimental instructions were implemented according to the plan. The students were indeed working under help conditions that differed in being focused on mathematical content and students' interactive discourse, respectively. Both conditions allowed students to raise their mathematical level on the subject matter of mathematical transformations. A count of teachers' interventions revealed that the collaboration was predominantly teacher-independent in both groups. The series of problems provided sufficient challenge and guidance for the students. Constructions based on perceptual conceptions of rotation were gradually transformed into constructions based on conceptions of aspects of the transformation, such as angle and rotation point. In both groups, too, we saw that the students were not able to complete the attempted construction. In the product-help group, one student asked the teacher for help. In the process-help group students decided to go on to the next problem.

There were also differences to be observed between the two help conditions. In the product-help condition we found a number of interrelated phenomena. First, the teacher, like the teacher in Brodie's study, was sometimes insufficiently informed about the students' work to intervene appropriately. Second, the teacher often communicated with just one student, often after having addressed the whole group first. Third, the teacher sometimes could not prevent a switch from the role of 'assistant' into the role of full participant, i.e. a person who actually performed a mathematical operation instead of just giving a hint. In conjunction and in the long

run, these phenomena may increase the tendency of one student asking the teacher for help and decrease the level of engagement of another.

In the process-help condition we observed the following phenomena. First, the teacher offered a substantial part of her contribution to the collaborative work at the very beginning of the lesson. Thus, interference during the process was minimized. Second, by its very nature, process help seemed to address more than one student. When the teacher addressed one student (“Is it okay what they are doing?”) another student took over (“Do you get it?”) and a third one attempted to give an explanation (“You just have to”). By addressing a student who fails to execute a regulating activity, the process teacher also raises awareness of all students of their role in the collaborative process. In the process-help group, too, we saw the long-term effects of teacher interventions. After the starting phase the students worked as a group, were not looking for teacher help, and rarely tended to drop out of the conversation.

We can conclude that our two hypotheses were confirmed: pre-test – post-test differences were larger in the process-help condition than in the product-help condition and teacher interventions interfered less with the students’ interaction and learning processes in the process help condition than in the product-help condition.

## 9. DISCUSSION

More research is needed before we can generalize the conclusion that process help is to be preferred over product help. Our research is restricted to one mathematical topic, which is treated during a few lessons, and to one step in the process of mathematical level raising. A relatively small number of students (35) participated in the study and the transcripts of discussions in only two groups were analyzed in depth.

Aspects of the setting that may have influenced our findings are the use of carefully designed materials and the fact that the students were used to working together in small groups. Both kinds of help were given by teachers who were well informed and capable of providing the intended kind of help.

One thing to investigate, for instance, is what it takes for a mathematics teacher who usually assists students in matters of mathematical content as well as collaborative discourse to take on the role of a process teacher. Another interesting question for further research is whether the relative success of the process help can be attributed to the strict adherence of the process teacher to the key and regulating activities for interaction and

mathematical level raising (see Figure 1), or similar results can be reached with an alternative selection of rules for the interactive discourse.

#### REFERENCES

- Brodie, K.: 2001, 'Teacher intervention in small-group work', *For the Learning of Mathematics* 20(1), 9–16.
- Cohen, E.G.: 1994, *Designing Groupwork: Strategies for Heterogeneous Classrooms*, Teacher College Press, New York.
- Dekker, R.: 1991, *Wiskunde leren in Kleine Heterogene Groepen* [Learning Mathematics in Small Heterogeneous Groups], Academisch Boeken Centrum, De Lier.
- Dekker, R. and Elshout-Mohr, M.: 1998, 'A process model for interaction and mathematical level raising', *Educational Studies in Mathematics* 35(3), 303–314.
- Dekker, R., Elshout-Mohr, M. and Wood, T.: (2004), 'Working together on assignments: multiple analysis of learning events', in J. van der Linden and P. Renshaw (eds.), *Dialogic Learning*, Kluwer Academic Publishers, Dordrecht, pp. 145–170.
- Elshout-Mohr, M. and Dekker, R.: 2000, 'Opdrachten om samen van te leren [Assignments for collaborative learning]', in J. van der Linden and R. Roelofs (eds.), *Leren in Dialoog*, Wolters-Noordhoff, Groningen, pp. 35–63.
- Freudenthal, H.: 1978, *Weeding and Sowing*, Reidel, Dordrecht.
- Freudenthal, H.: 1991, *Revisiting Mathematics Education*, Kluwer Academic Publishers, Dordrecht.
- Grugnetti, L. and Jaquet, F.: 1996, 'Senior secondary school practices', in A. Bishop, K. Clements, C. Keitel, J. Kilpatrick and C. Laborde (eds.), *International Handbook of Mathematics Education*, Part 1, Kluwer Academic Publishers, Dordrecht, pp. 615–645.
- Hershkowitz, R., Schwarz, B.B. and Dreyfus, T.: 2001, 'Abstraction in context: epistemic actions', *Journal for Research in Mathematics Education* 32(2), 195–222.
- Kieran C. and Dreyfus, T.: 1998, 'Collaboration versus individual problem solving: entering another's universe of thought', in A. Olivier and K. Newstead (eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education*, Vol. 3, Stellenbosch, South Africa, pp. 112–119.
- Kramarski, B., Mevarech, Z.R. and Arami, M.: 2002, 'The effects of metacognitive instruction on solving mathematical authentic tasks', *Educational Studies in Mathematics* 49(2), 225–250.
- Pea, R.D.: 1993, 'Learning scientific concepts through material and social activities: conversational analysis meets conceptual change', *Educational Psychologist* 28(3), 265–279.
- Pijls, M.H.J.: 1996. *Spiegeltaal* [Mirror language], Universiteit van Amsterdam, Faculteit WINS, Amsterdam.
- Sfard, A.: 2001, 'There is more to discourse than meets the ears: Looking at thinking as communicating to learn more about mathematical learning', *Educational Studies in Mathematics* 46(1), 13–57.
- Van Boxtel, C., Van der Linden, J.L. and Kanselaar, G.: 1997, 'Collaborative construction of conceptual understanding: interaction processes and learning outcomes emerging from a concept mapping and poster task', *Journal for Interactive Learning Research* 8(3/4), 341–361.
- Van der Linden, J. and Renshaw, P. (eds.): (2004), *Dialogic Learning*, Kluwer Academic Publishers, Dordrecht.

- Van Hiele, P.M.: 1986, *Structure and Insight*, Academic Press, Inc., Orlando.
- Webb, N.M.: 1989, 'Peer interaction and learning in small groups', *International Journal of Educational Research* 1, 21–39.
- Wood, T.: 1999, 'Creating a context for argument in mathematics class', *Journal for Research in Mathematics Education* 30(2), 171–191.
- Yackel, E., Cobb, P. and Wood, T.: 1991, 'Small-group interactions as a source of learning opportunities in second-grade mathematics', *Journal for Research in Mathematics Education* 22(5), 390–408.

*Universiteit van Amsterdam,  
Graduate School of Teaching and Learning,  
Wibautstraat 2–4,  
1091 GM Amsterdam,  
The Netherlands,  
E-mail: r.dekker@uva.nl*

